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Laurent Boudin, Céline Grandmont, Ayman Moussa

GLOBAL EXISTENCE OF SOLUTIONS TO THE INCOMPRESSIBLE NAVIER-STOKES-VLASOV EQUATIONS IN A TIME-DEPENDENT DOMAIN

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Abstract. In this article, we prove the existence of global weak solutions for the incompressible Navier-Stokes-Vlasov system in a three-dimensional time-dependent domain with absorption boundary conditions for the kinetic part. This model arises from the study of respiratory aerosol in the human airways. The proof is based on a regularization and approximation strategy designed for our time-dependent framework.

1. Introduction

The collective motion of a dispersed phase of small particles inside a fluid is often described using the so-called *spray* or fluid-kinetic models, first introduced in the combustion theory framework [38] (see also [34, 18, 16]). In such models, one couples a kinetic equation with fluid mechanics equations. The fluid unknowns are the standard macroscopic quantities (mass density, velocity, for instance), and the dispersed phase is represented thanks to a distribution function. When one considers *thin* sprays, the fluid and kinetic equations are coupled through a drag term. This term depends on the fluid unknowns, the distribution function and their variables, and allows momentum and energy exchanges between both continuous and dispersed phases. Note that, with more dense (*thick*) sprays, the fluid volume fraction also appears in the equations, see [8] for instance.

The model investigated in this article, the incompressible Navier-Stokes-Vlasov system in a time-dependent domain, arises to study the transport and deposition of a therapeutic aerosol in a Newtonian, viscous, incompressible airflow, inside the human upper airways. The blueprints of the model were presented, in the aerosol therapy context, in [2, 20], and the extensive model was eventually written in [9].

The fluid-kinetic models have been studied from a mathematical point of view for two decades or so. They depend on the physical phenomena we take into account, such as the fluid compressibility or viscosity, the particle transport, the interactions of the particles with the fluid or the wall, the time dependence of the domain, *etc.* We focus on systems where only mass and momentum are exchanged. The case with energy exchanges is, for instance, discussed in [6].

In the compressible case, up to our knowledge, [3] is the first contribution, for the coupled compressible Euler-Vlasov system, where Baranger and Desvillettes obtain the local-in-time existence of classical solutions. Mathiaud [28] obtains the same kind of result for the Euler-Vlasov-Boltzmann system. For the compressible Navier-Stokes-Vlasov-Fokker-Planck system, in [29], Mellet and Vasseur prove global existence of weak solutions, and provide an asymptotic analysis for the system in [30]. More recently, Chae, Kang and Lee [14] and Li, Mu and Wang [26] investigate, among other topics, the existence of global strong solutions close to the equilibrium.

There has been more mathematical contributions in the incompressible case, starting with [1, 25]. In [7], the authors of the present article prove, with Desvillettes, the global existence of weak solutions to the incompressible Navier-Stokes-Vlasov equations in a periodic framework. The result is extended to the bounded domain case in [39]. The present work then appears as a natural continuation of those previous articles, with the

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additional difficulty of the time-dependent domain, which was not tackled yet, up to our knowledge. It also involves absorption boundary conditions for the distribution function, which are expected when dealing with depositing aerosols. Those boundary conditions induce extra difficulties to deal with, when compared to the periodic or whole-space cases. Note that Goudon, Jabin and Vasseur [23, 24] investigate the hydrodynamic limit of the Vlasov-Navier-Stokes system, for some particular regimes of the dispersed phase. We also mention a particular model [33] of inviscid fluid-particles, where the drag force is replaced by a lift force.

Various other problems in the incompressible case have also been investigated. Goudon, He, Moussa and Zhang [22] establish the global existence of strong solutions near the equilibrium for the incompressible Navier-Stokes-Vlasov-Fokker-Planck system, whereas Carrillo, Duan and Moussa [12] studied the corresponding inviscid case. The global existence of weak and classical solutions for the Navier-Stokes-Vlasov-Fokker-Planck equations in a torus is investigated by Chae, Kang and Lee in [13]. If the fluid is inhomogeneous, Wang and Yu [37] prove the existence of global weak solutions to the Navier-Stokes-Vlasov equations. Eventually, Benjelloun, Desvillettes and Moussa [5] consider the formal asymptotic limit of the incompressible Vlasov-Navier-Stokes system with a fragmentation kernel, and prove the existence of global weak solutions for the resulting system, which shares some ties with the one considered in [37].

The article is organized as follows. In the next section, we present our problem in the time-dependent domain, state the main result, and discuss the proof strategy such as how we build an approximated problem to first deal with. Then, in Section 3, we recall some standard results on transport equation of Vlasov type in bounded domains and extend a result from [11] to the case of a moving and unbounded domain. Section 4 is dedicated to the proof of existence of solutions to the approximated problem. Finally, in the last section, we go back to the whole problem by passing to the limit in the approximated one as the regularization parameters vanish, using, in particular, an Aubin-Lions lemma-like result from [32].

2. Presentation of the problem

We investigate the interaction of a size-monodispersed aerosol with a Newtonian, viscous and incompressible fluid. The evolution of this system is studied in a time-dependent domain.

Consider a finite constant $T > 0$, bounded an open set $\Omega \subset \mathbb{R}^3$, and an open ball $\mathcal{B}$ such that $\overline{\Omega} \subset \mathcal{B}$. We assume that Ω has a Lipschitz boundary.

Define a mapping $\mathcal{A} \in \mathscr{C}^2(\mathbb{R}_+ \times \mathbb{R}^3; \mathbb{R}^3)$, $(t, \boldsymbol{x}) \mapsto \mathcal{A}(t, \boldsymbol{x}) = \mathcal{A}_t(\boldsymbol{x})$ such that, for each $t \geq 0$, $\mathcal{A}_t$ is a $\mathscr{C}^1$-diffeomorphism, $\mathcal{A}_t = \mathrm{Id}_{\mathbb{R}^3}$ on $\mathcal{B}^c$, and $\mathcal{A}_0 = \mathrm{Id}_{\mathbb{R}^3}$. The time-dependent bounded open domains Ω_t, $0 \leq t \leq T$, are obtained as $\Omega_t = \mathcal{A}_t(\Omega)$, and clearly satisfy $\overline{\Omega_t} \subset \mathcal{B}$ for all $t \in [0, T]$. We denote

$$\widehat{\Omega} = \bigcup_{0<t<T} \{t\} \times \Omega_t \subset \widehat{\mathcal{B}} = (0, T) \times \mathcal{B}, \qquad \widehat{\Gamma} = \bigcup_{0<t<T} \{t\} \times \partial\Omega_t.$$

Moreover, for all $t \in [0, T]$, we denote by $\boldsymbol{n}_t$ the outgoing unit normal vector field of $\partial\Omega_t$. Finally, we introduce $\boldsymbol{w}$, the Eulerian velocity associated to the flow $t \mapsto \mathcal{A}_t$, characterized by

$$\partial_t \mathcal{A}(t, \boldsymbol{x}) = \boldsymbol{w}(t, \mathcal{A}_t(\boldsymbol{x})), \quad (t, \boldsymbol{x}) \in [0, T] \times \mathbb{R}^3.$$

Note that the assumptions on $\mathcal{A}_t$ imply that $\boldsymbol{w} \equiv 0$ on $(0, T) \times \mathcal{B}^c$. Without loss of generality, we assume that $\mathrm{div}_{\boldsymbol{x}}\, \boldsymbol{w} = 0$, which is equivalent to assume that the Jacobian of the transformation $\mathcal{A}_t$ does not depend on t. Besides, we need to introduce the following phase-space boundaries for the aerosol

$$\begin{aligned}
\widehat{\Sigma} &= \widehat{\Gamma} \times \mathbb{R}^3, \\
\widehat{\Sigma}^{\pm} &= \{(t, \boldsymbol{x}, \boldsymbol{\xi}) \in \widehat{\Sigma} \,|\, \pm (\boldsymbol{\xi} - \boldsymbol{w}(t, \boldsymbol{x})) \cdot \boldsymbol{n}_t(\boldsymbol{x}) > 0\}, \\
\widehat{\Sigma}^{0} &= \{(t, \boldsymbol{x}, \boldsymbol{\xi}) \in \widehat{\Sigma} \,|\, (\boldsymbol{\xi} - \boldsymbol{w}(t, \boldsymbol{x})) \cdot \boldsymbol{n}_t(\boldsymbol{x}) = 0\},
\end{aligned}$$

and, for any t,

$$\widehat{\Sigma}_t^{\pm} = \{(\boldsymbol{x}, \boldsymbol{\xi}) \in \partial\Omega_t \times \mathbb{R}^3 \mid \pm (\boldsymbol{\xi} - \boldsymbol{w}(t, \boldsymbol{x})) \cdot \boldsymbol{n}_t(\boldsymbol{x}) > 0\}.$$

The fluid is described *via* macroscopic quantities, pressure $p(t, \boldsymbol{x})$ and velocity $\boldsymbol{u}(t, \boldsymbol{x})$, thanks to the classical incompressible Navier-Stokes equation, with constant density and viscosity both chosen equal to 1,

$$\partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \nabla_{\boldsymbol{x}} \boldsymbol{u} + \nabla_{\boldsymbol{x}} p - \Delta_{\boldsymbol{x}} \boldsymbol{u} = \mathbf{F} \quad \text{in } \widehat{\Omega}, \tag{1}$$

$$\operatorname{div}_{\boldsymbol{x}} \boldsymbol{u} = 0 \quad \text{in } \widehat{\Omega}, \tag{2}$$

where $\mathbf{F}$ is the action of the aerosol on the fluid that will be defined later on. We prescribe the following Dirichlet boundary conditions

$$\boldsymbol{u} = \boldsymbol{w} \quad \text{in } \widehat{\Gamma}. \tag{3}$$

The aerosol is described through a density function $f(t, \boldsymbol{x}, \boldsymbol{\xi})$, which solves a Vlasov-like equation

$$\partial_t f + \boldsymbol{\xi} \cdot \nabla_{\boldsymbol{x}} f + \nabla_{\boldsymbol{\xi}} \cdot [\mathbf{A} f] = 0 \quad \text{in } \widehat{\Omega} \times \mathbb{R}^3, \tag{4}$$

where $\mathbf{A}$ is the drag acceleration exerted by the fluid on the particles. We prescribe the following absorption boundary conditions for the aerosol:

$$f = 0 \quad \text{in } \widehat{\Sigma}^-. \tag{5}$$

The coupling terms $\mathbf{A}$ and $\mathbf{F}$, which respectively depend on $(t, \boldsymbol{x}, \boldsymbol{\xi})$ and $(t, \boldsymbol{x})$, are defined by

$$\mathbf{A} = \boldsymbol{u} - \boldsymbol{\xi}, \qquad \mathbf{F} = -\int_{\mathbb{R}^3} f \, \mathbf{A}. \tag{6}$$

The system is eventually supplemented with initial conditions for $\boldsymbol{u}$ and f which are

$$\boldsymbol{u}(0, \cdot) = \boldsymbol{u}_{\mathrm{in}}, \qquad \text{in } \Omega, \tag{7}$$

$$f(0, \cdot, \cdot) = f_{\mathrm{in}}, \qquad \text{in } \Omega \times \mathbb{R}^3. \tag{8}$$

As usual in the framework of fluid-kinetic coupling, we may compute, at least formally, an energy equality reflecting the dissipation of energy and the exchange between the two phases. Recall Reynolds' formula for any real-valued function $k : \widehat{\Omega} \to \mathbb{R}$:

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega_t} k = \int_{\Omega_t} \partial_t k + \int_{\partial\Omega_t} k \, \boldsymbol{w} \cdot \boldsymbol{n}_t.$$

We multiply equation (1) by $\boldsymbol{u} - \boldsymbol{w}$ and integrate over Ω_t (t is fixed). After integration by parts, using (2), (3), (6) and the Reynolds formula, we get

$$\begin{aligned} \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega_t} |\boldsymbol{u}|^2 + \int_{\Omega_t} |\nabla_{\boldsymbol{x}} \boldsymbol{u}|^2 = \frac{\mathrm{d}}{\mathrm{d}t} \left[\int_{\Omega_t} \boldsymbol{u} \cdot \boldsymbol{w} \right] - \int_{\Omega_t} \boldsymbol{u} \cdot \partial_t \boldsymbol{w} - \int_{\Omega_t} (\boldsymbol{u} \cdot \nabla_{\boldsymbol{x}} \boldsymbol{w}) \cdot \boldsymbol{u} \\ + \int_{\Omega_t} \nabla_{\boldsymbol{x}} \boldsymbol{u} : \nabla_{\boldsymbol{x}} \boldsymbol{w} - \int_{\Omega_t \times \mathbb{R}^3} (\boldsymbol{u} - \boldsymbol{w}) \cdot (\boldsymbol{u} - \boldsymbol{\xi}) f. \end{aligned}$$

We then multiply (4) by $|\boldsymbol{\xi}|^2/2$ and integrate on $\Omega_t \times \mathbb{R}^3$ to get after integration by parts, using (6),

$$\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega_t \times \mathbb{R}^3} f |\boldsymbol{\xi}|^2 = \int_{\Omega_t \times \mathbb{R}^3} \boldsymbol{\xi} \cdot (\boldsymbol{u} - \boldsymbol{\xi}) f + \frac{1}{2} \int_{\partial\Omega_t \times \mathbb{R}^3} f |\boldsymbol{\xi}|^2 (\boldsymbol{w} - \boldsymbol{\xi}) \cdot \boldsymbol{n}_t.$$

