Critical Point Computations on Smooth Varieties: Degree and Complexity bounds

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Abstract

Let $V \subset \mathbb{C}^n$ be an equidimensional algebraic set and g be an n-variate polynomial with rational coefficients. Computing the critical points of the map that evaluates g at the points of V is a cornerstone of several algorithms in real algebraic geometry and optimization. Under the assumption that the critical locus is finite and that the projective closure of V is smooth, we provide sharp upper bounds on the degree of the critical locus which depend only on $\deg(g)$ and the degrees of the generic polar varieties associated to V. Hence, in some special cases where the degrees of the generic polar varieties do not reach the worst-case bounds, this implies that the number of critical points of the evaluation map of g is less than the currently known degree bounds. We show that, given a lifting fiber of V, a slight variant of an algorithm due to Bank, Giusti, Heintz, Lecerf, Matera and Solernó computes these critical points in time which is quadratic in this bound up to logarithmic factors, linear in the complexity of evaluating the input system and polynomial in the number of variables and the maximum degree of the input polynomials.

1 Introduction

Problem statement. Let $\mathbf{f} = (f_1, \ldots, f_p) \subset \mathbb{Q}[X_1, \ldots, X_n]$ be a polynomial system defining a smooth and equidimensional algebraic set $V \subset \mathbb{C}^n$ of dimension d and $g \in \mathbb{Q}[X_1, \ldots, X_n]$ be a polynomial of degree D. We focus on the complexity of computing the critical points of the map evaluating g at the points of V. These critical points are defined by $f_1 = \cdots = f_p = 0$ and by the simultaneous vanishing of the $(n-d+1)$-minors of the jacobian matrix $\mathrm{jac}(\mathbf{f}, g)$

$$\begin{bmatrix} \frac{\partial f_1}{\partial X_1} & \cdots & \frac{\partial f_1}{\partial X_n} \\ \vdots & & \vdots \\ \frac{\partial f_p}{\partial X_1} & \cdots & \frac{\partial f_p}{\partial X_n} \\ \frac{\partial g}{\partial X_1} & \cdots & \frac{\partial g}{\partial X_n} \end{bmatrix}.$$

Usually, in symbolic computation, when the critical locus is finite, we aim at computing a rational parametrization of it, which is the data $((q, v_1, \dots, v_n), \lambda)$ where q and the v_i's lie in $\mathbb{Q}[T]$ (T is a new variable) and λ is a linear form in $X_1, \dots, X_n$ with q square-free, $\deg(v_i) < \deg(q)$, $\lambda(v_1, \dots, v_n) = T\frac{\partial q}{\partial T} \mod q$ and the set defined by

$$q(\tau) = 0, \qquad X_i = \frac{v_i}{\partial q/\partial T}(\tau) \quad 1 \leq i \leq n$$

coincides with the critical locus under consideration. Observe that the degree of q coincides with the number of critical points and the number of rational numbers required in such a rational parametrization is $O(n \deg(q))$.

Assuming again the critical locus to be finite, several bounds on its cardinality have been established (see [26] and references therein). These bounds depend on n, p and the degrees of $f_1, \dots, f_p$ and g. However, it has been remarked that when V is not a complete intersection, or when it has some special properties, the cardinality of the critical locus may be far less than these bounds and sometimes depends only on D and on some quantities attached to geometric objects. These latter objects are *polar varieties* (see [2, 4]); they may be understood as the critical loci of the restriction to V of projections on generic linear subspaces ; we define them further precisely.

Assuming the smoothness of the projective closure of V and the finiteness of the critical locus under study, this paper adresses the following topical questions.

- Provide a bound on the number of complex critical points depending on D and on the degrees of the generic polar varieties associated to V.
- Find an algorithm computing a rational parametrization of this critical locus within an arithmetic complexity which is essentially *quadratic* in the obtained bound and polynomial in p, n, the complexity of evaluating $\mathbf{f}, g$ and $\max_{1 \leq i \leq p}(\deg(f_i))$.

Motivations and prior works. Since local extrema of the evaluation map of g are reached at critical points, computing critical points is a basic and useful task for polynomial optimization (see *e.g.* [18, 19, 30]). Because of their topological properties related to Morse theory, computing critical points is also a subroutine for many modern algorithms in real algebraic geometry yielding asymptotically optimal complexities (see *e.g.* [20, 21, 29] and [6] for a textbook reference on this family of algorithms). Polar varieties have been introduced in [2] for computing sample points in each connected component of a real algebraic set and this technique has been developed in [3, 31]; they are also used for computing roadmaps for deciding connectivity queries [7, 8, 32, 33], for computing the real dimension of a real algebraic set (see [5] and references therein) or for variant quantifier elimination [23].

