LU based Beamforming schemes for MIMO systems

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***Abstract*—We present a time-domain broadband beamforming based on a Unimodular-Upper polynomial matrix decomposition. The unimodular factor is the product of elementary J-orthogonal matrices and a lower triangular matrix with ones on the diagonal, as in the constant matrix LU decomposition. This leads to a J-Orthogonal LU polynomial matrix decomposition, as a combination of two classical matrix factorization methods: Smith canonical form and LU Gaussian elimination. The inversion of the unimodular factor, for use as a pre/post filter in the beamforming scheme, is immediate and can be achieved with O(1) complexity. The resulting reduced MIMO channel is exactly diagonal, leading to separate single input single output (SISO) channels with no co-channel inteference. There is no need to model the MIMO channel as Laurent polynomial as usual, thus introducing unnecessary delays just for technical reasons. In addition, it turns out that each of the resulting SISO channels, except the last one, reduces to a simple additive noise channel, with no intersymbol interference, except for unprobable original MIMO channels. However, these very interesting features are to be balanced with the possible noise enhancement in the postfiltering step. The performance in terms of bit-error rate is studied and compared to the QR-based frequency-domain and time-domain broadband beamforming. In particular, the proposed beamforming scheme can be used both in OFDM and in single carrier MIMO systems, without cyclic prefix. Meanwhile, the QR based scheme requires a cyclic prefix extension.**

I. Introduction

The combination of Multiple-Input Multiple-Output (MIMO) and Orthogonal Frequency Division Multiplexing (OFDM) technologies (MIMO-OFDM) is now adopted in several communication standards, including the 5^{th} generation of mobile communication network (5G) [1], the IEEE WLAN 802.11ac [2] and the IEEE 802.16 standards (WiMax) [3].

On the one hand, OFDM is a worthwhile tradeoff between bit-error rate performance and spectral efficiency. OFDM consumes part of the channel bandwidth, but it is robust to frequency selective fading environment. In addition, it enables the use of several advanced technics to further enhance the system throughput, as for instance the bit loading technic [4] and the subcarriers allocation in orthogonal frequency division multiple access (OFDMA) [5].

On the other hand, the MIMO system has the potential to improve the system capacity. There are several MIMO transmission technics including beamforming, cyclic shift diversity, space time block/treillis coding, etc. Among these technics, only the beamforming enables to reach the highest throughput [6]. However, in the beamforming scheme, the MIMO channel matrix has to be diagonalized in space in order to eliminate the co-channel interferences (CCI). This is achieved for example when a singular value decomposition (SVD) is considered.

For many telecommunications standards, the frequency-domain broadband beamforming (FBBF) is adopted [1]–[3]. In FBBF, the diagonalization of the system is done in the frequency domain for each subcarrier. Nevertheless, when the system has a large number of subcarriers, which can reach 512 in 802.11ac or even 2048 in $5G$ and mobile WiMax, the use of the FBBF technic may be of high complexity due to the pre- and post-filtering on each subcarrier.

An alternative is to consider time-domain broadband beamforming (TBBF) for which the diagonalization of the temporal MIMO channel can be performed once for the entire system. This can be achieved by using QR polynomial matrix SVD (QR-PMSVD) [7], [8], [9] (see also [10] where a time domain decomposition is obtained from pointwise frequency domain SVD) or Jacobi's method as the second order sequential best rotation algorithm in [11]. However, since a polynomial matrix SVD with polynomial factors does not exist in general [12], the QR-PMSVD can not completely eliminate the CCI. See [13] for a study on the effect of the residual CCI.

In order to exactly cancel the CCI in TBBF, we consider here a J-orthogonal LU polynomial matrix decomposition (JO-LU PMD). This decomposition consists in writing a given polynomial matrix $H(z)$ in the form $H(z) = V(z)L(z)U(z)$ where all three factors are polynomial matrices such that $V(z)$ is a product of elementary J-orthogonal matrices, $L(z)$ is lower triangular with ones on the diagonal and $U(z)$ is upper triangular. The product $V(z)L(z)$ is then clearly a unimodular polynomial matrix. Therefore, the decomposition can also be seen as Unimodular-Upper polynomial matrix decomposition (UU-PMD) [14]. The JO-LU-PMD (or UU-PMD) algorithm is a direct combination of the classical Smith form and Gaussian elimination LU factorization. As is well-known, such decomposition always exists in any Bezout ring, as opposed to QR based polynomial matrix decomposition.

Note also that there is no need to consider Laurent polynomial model for the MIMO channel. Such model, which introduces additional processing delays, is as unnatural as unnecessary though it is on the basis of QR based polynomial matrix decomposition.

It is important to note that all the previously mentioned technics are used in OFDM systems. This is largely due to the fact that the channel equalization is more simple in OFDM systems than in single carrier (SC) one. In fact, unlike the SC,

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the OFDM where the maximum multipath delay is within the cyclic prefix (CP) enables, in a simple way, the ISI to be mitigated. Note that the CP is now also used in SC systems (CP-SC) to improve their robustness to multipath propagation [15]. However, the inserted CP in OFDM, as in CP-SC, results in a throughput loss. Meanwhile, the proposed decomposition is useful for both OFDM and SC systems. Indeed, the MIMO channel matrix is perfectly diagonalized in both contexts so that the CCI is completely eliminated. Let us mention that also, the residual CCI resulting from a QR based decomposition can be made as small as desired. The downside is that the degrees of the diagonal elements become very high, which results in general to more severe intersymbol interference (ISI) and requires a CP extension.

Now, we show that with the proposed decomposition, each of the resulting SISO channels, except the last one, reduces in general to a simple additive noise channel, with no ISI. This is the case except for some pathological and unprobable original MIMO channels. The proposed scheme thus enables to have an efficient MIMO-OFDM system without CP. It also enables to have a MIMO SC system with very simple equalizer.

Let us mention however that these very interesting features of the proposed method are mitigated by the potential enhancement of the noise, in the postfiltering step. This is because the postfilter is not paraunitary as in the QR-based beamforming.

The rest of this paper is organized as follows. The FBBF and TBBF methods for beamforming in MIMO-OFDM system are briefly recalled in section II. The proposed beamforming in TBBF context is presented in section III. Its performance are compared with that of the QR-based FBBF and TBBF through the simulation studies in section IV. Finally, section V shows the application of the proposed beamforming in MIMO SC.