Summing both previous equalities, we obtain the following energy balance

$$\begin{aligned}\frac{\mathrm{dE}_{\boldsymbol{u},f}}{\mathrm{d}t} &+ \int_{\Omega_t} |\nabla_{\boldsymbol{x}} \boldsymbol{u}|^2 + \int_{\Omega_t\times\mathbb{R}^3} f|\boldsymbol{u}-\boldsymbol{\xi}|^2 \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\left[\int_{\Omega_t} \boldsymbol{u}\cdot\boldsymbol{w}\right] - \int_{\Omega_t} \boldsymbol{u}\cdot\partial_t \boldsymbol{w} - \int_{\Omega_t} (\boldsymbol{u}\cdot\nabla_{\boldsymbol{x}}\boldsymbol{w})\cdot\boldsymbol{u} \\ &\quad + \int_{\Omega_t} \nabla_{\boldsymbol{x}}\boldsymbol{u} : \nabla_{\boldsymbol{x}}\boldsymbol{w} + \int_{\Omega_t\times\mathbb{R}^3} \boldsymbol{w}\cdot(\boldsymbol{u}-\boldsymbol{\xi})f + \frac{1}{2}\int_{\partial\Omega_t\times\mathbb{R}^3} f|\boldsymbol{\xi}|^2(\boldsymbol{w}-\boldsymbol{\xi})\cdot\boldsymbol{n}_t,\end{aligned}$$

where $\mathrm{E}_{\boldsymbol{u},f}$ denotes the total kinetic energy of the system and writes

$$\mathrm{E}_{\boldsymbol{u},f}(t) = \frac{1}{2}\int_{\Omega_t} |\boldsymbol{u}|^2 + \frac{1}{2}\int_{\Omega_t\times\mathbb{R}^3} f|\boldsymbol{\xi}|^2.$$

Assuming f to be nonnegative, we note that

$$\int_{\Omega_t\times\mathbb{R}^3} \boldsymbol{w}\cdot(\boldsymbol{u}-\boldsymbol{\xi})f \le \frac{1}{2}\int_{\Omega_t\times\mathbb{R}^3} f|\boldsymbol{u}-\boldsymbol{\xi}|^2 + \frac{1}{2}\int_{\Omega_t\times\mathbb{R}^3} f|\boldsymbol{w}|^2.$$

Thanks to the absorption boundary condition (5), we may eventually write

$$\begin{aligned}&\frac{\mathrm{dE}_{\boldsymbol{u},f}}{\mathrm{d}t} + \int_{\Omega_t} |\nabla_{\boldsymbol{x}} \boldsymbol{u}|^2 + \frac{1}{2}\int_{\Omega_t\times\mathbb{R}^3} f|\boldsymbol{u}-\boldsymbol{\xi}|^2 \\ &\le \frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\Omega_t} \boldsymbol{u}\cdot\boldsymbol{w}\right) - \int_{\Omega_t} \boldsymbol{u}\cdot\partial_t \boldsymbol{w} - \int_{\Omega_t} (\boldsymbol{u}\cdot\nabla_{\boldsymbol{x}}\boldsymbol{w})\cdot\boldsymbol{u} + \int_{\Omega_t} \nabla_{\boldsymbol{x}}\boldsymbol{u} : \nabla_{\boldsymbol{x}}\boldsymbol{w} + \frac{1}{2}\int_{\Omega_t\times\mathbb{R}^3} f|\boldsymbol{w}|^2.\end{aligned}$$

We deduce, thanks to Young's inequality, using the regularity of $\boldsymbol{w}$, and integrating over $[0,t]$,

$$\begin{aligned}\mathrm{E}_{\boldsymbol{u},f}(t) &+ \int_0^t\int_{\Omega_s} |\nabla_{\boldsymbol{x}}\boldsymbol{u}|^2 + \int_0^t\int_{\Omega_s\times\mathbb{R}^3} f|\boldsymbol{u}-\boldsymbol{\xi}|^2 \\ &\le \mathrm{C}\,\mathrm{E}_{\boldsymbol{u},f}(0) + \mathrm{C}_{\boldsymbol{w}} + \mathrm{C}_{\boldsymbol{w}}\int_0^t\int_{\Omega_s} |\boldsymbol{u}|^2 + \mathrm{C}_{\boldsymbol{w}}\int_0^t\int_{\Omega_s\times\mathbb{R}^3} f,\end{aligned} \tag{9}$$

where $\mathrm{C} > 0$ is a universal constant and $\mathrm{C}_{\boldsymbol{w}} > 0$ is a constant depending on the domain velocity $\boldsymbol{w}$. Moreover, integrating (4) on $\Omega_t\times\mathbb{R}^3$, we get, using once again (5),

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\Omega_t\times\mathbb{R}^3} f\right) = \int_{\partial\Omega_t\times\mathbb{R}^3} f(\boldsymbol{w}-\boldsymbol{\xi})\cdot\boldsymbol{n} \le 0,$$

and consequently,

$$\int_{\Omega_t\times\mathbb{R}^3} f \le \int_{\Omega\times\mathbb{R}^3} f_{\mathrm{in}}.$$

All in all, going back to (9) we have, for some constant $\mathrm{C}_{\boldsymbol{w},f_{\mathrm{in}}} > 0$ depending on $\boldsymbol{w}$ and f_{in},

$$\mathrm{E}_{\boldsymbol{u},f}(t) + \int_0^t\int_{\Omega_s} |\nabla_{\boldsymbol{x}}\boldsymbol{u}|^2 + \int_0^t\int_{\Omega_s\times\mathbb{R}^3} f|\boldsymbol{u}-\boldsymbol{\xi}|^2 \le \mathrm{C}\,\mathrm{E}_{\boldsymbol{u},f}(0) + \mathrm{C}_{\boldsymbol{w},f_{\mathrm{in}}}\left[T + \int_0^t\int_{\Omega_s} |\boldsymbol{u}|^2\right],$$

so that, thanks to Gronwall's lemma, we infer

$$\mathrm{E}_{\boldsymbol{u},f}(t) + \int_0^t\int_{\Omega_s} |\nabla_{\boldsymbol{x}}\boldsymbol{u}|^2 + \int_0^t\int_{\Omega_s\times\mathbb{R}^3} f|\boldsymbol{u}-\boldsymbol{\xi}|^2 \le (\mathrm{C}\,\mathrm{E}_{\boldsymbol{u},f}(0) + \mathrm{C}_{\boldsymbol{w},f_{\mathrm{in}}}T)e^{T^2\mathrm{C}_{\boldsymbol{w},f_{\mathrm{in}}}}. \tag{10}$$

The previous energy inequality, together with equation (2) motivates the introduction of several function spaces. Set, for the fluid part,

$$\begin{aligned}\mathrm{L}^2(0,T;\mathrm{H}^1(\Omega_t)) &= \{\boldsymbol{\psi}\in\mathrm{L}^2(\widehat{\Omega})\,|\,\nabla_{\boldsymbol{x}}\boldsymbol{\psi}\in\mathrm{L}^2(\widehat{\Omega})\}, \\ \mathrm{V} &= \{\boldsymbol{\psi}\in\mathrm{L}^2(0,T;\mathrm{H}^1(\Omega_t))\,|\,\mathrm{div}_{\boldsymbol{x}}\,\boldsymbol{\psi}=0\}, \\ \mathrm{V}_0 &= \{\boldsymbol{\psi}\in\mathrm{V}\,|\,\boldsymbol{\psi}=0 \text{ on } \widehat{\Gamma}\}, \\ \mathscr{V} &= \{\boldsymbol{\psi}\in\mathscr{C}^1(\overline{\widehat{\Omega}})\,|\,\mathrm{div}_{\boldsymbol{x}}\,\boldsymbol{\psi}=0 \text{ on } \widehat{\Omega},\ \boldsymbol{\psi}=0 \text{ on } \widehat{\Gamma}\}.\end{aligned}$$

Note that V_0 is the closure of $\mathscr{V}$ in $\mathrm{L}^2(0,T;\mathrm{H}^1(\Omega_t))$. For any function (scalar or vector-valued) g defined on $\widehat{\Omega}$, we denote by $\overline{g}$ the extension (in the space variable) by 0 of g on $(0,T)\times\mathbb{R}^3$. For the kinetic part, we now set, for any $p\in[1,\infty]$,

$$\mathrm{L}^\infty(0,T;\mathrm{L}^p(\Omega_t\times\mathbb{R}^3)) = \{f \text{ measurable} \,|\, \overline{f}\in \mathrm{L}^\infty(0,T;\mathrm{L}^p(\mathbb{R}^3\times\mathbb{R}^3))\},$$

$$\mathscr{W} = \{\phi\in\mathscr{C}^1_c(\overline{\widehat{\Omega}}\times\mathbb{R}^3)\,|\,\phi=0 \text{ on } \widehat{\Sigma}^+\cup\widehat{\Sigma}^0\}.$$

For any function $h:\widehat{\Omega}\times\mathbb{R}^3\to\mathbb{R}_+$ and $\alpha\in\mathbb{R}_+$, we further introduce the following notations on the moments of h which will be useful in the study of the Vlasov equation. For a.e. $(t,\boldsymbol{x})$, we set

$$m_\alpha h(t,\boldsymbol{x}) = \int_{\mathbb{R}^3} h\,|\boldsymbol{\xi}|^\alpha, \qquad M_\alpha h(t) = \int_{\Omega_t\times\mathbb{R}^3} h\,|\boldsymbol{\xi}|^\alpha = \int_{\Omega_t} m_\alpha h.$$

In particular, we recall a standard interpolation estimate, see [25] for instance:

Lemma 2.1. *Let $\beta>0$, and h be a nonnegative function in $\mathrm{L}^\infty(\widehat{\Omega}\times\mathbb{R}^3)$ such that $m_\beta h(t,\boldsymbol{x})<+\infty$ for a.e. $(t,\boldsymbol{x})$. The following estimate holds for any $\alpha\in[0,\beta)$ and a.e. $(t,\boldsymbol{x})$:*

$$m_\alpha h(t,\boldsymbol{x}) \le \left(\frac{4}{3}\pi\|h\|_{\mathrm{L}^\infty(\widehat{\Omega}\times\mathbb{R}^3)}+1\right) m_\beta h(t,\boldsymbol{x})^{\frac{\alpha+3}{\beta+3}}.$$

Let us now give the assumptions on the initial data and the domain motion:

Assumption 2.1. $\boldsymbol{u}_{\mathrm{in}}\in\mathrm{L}^2(\Omega)$, $\mathrm{div}_{\boldsymbol{x}}\,\boldsymbol{u}_{\mathrm{in}}=0$, *and* $\boldsymbol{u}_{\mathrm{in}}\cdot\boldsymbol{n}_0=\boldsymbol{w}(0,\cdot)\cdot\boldsymbol{n}_0$ *on* $\partial\Omega$.

Assumption 2.2. f_{in} *is nonnegative,* $f_{\mathrm{in}}\in\mathrm{L}^\infty(\Omega\times\mathbb{R}^3)$, $f_{\mathrm{in}}(1+|\boldsymbol{\xi}|^2)\in\mathrm{L}^1(\Omega\times\mathbb{R}^3)$.

Weak solutions of the coupled problem are defined as follows.

Definition 2.1. *We say that a couple $(\boldsymbol{u},f)$ is a weak solution of system (1)–(5) with initial datum $(\boldsymbol{u}_{\mathrm{in}},f_{\mathrm{in}})$ if the following conditions are satisfied:*

- $\overline{\boldsymbol{u}}\in\mathrm{L}^\infty(0,T;\mathrm{L}^2(\mathbb{R}^3))$,
- $\boldsymbol{u}-\boldsymbol{w}\in\mathrm{V}_0$,
- $\overline{f}\in\mathrm{L}^\infty(0,T;\mathrm{L}^\infty(\mathbb{R}^3\times\mathbb{R}^3))\cap\mathscr{C}^0([0,T];\mathrm{L}^p(\mathbb{R}^3\times\mathbb{R}^3))$, *for all* $p\in[1,\infty)$,
- $\overline{f}(1+|\boldsymbol{\xi}|^2)\in\mathrm{L}^\infty(0,T;\mathrm{L}^1(\mathbb{R}^3\times\mathbb{R}^3))$,
- *for all* $\boldsymbol{\psi}\in\mathscr{V}$, *for a.e.* $t\in[0,T]$,

$$\int_{\Omega_t}\boldsymbol{u}(t)\cdot\boldsymbol{\psi}(t) - \int_0^t\int_{\Omega_s}\boldsymbol{u}\cdot(\partial_t\boldsymbol{\psi}+\boldsymbol{u}\cdot\nabla_{\boldsymbol{x}}\boldsymbol{\psi}) - \int_0^t\int_{\Omega_s}\nabla_{\boldsymbol{x}}\boldsymbol{u}:\nabla_{\boldsymbol{x}}\boldsymbol{\psi} - \int_0^t\int_{\Omega_s}\boldsymbol{u}\cdot\left(\int_{\mathbb{R}^3}(\boldsymbol{\xi}-\boldsymbol{u})f\right) = \int_\Omega \boldsymbol{u}_{\mathrm{in}}\cdot\boldsymbol{\psi}(0),$$

- *for all* $\phi\in\mathscr{W}$, *for all* $t\in[0,T]$,

$$\int_{\Omega_t\times\mathbb{R}^3} f(t)\phi(t) - \int_0^t\int_{\Omega_s\times\mathbb{R}^3} f\,(\partial_t\phi+\boldsymbol{\xi}\cdot\nabla_{\boldsymbol{x}}\phi+(\boldsymbol{u}-\boldsymbol{\xi})\cdot\nabla_{\boldsymbol{\xi}}\phi) = \int_{\Omega\times\mathbb{R}^3} f_{\mathrm{in}}\phi(0).$$

The main result of this paper is the following

Theorem 2.1. *Under Assumptions 2.1–2.2, there exists at least one weak solution $(\boldsymbol{u},f)$ to (1)–(8) in the sense of Definition 2.1.*

The proof follows a standard scheme: introduction of an approximated system, itself solved through a fixed-point procedure.