Some bounds on the cardinality of the critical locus under consideration are given in [26] when $f_1, \dots, f_p$ is a regular sequence. These bounds depend on the degrees of the f_i's, D, n and p.

Since polynomial systems appearing in applications arise most of the time with a special structure, a natural question to ask for is to identify situations where the cardinality of the considered critical locus is less than what the worst case bounds predicted in [26].

Such situations have been exhibited in [9] where critical points are used in computational statistics via the notion of ML degree. When $\deg(g) = 2$ the cardinality of the critical locus is bounded by the *generic ED degree* of V which depends only on the degrees of the generic polar varieties associated to V [10]. These bounds do not require any smoothness assumption. The results on polar varieties in [22, 28] play a central role in this setting.

On the algorithmic side, many recent works have focused on the complexity of computing critical loci. The results in [14, 35] provide complexity bounds for computing critical points using Gröbner bases under genericity assumptions on the input polynomials $\mathbf{f}$. The obtained complexity bounds are not quadratic in the generic number of critical points and the genericity assumptions are not well-suited to the situations we are willing to consider. The results in [1] provide complexity bounds for a probabilistic algorithm computing degeneracy loci in time quadratic in an intrinsic quantity called the *system degree*. This work is strongly related to the algorithmic framework of the solver proposed in [17] and to computational aspects of polar varieties which have been deeply investigated in the last decades, see [2, 3, 4] and references therein. We will use a slight variant of [1] for our algorithmic contribution.

Main results. Under some smoothness assumptions which are precised below, we prove a bound on the number of complex critical points of the map $x \in V \to g(x)$ depending on the degree of g and integers $\delta_1(V), \ldots, \delta_{d+1}(V)$. The number $\delta_{i+1}(V)$ is the degree of the polar variety of V associated to a generic linear projection on $\mathbb{C}^{i+1}$. In the sequel, given $a = (a_1, \ldots, a_i) \subset \mathbb{C}^{ni}$, we denote by $W((g, a), V)$ the critical locus of the map $x \in V \to (g(x), a_1 \cdot x, \ldots, a_i \cdot x)$. We also denote by $\mathbb{C}[X_1, \ldots, X_n]_{\leq D}$ the set of polynomials in $\mathbb{C}[X_1, \ldots, X_n]$ of total degree $\leq D$.

Theorem 1. *Let $V \subset \mathbb{C}^n$ be a d-equidimensional algebraic set whose projective closure is smooth and $D \geq 1$ and $i \in \{0, \ldots, d\}$. There exists a Zariski dense subset $\Omega_i \subset \mathbb{C}[X_1, \ldots, X_n]_{\leq D} \times \mathbb{C}^{i \times n}$ such that for any $(g, a) \in \Omega$, the degree of $W((g, a), V) \subset \mathbb{C}^n$ is bounded by*

$$\deg(W((g, a), V)) \leq \begin{cases} \delta_{i+1}(V) \text{ if } D = 1 \\ \sum_{j=i}^{d} \delta_{j+1}(V)(D-1)^{j-i} \text{ otherwise.} \end{cases}$$

One of the central ingredient of the proof is an algebraic version of Thom's weak transversality Theorem. We use a formalism and notation similar to [33, Sec 4.2] which provides a proof of this result using charts and atlases. We also show that this degree bound holds under milder assumptions on g: it is sufficient to assume that the evaluation map of g has finitely many critical points on V. Let us see how such a bound behaves on an example.