II. Beamforming in MIMO-OFDM System

A. FBBF in MIMO-OFDM System

Consider a MIMO channel with N_t transmit and N_r receive antennas. In classical MIMO-OFDM context, for each subcarrier k, the $N_r \times 1$ received signal y_k reads as:

$$y_k = H_k x_k + n, \quad k \in \{1, \cdots, N_s\} \tag{1}$$

where H_k is the $N_r \times N_t$ narrowband frequency channel coefficients for the subcarrier k, N_s is the total number of subcarriers, $x_k = [x_{1,k}, \cdots, x_{N_t,k}]^T$ is the $N_t \times 1$ transmitted signal, n stands for the $N_r \times 1$ white gaussian noise with covariance $\mathbb{E}(nn^\dagger) = \sigma_n^2 I$ and the † stands for the transpose-conjugation. Consider a SVD of the matrix H_k: $H_k = U_k D_k V_k$, where $D_k \in \mathbb{C}^{N_r \times N_t}$ is diagonal and $U_k \in \mathbb{C}^{N_r \times N_r}$ and $V_k \in \mathbb{C}^{N_t \times N_t}$ are unitary matrices. In MIMO-OFDM with beamforming context, the transmit and receive signals are pre- and post-coded respectively into $\widehat{x}_k = [\widehat{x}_{1,k}, \cdots, \widehat{x}_{N_t,k}]^T = V_k^\dagger x_k$ and $\widehat{y}_k = [\widehat{y}_{1,k}, \cdots, \widehat{y}_{N_t,k}]^T = U_k^\dagger y_k$. Equation (1) then updates to:

$$\begin{aligned}\widehat{y}_k &= U_k^\dagger H_k \widehat{x}_k + U_k^\dagger n = U_k^\dagger U_k D_k V_k V_k^\dagger x_k + U_k^\dagger n \\ &= D_k x_k + U_k^\dagger n.\end{aligned} \tag{2}$$

Observe that the additive white noise setting is not altered by the post-coding because U_k is unitary. Since D_k is exactly diagonal, there is no CCI and the MIMO channel matrix is reduced on $\min(N_r, N_t)$ separate and independent SISO channels. In all the sequel, we assume that $N_r \geqslant N_t$.

B. TBBF in MIMO-OFDM System

Unlike the FBBF, where the post-coding and the pre-coding are carried out in the frequency domain for each subcarrier, the TBBF is performed in the time domain. Thence, we can represent the MIMO channel by

$$H(z) = \begin{bmatrix} h_{1,1}(z) & h_{1,2}(z) & \cdots & h_{1,N_t}(z) \\ h_{2,1}(z) & h_{2,2}(z) & \cdots & h_{2,N_t}(z) \\ \vdots & \vdots & \cdots & \vdots \\ h_{N_r,1}(z) & h_{N_r,2}(z) & \cdots & h_{N_r,N_t}(z) \end{bmatrix} \tag{3}$$

where each $h_{i,j}(z)$ is a polynomial representing the sub-channel from the transmit antenna i to the receive antenna j. To perform the TBBF, $H(z)$ is decomposed in the form:

$$H(z) = U(z)D(z)V(z), \tag{4}$$

where the factors $V(z)$, $D(z)$ and $U(z)$ are simple and structured polynomial matrices. Then, the inverse of the factors $U(z)$ and $V(z)$, noted by $U_{po}(z)$ and $V_{pr}(z)$, are used as post- and pre-coders respectively in order to reduce the MIMO channel to a simpler form $D(z)$.

Denote by $x_i(z)$, $1 \leqslant i \leqslant N_t$, the polynomial of degree $N_s + CP - 1$, representing the z-transform of the OFDM time symbol with cyclic prefix on the transmit antenna i. These OFDM symbols are grouped before and after the pre-coder respectively in the vectors $x(z) = [x_1(z) \ \cdots \ x_{N_t}(z)]^T$ and $\widehat{x}(z) = [\widehat{x}_1(z) \ \cdots \ \widehat{x}_{N_t}(z)]^T = V_{pr}(z)x(z)$. At the receiver side, the corresponding signal $\widehat{y}(z) = [\widehat{y}_1(z) \ \cdots \ \widehat{y}_{N_r}(z)]^T$, after the post-coding, can be expressed as:

$$\begin{aligned}\widehat{y}(z) &= U_{po}(z)H(z)\widehat{x}(z) + U_{po}(z)n(z) \\ &= U_{po}(z)U(z)D(z)V(z)V_{pr}(z)x(z) + U_{po}(z)n(z) \\ &= D(z)x(z) + U_{po}(z)n(z)\end{aligned} \tag{5}$$

where $n(z)$ is the z-transform of the sample realization of the additive white Gaussian noise. Ideally, $D(z)$ has the form

$$D(z) = \begin{bmatrix} d_{1,1}(z) & & \\ & \ddots & \\ & & d_{N_t,N_t}(z) \\ \hdashline & \mathbf{0}_{N_r-N_t,N_t} & \end{bmatrix}$$

and in this case, (5) gives:

$$\widehat{y}_i(z) = d_{i,i}(z)x_i(z) + \widehat{n}_i(z), \quad i = 1, \cdots, N_t \tag{6}$$

where $\widehat{n}(z) = [\widehat{n}_1(z) \ \cdots \ \widehat{n}_{N_r}(z)]^T = U_{po}(z)n(z)$. Thereby, the MIMO channel is transformed into *i)* N_t separate and independent SISO channels, so that there is no CCI and *ii)* $N_r - N_t$ remaining noise-only channels due to the fact that $\widehat{y}_i(z) = \widehat{n}_i(z)$ for $i = N_t + 1, \cdots, N_r$.

The channel matrix decomposition represented in (4) can be preformed by using a QR-PMSVD [7] or the method proposed in section III below (see also [14]). Let us briefly recall the QR-PMSVD method for comparison purpose. The method was

proposed by Foster, McWhirter *et al.* [7]. In the sequel, the tilde sign will denote the para-Hermitian conjugate of a matrix valued function $F(z)$, defined as $\widetilde{F}(z) \triangleq [F(1/z^*)]^\dagger$.
The algorithm begins with a first tentative of QR decomposition of $H(z)$ as:

$$\widetilde{U}_1(z)H(z) = R_1(z), \tag{7}$$

where $\widetilde{U}_1(z)$ is obtained by applying successive elementary Laurent polynomial Givens rotations to the rows of $H(z)$ to put the result $R_1(z)$ in a upper-triangular form. The Laurent polynomial matrix $\widetilde{U}_1(z)$ is paraunitary by construction *i.e.* $\widetilde{U}_1(z) = [U_1(z)]^{-1}, \ \forall z \in \mathbb{C}$. However, $R_1(z)$ is only approximately upper-triangular because polynomial matrix QR decomposition does not exist in general [12]. Next, the same decomposition is applied to the matrix $\widetilde{R}_1(z)$ and the resulting factors are denoted by $V_1(z)$ and $\widetilde{H}_1(z)$ as:

$$V_1(z)\widetilde{R}_1(z) = \widetilde{H}_1(z). \tag{8}$$

By combining (7) and (8), one obtains the result of the first iteration which reads as:

$$\widetilde{U}_1(z)H(z)\widetilde{V}_1(z) = H_1(z). \tag{9}$$

These two steps are then repeated replacing $H(z)$ with $H_1(z)$. This leads to the recursion $\widetilde{U}_k(z)H_{k-1}(z)\widetilde{V}_k(z) = H_k(z)$, which runs until the largest off-diagonal coefficient (in magnitude) of $H_k(z)$ is less than a prescribed tolerance parameter ε. In contrast, the orders of the polynomial matrices $U_k(z)$, $V_k(z)$ and $H_k(z)$ will grow fast at each iteration. A truncation step with a parameter μ is introduced in order to limit this growth. At convergence, say at iteration $k = K$, one gets a factorization of $H(z)$ as in (4), with $D(z) = H_K(z)$, $U(z) = U_1(z)U_2(z)\cdots U_K(z)$ and $V(z) = V_K(z)\cdots V_2(z)V_1(z)$.

It is important to note that:

- despite the truncation step, the orders of the three Laurent polynomial factors will still be high because the parameter μ must be small to avoid losing the paraunitary property of the matrices $U(z)$ and $V(z)$.
- the polynomial matrix $D(z)$ is only approximately diagonal. Thus, equation (6) becomes:

$$\widehat{y}_i(z) = d_{i,i}(z)x_i(z) + \sum_{j=1, j\neq i}^{N_t} d_{i,j}(z)x_j(z) + \widehat{n}_i(z). \tag{10}$$

  In that case, the CCI represented by $\sum_{j=1,i\neq j}^{N_t} d_{i,j}(z)x_j(z)$ is not completely eliminated. However, the tolerance parameter ε enables this residual CCI to be made as small as desired.

The effect of ε and the order of $D(z)$ in the system performance are studied in section IV. Note that one can also consider only one QR decomposition and settle for an equivalent channel in a triangular form rather than diagonal [16].

III. LU Beamforming

The MIMO equivalent baseband channel is represented by (3). The J-orthogonal LU decomposition follows almost the same steps as the classical LU (or Gauss elimination) factorization. However in each step, a preprocessing by the first step of the decomposition in Smith canonical form is considered. This preprocessing reduces the resulting pivot element to a constant. The procedure amounts to finding a set of elementary transformations to successively zero the elements of the columns beneath the diagonal.

To begin, consider that, after applying $k-1$ iterations, one obtains the polynomial matrix of the form

$$H_{k-1}(z) = \left[\begin{array}{cccc:ccc} d_1(z) & h_{1,2}^{(k)}(z) & \cdots & h_{1,k-1}^{(k)}(z) & h_{1,k}^{(k)}(z) & \cdots & h_{1,N_t}^{(k)}(z) \\ 0 & d_2(z) & \ddots & \vdots & \vdots & \cdots & \\ \vdots & & \ddots & h_{k-2,k-1}^{(k)}(z) & \vdots & \cdots & \\ 0 & & \ddots & d_{k-1}(z) & \vdots & \cdots & \\ \hdashline 0 & & \ddots & 0 & h_{k,k}^{(k)}(z) & \cdots & \vdots \\ \vdots & & & \vdots & \vdots & & \\ 0 & & \cdots & 0 & h_{N_r,k}^{(k)}(z) & \cdots & h_{N_r,N_t}^{(k)}(z) \end{array}\right] \tag{11}$$

Then the iteration k will follow two steps:

- a reduction step, in which the pivot is reduced to 1.
- an LU (Gaussian elimination) step to zero the elements of the column k beneath the diagonal.

A. Reduction step

We begin as in a Smith canonical form decomposition. Assume that the subchannels $\boldsymbol{h}_k(z) = [h_{k,k}^{(k)}(z) \ h_{k+1,k}^{(k)}(z)]$ linking the transmit antennas k and $k+1$ to the receive antenna k do not share any common zero. If necessary, we may replace $h_{k+1,k}^{(k)}(z)$ by another subchannel $h_{k+\ell,k}^{(k)}(z)$, as for instance the one issuing from the most distant transmit antenna from antenna k. Such operation simply amounts to applying a permutation of rows $k+1$ and $k+\ell$ in $H(z)$. This may however be insufficient, *e.g.* when the polynomials $\{h_{j,k}^{(k)}(z), j = k, \ldots, N_r\}$ do not form an irreducible set. And even though this set is irreducible, it may happen that any chosen pair of polynomials share some common zeros. Therefore, different possible scenarii need to be considered.

1) Case 1: We begin with the simplest situation where $\boldsymbol{h}_k(z) = [h_{k,k}^{(k)}(z) \ h_{k+1,k}^{(k)}(z)]$ is irreducible. Then, the Bezout equation

$$h_{k,k}^{\sharp}(z)h_{k,k}^{(k)}(z) + h_{k+1,k}^{\sharp}(z)h_{k+1,k}^{(k)}(z) = 1, \tag{12}$$

is solvable for some pair $\boldsymbol{h}_k^{\sharp}(z) = [h_{k,k}^{\sharp}(z), \ h_{k+1,k}^{\sharp}(z)]$ such that $\deg \boldsymbol{h}_k^{\sharp}(z) < \deg \boldsymbol{h}_k(z)$. Setting

$$\overline{B}_k(z) = \begin{bmatrix} h_{k,k}^{\sharp}(z) & h_{k+1,k}^{\sharp}(z) \\ -h_{k+1,k}^{(k)}(z) & h_{k,k}^{(k)}(z) \end{bmatrix} \tag{13}$$

we define the block diagonal polynomial matrix

$$\overline{A}_k(z) = \left[\begin{array}{c:c:c} I_{k-1} & & \\ \hdashline & \overline{B}_k(z) & \\ \hdashline & & I_{N_r-k-1} \end{array}\right]. \tag{14}$$

Recall that a real square matrix Q is called J-orthogonal if it satisfies $Q^tJQ = J = QJQ^t$ where J is a signature matrix.

Lemma 1. *For all z, the matrix $\overline{B}_k(z)$ in* (13) *satisfies*

$$\overline{B}_k(z)^t J \overline{B}_k(z) = J = \overline{B}_k(z) J \overline{B}_k(z)^t, \text{ with } J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \tag{15}$$

Proof: The proof is by direct verification and is left to the reader. ■

For z real, $\overline{B}_k(z)$ is thus J-orthogonal. Consequently, the matrix $\overline{A}_k(z)$ is $\widehat{J}$-orthogonal with

$$\widehat{J} = \left[\begin{array}{c:c:c} J_1 & & \\ \hdashline & J & \\ \hdashline & & J_2 \end{array}\right]$$

for any J_1 and J_2 such that $J_1^2 = I_{k-1}$ and $J_2^2 = I_{Nr-k-1}$.