Let us explain how we obtain the approximated system. First, we regularize the convection term in the Navier-Stokes equations. Then, in order to work in the cylindrical domain $\widehat{\mathcal{B}}$ containing the non-cylindrical domain $\widehat{\Omega}$, we add a penalty term in (1), as in [19]. Since the relative velocity term $\boldsymbol{u}-\boldsymbol{\xi}$ governs the coupling of the system through (6), to be able to apply standard existence results for the Navier-Stokes equations, we are led to truncate the right hand side of (1). Moreover, to preserve the energy estimate of the coupled approximated system, the same truncation is performed in (4). We thus introduce an odd, increasing and bounded function $\chi \in \mathscr{C}^\infty(\mathbb{R})$, satisfying $0 \leq \chi(z) \leq z$ for any $z \geq 0$, and we use the abuse of notation $\chi(\boldsymbol{z}) = (\chi(z_1), \chi(z_2), \chi(z_3))$, when $\boldsymbol{z} \in \mathbb{R}^3$. Hence, the resulting problem system writes

$$\partial_t \boldsymbol{u} + (\boldsymbol{u} \star \varphi) \cdot \nabla_{\boldsymbol{x}} \boldsymbol{u} + \nabla_{\boldsymbol{x}} p - \Delta_{\boldsymbol{x}} \boldsymbol{u} + \lambda \mathbb{1}_{\widehat{\mathcal{B}} \setminus \widehat{\Omega}} (\boldsymbol{u} - \boldsymbol{w}) = \int_{\mathbb{R}^3} \overline{f} \chi(\boldsymbol{\xi} - \boldsymbol{u}) \qquad \text{in } \widehat{\mathcal{B}}, \tag{11}$$

$$\operatorname{div}_{\boldsymbol{x}} \boldsymbol{u} = 0 \qquad \text{in } \widehat{\mathcal{B}}, \tag{12}$$

$$\partial_t f + \boldsymbol{\xi} \cdot \nabla_{\boldsymbol{x}} f + \nabla_{\boldsymbol{\xi}} \cdot [\chi(\boldsymbol{u} - \boldsymbol{\xi}) f] = 0 \qquad \text{in } \widehat{\Omega} \times \mathbb{R}^3. \tag{13}$$

In (11), $\lambda > 0$ is the penalty parameter and φ denotes an element of $\mathscr{D}(\mathbb{R}^3)$, the convolution product of the convection velocity being made in $\boldsymbol{x}$ only. The previous system is completed with the following boundary conditions: homogeneous Dirichlet boundary conditions for $\boldsymbol{u}$ on $(0,T) \times \partial\mathcal{B}$ and (5) for f. The initial data are also extended and/or regularized, to ensure that, at least, they satisfy the following assumptions.

Assumption 2.3. $\boldsymbol{u}_{\mathrm{in}} \in \mathrm{H}^1_0(\mathcal{B})$ *and* $\operatorname{div}_{\boldsymbol{x}} \boldsymbol{u}_{\mathrm{in}} = 0$*,*

Assumption 2.4. f_{in} *is nonnegative, has a compact support in* $\boldsymbol{\xi}$*,* $f_{\mathrm{in}} \in \mathrm{L}^\infty(\Omega \times \mathbb{R}^3) \cap \mathrm{L}^1(\Omega \times \mathbb{R}^3)$*.*

The existence of weak solutions to (11)–(13) is obtained thanks to Schaefer's fixed point theorem (for the proof, see, for instance, [21, Theorem 11.6, p.286]) which we here recall, for the sake of completeness.

Theorem 2.2. *Let* E *be a Banach space and* $\Theta : E \times [0,1] \to E$ *a continuous mapping sending bounded subsets of* $E \times [0,1]$ *on relatively compact subsets of* E*. Denoting* $\Theta_\sigma = \Theta(\cdot, \sigma)$*, if* $\Theta_0 = 0$ *and the set of all the fixed points of the family* $(\Theta_\sigma)_{\sigma \in [0,1]}$ *is bounded in* E*, then* Θ_1 *has at least one fixed point in* E*.*

We choose $E = \mathrm{L}^2(0,T; \mathrm{H}^1_0(\mathcal{B}))$, and define the mapping $\Theta : E \times [0,1] \to E$ in the following way. Starting from $\boldsymbol{u} \in E$, we define $f_{\boldsymbol{u}}$ as the unique weak solution of (13), with absorption boundary conditions and initial datum f_{in} satisfying Assumption 2.4, and we set

$$\mathbf{S}_\chi(\boldsymbol{u}) = \int_{\mathbb{R}^3} \overline{f_{\boldsymbol{u}}} \chi(\boldsymbol{\xi} - \boldsymbol{u}). \tag{14}$$

Then, for each $\sigma \in [0,1]$, consider $\widetilde{\boldsymbol{u}}_\sigma$ the unique solution of

$$\partial_t \widetilde{\boldsymbol{u}}_\sigma + (\widetilde{\boldsymbol{u}}_\sigma \star \varphi) \cdot \nabla_{\boldsymbol{x}} \widetilde{\boldsymbol{u}}_\sigma + \nabla_{\boldsymbol{x}} p - \Delta_{\boldsymbol{x}} \widetilde{\boldsymbol{u}}_\sigma = \sigma \mathbf{S}_\chi(\boldsymbol{u}) - \lambda \sigma \mathbb{1}_{\widehat{\mathcal{B}} \setminus \widehat{\Omega}} (\boldsymbol{u} - \boldsymbol{w}) \qquad \text{in } \widehat{\mathcal{B}}, \tag{15}$$

$$\nabla_{\boldsymbol{x}} \cdot \widetilde{\boldsymbol{u}}_\sigma = 0 \qquad \text{in } \widehat{\mathcal{B}}, \tag{16}$$

$$\widetilde{\boldsymbol{u}}_\sigma = 0 \qquad \text{on } (0,T) \times \partial\mathcal{B}, \tag{17}$$

$$\widetilde{\boldsymbol{u}}_\sigma(0) = \sigma \boldsymbol{u}_{\mathrm{in}} \qquad \text{in } \mathcal{B}, \tag{18}$$

where $\boldsymbol{u}_{\mathrm{in}}$ satisfies Assumption 2.3. Eventually, Theorem 2.2 is used for the mapping $\Theta : E \times [0,1] \to E$, $(\boldsymbol{u}, \sigma) \mapsto \widetilde{\boldsymbol{u}}_\sigma$.

Now that we have introduced the approximated problems, let us describe more precisely the main steps and the difficulties in the existence proof. We first deal with the Vlasov equation (13) with absorption boundary conditions, and a non smooth drag acceleration in a time-dependent domain. Consequently, the solution to (13), together with its trace on the boundary $\widehat{\Sigma}^-$, can be defined in a weak sense, following the arguments developped in [10, 11] and extended to our time-dependent domain setting. In particular, the use of renormalization theory is crucial to ensure uniqueness of solutions, and strongly relies on the regularity of $\chi(\boldsymbol{u} - \boldsymbol{\xi})$. The study of the incompressible Navier-Stokes system in a time-dependent domain is more standard, see for instance [19, 35]. Nevertheless, because of the time dependence of the domain, a special care to apply or adapt the Aubin-Lions lemma is required to prove the needed compactness on the fluid velocity field and to take the limit in the sequence of approximated problems as the regularization parameters vanish.

Here we choose to apply a result obtained in [32]. Note finally that, as in [1, 13], the approximation strategy preserves the energy estimate satisfied by the full coupled system. In particular, it ensures that the set of fixed points of the mappings Θ_σ is bounded in E.

3. Vlasov's equation in a moving domain

We here consider the following Vlasov equation

$$\partial_t f + \boldsymbol{\xi} \cdot \nabla_{\boldsymbol{x}} f + \nabla_{\boldsymbol{\xi}} \cdot (f\mathbf{G}) = 0 \quad \text{in } \widehat{\Omega} \times \mathbb{R}^3, \tag{19}$$

$$f = 0 \quad \text{in } \widehat{\Sigma}^-, \tag{20}$$

$$f(0,\cdot) = f_0 \quad \text{in } \Omega \times \mathbb{R}^3, \tag{21}$$

where the vector field $\mathbf{G}(t,\boldsymbol{x},\boldsymbol{\xi})$ and the initial datum $f_0(\boldsymbol{x},\boldsymbol{\xi})$ are given. In this section, we aim to obtain an appropriate functional setting in order to define weak solutions to (19)–(21) and prove their existence and uniqueness. For uniqueness and to give a weak sense to the trace on $\widehat{\Sigma}$, we closely follow the approach developped by Boyer in [10] and adapt it to our time-dependent domain. As for the existence, we follow the standard characteristic method together with a regularization procedure of the field $\mathbf{G}$.

The first subsection is devoted to the definition and existence of weak solutions to (19)–(21). Next, we prove that the trace on $\widehat{\Sigma}$ of such solutions can properly be defined together with a renormalized weak formulation satisfied by any bounded weak solution. Uniqueness is then obtained, strongly relying on the renormalized weak formulation satisfied by the trace.

3.1. **Existence of solution.**

We only consider the case of bounded solutions and assume that $\mathbf{G} \in \mathrm{L}^1(0,T;\mathrm{W}^{1,1}_{\mathrm{loc}}(\mathcal{B}\times\mathbb{R}^3))$, $\operatorname{div}_{\boldsymbol{\xi}} \mathbf{G} \in \mathrm{L}^\infty(0,T;\mathrm{L}^\infty(\mathcal{B}\times\mathbb{R}^3))$ and $f_0 \in \mathrm{L}^1(\Omega\times\mathbb{R}^3)\cap \mathrm{L}^\infty(\Omega\times\mathbb{R}^3)$. We introduce the following definition:

Definition 3.1. *A function $f \in \mathrm{L}^\infty(\widehat{\Omega}\times\mathbb{R}^3)$ is a weak solution of (19)–(21) if, for any test function $\phi \in \mathscr{W}$ such that $\phi(T,\cdot) = 0$, one has*

$$-\int_0^T \int_{\Omega_t\times\mathbb{R}^3} f\,(\partial_t\phi + \boldsymbol{\xi}\cdot\nabla_{\boldsymbol{x}}\phi + \mathbf{G}\cdot\nabla_{\boldsymbol{\xi}}\phi) = \int_{\Omega\times\mathbb{R}^3} f_0\,\phi(0). \tag{22}$$

Remark 3.1. *Note that the previous kind of solution is weaker than the one introduced for f in Definition 2.1. Indeed, the latter included some continuity in time and a larger set of test functions. Nevertheless, the arguments of Subsection 3.2 enable to prove that both definitions are in fact equivalent.*

Let us first consider a vector field $\mathbf{G}$ defined in the whole space and regular enough, namely $\mathbf{G} \in \mathscr{C}^0(\mathbb{R};\mathrm{W}^{1,\infty}(\mathbb{R}^3\times\mathbb{R}^3))$. With this regularity, the Cauchy-Lipschitz theorem allows us to define global characteristic curves associated to the field $(\boldsymbol{\xi},\mathbf{G})$. More precisely, for any $s\in\mathbb{R}$ and any $(\boldsymbol{x},\boldsymbol{\xi}) \in \mathbb{R}^3\times\mathbb{R}^3$, there exists a unique global solution of the differential system

$$\begin{cases} \dfrac{\mathrm{d}\boldsymbol{x}_s}{\mathrm{d}t}(t) = \boldsymbol{\xi}_s(t), \\ \dfrac{\mathrm{d}\boldsymbol{\xi}_s}{\mathrm{d}t}(t) = \mathbf{G}(t,\boldsymbol{x}_s(t),\boldsymbol{\xi}_s(t)), \\ (\boldsymbol{x}_s(s),\boldsymbol{\xi}_s(s)) = (\boldsymbol{x},\boldsymbol{\xi}). \end{cases}$$

We denote the associated curve by $t\mapsto \mathbf{T}_{s,t}(\boldsymbol{x},\boldsymbol{\xi}) = (\boldsymbol{x}_s(t),\boldsymbol{\xi}_s(t))$. For $(t,\boldsymbol{x},\boldsymbol{\xi}) \in \widehat{\Omega}\times\mathbb{R}^3$, let us then consider the retrograde outgoing time

$$\tau^-(t,\boldsymbol{x},\boldsymbol{\xi}) = \inf\{s\le t \mid \forall\sigma\in[s,t], \mathbf{T}_{t,\sigma}(\boldsymbol{x},\boldsymbol{\xi}) \in \Omega_\sigma\times\mathbb{R}^3\}.$$

In that case, we can define, for $(t,\boldsymbol{x},\boldsymbol{\xi}) \in \widehat{\Omega}\times\mathbb{R}^3$,

$$f(t,\boldsymbol{x},\boldsymbol{\xi}) = e^{h(t,\boldsymbol{x},\boldsymbol{\xi})} f_0(\mathbf{T}_{t,0}(\boldsymbol{x},\boldsymbol{\xi}))\mathbb{1}_{\tau^-<0}(t,\boldsymbol{x},\boldsymbol{\xi}), \tag{23}$$

where h is the solution in $(0,T)\times\mathbb{R}^3\times\mathbb{R}^3$ of the following transport equation with a source term

$$\partial_t h + \boldsymbol{\xi}\cdot\nabla_{\boldsymbol{x}} h + \mathbf{G}\cdot\nabla_{\boldsymbol{\xi}} h = \operatorname{div}_{\boldsymbol{\xi}} \mathbf{G}$$

with initial condition $h(0,\cdot)=0$. One can check, thanks to quite straightforward though tedious computations, that the function defined in (23) is a weak solution of (19)–(21) in the sense of Definition 3.1. The existence of weak solutions to the transport equation with absorption boundary conditions based on formula (23) is investigated in [4] when the domain is bounded and the field $\mathbf{G}$ is regular using the semigroup theory, whereas the Vlasov equation case is studied in [31] when $\mathbf{G}$ is not regular, but only depends on t and $\boldsymbol{x}$. Note that Boyer and Fabrie [11] follow another strategy to obtain the existence of weak solutions, which is based on a parabolic regularization of the transport equation.

