Dynamics of a developable shell with uniform curvatures

Alejandro Blumentals, Florence Bertails-Descoubes, Romain Casati

Dynamics of a developable shell with uniform curvatures

Alejandro Blumentals, Florence Bertails-Descoubes and Romain Casati
INRIA Grenoble and Laboratoire Jean Kuntzmann, {alejandro.blumentals,florence.descoubes,romain.casati}@inria.fr

Many surface-like objects around us such as leaves, garments, or boat sails, may easily bend but hardly stretch. One is thus faced with the need for numerical models able to handle inextensibility constraints properly. In the present work we restrict ourselves to the modeling of elastic *developable* surfaces, i.e., surfaces which always remain isometric to a planar configuration. Our surfaces of interest may however take a non-planar rest configuration, hence we shall model them as *developable thin elastic shells*. Our goal is to design a both robust and efficient discrete model for simulating the motion of such objects. This work presents a first step towards this direction, by introducing a perfectly inextensible patch for a developable thin elastic shell.

From inextensible rods to inextensible shells The approach we follow is directly inspired by the super-helix model for thin elastic rods [1]. In this model, each rod element is characterized by uniform material curvatures and twist (i.e., uniform bending and twisting strains); a multibody rod dynamical system is then derived with curvatures and twist as degrees of freedom, leading to a perfectly inextensible dynamical rod model. Similarly, we build here an inextensible shell patch by taking as degrees of freedom the *curvatures* of its mid-surface, expressed in the local frame; equations of motion are then directly solved for the curvatures. Compared to the 1D (rod) case however, some difficulties arise in the 2D (plate/shell) case, where compatibility conditions are to be treated carefully. In the remainder of this abstract, we first show how to build a consistent kinematic model for our shell patch, then explain how the dynamical equations may be computed exactly, and finally present main implementation features and some simulation results.

Inextensible kinematics of a developable shell patch In the context of thin elastic shells, the mid-surface is parametrized by a function $r(s_1,s_2) \in \mathbb{R}^3$ of two spatial parameters (s_1,s_2) living on a planar domain $D \subset \mathbb{R}^2$. One way of enforcing developability of the surface throughout its deformation is to require r be a local isometry between the planar configuration D and the actual mid-surface. That is, the frame defined by $R(s_1,s_2) = (\partial_1 r, \partial_2 r, \partial_1 r \times \partial_2 r)$ is constrained to remain a rotation matrix $R(s_1,s_2) \in SO(3)$. By taking the derivative of the frame with respect to s_1 and s_2, one obtains the reconstruction equations for the mid-surface of the shell, which can be conveniently expressed on the special euclidean group $SE(3)$ as

$$\text{for } i = 1,2, \quad \frac{\partial}{\partial s_i}\begin{bmatrix} R(s_1,s_2) & r(s_1,s_2) \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} R(s_1,s_2) & r(s_1,s_2) \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \hat{\Omega}_i(s_1,s_2) & e_i \\ 0 & 0 \end{bmatrix}, \tag{1}$$

where the vectors $e_1 = (1,0,0)^T$ and $e_2 = (0,1,0)^T$ are the prescribed in-plane strains, and the entries of the instantaneous rotation vectors Ω_i are the out-of-plane bending strains. The hat map takes a vector v to the skew symmetric matrix $\hat{v}$ such that $\hat{v}u = v \times u$ for any vector $u \in \mathbb{R}^3$.

If one is given functions Ω_i on a simply connected domain D satisfying the compatibility conditions

$$\begin{cases} \partial_2\Omega_1 - \partial_1\Omega_2 = \Omega_1 \times \Omega_2 \\ \Omega_1 \times e_2 = \Omega_2 \times e_1 \end{cases}, \tag{2}$$

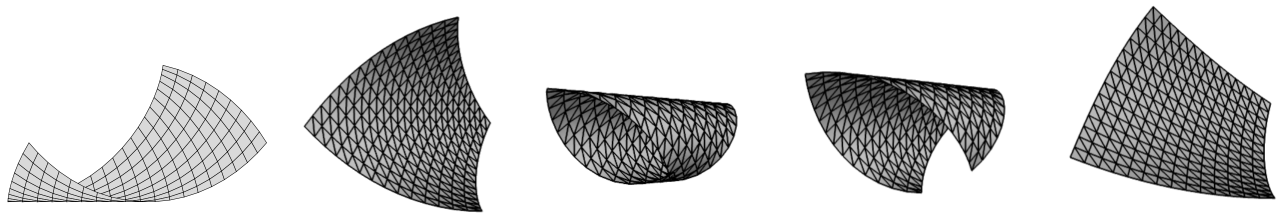
it can be shown [2] that the solution of (1) exists on the entire domain D and is unique up to a rigid body motion.