Multiplying $\overline{A}_k(z)$ in the left of $H_{k-1}(z)$ (11), results in $\overline{H}_k(z) = \overline{A}_{k-1}(z)H_{k-1}(z)$ which reads as

$$\overline{H}_k(z) =$$
$$\left[\begin{array}{ccc:c:ccc} d_1(z) & \cdots & h^{(k)}_{1,k-1}(z) & h^{(k)}_{1,k}(z) & h^{(k)}_{1,k+1}(z) & \cdots & h^{(k)}_{1,N_t}(z) \\ 0 & \ddots & \vdots & \vdots & \vdots & & \vdots \\ 0 & \ddots & d_{k-1}(z) & h^{(k)}_{k-1,k}(z) & h^{(k)}_{k-1,k+1}(z) & \cdots & h^{(k)}_{k-1,N_t}(z) \\ \hdashline 0 & \cdots & 0 & 1 & \overline{h}^{(k)}_{k,k+1}(z) & \cdots & \overline{h}^{(k)}_{k,N_t}(z) \\ 0 & \cdots & 0 & 0 & \overline{h}^{(k)}_{k+1,k+1}(z) & \cdots & \overline{h}^{(k)}_{k+1,N_t}(z) \\ \hdashline 0 & & 0 & h^{(k)}_{k+2,k}(z) & h^{(k)}_{k+2,k+1}(z) & \cdots & h^{(k)}_{k+2,N_t}(z) \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & & 0 & h^{(k)}_{N_r,k}(z) & h^{(k)}_{N_t,k+1}(z) & \cdots & h^{(k)}_{N_t,N_t}(z) \end{array}\right] \tag{16}$$

Observe that $\overline{H}_k(z)$ has exactly the same form as $H_{k-1}(z)$ given in (11). The initial setting of iteration k is thus recovered, with $h^{(k)}_{k,k}(z)$ replaced by 1 and $h^{(k)}_{k+1,k}(z)$ by 0.

2) Case 2: If $\boldsymbol{h}_k(z) = [h^{(k)}_{k,k}(z)\ h^{(k)}_{k+1,k}(z)]$ is not irreducible then one may retry *Case 1* with $h^{(k)}_{k+1,k}(z)$ replaced by $h^{(k)}_{k+\ell,k}(z)$ for some $\ell > 1$, as mentioned earlier. However, this is not necessary and we consider instead, the following more pragmatic solution. Let $d_{k,1}(z)$ be the greatest common divisor (*gcd*) of the two subchannels so that we may write

$$\boldsymbol{h}_k(z) = d_{k,1}(z)[\widehat{h}^{(k)}_{k,k}(z)\ \widehat{h}^{(k)}_{k+1,k}(z)] \triangleq d_{k,1}(z)\widehat{\boldsymbol{h}}_k(z). \tag{17}$$

The solution of the Bezout equation (12) where $\boldsymbol{h}_k(z)$ is now replaced by $\widehat{\boldsymbol{h}}_k(z)$ is still denoted by $\boldsymbol{h}^\sharp_k(z)$. Let the matrix $\overline{B}_k(z)$ in (13), and incidentally $\overline{A}_k(z)$ in (14), be updated accordingly. Then, it is straightforward to see that the updated product $\overline{H}_{k,1}(z) = \overline{A}_{k-1}(z)H_{k-1}(z)$ is identical to the expression in (16) except that the value 1 of the (k,k) entry is now replaced by $\left(\overline{H}_k(z)\right)_{k,k} = d_{k,1}(z)$. The iteration then proceeds with the subchannels pair

$$\boldsymbol{h}_{k,1}(z) = [d_{k,1}(z)\ h^{(k)}_{k+2,k}(z)] \triangleq d_{k,2}(z)[\widehat{d}_{k,1}(z)\ \widehat{h}^{(k)}_{k+2,k}(z)],$$

where $\widehat{\boldsymbol{h}}_{k,1}(z) \triangleq [\widehat{d}_{k,1}(z)\ \widehat{h}^{(k)}_{k+2,k}(z)]$ is irreducible and $d_{k,2}(z)$ denotes the *gcd* of the pair.

This process is repeated, leading to sub iterations for the current iteration k, with initial setting $\overline{H}_{k,0}(z) \triangleq \overline{H}_k(z)$. The corresponding internal loop reads as follow. Given $\boldsymbol{h}_{k,n-1}(z) = d_{k,n}(z)[\widehat{d}_{k,n-1}(z)\ \widehat{h}^{(k)}_{k+n,k}(z)]$ at the sub iteration n, $1 \leqslant n \leqslant N_r - k$, we solve the Bezout equation

$$h^\sharp_{k,k}(z)\widehat{d}_{k,n-1}(z) + h^\sharp_{k,n}(z)\widehat{h}^{(k)}_{k+n,k}(z) = 1.$$

Next, the following matrix is formed:

$$\overline{A}_{k,n}(z) = \left[\begin{array}{c:c:c:c:c} I_{k-1} & & & & \\ \hdashline & h^\sharp_{k,k}(z) & & h^\sharp_{k+n,k}(z) & \\ \hdashline & & I_n & & \\ \hdashline & -\widehat{d}_{k,n-1}(z) & & \widehat{h}^{(k)}_{k+n,k}(z) & \\ \hdashline & & & & I_{N_r-k-1} \end{array}\right]. \tag{18}$$

Based on Lemma 1 and by construction, this polynomial matrix is $\widehat{J}$-orthogonal for some appropriate $\widehat{J}$, for all z real. Then we compute $\overline{H}_{k,n}(z) = \overline{A}_{k,n}(z)\overline{H}_{k,n-1}(z)$ as

$$\overline{H}_{k,n}(z) =$$
$$\left[\begin{array}{c:c:ccc} & h^{(k)}_{1,k}(z) & h^{(k)}_{1,k+1}(z) & \cdots & h^{(k)}_{1,N_t}(z) \\ G(z) & \vdots & \vdots & & \vdots \\ & h^{(k)}_{k-1,k}(z) & h^{(k)}_{k-1,k+1}(z) & \cdots & h^{(k)}_{k-1,N_t}(z) \\ \hdashline & d_{k,n}(z) & h^{(k,n)}_{k,k+1}(z) & \cdots & h^{(k,n)}_{k,N_t}(z) \\ & \mathbf{0}_n & \vdots & \cdots & \vdots \\ & & h^{(k,n)}_{k+n,k+1}(z) & \cdots & h^{(k,n)}_{k+n,N_t}(z) \\ \hdashline & h^{(k)}_{k+n+1,k}(z) & h^{(k)}_{k+n+1,k+1}(z) & \cdots & h^{(k)}_{k+n+1,N_t}(z) \\ & \vdots & \vdots & & \vdots \\ & h^{(k)}_{N_r,k}(z) & h^{(k)}_{N_t,k+1}(z) & \cdots & h^{(k)}_{N_t,N_t}(z) \end{array}\right] \tag{19}$$

where $G(z)$ is upper triangular and corresponds to the $(k-1)^{\text{th}}$ leading principal submatrix of $\overline{H}_k(z)$ in (16).