The function defined in (23) furthermore satisfies

$$\|\overline{f}\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^1(\mathbb{R}^3\times\mathbb{R}^3)\cap\mathrm{L}^\infty(\mathbb{R}^3\times\mathbb{R}^3))} \le \exp\Big(T\|\operatorname{div}_{\boldsymbol{\xi}}\mathbf{G}\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^\infty(\mathcal{B}\times\mathbb{R}^3))}\Big)\|f_0\|_{\mathrm{L}^1(\Omega\times\mathbb{R}^3)\cap\mathrm{L}^\infty(\Omega\times\mathbb{R}^3)}.$$

If $\mathbf{G}\in\mathrm{L}^1(0,T;\mathrm{W}^{1,1}_{\mathrm{loc}}(\mathcal{B}\times\mathbb{R}^3))$, with $\operatorname{div}_{\boldsymbol{\xi}}\mathbf{G}\in\mathrm{L}^\infty(0,T;\mathrm{L}^\infty(\mathcal{B}\times\mathbb{R}^3))$, it can be approximated in $\mathrm{L}^1_{\mathrm{loc}}(\widehat{\mathcal{B}}\times\mathbb{R}^3)$ by a sequence $(\mathbf{G}_k)$ lying in $\mathscr{D}([0,T]\times\mathbb{R}^3\times\mathbb{R}^3)$. Furthermore, $(\mathbf{G}_k)$ can be chosen such that $\|\operatorname{div}_{\boldsymbol{\xi}}\mathbf{G}_k\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^\infty(\mathbb{R}^3\times\mathbb{R}^3))}\le\|\operatorname{div}_{\boldsymbol{\xi}}\mathbf{G}\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^\infty(\mathcal{B}\times\mathbb{R}^3))}$. Letting k go to $+\infty$, we obtain the following

Proposition 3.1. *If* $\mathbf{G}\in\mathrm{L}^1(0,T;\mathrm{W}^{1,1}_{\mathrm{loc}}(\mathcal{B}\times\mathbb{R}^3))$ *is such that* $\operatorname{div}_{\boldsymbol{\xi}}\mathbf{G}\in\mathrm{L}^\infty(0,T;\mathrm{L}^\infty(\mathcal{B}\times\mathbb{R}^3))$*, then, for any* $f_0\in\mathrm{L}^1(\Omega\times\mathbb{R}^3)\cap\mathrm{L}^\infty(\Omega\times\mathbb{R}^3)$*, there exists at least one weak solution to the system* (19)*–*(21) *in the sense of Definition 3.1. Furthermore, this solution satisfies*

$$\|\overline{f}\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^1(\mathbb{R}^3\times\mathbb{R}^3)\cap\mathrm{L}^\infty(\mathbb{R}^3\times\mathbb{R}^3))} \le \exp\Big(T\|\operatorname{div}_{\boldsymbol{\xi}}\mathbf{G}\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^\infty(\mathcal{B}\times\mathbb{R}^3))}\Big)\|f_0\|_{\mathrm{L}^1(\Omega\times\mathbb{R}^3)\cap\mathrm{L}^\infty(\Omega\times\mathbb{R}^3)}. \tag{24}$$

Remark 3.2. *It is clear from the previous proof that the assumption* $\mathbf{G}\in\mathrm{L}^1(0,T;\mathrm{W}^{1,1}_{\mathrm{loc}}(\mathcal{B}\times\mathbb{R}^3))$ *can be replaced by* $\mathbf{G}\in\mathrm{L}^1(0,T;\mathrm{L}^1_{\mathrm{loc}}(\mathcal{B}\times\mathbb{R}^3))$*. However, the stronger assumption on* $\mathbf{G}$ *is crucial in the next subsection, where we define the trace of the solution on the boundary.*

Remark 3.3. *If* $\mathbf{G}\in\mathrm{L}^1(0,T;\mathrm{L}^\infty(\mathcal{B}\times\mathbb{R}^3))$ *and if* f_0 *has a compact support in* $\boldsymbol{\xi}$*, then the weak solution* f *to* (19)*–*(21) *also has a compact support in* $\boldsymbol{\xi}$*. It is clear in the regularized case, when* f *is given by* (23)*. Moreover, the support of* f *in* $\boldsymbol{\xi}$ *can be chosen independently from the regularization parameter thanks to the assumption* $\mathbf{G}\in\mathrm{L}^1(0,T;\mathrm{L}^\infty(\mathcal{B}\times\mathbb{R}^3))$*. Consequently, the property still holds at the limit. It will be useful to obtain a priori estimates for the regularized coupled problem involving moments of* f*.*

3.2. **Trace theorem.** Let us first state a trace theorem which holds for bounded solutions of transport equations in bounded domains. A first version of this theorem has been established by Boyer [10, Theorem 3.1], we here use the more general version presented in [11]. Note nevertheless that we do not state that theorem in its whole generality. We choose to adapt it to our framework.

Theorem 3.1. *Consider a Lipschitz bounded open set* $O\subset\mathbb{R}^d$*. Assume that* $\mathbf{a}\in\mathrm{L}^1(0,T;\mathrm{W}^{1,1}(O))$ *such that* $\operatorname{div}\mathbf{a}\in\mathrm{L}^1(0,T;\mathrm{L}^\infty(O))$*. Let* $\rho\in\mathrm{L}^\infty((0,T)\times O)$ *be a distributional solution of the following transport equation*

$$\partial_t\rho+\operatorname{div}(\mathbf{a}\rho)=0, \ \text{ in } (0,T)\times O.$$

Then $\rho\in\mathscr{C}^0([0,T];\mathrm{L}^p(O))$ *for all* $p\in[1,\infty)$*, and admits a well-defined trace* $\gamma\rho$ *on* $(0,T)\times\partial O$*: it is the only element of* $\mathrm{L}^\infty((0,T)\times\partial O,|\mathrm{d}\mu_{\mathbf{a}}|)$ *with* $\mathrm{d}\mu_{\mathbf{a}}=\mathbf{a}\cdot\boldsymbol{n}\,\mathrm{d}\sigma\,\mathrm{d}t$*, which satisfies, for any test function* $\varphi\in\mathscr{D}(\mathbb{R}\times\mathbb{R}^d)$*, any real-valued function* $\beta\in\mathscr{C}^1(\mathbb{R})$*, and* *all* t_0, t_1 *such that* $0 \le t_0 \le t_1 \le T$,

$$\int_{t_0}^{t_1}\int_O \beta(\rho)(\partial_t\varphi + \mathbf{a}\cdot\nabla\varphi)) - \int_{t_0}^{t_1}\int_O (\rho\beta'(\rho) - \beta(\rho))\varphi\,\mathrm{div}\,\mathbf{a} = \int_O \beta(\rho(t_1))\varphi(t_1) - \int_O \beta(\rho(t_0))\varphi(t_0) + \int_{t_0}^{t_1}\int_{\partial O}\beta(\gamma\rho)\,\varphi\;\mathbf{a}\cdot\boldsymbol{n}. \tag{25}$$

Here we need to generalize this result to our non-cylindrical and unbounded setting. In this case, the theorem we obtain writes

Theorem 3.2. *Assume that* $f \in \mathrm{L}^\infty(\widehat{\Omega}\times\mathbb{R}^3)$ *is a solution of*

$$\partial_t f + \boldsymbol{\xi}\cdot\nabla_{\boldsymbol{x}} f + \mathrm{div}_{\boldsymbol{\xi}}(\mathbf{G}f) = 0 \text{ in } \mathscr{D}'(\widehat{\Omega}\times\mathbb{R}^3), \tag{26}$$

where $\mathbf{G} \in \mathrm{L}^1(0,T;\mathrm{W}^{1,1}_{\mathrm{loc}}(\mathcal{B}\times\mathbb{R}^3))$. *Then, for all* $p\in[1,\infty)$, $\overline{f}\in\mathscr{C}^0([0,T];\mathrm{L}^p_{\mathrm{loc}}(\mathbb{R}^3\times\mathbb{R}^3))$. *Moreover the trace of* f *on* $\widehat{\Gamma}$ *is well-defined: it is the unique element* $\widehat{\gamma}f\in\mathrm{L}^\infty(\widehat{\Gamma})$ *satisfying for any test function* $\psi\in\mathscr{D}(\mathbb{R}\times\mathbb{R}^3\times\mathbb{R}^3)$, *all real-valued function* $\beta\in\mathscr{C}^1(\mathbb{R})$, *and all* $0\le t_0\le t_1\le T$:

$$\int_{t_0}^{t_1}\int_{\Omega_t\times\mathbb{R}^3}\beta(f)\,(\partial_t\psi + \boldsymbol{\xi}\cdot\nabla_{\boldsymbol{x}}\psi + \mathbf{G}\cdot\nabla_{\boldsymbol{\xi}}\psi) - \int_{t_0}^{t_1}\int_{\Omega_t\times\mathbb{R}^3}(f\beta'(f)-\beta(f))\psi\,\mathrm{div}_{\boldsymbol{\xi}}\,\mathbf{G} = \int_{\Omega_{t_1}\times\mathbb{R}^3}\beta(f(t_1))\psi(t_1) - \int_{\Omega_{t_0}\times\mathbb{R}^3}\beta(f(t_0))\psi(t_0) + \int_{t_0}^{t_1}\int_{\partial\Omega_t\times\mathbb{R}^3}\beta(\widehat{\gamma}f)\,\psi\,(\boldsymbol{\xi}-\boldsymbol{w})\cdot\boldsymbol{n}_t. \tag{27}$$

Remark 3.4. *Following* [11, Corollary VI.1.5], β *can be chosen continuous and piecewise* $\mathscr{C}^1$. *Indeed, from* (26), *we can prove that, for any* α *and for any test function* $\psi\in\mathscr{D}(\mathbb{R}\times\mathbb{R}^3\times\mathbb{R}^3)$,

$$\int_{\Omega_t\times\mathbb{R}^3}\psi\,\mathbb{1}_{f=\alpha}\,f\;\mathrm{div}_{\boldsymbol{\xi}}\,\mathbf{G} = 0.$$

Remark 3.5. *Note that, in the boundary term* (27), *the velocity* $\boldsymbol{w}$ *appears because of our time-dependent domain setting. However, the case* $\boldsymbol{w}=0$, *which corresponds to a cylindrical setting, does not fall directly into the scope of Boyer's theorem because* $\Omega\times\mathbb{R}^3$ *is not bounded. Consequently, one of the main step of the proof will be to generalize Theorem 3.1 for* $O=\Omega\times\mathbb{R}^3$.