Example 2. *Let $V \subset \mathbb{C}^8$ be the set of points $(x_1, \ldots, x_8)$ where the matrix*

$$\begin{bmatrix} 1 & x_1 & x_2 \\ x_3 & x_4 & x_5 \\ x_6 & x_7 & x_8 \end{bmatrix}$$

has rank 1. This variety has dimension 4, degree 6 and $(\delta_1, \ldots, \delta_5) = (1, 4, 10, 12, 6)$. Consider $g = \sum_{i=1}^{8} i x_i^3$. Representing an open subset of V as the zero locus of a reduced regular sequence of quadratic polynomials $f_1, \ldots, f_4$, bounds depending on the degrees of $f_1, \ldots, f_4, g$ (see e.g. *[26, Thm. 2.2]) would give the upper bound 2608 for the number of complex critical points. Theorem 1 and its variant for non-generic objective functions (see Prop. 12 below) yield the bound $241 = \delta_1 + 2\delta_2 + 4\delta_3 + 8\delta_4 + 16\delta_5$. Computations show that the evaluation map of g restricted to V has actually exactly 241 complex critical points on V.*

A convenient representation of an equidimensional variety V of dimension d is a *lifting fiber* for V (see [17]). Roughly speaking, this lifting fiber consists in a rational parametrization of the (finite) set of points in a section of V by a $(n - d)$-dimensional affine plane, together with a *lifting system* which allows to reconstruct a curve in the variety by symbolic Newton-Hensel iteration. Assuming that a lifting fiber of V has been precomputed and that $\deg(g) \geq 2$, we use the algorithm proposed in [1] to compute the critical points. Since this algorithm handles the more general case of quasi-affine varieties, so does the proposed variant. However, our main complexity results hold only under the assumption that the projective closure of V is smooth and that the evaluation map of g has finitely many critical points on V. Our second main result is a proof that the arithmetic complexity is quadratic up to logarithmic factors in the degree bound from Theorem 1, polynomial in n, the maximum of the degrees of the lifting system, $\deg(g)$, and the complexity of evaluating the lifting system and g.

Organization of the paper. Section 2 describes notation and preliminary results used throughout this paper. Section 3 is devoted to the proof of Theorem 1. It relies on a transversality result which is proved in Section 4. Section 5 deals with nongeneric objective functions. Finally, Section 6 discusses algorithmic aspects and complexity bounds.

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2 Preliminaries

2.1 Notation and basic definitions

We refer to [34] and [12] for basic definitions about algebraic sets and polynomial ideals. Given an algebraic set $V \subset \mathbb{C}^n$, we denote by $I(V)$ the ideal associated to V. Given $\mathbf{f} = (f_1, \dots, f_p)$ in $\mathbb{Q}[X_1, \dots, X_n]$, the set of their common solutions in $\mathbb{C}^n$ is denoted by $Z(\mathbf{f})$ and the ideal generated by $\mathbf{f}$ is denoted by $\langle \mathbf{f} \rangle$. We say that $\mathbf{f} = (f_1, \dots, f_p)$ is a reduced sequence when the ideal $\langle \mathbf{f} \rangle$ generated by $\mathbf{f}$ is radical.

Tangent spaces, regular and singular points. Let $V \subset \mathbb{C}^n$ be an algebraic set. For $x \in V$, the tangent space $T_x V$ at x to V is the vector space defined by $\sum_{i=1}^{n} \frac{\partial f}{\partial X_i}(x) Y_i = 0$ for any $f \in I(V)$. Also, given a finite set of generators $\mathbf{f} = (f_1, \dots, f_p)$ of $I(V)$, $T_x V$ is the kernel of the jacobian matrix $\mathrm{jac}(\mathbf{f}) = \left(\frac{\partial f_i}{\partial X_j}\right)_{\substack{1 \leq i \leq p \\ 1 \leq i \leq n}}$. We denote by $N_x V$ the orthogonal complement to $T_x V$.

Assume now that V is d-equidimensional. The set of points $x \in V$ where $\dim(T_x V) = d$ is the set of regular points of V; we denote it by $\mathrm{reg}(V)$. The subset of singular points $\mathrm{sing}(V)$ is the complement of $\mathrm{reg}(V)$ in V; it has dimension less than d. Observe that given a finite set of generators $\mathbf{f}$ of $I(V)$, $\mathrm{jac}(\mathbf{f})$ has rank $n - d$ at all $x \in \mathrm{reg}(V)$. Also, $N_x V$ is generated by the gradient vectors of the polynomials in $\mathbf{f}$ evaluated at x. An equidimensional algebraic set V is said to be smooth when $\mathrm{sing}(V)$ is empty.

Zariski topology. The Zariski topology over $\mathbb{C}^n$ is the topology for which the closed sets are the algebraic sets of $\mathbb{C}^n$. Let $f \in \mathbb{C}[X_1, \dots, X_n]$; we denote by $\mathcal{O}(f) \subset \mathbb{C}^n$ the subset defined by $f \neq 0$; it is a Zariski open set, which is non-empty when f is not identically 0. Further, we will prove some properties depending on parameters that are generically chosen. That means that, in the parameter space, there exists a non-empty Zariski open set such that the property is satisfied for any choice of the parameter values in this set.