Lemma 2. *For all k and n, we have*

$$d_{k,n}(z) = gcd\left(h^{(k)}_{k,k}(z), h^{(k)}_{k+1,k}(z), \cdots, h^{(k)}_{k+n,k}(z)\right) \tag{20}$$

Proof: We have $d_{k,1}(z) = \gcd\left(h^{(k)}_{k,k}(z), h^{(k)}_{k+1,k}(z)\right)$ and $d_{k,j}(z) = \gcd\left(d_{k,j-1}(z), h^{(k)}_{k+j,k}(z)\right)$ by construction. The rest is plain. ■

The end of this internal loop is reached after $n_k \leqslant N_r - k$ sub iterations when either

Case 2-1: $d_{k,n_k}(z) = 1$, $\forall z$. The context of *Case 1* above is thus recovered with $n_k < N_r - k$.

Case 2-2: $n_k = N_r - k$.

To conclude, let us observe that the whole reduction step of iteration k can be summarized by

$$\begin{aligned} \overline{H}_k(z) &\triangleq \overline{H}_{k,n_k}(z) = \overline{A}_{k,n_k}(z)\overline{H}_{k,n_k-1}(z) \\ &= \overline{A}_k(z)H_{k-1}(z), \end{aligned} \tag{21}$$

where

$$\overline{A}_k(z) = \overline{A}_{k,n_k}(z)\overline{A}_{k,n_k-1}(z)\cdots\overline{A}_{k,1}(z). \quad (22)$$

B. Gaussian elimination-LU step

This step applies an elementary Gaussian elimination transformation to the matrix $\overline{H}_k(z)$ from the above reduction step in order to zero all the elements of column k beneath the diagonal. Obviously, the step is void for iteration k, if the reduction step ended in *Case 2-2*, with $n_k = N_r - k$. Otherwise, consider $\overline{H}_k(z) = \overline{H}_{k,n_k}(z)$, $1 \leqslant n_k < N_r - k$ in (19) with $d_{k,n_k}(z) = 1$. Since the pivot, $d_{k,n_k}(z)$, is a constant a classical LU step can be applied. This amounts to left multiplying $\overline{H}_k(z)$ by the lower triangular matrix

$$\overline{L}_k(z) = \left[\begin{array}{c:c:c} I_{k-1} & & \\ \hdashline & 1 & \\ \hdashline & \boldsymbol{f}_k(z) & I_{N_r-k} \end{array}\right] \quad (23)$$

where $\boldsymbol{f}_k(z) = \left[0 \ \cdots \ 0 \ h^{(k)}_{k+n_k,k}(z) \cdots h^{(k)}_{N_r,k}(z)\right]^t$ is the polynomial vector formed by the last $N_r - k$ entries of the k^{th} column of $\overline{H}_k(z)$. Now a direct verification shows that the result of this multiplication has the same form as $H_{k-1}(z)$ in (11), for k incremented. This ends the k^{th} iteration by setting

$$H_k(z) = \overline{L}_k(z)\overline{H}_k(z).$$

The k^{th} diagonal entry of this polynomial matrix is given by

$$d_k(z) = \gcd\left(h^{(k)}_{\ell,k}(z),\ \ell = k, k+1, \cdots, N_r\right), \quad (24)$$

as shown by Lemma 2.

The reduction and Gaussian elimination steps are repeated for all k, from $k = 1$ to $k = N = \min(N_r - 1, N_t)$.

At the last reduction step, the final matrix obtained is an upper triangular polynomial matrix which reads as:

$$H_N(z) = \overline{A}_N(z)\overline{L}_{N-1}(z)\overline{A}_{N-1}(z)\cdots\overline{L}_1(z)\overline{A}_1(z)\cdot H(z). \quad (25)$$

C. Factorisation step

To complete the decomposition, we first observe that the polynomial matrix $\overline{A}_N(z)\overline{L}_{N-1}(z)\overline{A}_{N-1}(z)\cdots\overline{L}_1(z)\overline{A}_1(z)$ appearing above is unimodular and we show that the computation of its inverse is rather straightforward. We define

$$\boldsymbol{\ell}_k(z) = \left[\begin{array}{c} \mathbf{0}_k \\ \hdashline \\ \boldsymbol{f}_k(z) \\ \\ \end{array}\right] = \left[\begin{array}{c} \mathbf{0}_{k+n_k} \\ \hdashline h^{(k)}_{k+n_k+1,k}(z) \\ \vdots \\ h^{(k)}_{N_r,k}(z) \end{array}\right]. \quad (26)$$

Then, one may readily check the following easy facts:

Lemma 3. *In the iteration k,*

1) *$\overline{L}_k(z)$ as in (23) is a lower triangular, unimodular polynomial matrix which can be expressed as*
$$\overline{L}_k(z) = I_{N_r} + \boldsymbol{\ell}_k(z)\boldsymbol{e}^t_k,$$
where $\boldsymbol{e}^t_k$ is the k^{th} vector of the canonical basis of $\mathbb{R}^{N_r}$.
2) $L_k(z) \stackrel{\triangle}{=} \overline{L}_k(z)^{-1} = I_{N_r} - \boldsymbol{\ell}_k(z)\boldsymbol{e}^t_k$.
3) $A_k(z) \stackrel{\triangle}{=} \overline{A}_k(z)^{-1} = A_{k,1}(z)A_{k,2}(z)\cdots A_{k,n_k}(z)$ *where $A_{k,m}(z) = J^t_m\overline{A}_{k,m}(z)^t J_m$, for some appropriate signature matrice J_m.*

Finally, by using the properties above and equation (25) the preceding steps lead to the factorisation

$$H(z) = U(z)H_N(z) \quad (27)$$

where

$$U(z) = A_1(z)L_1(z)\cdots A_{N-1}(z)L_{N-1}(z)A_N(z) \quad (28)$$

is a unimodular polynomial matrix. A unimodular-upper decomposition of the original polynomial matrix $H(z)$ is therefore obtained. Moreover, the unimodular factor may be further decomposed. Indeed, we have:

Lemma 4. *The unimodular matrix in* (28) *reads as:*

$$U(z) = A(z)L(z) \quad (29)$$

where $A(z)$ is the polynomial matrix given by $A(z) = A_1(z)A_2(z)\cdots A_N(z)$ and $L(z)$ is a lower triangular polynomial matrix with 1 on the diagonal.