Proof. Since we deal with a time-dependent domain, we perform a change of variables in order to work in a cylindrical domain. We associate to f a function g defined in the reference domain as follows:

$$g(t,\boldsymbol{y},\boldsymbol{\xi}) = f(t,\mathcal{A}_t(\boldsymbol{y}),\boldsymbol{\xi}), \quad \text{a.e. } (t,\boldsymbol{y},\boldsymbol{\xi})\in(0,T)\times\Omega\times\mathbb{R}^3. \tag{28}$$

The fact that $f\in\mathrm{L}^\infty(\widehat{\Omega}\times\mathbb{R}^3)$ implies that $g\in\mathrm{L}^\infty((0,T)\times\Omega\times\mathbb{R}^3)$. In the same way as in (28), we set

$$\mathbf{K}(t,\boldsymbol{y},\boldsymbol{\xi}) = \mathbf{G}(t,\mathcal{A}_t(\boldsymbol{y}),\boldsymbol{\xi}), \quad \text{a.e. } (t,\boldsymbol{y},\boldsymbol{\xi})\in(0,T)\times\Omega\times\mathbb{R}^3.$$

Since f solves (26), we have

$$\partial_t g + (\boldsymbol{\xi}-\boldsymbol{v})\cdot(\mathrm{Cof}(\nabla_{\boldsymbol{y}}\mathcal{A}_t)\nabla_{\boldsymbol{y}})\,g + \mathrm{div}_{\boldsymbol{\xi}}(\mathbf{K}g) = 0 \quad \text{in } \mathscr{D}'((0,T)\times\Omega\times\mathbb{R}^3),$$

where $\boldsymbol{v}$ is the domain Lagrangian velocity, *i.e.*

$$\boldsymbol{v}(t,\boldsymbol{y}) = \boldsymbol{w}(t,\mathcal{A}_t(\boldsymbol{y})), \quad \text{for any } (t,\boldsymbol{y})\in(0,T)\times\Omega,$$

and $\mathrm{Cof}(\nabla_{\boldsymbol{y}}\mathcal{A}_t)$ denotes the cofactor matrix of $\nabla_{\boldsymbol{y}}\mathcal{A}_t$. Thanks to the additional assumption $\mathrm{div}_{\boldsymbol{x}}\,\boldsymbol{w}=0$ on the domain Eulerian velocity $\boldsymbol{w}$, we have $\mathrm{div}_{\boldsymbol{y}}(\mathrm{Cof}(\nabla_{\boldsymbol{y}}\mathcal{A}_t)^\intercal\boldsymbol{v})=0$. Moreover, Piola's identity ensures that $\mathrm{div}_{\boldsymbol{y}}(\mathrm{Cof}(\nabla_{\boldsymbol{y}}\mathcal{A}_t))=\mathbf{0}$ (see [15]). Consequently, introducing the new variable $\boldsymbol{z}=(\boldsymbol{y},\boldsymbol{\xi})\in\Omega\times\mathbb{R}^3$ and considering the vector field

$$\mathbf{a}(t,\boldsymbol{z}) = ((\mathrm{Cof}(\nabla_{\boldsymbol{y}}\mathcal{A}_t))^\intercal(\boldsymbol{\xi}-\boldsymbol{v}),\mathbf{K})^\intercal, \tag{29}$$

the previous equation may be written, in $\mathscr{D}'((0,T)\times\Omega\times\mathbb{R}^3)$,

$$\partial_t g + \operatorname{div}_{\boldsymbol{z}}(\mathbf{a}g) = 0. \tag{30}$$

We cannot directly apply Theorem 3.1 because $\Omega\times\mathbb{R}^3$ is not bounded. Let us use Boyer's theorem for $O = O_R$, where $R>0$, $O_R = \Omega\times B_R$ and B_R is the open ball of radius $R>0$ centred at 0. It is possible because g also solves (30) in $\mathscr{D}'((0,T)\times O_R)$ and $g\in \mathrm{L}^\infty((0,T)\times O_R)$. We thus infer that $g\in\mathscr{C}^0([0,T];\mathrm{L}^p(O_R))$ for $p\in[1,\infty)$ and for all $R>0$, so that $g\in\mathscr{C}^0([0,T];\mathrm{L}^p_{\mathrm{loc}}(\overline{\Omega}\times\mathbb{R}^3))$. Consequently, going back to the non-cylindrical domain, we get $\overline{f}\in\mathscr{C}^0([0,T];\mathrm{L}^p_{\mathrm{loc}}(\mathbb{R}^3\times\mathbb{R}^3))$.

Next, we would like to define the trace of g on $(0,T)\times\partial\Omega\times\mathbb{R}^3$. Using, once again, Theorem 3.1 in $(0,T)\times O_R$, we get existence and uniqueness of the trace $\gamma_R g$ in $\mathrm{L}^\infty((0,T)\times\partial O_R, |\mathrm{d}\mu_{\mathbf{a},R}|)$ such that (25) holds with $\rho = g$ and $O = O_R$. Note that $\partial O_R = (\partial\Omega\times\overline{B_R})\cup(\overline{\Omega}\times\partial B_R)$. We restrict ourselves to the space part of the boundary $\partial\Omega\times\overline{B_R}$, on which $\mathrm{d}\mu_{\mathbf{a},R}$ does not depend on R and equals

$$\mathrm{d}\mu_{\mathbf{a}} = (\boldsymbol{n}_0,\mathbf{0})^\intercal\cdot\mathbf{a}\,\mathrm{d}\sigma\,\mathrm{d}\boldsymbol{\xi}\,\mathrm{d}t = (\boldsymbol{\xi}-\boldsymbol{v})\cdot\mathrm{Cof}(\nabla_{\boldsymbol{y}}\mathcal{A}_t)\boldsymbol{n}_0\,\mathrm{d}\sigma\,\mathrm{d}\boldsymbol{\xi}\,\mathrm{d}t. \tag{31}$$

The uniqueness of the trace implies that, for any $R'>R$,

$$(\gamma_R g)_{|(0,T)\times\partial\Omega\times\overline{B_R}} = (\gamma_{R'} g)_{|(0,T)\times\partial\Omega\times\overline{B_R}} \quad \text{in } \mathrm{L}^\infty((0,T)\times\partial\Omega\times\overline{B_R}, |\mathrm{d}\mu_{\mathbf{a}}|).$$

Since $\mathrm{d}\sigma\,\mathrm{d}\boldsymbol{\xi}\,\mathrm{d}t$-almost everywhere, we have $(\boldsymbol{\xi}-\boldsymbol{v})\cdot\mathrm{Cof}(\nabla_{\boldsymbol{y}}\mathcal{A}_t)\boldsymbol{n}_0\neq 0$ on $(0,T)\times\partial\Omega\times\overline{B_R}$, we have $\mathrm{L}^\infty((0,T)\times\partial\Omega\times\overline{B_R},|\mathrm{d}\mu_{\mathbf{a}}|)\subset\mathrm{L}^\infty((0,T)\times\partial\Omega\times\overline{B_R})$ and the previous equality also holds in the latter space. It is hence possible to define, without ambiguity, an element $\gamma g\in\mathrm{L}^\infty_{\mathrm{loc}}([0,T]\times\partial\Omega\times\mathbb{R}^3)$ by

$$\gamma g = \gamma_R g \quad \text{on } [0,T]\times\partial\Omega\times\overline{B_R}, \quad \forall R>0.$$

From the trace of g on the boundary of the reference domain, we define the element $\widehat{\gamma} f$ of $\mathrm{L}^\infty_{\mathrm{loc}}(\widehat{\Gamma})$ by

$$\gamma g(t,\boldsymbol{y},\boldsymbol{\xi}) = \widehat{\gamma} f(t,\mathcal{A}_t(\boldsymbol{y}),\boldsymbol{\xi}).$$

The next step consists in proving the weak renormalized formulation (27), which ensures that $\widehat{\gamma} f$ is, in fact, the trace of f on $\widehat{\Gamma}$. We first check that g satisfies the same type of equality as (25) on $(0,T)\times\Omega\times\mathbb{R}^3$. Note that, for any $\varphi\in\mathscr{D}(\mathbb{R}\times\mathbb{R}^3\times\mathbb{R}^3)$, the support of φ lies in fact in $\mathbb{R}^3\times\mathbb{R}^3\times B_R$, and thus we get

$$\begin{aligned}&\int_{t_0}^{t_1}\int_{\Omega\times\mathbb{R}^3}\beta(g)(\partial_t\varphi + \mathbf{a}\cdot\nabla_{\boldsymbol{z}}\varphi) - \int_{t_0}^{t_1}\int_{\Omega\times\mathbb{R}^3}(g\beta'(g)-\beta(g))\varphi\,\operatorname{div}_{\boldsymbol{z}}\mathbf{a}\\ &= \int_{\Omega\times\mathbb{R}^3}\beta(g(t_1))\varphi(t_1) - \int_{\Omega\times\mathbb{R}^3}\beta(g(t_0))\varphi(t_0) + \int_{t_0}^{t_1}\int_{\partial\Omega\times\mathbb{R}^3}\beta(\gamma g)\varphi\,(\boldsymbol{\xi}-\boldsymbol{v})\cdot\mathbf{a}\cdot(\boldsymbol{n}_0,\mathbf{0})^\intercal.\end{aligned} \tag{32}$$

From the definition (29) of $\mathbf{a}$, we obtain $\operatorname{div}_{\boldsymbol{z}}\mathbf{a} = \operatorname{div}_{\boldsymbol{\xi}}\mathbf{K}$. Thus, taking (31) into account, (32) writes

$$\begin{aligned}\int_{t_0}^{t_1}\int_{\Omega\times\mathbb{R}^3}&\beta(g)(\partial_t\varphi + \boldsymbol{\xi}\cdot(\mathrm{Cof}(\nabla_{\boldsymbol{y}}\mathcal{A}_t)\nabla_{\boldsymbol{y}})\varphi + \mathbf{K}\cdot\nabla_{\boldsymbol{\xi}}\varphi)\\ &- \int_{t_0}^{t_1}\int_{\Omega\times\mathbb{R}^3}(g\beta'(g)-\beta(g))\varphi\,\operatorname{div}_{\boldsymbol{\xi}}\mathbf{K}\\ &\qquad = \int_{\Omega\times\mathbb{R}^3}\beta(g(t_1))\varphi(t_1) - \int_{\Omega\times\mathbb{R}^3}\beta(g(t_0))\varphi(t_0)\\ &\qquad\qquad + \int_{t_0}^{t_1}\int_{\partial\Omega\times\mathbb{R}^3}\beta(\gamma g)\varphi\,(\boldsymbol{\xi}-\boldsymbol{v})\cdot\mathrm{Cof}(\nabla_{\boldsymbol{y}}\mathcal{A}_t)\boldsymbol{n}_0.\end{aligned} \tag{33}$$

Thanks to a density argument, (32) is still satisfied if φ is only $\mathscr{C}^1_c(\mathbb{R}\times\mathbb{R}^3\times\mathbb{R}^3)$.

The final step consists in going back to the time-dependent domain in order to define the trace of f and altogether obtain the weak formulation (27). For any fixed $t\geq 0$ and any given function ζ defined on $\partial\Omega_t$, thanks to [15], and with obvious notations, we have (whenever the integrals make sense)

$$\int_{\partial\Omega}(\zeta\circ\mathcal{A}_t)\,\mathrm{Cof}(\nabla_{\boldsymbol{y}}\mathcal{A}_t)\boldsymbol{n}_0\,\mathrm{d}\sigma=\int_{\partial\Omega_t}\zeta\,\boldsymbol{n}_t\,\mathrm{d}\sigma_t.$$

Consequently, since g is the image of f by the mapping $\mathcal{A}_t$, from (33), we easily obtain (27) by applying the change of variables $\boldsymbol{x}=\mathcal{A}_t(\boldsymbol{y})$. At this stage, this weak formulation (27) is satisfied for any function ψ such that $\varphi(t,\boldsymbol{y},\boldsymbol{\xi})=\psi(t,\mathcal{A}_t(\boldsymbol{y},\boldsymbol{\xi}))$, with $\varphi\in\mathscr{C}^1_c(\mathbb{R}\times\mathbb{R}^3\times\mathbb{R}^3)$. Since $\mathcal{A}_t$ is a $\mathscr{C}^1$-diffeomorphism of $\mathbb{R}^3$, (27) actually holds for any test function $\psi\in\mathscr{C}^1_c(\mathbb{R}\times\mathbb{R}^3\times\mathbb{R}^3)$. To end the proof of the theorem, we just have to check that the trace $\widehat{\gamma}f$ belongs to $\mathrm{L}^\infty(\widehat{\Gamma})$. In the same way as in [10], we choose an appropriate renormalization function β. Since $f\in\mathrm{L}^\infty(\widehat{\Omega}\times\mathbb{R}^3)$, there exist two real numbers c_1, c_2 such that $c_1\le f\le c_2$ a.e. on $\widehat{\Omega}\times\mathbb{R}^3$. We choose $\beta\in\mathscr{C}^1(\mathbb{R})$ such that $\beta(x)=0$ if and only if $x\in[c_1,c_2]$. For such a β, the weak formulation (27) leads to

$$\int_{\widehat{\Gamma}}\beta(\widehat{\gamma}f)\,\psi\,(\boldsymbol{\xi}-\boldsymbol{w})\cdot\boldsymbol{n}_t=0,\quad\forall\psi\in\mathscr{C}^1_c(\mathbb{R}\times\mathbb{R}^3\times\mathbb{R}^3).$$

This implies that $\beta(\widehat{\gamma}f)(\boldsymbol{\xi}-\boldsymbol{w})\cdot\boldsymbol{n}_t=0$ a.e. on $\widehat{\Gamma}$. But $(\boldsymbol{\xi}-\boldsymbol{w})\cdot\boldsymbol{n}_t\neq0$ a.e. (this is the counterpart of the already used property $(\boldsymbol{\xi}-\boldsymbol{v})\cdot\mathrm{Cof}(\nabla_{\boldsymbol{y}}\mathcal{A}_t)\boldsymbol{n}_0\neq0$ a.e.), so that we get $\beta(\widehat{\gamma}f)=0$ a.e., and thus $c_1\le\widehat{\gamma}f\le c_2$ a.e. Finally, the uniqueness of $\widehat{\gamma}f$ in $\mathrm{L}^\infty(\widehat{\Omega}\times\mathbb{R}^3)$ is straightforward. ☐

Remark 3.6. *If $f\ge0$ almost everywhere, following the above argument and taking $c_1=0$, we can prove that $\widehat{\gamma}f\ge0$ almost everywhere.*

Remark 3.7. *Note that, thanks to the just proved renormalized weak formulation* (27)*, any weak solution f of* (19)–(21) *in the sense of Definition 3.1 satisfies $\hat{\gamma}f=0$ on $\widehat{\Sigma}^-$.*

3.3. **Uniqueness.** In this subsection, we prove uniqueness of the weak solution of (19)–(21). The following result holds.