Projective varieties. We will consider algebraic sets in the projective space $\mathbb{P}^n(\mathbb{C})$ defined by homogeneous polynomials. In the sequel, we use the shorthand $\mathbb{P}^n$ for $\mathbb{P}^n(\mathbb{C})$.

Let $V \subset \mathbb{P}^n$ be a projective variety. Notions of dimension, tangent space and regular (resp. singular) spaces extend to projective varieties. We denote by $\mathsf{aff}(V) \subset \mathbb{C}^{n+1}$ the Zariski closure of the set

$\{(x_0, \ldots, x_n) \subset \mathbb{C}^{n+1} \mid (x_0 : \cdots : x_n) \in V\}$. The variety $\mathsf{aff}(V)$ is an *affine cone* (for all $x \in \mathsf{aff}(V)$, $\lambda \in \mathbb{C}$ we have $\lambda x \in \mathsf{aff}(V)$). By a slight abuse of notation, when V is an algebraic set of $\mathbb{C}^n$, we also denote by $\mathsf{aff}(V)$ the affine cone of the projective closure of V. Let now $V' \subset \mathbb{C}^{n+1}$ be an affine cone. Observe that the map $\mathsf{proj} : (x_0, \ldots, x_n) \in V' \setminus \{\mathbf{0}\} \to (x_0 : \cdots : x_n) \in \mathbb{P}^n$ sends $V' \setminus \{\mathbf{0}\}$ to a projective set. Besides, for a projective variety $V \subset \mathbb{P}^n$, $\mathsf{proj}(\mathsf{aff}(V)) = V$. We also consider bi-projective varieties lying in $\mathbb{P}^n \times \mathbb{P}^n$. The above constructions extend similarly: to any variety $V \subset \mathbb{P}^n \times \mathbb{P}^n$ can be associated a cone $\mathsf{aff}(V) \subset \mathbb{C}^{n+1} \times \mathbb{C}^{n+1}$ which is the Zariski closure of the set of points $(x_0, \ldots, x_n, y_0, \ldots, y_n)$ such that $((x_0 : \cdots : x_n), (y_0 : \cdots : y_n)) \in V$. The map proj is extended in the following way: $\mathsf{proj} : (x, y) \in (\mathbb{C}^{n+1} \setminus \{\mathbf{0}\}) \times (\mathbb{C}^{n+1} \setminus \{\mathbf{0}\}) \to (x, y) \in \mathbb{P}^n \times \mathbb{P}^n$.

2.2 Atlases and transversality

Let $V \subset \mathbb{C}^n$ be a d-equidimensional algebraic set of codimension c and $S \subset V$ be a subset. Following the terminology in [33, Chap. 5], an atlas for (V, S) is a finite sequence $\boldsymbol{\psi} = ((\mathbf{h}_j, m_j))_{1 \leq j \leq \ell}$, with $\mathbf{h}_j = (h_{j,1}, \ldots, h_{j,c}) \subset \mathbb{C}[X_1, \ldots, X_n]$ and $m_j \in \mathbb{C}[X_1, \ldots, X_n]$ such that for all $1 \leq j \leq \ell$ the following holds:

- P_1 $\mathcal{O}(m_j) \cap (V \setminus S) = \mathcal{O}(m_j) \cap (Z(\mathbf{h}_j) \setminus S)$;
- P_2 $\mathcal{O}(m_j) \cap (V \setminus S)$ is not empty;
- P_3 for all $x \in \mathcal{O}(m_j) \cap V \setminus S$, $\mathrm{jac}(\mathbf{h}_j)$ has full rank c at x;
- P_4 the open sets $\mathcal{O}(m_j)$ cover $V \setminus S$.