Proof: We first establish the following relation:

$$U(z) = A_1(z)A_2(z)\cdots A_N(z) + \sum_{k=1}^{N-1}\boldsymbol{\ell}_k(z)\boldsymbol{e}^t_k \quad (30)$$

To see this, observe that for all $i \leqslant j$, $A_i(z)$ can be written in 2×2 block diagonal form with the second block given by I_{N_r-j-1}. Hence, the dimension of the first block (top-left) is $j+1$. Now, the first $j+1$ components of $\boldsymbol{\ell}_j(z)$ vanish as shown in (26). Therefore we have

$$A_i(z)\boldsymbol{\ell}_j(z) = \boldsymbol{\ell}_j(z). \quad (31)$$

For all $i \leqslant j$, we also have $\boldsymbol{e}^t_i A_j(z) = \boldsymbol{e}^t_i$ and $\boldsymbol{\ell}_i(z)\boldsymbol{e}^t_i\boldsymbol{\ell}_j(z)\boldsymbol{e}^t_j = 0$. The relation (30) then follows from (28) by replacing therein the matrices $L_i(z)$ by the expression given in Lemma 3. For the remaining, let us rewrite (30) as

$$U(z) = A(z)\{I_{N_r} + \overline{A}_N(z)\cdots\overline{A}_1(z)\sum_{k=1}^{N-1}\boldsymbol{\ell}_k(z)\boldsymbol{e}^t_k\}. \quad (32)$$

From (31), we may write $\overline{A}_N(z)\cdots\overline{A}_j(z)\boldsymbol{\ell}_i(z) = \boldsymbol{\ell}_i(z)$ for $i \leqslant j$. Now observe again that for any $i > j$, $A_i(z)$ can be written in 2×2 block with I_j in the first block. Therefore, the first j components of $\boldsymbol{\ell}'_j(z) = A_i(z)\boldsymbol{\ell}_j(z)$ vanish. This shows that $\boldsymbol{\ell}'_j(z)$ has the same structure as $\boldsymbol{\ell}_j(z)$. Hence, the first j components of the product $\overline{A}_N(z)\cdots\overline{A}_1(z)\boldsymbol{\ell}_j(z)$ vanish. Setting $\boldsymbol{\ell}'_i(z) = \overline{A}_N(z)\cdots\overline{A}_1(z)\boldsymbol{\ell}_j(z)$, we rewrite (32) as

$$U(z) = A(z)\left(I_{N_r} + \sum_{k=1}^{N-1}\widehat{\boldsymbol{\ell}}_k(z)\boldsymbol{e}^t_k\right). \quad (33)$$

Note finally that if we define $\widehat{L}_k(z) = I_{N_r} + \widehat{\boldsymbol{\ell}}_k(z)\boldsymbol{e}^t_k$, then the following relation is easy to verify:

$$\sum_{k=1}^{N-1}\widehat{\boldsymbol{\ell}}_k(z)\boldsymbol{e}^t_k = \widehat{L}_1(z)\widehat{L}_2(z)\cdots\widehat{L}_{N-1}(z) = L(z) \quad (34)$$

and the resulting polynomial matrix $L(z)$ is upper triangular with 1's on the diagonal. This concludes the proof. ■

In summary, we have given an explicit construction of the

Theorem 1. *Every $p \times q$ polynomial matrix $H(z)$ admits a decomposition of the form*

$$H(z) = A(z)L(z)R(z) = U(z)R(z) \tag{35}$$

where $A(z)$ is a product of J-orthogonal polynomial matrices, $L(z)$ is a lower triangular polynomial matrix with 1's on the diagonal and $R(z)$ is an upper triangular matrix.

The matrix $U(z) = A(z)L(z)$ is unimodular.

Moreover, the k^{th} diagonal element of $R(z)$ is equal to the gcd of the last $p-k+1$ polynomials in the column k.

Applying the same decomposition to $R(z)^t$, namely $R(z)^t = V(z)D(z)$, provides a factorization of $H(z)$ as $H(z) = U(z)D(z)V(z)^t$ where $V(z)$ and $U(z)$ are two unimodular matrices and $D(z)$ is a diagonal polynomial matrix.

Remark 1. *The algorithm for the decomposition* (35) *is effective and it does not require any tuning parameter. The number of steps is exactly $p-1$. The k^{th} step consists of successive resolutions of n_k Bezout equations and one multiplication by a matrix of the form $L_k(z)$. In the best case, $n_k = 1$ ($h_{k,k}(z)$ and $h_{k,k+1}(z)$ are coprime) while $n_k = p-k-1$ in the worst case (all the polynomials $h_{k,j}(z), j = k, \cdots, p$ share some common zeros). In this worst case, $L_k(z) = I$.*

Remark 2. *Since $D(z)$ is diagonal, unlike the classical QR-PMSVD, the proposed UU-PMD method allows the CCI to be completely canceled. However, the matrices $V(z)$ and $U(z)$ are not paraunitary, therefore the noise component in (5) will be possibly enhanced. Indeed, assuming a spatial-temporal unitary white noise $\boldsymbol{n}$, the output noise power after post-coding reads as $\mathsf{E}(\|\widehat{\boldsymbol{n}}\|^2) = \|V(z)\|_2^2 = \frac{1}{2\pi}\int_0^{2\pi} \mathsf{Tr}\left[V(e^{i\omega})V(e^{-i\omega})^\dagger\right] d\omega$. The noise is thus amplified whener this norm is greater than 1. Note however that there are various directions where this amplification could be mitigated. One such direction is to consider a minimum phase weighting matrix $W(z)$ such that the norm $\|W(z)V(z)^t\|_2^2$ is minimized under the constrain that $D(z)W^{-1}(z)$ is both CCI and ISI free. Then, it suffices to replace the postfilter by $W(z)V(z)^t$ and to note that this would correspond to updating the equivalent channel $D(z)$ by $D(z)W^{-1}(z)$. One simple example is to choose for $W(z)$ a constant diagonal matrix. This solution is studied in a forthcoming paper.*