Proposition 3.2. *Consider a weak solution $f\in\mathrm{L}^\infty(0,T;\mathrm{L}^1(\mathbb{R}^3\times\mathbb{R}^3))\cap\mathrm{L}^\infty(0,T;\mathrm{L}^\infty(\mathbb{R}^3\times\mathbb{R}^3))$ to the system* (19)–(21) *in the sense of Definition 3.1 with $f_0\equiv0$. Assume that $\mathbf{G}\in\mathrm{L}^1(0,T;\mathrm{W}^{1,1}_{\mathrm{loc}}(\mathcal{B}\times\mathbb{R}^3))$, $\mathrm{div}_{\boldsymbol{\xi}}\,\mathbf{G}\in\mathrm{L}^\infty(0,T;\mathrm{L}^\infty(\mathcal{B}\times\mathbb{R}^3))$, and*

$$\frac{\mathbf{G}}{1+|\boldsymbol{\xi}|}\in\mathrm{L}^1(0,T;\mathrm{L}^1(\mathcal{B}\times\mathbb{R}^3))+\mathrm{L}^1(0,T;\mathrm{L}^\infty(\mathcal{B}\times\mathbb{R}^3)).\tag{34}$$

Then $f\equiv0$.

Proof. The proof follows the strategy developped in [17] and later used for instance in [31, 10]. The main idea is to use the renormalized weak formulation obtained in Theorem 3.2.

Let f be a bounded weak solution of (19)–(21) in the sense of Definition 3.1 with $f_0\equiv0$. Following Remark 3.7, $\hat{\gamma}f=0$ on $\widehat{\Sigma}^-$. Next, we choose appropriate test and renormalized functions in formulation (27). Thanks to Remark 3.4, we can choose $\beta(z)=|z|$. Let us also consider a nonnegative test function $\varphi\in\mathscr{D}(\mathbb{R}^3)$, only depending on the velocity unknown $\boldsymbol{\xi}$, such that $\varphi\equiv1$ if $|\boldsymbol{\xi}|\le1$ and $\varphi\equiv0$ if $|\boldsymbol{\xi}|\ge2$. Then we set $\varphi_R(\boldsymbol{\xi})=\varphi(\boldsymbol{\xi}/R)$ for any $R\ge1$, and use the weak formulation (27) to write, for any $t_1\in[0,T]$,

$$\int_{\Omega_{t_1}\times\mathbb{R}^3}|f(t_1)|\varphi_R+\int_0^{t_1}\int_{\widehat{\Sigma}^+_t}|\hat{\gamma}f|\varphi_R\,(\boldsymbol{\xi}-\boldsymbol{w})\cdot\boldsymbol{n}_t=\int_0^{t_1}\int_{\Omega_t\times\mathbb{R}^3}|f|\,\mathbf{G}\cdot\nabla_{\boldsymbol{\xi}}\varphi_R.$$

The boundary term is clearly nonnegative. We can then successively write

$$\begin{aligned}\int_{\Omega_{t_1}\times\mathbb{R}^3}|f(t_1)|\varphi_R&\le\int_0^{t_1}\int_{\Omega_t\times\mathbb{R}^3}|f|\,\mathbf{G}\cdot\nabla_{\boldsymbol{\xi}}\varphi_R\\&\le\frac{C}{R}\int_0^{t_1}\int_{\Omega_t\times\mathbb{R}^3}|f|\,|\mathbf{G}|\,\mathbb{1}_{R\le|\boldsymbol{\xi}|\le2R}\le C\int_0^{t_1}\int_{\Omega_t\times\mathbb{R}^3}|f|\,\frac{|\mathbf{G}|}{1+|\boldsymbol{\xi}|}\,\mathbb{1}_{R\le|\boldsymbol{\xi}|}.\end{aligned}$$

The last integral vanishes when R goes to $+\infty$, thanks to the assumptions on both f and $\mathbf{G}$. That eventually implies that $f(t_1,\cdot)=0$ for any t_1, and consequently $f\equiv0$. ☐

In the next section, we apply the previous results to Eqn. (13) with $\mathbf{G}(t,\boldsymbol{x},\boldsymbol{\xi}) = \chi(\boldsymbol{u}(t,\boldsymbol{x})-\boldsymbol{\xi})$.

4. Existence of solutions to the approximated problem

We want to prove that the approximated problem (11)–(13), supplemented with homogeneous Dirichlet boundary conditions for $\boldsymbol{u}$ on $(0,T)\times\partial\mathcal{B}$ and (5) for f, and initial data satisfying Assumptions 2.3–2.4, has at least one weak solution.

We must check that the mapping Θ_σ is well defined for any $\sigma\in[0,1]$. Choose $\boldsymbol{u}\in E$. We first prove that there exists a unique $f_{\boldsymbol{u}}$ solution of (13), with absorption boundary conditions and initial datum satisfying Assumption 2.4, such that $\overline{f_{\boldsymbol{u}}}\in \mathrm{L}^\infty(0,T;\mathrm{L}^1(\mathbb{R}^3\times\mathbb{R}^3))\cap\mathrm{L}^\infty(0,T;\mathrm{L}^\infty(\mathbb{R}^3\times\mathbb{R}^3))$. To do so, we just have to verify that $\mathbf{G}(t,\boldsymbol{x},\boldsymbol{\xi})=\chi(\boldsymbol{u}(t,\boldsymbol{x})-\boldsymbol{\xi})$ has the required regularity properties. Since $\boldsymbol{u}\in E$ and thanks to the properties of χ, it is clear that $\mathbf{G}\in\mathrm{L}^1(0,T;\mathrm{W}^{1,1}_{\mathrm{loc}}(\mathcal{B}\times\mathbb{R}^3))$. Because of the asymptotical properties of χ, χ' is bounded on $\mathbb{R}$, which obviously implies that $\mathrm{div}_{\boldsymbol{\xi}}\,\mathbf{G}\in\mathrm{L}^\infty(0,T;\mathrm{L}^\infty(\mathcal{B}\times\mathbb{R}^3))$. Eventually, since $E\hookrightarrow\mathrm{L}^2(0,T;\mathrm{L}^6(\mathcal{B}))$, we can write

$$\frac{|\chi(\boldsymbol{u}-\boldsymbol{\xi})|}{1+|\boldsymbol{\xi}|}\le\frac{|\boldsymbol{u}|}{1+|\boldsymbol{\xi}|}+1\le\frac{|\boldsymbol{u}|^6}{(1+|\boldsymbol{\xi}|)^6}+C,$$

for some constant C. The previous estimate clearly allows to obtain (34). Having existence and uniqueness of $f_{\boldsymbol{u}}$ allows to focus on the approximated Navier-Stokes equations (15)–(18).

In (15), we need to prove that the right-hand side term $\sigma\mathbf{S}_\chi(\boldsymbol{u})-\lambda\sigma\mathbb{1}_{\widehat{\mathcal{B}}\setminus\widehat{\Omega}}(\boldsymbol{u}-\boldsymbol{w})$ lies in $\mathrm{L}^2((0,T)\times\mathcal{B}))$. The penalty term clearly belongs to $\mathrm{L}^2((0,T)\times\mathcal{B}))$, because of the regularity of $\boldsymbol{u}$ and $\boldsymbol{w}$, and satisfies

$$\|\lambda\sigma\mathbb{1}_{\widehat{\mathcal{B}}\setminus\widehat{\Omega}}(\boldsymbol{u}-\boldsymbol{w})\|_{\mathrm{L}^2((0,T)\times\mathcal{B})}\le|\lambda|(\|\boldsymbol{u}\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathcal{B}))}+\|\boldsymbol{w}\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathcal{B}))}).\tag{35}$$

The regularity of $\mathbf{S}_\chi(\boldsymbol{u})$ is also clear. Indeed, $f_{\boldsymbol{u}}$ has a compact support in $\boldsymbol{\xi}$, see Remark 3.3, and lies in $\mathrm{L}^\infty(0,T;\mathrm{L}^1(\mathbb{R}^3\times\mathbb{R}^3))\cap\mathrm{L}^\infty(0,T;\mathrm{L}^\infty(\mathbb{R}^3\times\mathbb{R}^3))$, with

$$\begin{aligned}&\|\overline{f_{\boldsymbol{u}}}\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^1(\mathbb{R}^3\times\mathbb{R}^3))\cap\mathrm{L}^\infty(0,T;\mathrm{L}^\infty(\mathbb{R}^3\times\mathbb{R}^3))}\\&\qquad\le\exp(3\|\chi'\|_{\mathrm{L}^\infty(\mathbb{R})}T)\|f_{\mathrm{in}}\|_{\mathrm{L}^1(\Omega\times\mathbb{R}^3)\cap\mathrm{L}^\infty(\Omega\times\mathbb{R}^3)}.\end{aligned}\tag{36}$$

Using (24) with $\mathbf{G}=\chi(\boldsymbol{u}-\boldsymbol{\xi})$, and since χ is a bounded function, that allows to state that $\mathbf{S}_\chi(\boldsymbol{u})\in\mathrm{L}^2((0,T)\times\mathcal{B}))$, and

$$\|\mathbf{S}_\chi(\boldsymbol{u})\|_{\mathrm{L}^2((0,T)\times\mathcal{B})}\le C_T\|\chi\|_{\mathrm{L}^\infty(\mathbb{R})}\exp(3\|\chi'\|_{\mathrm{L}^\infty(\mathbb{R})}T)\|f_{\mathrm{in}}\|_{\mathrm{L}^1(\Omega\times\mathbb{R}^3)\cap\mathrm{L}^\infty(\Omega\times\mathbb{R}^3)}.\tag{37}$$

Note that the proof of the previous estimate strongly uses the compact support assumption for f_{in}. Consequently, since the convection velocity in (15) is regularized, we can apply standard existence and regularity results which hold for the Stokes system [27, 36], namely, there exists a unique solution $\widetilde{\boldsymbol{u}}_\sigma\in\mathrm{H}^1(0,T;\mathrm{L}^2(\mathcal{B}))\cap\mathrm{L}^2(0,T;\mathrm{H}^2(\mathcal{B}))\cap E$ to (15)–(18), which continuously depends on the data of the problem. That allows to define Θ_σ for any $\sigma\in[0,1]$, and is useful to check the required properties to apply Schaefer's fixed point theorem to the mapping Θ.

Let us first prove that Θ sends bounded subsets on relatively compact subsets. Indeed, using estimates (35)–(37) on the right-hand side of the regularized Navier-Stokes equation (15), we have

$$\begin{aligned}&\|\widetilde{\boldsymbol{u}}_\sigma\|_{\mathrm{L}^2(0,T;\mathrm{H}^2(\mathcal{B}))\cap\mathrm{H}^1(0,T;\mathrm{L}^2(\mathcal{B}))}\\&\qquad\le C\left(\|\boldsymbol{u}\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathcal{B}))},\|\boldsymbol{u}_{\mathrm{in}}\|_{\mathrm{H}^1(\mathcal{B})},\|f_{\mathrm{in}}\|_{\mathrm{L}^1(\Omega\times\mathbb{R}^3)\cap\mathrm{L}^\infty(\Omega\times\mathbb{R}^3)}\right),\end{aligned}\tag{38}$$

where C also depends on χ, φ, λ, $\boldsymbol{w}$ and T. Consequently, the previous estimate and the compact embedding of $\mathrm{H}^1(0,T;\mathrm{L}^2(\mathcal{B}))\cap\mathrm{L}^2(0,T;\mathrm{H}^2(\mathcal{B}))\cap E$ in E allow to obtain the required relative compactness property.