We say that $\mathbf{h}_j$ is a set of local equations over $\mathcal{O}(m_j)$. [33, Lemma 5.2.4] establishes that there exists an atlas for $(V, \mathrm{sing}(V))$. Also, observe that $\mathrm{sing}(V) \subset Z(m_1 \cdots m_\ell)$.
Further, we use the notion of transverse intersection for algebraic sets and projective varieties. Let V and W be equidimensional algebraic sets in $\mathbb{C}^n$. As in [13, pp. 21], we say that V and W intersect transversely at x if $x \in \mathrm{reg}(V) \cap \mathrm{reg}(W)$ and $T_x V + T_x W = \mathbb{C}^n$. They intersect generically transversely if they meet transversely at a generic point of each irreducible component of $V \cap W$. This definition is naturally extended to projective varieties.
We say that two sets V and W intersect transversely over an open set U if V and W intersect tranversely at any point of $V \cap W \cap U$.

Lemma 3. *Let V_1 and V_2 be equidimensional algebraic sets of codimensions c_1 and c_2. Consider atlases $\boldsymbol{\alpha}_1 = ((\mathbf{h}_j, m_j))_{1 \leq j \leq \ell}$ and $\boldsymbol{\alpha}_2 = ((\mathbf{g}_j, n_j))_{1 \leq j \leq k}$ for $(V_1, \mathrm{sing}(V_1))$ and $(V_2, \mathrm{sing}(V_2))$. Assume that $V_1 \cap V_2$ is either empty or that for any irreducible component Z of $V_1 \cap V_2$, there exist $r \in \{1, \ldots, \ell\}$ and $s \in \{1, \ldots, k\}$ such that*

- T_1 *$Z \cap \mathcal{O}(m_r n_s)$ is not empty;*
- T_2 *At any point of $\mathrm{reg}(Z) \cap \mathcal{O}(m_r n_s)$, the matrix $\mathrm{jac}(\mathbf{h}_r, \mathbf{g}_s)$ has rank $c_1 + c_2$.*

Then V_1 and V_2 intersect generically transversely.

Proof. The equality $\mathrm{rank}(\mathrm{jac}(\mathbf{h}_r, \mathbf{g}_s)) = \mathrm{rank}(\mathrm{jac}(\mathbf{h}_r)) + \mathrm{rank}(\mathrm{jac}(\mathbf{g}_s))$ implies that at any point $x \in \mathrm{reg}(Z) \cap \mathcal{O}(m_r n_s)$, $N_x V_1 \cap N_x V_2 = 0$. Consequently, $T_x V_1 + T_x V_2 = (N_x V_1 \cap N_x V_2)^\perp = \mathbb{C}^n$. Finally, noticing that $\mathrm{reg}(Z) \cap \mathcal{O}(m_r n_s)$ is dense in $Z \cap \mathcal{O}(m_r n_s)$, which is dense in Z (by T_1) concludes the proof. ∎

We also need to prove that the intersection of bi-projective varieties is transverse. This is done via their associated affine cones. In the sequel, the set $\{(x, y) \mid x = \mathbf{0}\} \subset \mathbb{C}^{n+1} \times \mathbb{C}^{n+1}$ is denoted by $\mathscr{X}$ and the set $\{(x, y) \mid y = \mathbf{0}\} \subset \mathbb{C}^{n+1} \times \mathbb{C}^{n+1}$ is denoted by $\mathscr{Y}$.

Lemma 4. *Let V_1 and V_2 be projective varieties in $\mathbb{P}^n \times \mathbb{P}^n$. Then V_1 and V_2 intersect transversely at every point $(x, y) = ((x_0 : \ldots : x_n), (y_0 : \ldots : y_n)) \in \mathbb{P}^n \times \mathbb{P}^n$ iff $\mathsf{aff}(V_1)$ and $\mathsf{aff}(V_2)$ intersect transversely over $\mathbb{C}^{n+1} \times \mathbb{C}^{n+1} \setminus (\mathscr{X} \cup \mathscr{Y})$.*