Remark 3. *From the last statement of Theorem 1, $(D(z))_{1,1}$ is formed by the common zeros of 1) all subchannels issuing from the first transmit antenna and 2) all subchannels terminating to the first receive antenna. Obviously, unless for some pathological MIMO channels, this set is most likely empty, resulting to $(D(z))_{1,1} = 1$. Likewise, for $k = 2, \cdots, N_r - 1$, $(D(z))_{k,k} = 1$, unless the subchannels from the transmit antenna k to the last $N_r - k + 1$ receive antennas and the subchannels from the last $N_t - k + 1$ transmit antennas to the receive antenna k, all share some common zeros. It is therefore realistic to consider that all resulting equivalent SISO channels reduce to Gaussian channels, except for the last one which, for $N_r = N_t$, would read as $(D(z))_{N_r,N_r} = \det H(z)$.*

IV. Performance comparison

In this section, we compare the three beamforming technics (FBBF, TBBF with QR-PMSVD and UU-PMD respectively) described above, in terms of bit error rate (BER) in the IEEE 802.16e context (mobile WiMax) [3]. For the simulation, we consider the MIMO-OFDM beamforming profile selected in WiMax Forum, with the ITU Pedestrian A channel model with parameters: 5MHz of bandwidth, $N_s = 512$ subcarriers, $CP = N_s/8 = 64$ with a 4-QAM modulation.

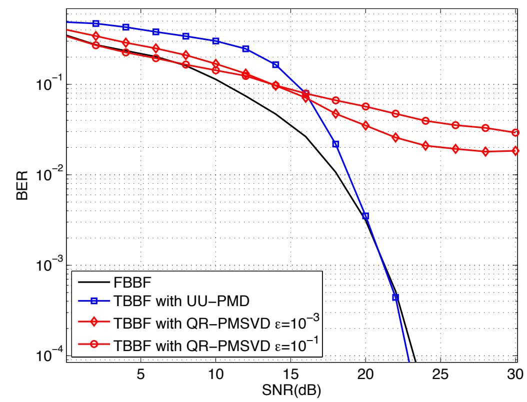

Fig. 1. BER comparison of FBBF, TBBF with UU-PMD and TBBF with QR-PMSVD in MIMO 2×2 setting. $\mu = 10^{-6}$ for QR-PMSVD.

Figure 1 shows the BER comparison of FBBF, TBBF with UU-PMD and QR-PMSVD respectively in MIMO 2×2 configuration. The tolerance parameter ε in TBBF with QR-PMSVD is set to 10^{-1} then to 10^{-3}. We see that, the FBBF presents better performance than the TBBF with UU-PMD in presence of low signal to noise ratio (SNR). However, the gap between these two performance decreases when the SNR increases and it eventually disappears. This is because the post-filter is not paraunitary in UU-PMD and therefore, the noise component is most likely enhanced. We also see that the TBBF with QR-PMSVD presents poor performance for both values of ε. This could be explained by the effect of the CCI for $\varepsilon = 10^{-1}$. However, when $\varepsilon = 10^{-3}$ the residual CCI is insignificant. In this context, the observed performance loss is rather due the effect of ISI. Indeed, figure 2 shows that not less than 140 iterations are required in order to bring the magnitude of the off-diagonal coefficients of $D(z)$ under the level $\varepsilon = 10^{-3}$. Now, as figure 3 shows, a large number of iterations translates into a very high degree for the polynomials in $D(z)$. The resulting SISO channels' impulse responses' durations are thus so important that the selected CP length $CP = \frac{N_s}{8} = 64$ does not enable the ISI to be sufficiently mitigated. This is confirmed in the next experiment, where we extend the CP length. Indeed, figure 4 shows that CP extension improves the performance of the TBBF system with QR-PMSVD. It is even possible to reach the same performance as that of FBBF by extended the CP to 75%. Note that unfortunately, the extension of the CP length has a direct impact on the system throughput as it reduces

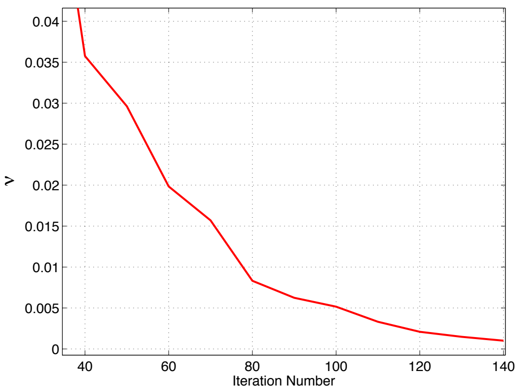

Fig. 2. Magnitude of the largest off diagonal coefficient of $D(z)$, noted ν *vs* iterations, for QR-PMSVD with $\varepsilon = 10^{-3}$.

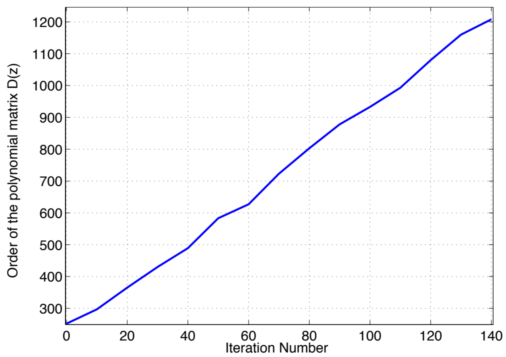

Fig. 3. Order of the polynomial matrix $D(z)$, QR-PMSVD with $\varepsilon = 10^{-3}$.

the spectral efficiency by consuming a part of the available bandwidth.

In TBBF with UU-PMD the ISI problem can be solved in view of Remark 3, by simply ignoring the last SISO channel:

- On the transmitter side, the data to be transmitted are multiplexed on the first $N_t - 1$ transmit antennas. One can transmit redundant data on the last one.
- On the receiver side, after the post-coding filter, the signal on the first $N_t - 1$ receive antennas only are considered.

In this case, there is no ISI to eliminate and thus the CP is no longer necessary. The resulting modified scheme is named

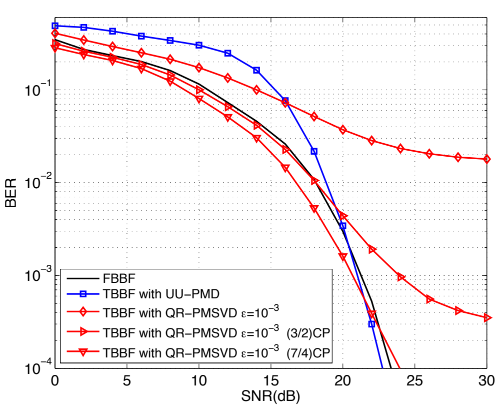

Fig. 4. BER comparison: FBBF, TBBF with UU-PMD and TBBF with QR-PMSVD in MIMO 2×2 configuration. CP is extended to $\frac{3}{2}$CP then to $\frac{7}{4}$CP. The truncation parameter in the QR-PMSVD is set to $\mu = 10^{-6}$.