Discrete case: uniform curvatures If one assumes $D = [0,L] \times [0,W]$ to be rectangular and the Ω_i uniform, i.e., constant with respect to (s_1,s_2), then the unique solution to the reconstruction equation (1) such that $R(0,0) = R_0 \in SO(3)$ and $r(0,0) = r_0 \in \mathbb{R}^3$ is explicitly given by

$$\begin{bmatrix} R(s_1,s_2) & r(s_1,s_2) \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} R_0 & r_0 \\ 0 & 1 \end{bmatrix} \exp\left(\begin{bmatrix} s_1\hat{\Omega}_1 + s_2\hat{\Omega}_2 & s_1 e_1 + s_2 e_2 \\ 0 & 0 \end{bmatrix}\right). \tag{3}$$

Compatibility conditions (2) boil down to saying that there is a total of three curvatures $q = (q_0, q_1, q_2)^T$ to keep track of, that they obey $q_0 q_1 = q_2^2$ (which expresses the developability constraint as the vanishing of the Gaussian curvature of the mid-surface), and that the Ω_i vectors are given by $\Omega_1 = [q_2, -q_0, 0]^\top$ and $\Omega_2 = [q_1, -q_2, 0]^\top$.

Luckily the exponential map on $\mathfrak{se}(3)$ admits a closed form expression and one may compute the position of the mid-surface exactly as a function of the curvatures, $r = r(q, s_1, s_2)$, where q is independent of (s_1, s_2). It is noteworthy that the reconstructed shell geometry actually corresponds to a cylindrical patch (see figure 1(a)).



(a) (b) (c) (d) (e)

Fig. 1: (a) Geometry of our shell patch for $q_0 = q_1 = q_2 = 1$. (c-e) Dynamical oscillations of our inextensible shell patch clamped at bottom right corner and subject to gravity, with rest shape (b) and initial shape (c).

Dynamics of our inextensible shell patch To compute the pure bending dynamics of our shell patch, we build a constrained Lagrangian mechanical system with generalized coordinates q and Lagrangian of the form $L(q, \dot{q}) = \frac{1}{2}\dot{q}^T M(q)\dot{q} - U_g(q) - U_b(q)$, subject to the holonomic constraint $q_0 q_1 - q_2^2 = 0$. The mass matrix is a dense 3×3 matrix defined as $M(q) = \iint_D \rho h (\frac{\partial r}{\partial q}(q, s_1, s_2))^T \frac{\partial r}{\partial q}(q, s_1, s_2) \mathrm{d}s_1 \mathrm{d}s_2$, where ρ is the mass density and h the thickness of the shell. The potential energy due to gravity can be expressed as $U_g(q) = \iint_D \rho h g e_3^T r(q, s_1, s_2) \mathrm{d}s_1 \mathrm{d}s_2$. The elastic energy is entirely due to bending, and assuming a Hookean isotropic solid of Poisson ratio ν and Young modulus E, it is simply quadratic in q and may be formulated as $U_b(q) = \frac{1}{2}(q - \bar{q})^T K (q - \bar{q})$, where $\bar{q}$ is the vector of curvatures of the shell at rest (the so-called *natural* curvatures), and K the stiffness matrix defined as $K = \mathrm{area}(D) \frac{Eh^3}{12(1-\nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & 2(1-\nu) \end{bmatrix}$.

Implementation and results In order to compute all terms involved in the dynamics of the shell patch, one must carry out integrations over the domain D which take the form of a sum of terms like $\iint_D s_1^{p_1} s_2^{p_2} \cos(a_1(q)s_1 + a_2(q)s_2)^{p_3} \sin(a_1(q)s_1 + a_2(q)s_2)^{p_4} \mathrm{d}s_1 \mathrm{d}s_2$, where the p_i are natural numbers ranging from 0 to 4, and the $a_i(q)$ are functions of the curvatures only. These computations were carried out symbolically using the open source softwares Sympy and PyDy [3], generating efficient C code for the numerical evaluation of the above integrals. The resulting DAE is numerically solved by reducing to index 1 and using Baumgarte stabilization. The computational cost of 10,000 time steps is of about 1s on a 2.6 GHz processor. Snaphots from a simulation are depicted in figure 1(b-e).

Conclusion Though limited to cylindrical configurations, our inextensible shell patch already features some rich motion and could serve as an interactive model for simulating small flexible surfaces such as leaves or feathers. In the longer run, we would like to study how such shell patches could be assembled together so as to build a purely inextensible finite element model for developable thin elastic shells.

Acknowledgments This work was supported in part by the ERC grant GEM (StG-2014-639139).

References

[1] F. Bertails, B. Audoly, M.-P. Cani, B. Querleux, F. Leroy, and J.-L. Lévêque, “Super-helices for predicting the dynamics of natural hair,” *ACM Transactions on Graphics*, vol. 25, pp. 1180–1187, July 2006.

[2] P. Ciarlet, L. Gratie, and C. Mardare, “A new approach to the fundamental theorem of surface theory,” *Archive for Rational Mechanics and Analysis*, vol. 188, no. 3, pp. 457–473, 2008.

[3] G. Gede, D. L. Peterson, A. S. Nanjangud, J. K. Moore, and M. Hubbard, “Constrained multibody dynamics with python: From symbolic equation generation to publication,” in *ASME International Design Engineering Technical Conferences and Computers and Information in Engineering Conference*, vol. 7B, 2013.