Proof. Let i, j be such that $x_i \neq 0$ and $y_j \neq 0$. W.l.o.g., we assume that $i = j = 0$. Consider the affine chart $U \subset \mathbb{P}^n \times \mathbb{P}^n$ defined by $x_0 \neq 0, y_0 \neq 0$. Let $H_1 \subset \mathbb{C}^{n+1} \times \mathbb{C}^{n+1}$ (resp. H_2) be the hyperplane defined by $x_0 = 1$ (resp. $y_0 = 1$). For $\ell \in \{1, 2\}$, the variety $V_\ell \cap U$ can be identified to $\mathsf{aff}(V_\ell) \cap H_1 \cap H_2$. By definition of transversality, the varieties V_1 and V_2 intersect transversely at $(x, y) \in \mathbb{P}^n \times \mathbb{P}^n$ if and only if so do $V_1 \cap U$ and $V_2 \cap U$. By the previous identification, this is equivalent to saying that $\mathsf{aff}(V_1) \cap H_1 \cap H_2$ and $\mathsf{aff}(V_2) \cap H_1 \cap H_2$ intersect transversely at $(1, x_1, \ldots, x_n, 1, y_1, \ldots, y_n)$. Finally, direct tangent space computations show that for z_1, z_2 in $\mathbb{C} \setminus \{0\}$ and for $\ell \in \{1, 2\}$, $T_{(z_1, z_1x_1, \ldots, z_1x_n, z_2, z_2y_1, \ldots, z_2y_n)}\mathsf{aff}(V_\ell) = T_{(1, x_1, \ldots, x_n, 1, y_1, \ldots y_n)}\mathsf{aff}(V_\ell)$. ☐

2.3 Critical points and modified polar varieties

Let $V \subset \mathbb{C}^n$ be an equidimensional algebraic set of codimension c and $g \in \mathbb{Q}[X_1, \ldots, X_n]$. Consider the evaluation map $\varphi_g : x \in V \to g(x)$. We denote by $w(\varphi_g, V)$ the set $\{x \in \operatorname{reg}(V) \mid \operatorname{rank}(\operatorname{jac}_x(\mathbf{f}, g)) < c + 1\}$. This is a locally closed constructible set and it coincides with the critical locus of the map φ_g. Its Zariski closure is denoted by $W(\varphi_g, V)$. This construction can be generalized as follows.
Let $a_1, \ldots, a_n$ be linearly independent vectors in $\mathbb{C}^n$ and for $1 \leq i \leq n$, set $\mathbf{a}_i = (a_1, \ldots, a_i) \in \mathbb{C}^{i \times n}$. Then, for $1 \leq i \leq n$, let $W((g, \mathbf{a}_i), V)$ denote the algebraic set

$$\{x \in V \mid \operatorname{rank}(\operatorname{jac}_x(\mathbf{f}, g, \varphi_{\mathbf{a}_i})) < c + i + 1\},$$

where $\mathbf{f}$ is a set of generators of $I(V)$, and $\varphi_{\mathbf{a}_i}$ is the set of linear forms $(a_j \cdot X)_{j \in \{1, \ldots, i\}}$ (with $X = (X_1, \ldots, X_n)$). Reusing the terminology of [19], we call these sets *modified polar varieties* associated to g and V, the i-th one being $W((g, \mathbf{a}_i), V)$. We let $W(\mathbf{a}_i, V)$ be the classical polar variety $\{x \in V \mid \operatorname{rank}(\operatorname{jac}_x(\mathbf{f}, \varphi_{\mathbf{a}_i}) < c + i\}$, reusing the letter W for the sake of simplicity.

Proposition 5. *Let V be a d-equidimensional algebraic set, and $i \in \{1, \ldots, d\}$. There exists a Zariski dense subset $\mathscr{O} \subset \mathbb{C}^{in}$ and an integer integer numbers δ_i such that for any $a \in \mathscr{O}$, the following holds:*

- *$W(a, V)$ is either empty or equidimensional of dimension $i - 1$;*
- *$W(a, V)$ has degree at most δ_i.*

Proof. The first statement follows directly from [1, Prop.3]. For the second statement, we refer to the definition of $\delta_{classic}$ in [4, Sec. 4]. ☐

The integers δ_i are denoted by $\delta_i(V)$ in the sequel. By convention, we set $\delta_{d+1} = \deg(V)$. These numbers are also called *projective characters* of V (see [16, Example 14.3.3]).

3 Proof of Theorem 1

We start by introducing some objects which play a central role in the proof. As before, V is a d-equidimensional algebraic set and $\mathsf{aff}(V)$ denotes the affine cone over the projective closure of V. Let $\mathcal{N}_V \subset \mathbb{C}^{n+1} \times \mathbb{C}^{n+1}$ be the Zariski closure of the set

$$\{(x, y) \in \mathbb{C}^{n+1} \times \mathbb{C}^{n+1} \mid x \in \mathsf{aff}(V) \setminus \{\mathbf{0}\},\, y \in N_x\mathsf{aff}(V) \setminus \{\mathbf{0}\}\}.$$