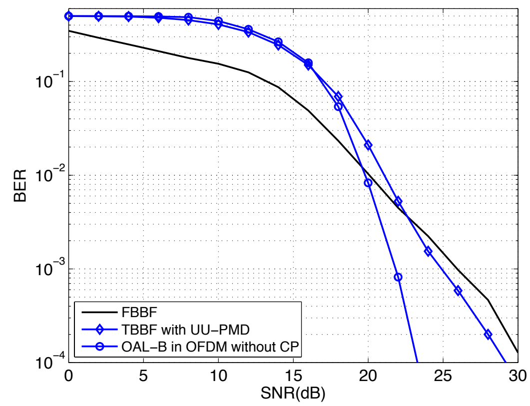

Fig. 5. BER comparison: FBBF, TBBF with UU-PMD and OAL-B in OFDM without CP in MIMO 3×3 configuration.

one antenna loss Beamforming (OAL-B).

Figure 5 shows the BER comparison of FBBF, TBBF with UU-PMD and OAL-B in MIMO 3×3 configuration. As expected, OAL-B presents better performance than the other beamforming schemes. This performance improvement is due to the absence of the ISI in OAL-B.

To summarise we can note the following points:

- TFFB with QR-PMSVD requires a CP extension. This is not suitable for realistic systems where the demand in terms of throughput is steadily increasing.
- In FBBF, the MIMO channel is perfectly diagonalized in space. Therefore, there is no CCI and the ISI can be mitigated by the conventional CP. However, unlike TBBF, the post-coding and the pre-coding must be carried out in the frequency domain for each subcarrier. This is problematic when the number of subcarriers is important.
- As in FBBF, the proposed TBBF with UU-PMD allows the CCI to be completely canceled. However, the noise is coloured and possibly enhanced due to the non paraunitary property of the corresponding post and pre filter. It is important to note that the post-coding and the pre-coding are carried out at one time for the entire system independently of the number of subcarriers. This constitutes a considerable advantage over FBBF. In addition, the first $N_t - 1$ separate SISO channels are thus equivalent to Gaussian channels.

Since ISI is not a critical issue for the proposed UU-PMD, we investigate its application in single carrier MIMO (MIMO-SC) systems in the next section.

V. Beamforming in MIMO-SC

A. *Principle of Beamforming in MIMO-SC*

In this context, equations (6) and (10) can be used for UU-PMD and QR-PMSVD based beamforming schemes respectively. The polynomials $\hat{y}_i(z)$ and $x_i(z)$ represent now the z-transform of the transmitted and received single carrier symbols before the precoder and after the postcoder respectively.

B. *Performance analysis*

We consider the WiMax simulation context of section IV.

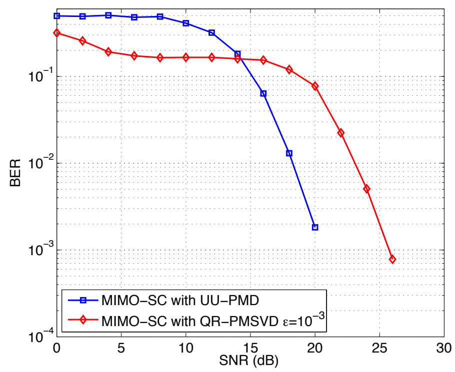

Fig. 6. BER comparison: MIMO-SC with UU-PMD and MIMO-SC with QR-PMSVD. The truncation parameter in the QR-PMSVD is $\mu = 10^{-6}$.

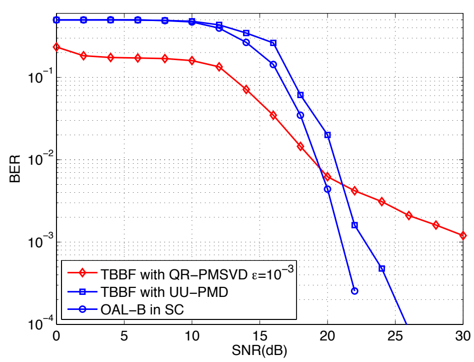

Fig. 7. BER comparison of MIMO-SC with UU-PMD, MIMO-SC with QR-PMSVD and OAL-B SC in MIMO 3×3 configuration. The truncation parameter in the QR-PMSVD is set to $\mu = 10^{-6}$.

Figure 6 shows the BER comparison of MIMO-SC with UU-PMD and QR-PMSVD with $\varepsilon = 10^{-3}$ in order to minimize the residual CCI. We notice that:

- As expected, the performance of the MIMO-SC with QR-PMSVD is degraded because of the presence of ISI due to the long duration of the impulse responses of the separate SISO channels (see figures 2 and 3). Indeed, the equalization is difficult and the residual ISI is important.
- The MIMO-SC with UU-PMD presents better performance for high and moderate SNR because all the resulting separate SISO channels, except the last one, are additive noise channels, with no ISI.

The performance of the UU-PMD based beamforming can be further improved with the ISI free OAL-B scheme described in the section IV. These observations are confirmed in the experiment in figure 7 below, showing the BER comparison of TBBF with QR-PMSVD, TBBF with UU-PMD and OAL-B SC in MIMO 3×3 configuration.

Let us mention that SC scheme is still adopted in telecommunication standards. Unlike OFDM, the peak-to-average power ratio (PAPR) is low in SC. Thence, as in OFDM, a CP is now used in order to improve its robustness to ISI [15]. This CP-SC is for example adopted in WiMax for $10-66$ GHz of frequency range and in the uplink of the 5G [1], [3]. The modified OAL-B, which does not need a CP, can be considered as a very good alternative to the CP-SC scheme.

VI. Conclusion

A time-domain broadband beamforming (TBBF) scheme based on a Unimodular Upper polynomial matrix decomposition algorithm is proposed. As in the classical frequency-domain broadband beamforming (FBBF), the proposed scheme completely eliminates the co-channel interferences while minimizing at the same time the ISI. In comparison, the QR based polynomial matrix SVD can not completely cancel the CCI unless a CP extension is considered in order to mitigate the ISI induced by the long duration of the resulting equivalent channel impulse responses.

Unlike the classical FBBF, which can only be used in OFDM systems, the proposed TBBF is suitable for both OFDM and SC systems. However, the proposed TBBF suffers from possible noise amplification since the post-filter obtained from the decomposition is not paraunitary.

We show how a modification of the proposed TBBF enables the ISI to be completely eliminated in most case without using a CP. However, for a $N_t \times N_r$ MIMO beamforming system, this modified scheme requires one more antenna than the other beamforming schemes, both at the transmitter and the receiver side.

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