Let us now focus on the continuity of Θ. Consider a sequence $(\boldsymbol{u}_n, \sigma_n)_{n\in\mathbb{N}}$ converging towards $(\boldsymbol{u}, \sigma)$ in $E \times [0,1]$ when n goes to $+\infty$. Set $\widetilde{\boldsymbol{u}_n} = \Theta_{\sigma_n}(\boldsymbol{u}_n)$ for any n. We want to prove that $(\widetilde{\boldsymbol{u}_n})$ strongly converges in E towards $\widetilde{\boldsymbol{u}} = \Theta_\sigma(\boldsymbol{u})$. Thanks to (36), $(\overline{f_{\boldsymbol{u}_n}})$ is uniformly bounded in $\mathrm{L}^\infty(0,T;\mathrm{L}^1(\mathbb{R}^3 \times \mathbb{R}^3)) \cap \mathrm{L}^\infty(0,T;\mathrm{L}^\infty(\mathbb{R}^3 \times \mathbb{R}^3))$. Up to a subsequence, $(\overline{f_{\boldsymbol{u}_n}})$ weakly-$*$ converges in $\mathrm{L}^\infty(0,T;\mathrm{L}^\infty(\mathbb{R}^3 \times \mathbb{R}^3))$, and $(f_{\boldsymbol{u}_n})$ weakly converges in $\mathrm{L}^p_{\mathrm{loc}}(\widehat{\overline{\Omega}} \times \mathbb{R}^3)$, for any $p \in (1,+\infty)$. That allows us to pass to the limit in the weak formulation (22) of Definition 3.1, which, as stated in Remark 3.1, is equivalent to the one appearing in Definition 2.1. By uniqueness of the weak solution to the Vlasov equation, the weak limit is $f_{\boldsymbol{u}}$, and the whole sequence $(f_{\boldsymbol{u}_n})$ converges. Thanks to estimate (38), $(\widetilde{\boldsymbol{u}_n})$ is uniformly bounded in $\mathrm{H}^1(0,T;\mathrm{L}^2(\mathcal{B})) \cap \mathrm{L}^2(0,T;\mathrm{H}^2(\mathcal{B})) \cap E$, and thus compact in E. Hence, up to a subsequence, $(\widetilde{\boldsymbol{u}_n})$ strongly converges in E. We can then pass to the limit in the regularized Navier-Stokes system, also using the convergences of (u_n), (σ_n) and $(f_{\boldsymbol{u}_n})$. We obtain the continuity of Θ, again thanks to a uniqueness argument.

Eventually, let us prove that the set of all the fixed points of the family Θ_σ is bounded in E. Consider $\sigma \in [0,1]$, and $\boldsymbol{u} \in E$ such that $\boldsymbol{u} = \Theta_\sigma(\boldsymbol{u})$. To obtain the required boundedness property, we derive a priori bounds for both $f_{\boldsymbol{u}}$ and $\widetilde{\boldsymbol{u}}_\sigma = \boldsymbol{u}$. Since $f_{\boldsymbol{u}}$ is currently compactly supported in $\boldsymbol{\xi}$, we can rigorously apply (27) with $\mathbf{G} = \chi(\boldsymbol{u} - \boldsymbol{\xi})$, $\beta(z) = z$ and $\psi = |\boldsymbol{\xi}|^2$. We then obtain, also recalling the homogeneous boundary condition on each $\widehat{\Sigma}_s^-$,

$$2\int_0^t \int_{\Omega_s \times \mathbb{R}^3} f_{\boldsymbol{u}} \boldsymbol{\xi} \cdot \chi(\boldsymbol{u} - \boldsymbol{\xi}) = \int_{\Omega_t \times \mathbb{R}^3} f_{\boldsymbol{u}} |\boldsymbol{\xi}|^2 - \int_{\Omega_t \times \mathbb{R}^3} f_{\mathrm{in}} |\boldsymbol{\xi}|^2 + \int_0^t \int_{\widehat{\Sigma}_s^+} \gamma f_{\boldsymbol{u}} |\boldsymbol{\xi}|^2 (\boldsymbol{\xi} - \boldsymbol{w}) \cdot \boldsymbol{n}_s,$$

where the last term is clearly nonnegative. Consequently, we can write

$$\frac{1}{2} M_2 f_{\boldsymbol{u}}(t) \leq \frac{1}{2} M_2 f_{\mathrm{in}} + \int_0^t \int_{\Omega_s \times \mathbb{R}^3} f_{\boldsymbol{u}} \boldsymbol{\xi} \cdot \chi(\boldsymbol{u} - \boldsymbol{\xi}). \tag{39}$$

Besides, multiplying (15) by $\widetilde{\boldsymbol{u}}_\sigma = \boldsymbol{u}$, we get

$$\begin{aligned} \frac{1}{2}\|\boldsymbol{u}(t)\|^2_{\mathrm{L}^2(\mathcal{B})} + \int_0^t \|\nabla_{\boldsymbol{x}} \boldsymbol{u}\|^2_{\mathrm{L}^2(\mathcal{B})} &= \frac{\sigma}{2}\|\boldsymbol{u}_{\mathrm{in}}\|^2_{\mathrm{L}^2(\mathcal{B})} \\ &\quad + \sigma \int_0^t \int_{\mathcal{B}\times\mathbb{R}^3} \chi(\boldsymbol{\xi} - \boldsymbol{u}) \cdot \boldsymbol{u} \overline{f_{\boldsymbol{u}}} + \sigma\lambda \int_0^t \int_{\mathcal{B}} \mathbb{1}_{\widehat{\mathcal{B}} \setminus \widehat{\Omega}} (\boldsymbol{w} - \boldsymbol{u}) \cdot \boldsymbol{u}. \end{aligned} \tag{40}$$

Then, multiplying (39) by σ, and adding (40), we obtain, since $\sigma \in [0,1]$, and thanks to the properties of χ and Young's inequality,

$$\begin{aligned} \frac{\sigma}{2} M_2 f_{\boldsymbol{u}}(t) + \frac{1}{2}\|\boldsymbol{u}(t)\|^2_{\mathrm{L}^2(\mathcal{B})} + \int_0^t \|\nabla_{\boldsymbol{x}} \boldsymbol{u}\|^2_{\mathrm{L}^2(\mathcal{B})} + \sigma \int_0^t \int_{\mathcal{B}\times\mathbb{R}^3} \chi(\boldsymbol{u} - \boldsymbol{\xi}) \cdot (\boldsymbol{u} - \boldsymbol{\xi}) \overline{f_{\boldsymbol{u}}} \\ \leq \frac{\sigma}{2} M_2 f_{\mathrm{in}} + \frac{1}{2}\|\boldsymbol{u}_{\mathrm{in}}\|^2_{\mathrm{L}^2(\mathcal{B})} + \sigma\lambda \int_0^t \int_{\mathcal{B}} \mathbb{1}_{\widehat{\mathcal{B}} \setminus \widehat{\Omega}} (\boldsymbol{w} - \boldsymbol{u}) \cdot \boldsymbol{u} \\ \leq \frac{1}{2} M_2 f_{\mathrm{in}} + \frac{1}{2}\|\boldsymbol{u}_{\mathrm{in}}\|^2_{\mathrm{L}^2(\mathcal{B})} + \lambda \int_0^t \int_{\mathcal{B}} |\boldsymbol{w} \cdot \boldsymbol{u}| \\ \leq \frac{1}{2} M_2 f_{\mathrm{in}} + \frac{1}{2}\|\boldsymbol{u}_{\mathrm{in}}\|^2_{\mathrm{L}^2(\mathcal{B})} + \lambda^2 \|\boldsymbol{w}\|^2_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathcal{B}))} + \int_0^t \|\boldsymbol{u}\|^2_{\mathrm{L}^2(\mathcal{B})}. \end{aligned} \tag{41}$$

Gronwall's lemma then ensures that $\boldsymbol{u}$ is bounded in E by a quantity which only depends on the data of the problem and the regularization parameters.

Consequently, thanks to Schaefer's theorem, we can state that Θ_1 has a at least one fixed point, which leads to the required existence result.

5. Back to the whole system

The results obtained in Section 4 allows us to state the existence, for any $n \in \mathbb{N}^*$, of $(\boldsymbol{u}_n, f_n)$ weakly solving the following system

$$\partial_t \boldsymbol{u} + (\boldsymbol{u} \star \varphi_n) \cdot \nabla_{\boldsymbol{x}} \boldsymbol{u} + \nabla_{\boldsymbol{x}} p - \Delta_{\boldsymbol{x}} \boldsymbol{u} + n \mathbb{1}_{\widehat{\mathcal{B}} \setminus \widehat{\Omega}} (\boldsymbol{u} - \boldsymbol{w}) = \int_{\mathbb{R}^3} \chi_n(\boldsymbol{\xi} - \boldsymbol{u}) \overline{f} \quad \text{in } \widehat{\mathcal{B}}, \tag{42}$$

$$\operatorname{div}_{\boldsymbol{x}} \boldsymbol{u} = 0 \quad \text{in } \widehat{\mathcal{B}}, \tag{43}$$

$$\partial_t f + \boldsymbol{\xi} \cdot \nabla_{\boldsymbol{x}} f + \nabla_{\boldsymbol{\xi}} \cdot [\chi_n(\boldsymbol{u} - \boldsymbol{\xi}) f] = 0 \quad \text{in } \widehat{\Omega} \times \mathbb{R}^3, \tag{44}$$

where (φ_n) is a mollifying sequence. The truncation functions χ_n satisfy the same assumptions as χ in the previous sections and are furthermore chosen such that (χ_n) converges to the identity mapping on $\mathbb{R}$ when n goes to $+\infty$. However, note that χ_n is not uniformly bounded with respect to n. Moreover, the system is supplemented with regularized initial conditions (see Assumptions 2.3–2.4)

$$\boldsymbol{u}(0, \cdot) = \boldsymbol{u}_{\text{in}}^n \quad \text{in } \Omega, \qquad f(0, \cdot) = f_{\text{in}}^n \quad \text{in } \Omega \times \mathbb{R}^3.$$

The sequences $(\boldsymbol{u}_{\text{in}}^n)$ and (f_{in}^n) are chosen such that $(\boldsymbol{u}_{\text{in}|\Omega}^n)$ strongly converges to $\boldsymbol{u}_{\text{in}}$ in $\mathrm{L}^2(\Omega)$, that (f_{in}^n) strongly converges to f_{in} in $\mathrm{L}^p(\Omega \times \mathbb{R}^3)$ for all $1 \le p < +\infty$, weakly$*$ in $\mathrm{L}^\infty(\Omega \times \mathbb{R}^3)$, and that $(M_2 f_{\text{in}}^n)$ is uniformly bounded with respect to n and strongly converges to $M_2 f_{\text{in}}$ in $\mathrm{L}^\infty(\Omega)$.

Remark 5.1. *Note that, for a given n, $\boldsymbol{u}_n$ is in fact a strong solution to the Navier-Stokes equations, satisfied in $\mathrm{L}^2(0,T; L^2(\mathcal{B}))$, whereas f_n is only a weak solution of the Vlasov equation in the sense of Definition 2.1.*

We need to obtain, for the previous approximated coupled problem, an energy estimate similar to (41), but which does not depend on the regularization parameter n. It relies on a proper treatment of the penalty term. Instead of having

$$n \int_0^t \int_{\mathcal{B}} \mathbb{1}_{\widehat{\mathcal{B}} \setminus \widehat{\Omega}} (\boldsymbol{w} - \boldsymbol{u}_n) \cdot \boldsymbol{u}_n$$

as in the first inequality of (41), we bring out

$$n \int_0^t \int_{\mathcal{B}} \mathbb{1}_{\widehat{\mathcal{B}} \setminus \widehat{\Omega}} (\boldsymbol{w} - \boldsymbol{u}_n)^2,$$

which is nonnegative. To recover the previous term, we rewrite (41) as

$$\begin{aligned} \frac{1}{2} M_2 f_n(t) + \frac{1}{2} \|\boldsymbol{u}_n(t)\|_{\mathrm{L}^2(\mathcal{B})}^2 + \int_0^t \|\nabla_{\boldsymbol{x}} \boldsymbol{u}_n\|_{\mathrm{L}^2(\mathcal{B})}^2 \\ + \int_0^t \int_{\mathcal{B} \times \mathbb{R}^3} \chi_n(\boldsymbol{u}_n - \boldsymbol{\xi}) \cdot (\boldsymbol{u}_n - \boldsymbol{\xi}) \overline{f_n} + n \int_0^t \int_{\mathcal{B}} \mathbb{1}_{\widehat{\mathcal{B}} \setminus \widehat{\Omega}} (\boldsymbol{u}_n - \boldsymbol{w}) \cdot \boldsymbol{u}_n \\ \le \frac{1}{2} M_2 f_{\text{in}}^n + \frac{1}{2} \|\boldsymbol{u}_{\text{in}}^n\|_{\mathrm{L}^2(\mathcal{B})}^2. \end{aligned} \tag{45}$$

Then we multiply (42) by $\boldsymbol{w}$ and integrate on $\widehat{\mathcal{B}}$. After integration by parts on both variables t and $\boldsymbol{x}$, remembering that $\boldsymbol{w} = 0$ on $\partial \mathcal{B}$ and $\operatorname{div}_{\boldsymbol{x}}(\boldsymbol{u}_n \star \varphi_n) = 0$, we get

$$\begin{aligned} -n \int_0^t \int_{\mathbb{R}^3} \mathbb{1}_{\widehat{\mathcal{B}} \setminus \widehat{\Omega}} (\boldsymbol{u}_n - \boldsymbol{w}) \cdot \boldsymbol{w} = \int_{\mathcal{B}} \boldsymbol{u}_n(t) \cdot \boldsymbol{w}(t) - \int_{\mathcal{B}} \boldsymbol{u}_{\text{in}}^n \cdot \boldsymbol{w}(0) \\ - \int_0^t \int_{\mathcal{B}} \boldsymbol{u}_n \cdot \partial_t \boldsymbol{w} + \int_0^t \int_{\mathcal{B}} \nabla_{\boldsymbol{x}} \boldsymbol{u}_n : \nabla_{\boldsymbol{x}} \boldsymbol{w} \\ - \int_0^t \int_{\mathcal{B}} (\boldsymbol{u}_n \star \varphi_n) \otimes \boldsymbol{u}_n : \nabla_{\boldsymbol{x}} \boldsymbol{w} + \int_0^t \int_{\mathcal{B}} \left[\int_{\mathbb{R}^3} \overline{f_n} \chi_n(\boldsymbol{u}_n - \boldsymbol{\xi}) \right] \cdot \boldsymbol{w}. \end{aligned}$$

Cauchy-Schwarz and Young's inequalities imply, together with the nonnegativity of f_n, that

$$
\begin{aligned}
-n\int_0^t\int_{\mathbb{R}^3}\mathbb{1}_{\widehat{\mathcal{B}}\setminus\widehat{\Omega}}(\boldsymbol{u}_n-\boldsymbol{w})\cdot\boldsymbol{w} \leq{}& \frac{1}{4}\|\boldsymbol{u}_n(t)\|^2_{L^2(\mathcal{B})}+\frac{1}{2}\|\boldsymbol{u}^n_{\mathrm{in}}\|^2_{L^2(\mathcal{B})}\\
&+\mathrm{C}_{\boldsymbol{w}}\int_0^t\|\boldsymbol{u}_n\|^2_{L^2(\mathcal{B})}+\frac{1}{2}\int_0^t\|\nabla_{\boldsymbol{x}}\boldsymbol{u}_n\|^2_{L^2(\mathcal{B})}\\
&+\mathrm{C}_{\boldsymbol{w}}+\mathrm{C}_{\boldsymbol{w}}\int_0^t M_0f_n+\frac{1}{2}\int_0^t\int_{\mathcal{B}\times\mathbb{R}^3}\overline{f_n}|\chi_n(\boldsymbol{u}_n-\boldsymbol{\xi})|^2.
\end{aligned}
$$

If we add (45) to the previous estimate, we obtain, thanks to the inequality $\chi_n(z)^2\leq\chi_n(z)z$ for any $z\in\mathbb{R}$, and the nonnegativity of f_n,

$$
\begin{aligned}
&\frac{1}{2}M_2f_n(t)+\frac{1}{4}\|\boldsymbol{u}_n(t)\|^2_{\mathrm{L}^2(\mathcal{B})}+\frac{1}{2}\int_0^t\|\nabla_{\boldsymbol{x}}\boldsymbol{u}_n\|^2_{\mathrm{L}^2(\mathcal{B})}\\
&\qquad+\frac{1}{2}\int_0^t\int_{\mathcal{B}\times\mathbb{R}^3}\overline{f_n}|\chi_n(\boldsymbol{u}_n-\boldsymbol{\xi})|^2+n\int_0^t\int_{\mathcal{B}}\mathbb{1}_{\widehat{\mathcal{B}}\setminus\widehat{\Omega}}(\boldsymbol{u}_n-\boldsymbol{w})^2\\
&\qquad\qquad\leq\frac{1}{2}M_2f^n_{\mathrm{in}}+\|\boldsymbol{u}^n_{\mathrm{in}}\|^2_{\mathrm{L}^2(\mathcal{B})}+\mathrm{C}_{\boldsymbol{w}}\int_0^t\|u_n\|^2_{L^2(\mathcal{B})}+\mathrm{C}_{\boldsymbol{w}}\int_0^t M_0f_n+\mathrm{C}_{\boldsymbol{w}}.
\end{aligned}
$$

Since, for a given n, f_n is compactly supported, we straightforwardly have, from the weak formulation of the Vlasov equation in Definition 2.1,

$$
\int_0^t M_0f_n\leq TM_0f^n_{\mathrm{in}}.
$$

Taking into account the boundedness and convergence properties of the sequences of initial data, we deduce

$$
\begin{aligned}
&\frac{1}{2}M_2f_n(t)+\frac{1}{4}\|\boldsymbol{u}_n(t)\|^2_{\mathrm{L}^2(\mathcal{B})}+\frac{1}{2}\int_0^t\|\nabla_{\boldsymbol{x}}\boldsymbol{u}_n\|^2_{\mathrm{L}^2(\mathcal{B})}+n\int_0^t\int_{\mathcal{B}}\mathbb{1}_{\widehat{\mathcal{B}}\setminus\widehat{\Omega}}(\boldsymbol{u}_n-\boldsymbol{w})^2\\
&\qquad\leq\frac{1}{2}M_2f_{\mathrm{in}}+\|\boldsymbol{u}_{\mathrm{in}}\|^2_{\mathrm{L}^2(\mathcal{B})}+\mathrm{C}_{\boldsymbol{w}}\int_0^t\|\boldsymbol{u}_n\|^2_{L^2(\mathcal{B})}+T\mathrm{C}_{\boldsymbol{w}}M_0f_{\mathrm{in}}+\mathrm{C}_{\boldsymbol{w}}.
\end{aligned}
\tag{46}
$$

Hence, thanks to Gronwall's lemma, we get the following bound for $(\boldsymbol{u}_n)$ and (f_n):

$$
\|\boldsymbol{u}_n\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^2(\mathcal{B}))\cap\mathrm{L}^2(0,T;\mathrm{H}^1(\mathcal{B}))}+\|M_2f_n\|_{\mathrm{L}^\infty(0,T)}\leq\mathrm{C}_{\boldsymbol{w}}(T,\|\boldsymbol{u}_{\mathrm{in}}\|_{\mathrm{L}^2(\mathcal{B})},M_0f_{\mathrm{in}},M_2f_{\mathrm{in}}),
\tag{47}
$$

where the constant depends on $\boldsymbol{w}$, T, $\|\boldsymbol{u}_{\mathrm{in}}\|_{\mathrm{L}^2(\mathcal{B})}$, M_0f_{in} and M_2f_{in}.

Estimates (46)–(47) ensures that

$$
(\boldsymbol{u}_n)\to\boldsymbol{w}\quad\text{strongly in }\mathrm{L}^2(\widehat{\mathcal{B}}\setminus\overline{\widehat{\Omega}}).
$$

The previous bounds also imply that there exists $(\boldsymbol{u},f)$ such that $\boldsymbol{u}\in\mathrm{L}^\infty(0,T;\mathrm{L}^2(\mathcal{B}))\cap\mathrm{L}^2(0,T;\mathrm{H}^1(\mathcal{B}))$, $\overline{f}\in\mathrm{L}^\infty(0,T;\mathrm{L}^\infty(\mathbb{R}^3\times\mathbb{R}^3)\cap\mathrm{L}^1(\mathbb{R}^3\times\mathbb{R}^3))$, and that the following weak convergences hold, up to a subsequence:

$$
\boldsymbol{u}_n\rightharpoonup\boldsymbol{u},\qquad\nabla_{\boldsymbol{x}}\boldsymbol{u}_n\rightharpoonup\nabla_{\boldsymbol{x}}\boldsymbol{u}\quad\text{weakly in }\mathrm{L}^2(0,T;\mathrm{L}^2(\mathcal{B})),
$$

$$
\boldsymbol{u}_n\rightharpoonup\boldsymbol{u}\quad\text{weakly}*\text{ in }\mathrm{L}^\infty(0,T;\mathrm{L}^2(\mathcal{B})),
$$

and

$$
\overline{f_n}\rightharpoonup\overline{f}\quad\text{weakly}*\text{ in }\mathrm{L}^\infty(0,T;\mathrm{L}^\infty(\mathbb{R}^3\times\mathbb{R}^3)).
$$

Besides, to perform the asymptotics $n\to+\infty$ in the weak formulations of (42)–(44), in particular to handle the convection and coupling terms, we need some strong convergence on the fluid velocity. A standard way to obtain compactness is the Aubin-Lions lemma, which requires uniform bounds on $\partial_t\boldsymbol{u}_n$. Nevertheless, because of the time dependence of the domain and our approximation penalty strategy, this lemma cannot directly be applied to $(\boldsymbol{u}_n)$. Various compactness proofs of the fluid velocity can be found, in the very same context, for instance, in [19, 35]. Here we choose to apply [32, Theorem 4 and Remark 4.5].

More precisely, we know, thanks to the energy estimate, that $(\boldsymbol{u}_n)$ is bounded in $\mathrm{L}^\infty(0,T;\mathrm{L}^2(\mathcal{B}))\cap\mathrm{L}^2(0,T;\mathrm{H}^1_0(\mathcal{B}))$. In order to apply [32, Theorem 4], we have to prove two properties.

The first one is the compactness of the normal trace of $(\boldsymbol{u}_n)$ in $\mathrm{L}^2(0,T;\mathrm{H}^{-1/2}(\partial\Omega_t))$. This control is obtained since $(\boldsymbol{u}_n)$ strongly converges towards $\boldsymbol{w}$ in $\mathrm{L}^2(\widehat{\mathcal{B}}\setminus\overline{\widehat{\Omega}})$, and the normal trace operator on $\widehat{\Gamma}$ is continuous from $\{\boldsymbol{\psi}\in\mathrm{L}^2(\widehat{\mathcal{B}}\setminus\overline{\widehat{\Omega}})\mid \operatorname{div}_{\boldsymbol{x}}\boldsymbol{\psi}\in\mathrm{L}^2(\widehat{\mathcal{B}}\setminus\overline{\widehat{\Omega}})\}$ to $\mathrm{L}^2(0,T;\mathrm{H}^{-1/2}(\mathcal{B}\setminus\Omega_t))$.

The second property is the following: for any test function $\boldsymbol{\psi}\in\mathscr{D}(\widehat{\Omega})$ such that $\operatorname{div}_{\boldsymbol{x}}\boldsymbol{\psi}=0$ on $\widehat{\Omega}$,

$$\left|\int_0^T\int_{\mathcal{B}}\partial_t\boldsymbol{u}_n\,\boldsymbol{\psi}\right|\le \mathrm{C}\sum_{|\alpha|\le 2}\|\partial^\alpha_{\boldsymbol{x}}\boldsymbol{\psi}\|_{\mathrm{L}^2(\widehat{\Omega})},\tag{48}$$

where C is a constant only depending on the data of the problem. This bound is obtained using the weak form of the Navier-Stokes equations, and relies on the already derived energy estimate, and on a uniform bound of $(\mathbf{S}_{\chi_n}(\boldsymbol{u}_n))$. Since $\boldsymbol{\psi}$ is compactly-supported in $\widehat{\Omega}$, the integral involving the penalty term cancels. As for $(\mathbf{S}_{\chi_n}(\boldsymbol{u}_n))$, we proceed as follows. Recalling that $|\chi_n(z)|\le|z|$, we have, almost everywhere,

$$|\mathbf{S}_{\chi_n}(\boldsymbol{u}_n)|\le m_1f_n+|\boldsymbol{u}_n|m_0f_n.$$

We know that (M_2f_n) is bounded in $\mathrm{L}^\infty(0,T)$ thanks to the energy estimate (47). Hence, Lemma 2.1 ensures that (m_1f_n) and (m_0f_n) are respectively bounded in $\mathrm{L}^\infty(0,T;\mathrm{L}^{5/4}(\mathcal{B}))$ and $\mathrm{L}^\infty(0,T;\mathrm{L}^{5/3}(\mathcal{B}))$. Since $(\boldsymbol{u}_n)$ is bounded in $\mathrm{L}^2(0,T;\mathrm{H}^1(\mathcal{B}))$, which is continuously embedded in $\mathrm{L}^2(0,T;\mathrm{L}^6(\mathcal{B}))$, we eventually get the boundedness of $(\mathbf{S}_{\chi_n}(\boldsymbol{u}_n))$ in the space $\mathrm{L}^2(0,T;\mathrm{L}^1(\mathcal{B}))$. Estimate (48) follows from

$$\left|\int_0^T\int_{\mathcal{B}}\mathbf{S}_{\chi_n}(\boldsymbol{u}_n)\cdot\boldsymbol{\psi}\right|\le\|\mathbf{S}_{\chi_n}(\boldsymbol{u}_n)\|_{\mathrm{L}^2(0,T;\mathrm{L}^1(\mathcal{B}))}\|\boldsymbol{\psi}\|_{\mathrm{L}^2(0,T;\mathrm{L}^\infty(\mathcal{B}))}\le\mathrm{C}\sum_{|\alpha|\le 2}\|\partial^\alpha_{\boldsymbol{x}}\boldsymbol{\psi}\|_{\mathrm{L}^2(\widehat{\Omega})},$$

where the last inequality is a consequence of the embedding of H^2 in L^∞ in three dimensions.

This allows to let n go to $+\infty$ in the weak formulation of the regularized coupled system to obtain a weak solution to (1)–(5) as given in Definition 2.1. In particular, the strong convergence of $(\boldsymbol{u}_n)$ enables to pass to the limit in the nonlinear terms (convection and coupling terms). Moreover, we prove that M_2f lies in $\mathrm{L}^\infty(0,T)$ by performing the asymptotics in the energy estimate (47), following the very same method as in [7, p. 1268].

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L.B.: Sorbonne Universités, UPMC Univ Paris 06 & CNRS, UMR 7598 LJLL, Paris, F-75005, France & Inria, EPC Reo, BP 105, F-78153 Le Chesnay Cedex, France
E-mail address: laurent.boudin@upmc.fr

C.G.: Inria, EPC Reo, F-78153 Le Chesnay Cedex, France & Sorbonne Universités, UPMC Univ Paris 06 & CNRS, UMR 7598 LJLL, Paris, F-75005, France
E-mail address: celine.grandmont@inria.fr

A.M.: Sorbonne Universités, UPMC Univ Paris 06 & CNRS, UMR 7598 LJLL, Paris, F-75005, France
E-mail address: ayman.moussa@upmc.fr