It is called the conormal variety of $\mathsf{aff}(V)$. Consider $a = (a_1, \ldots, a_i) \in \mathbb{C}^{(n+1)i}$, a homogeneous polynomial $g \in \mathbb{C}[X_0, X_1, \ldots, X_n]$ of degree D and the matrix

$$\Sigma_i(g, a) = \begin{bmatrix} Y_0 & \cdots & Y_n \\ \partial g/\partial X_0 & \cdots & \partial g/\partial X_n \\ \text{———} & a_1 & \text{———} \\ & \vdots & \\ \text{———} & a_i & \text{———} \end{bmatrix}.$$

Let $S_i(g, a) \subset \mathbb{C}^{n+1} \times \mathbb{C}^{n+1}$ be the variety defined by the rank condition $\operatorname{rank}(\Sigma_i(g, a)) \leq i + 1$. Let Π be the projection

$$\Pi : \quad (x, y) \in \mathbb{C}^{n+1} \times \mathbb{C}^{n+1} \quad \to \quad x \in \mathbb{C}^{n+1}. \tag{3.1}$$

If a is generic, then $\mathcal{N}_V \cap S_i(g, a)$ is the Zariski closure of set of points (x, y) such that $y \in N_x\mathsf{aff}(V)$, and $(y_0, \ldots, y_n) \in \operatorname{Span}(a) + \nabla_x g$. In other words, $(x_0, x_1, \ldots, x_n)$ is a critical point of the map $(X_0, \ldots, X_n) \mapsto (g(X), a_1 \cdot X, \ldots, a_i \cdot X)$. Let $a = (a'_1, \ldots, a'_{i-1}) \in \mathbb{C}^{n(i-1)}$ be a basis of the vector space $\{(u_1, \ldots, u_n) \in \mathbb{C}^n \mid (0, u_1, \ldots, u_n) \in \operatorname{Span}(a_1, \ldots, a_i)\}$. Therefore, if the first coordinate of a_1 is nonzero, then the restriction of $\Pi(\mathcal{N}_V \cap S_i(g, a))$ to the chart $x_0 = 1$ is the modified polar variety $W((g_{|x_0=1}, a'), V)$. The set of homogeneous polynomials in $\mathbb{C}[X_0, \ldots, X_n]$ of degree D is a finite dimensional vector space; we denote by N its dimension and identify those homogeneous polynomials to points in $\mathbb{C}^N$. Assume for the moment the following result which is proved in Section 4.

Proposition 6. *Let $V \subset \mathbb{C}^n$ be a d-equidimensional algebraic set such that its projective closure is smooth and $i \in \{0, \ldots, d\}$. There exists a non-empty Zariski open set $\mathscr{O} \subset \mathbb{C}^{(n+1)i} \times \mathbb{C}^N$ such that for any $(g, a) \in \mathscr{O}$, $\mathcal{N}_V$ and $S_i(g, a)$ meet generically transversely over* $(\mathbb{C}^{n+1} \setminus \{\mathbf{0}\}) \times (\mathbb{C}^{n+1} \setminus \{\mathbf{0}\})$.

One can associate to any equidimensional variety $Z \subset \mathbb{P}^n \times \mathbb{P}^n$ of codimension c a bivariate homogenous polynomial $\operatorname{bideg}(Z) \in \mathbb{N}[T, U]$ of degree c, called the bidegree of Z [36, 37]. The coefficient of $T^k U^{c-k}$ in $\operatorname{bideg}(Z)$ is the number of points (counted with multiplicity) of $Z \cap (H_1 \times \mathbb{P}^n) \cap (\mathbb{P}^n \times H_2)$ where H_1 (resp. H_2) is a generic linear space of dimension $n - k$ (resp. $n - c + k$).
By [10, Sec. 5], the bidegree of $\mathcal{N}_V$ is $\sum_{k=0}^{d} \delta_{k+1}(V) T^{n-k} U^{k+1}$.
We focus now on the bidegree of $S_i(g, a)$.

Lemma 7. *There exists a non-empty Zariski open set $\mathscr{O}' \subset \mathbb{C}^N \times \mathbb{C}^{(n+1)i}$ such that for $(g, a) \in \mathscr{O}'$, $S_i(g, a) \subset \mathbb{C}^{n+1} \times \mathbb{C}^{n+1}$ has codimension $n - i$ and its bidegree is $\sum_{k=0}^{n-i} (D-1)^k T^k U^{n-k-i}$. Moreover,* $\operatorname{reg}(S_i(g, a))$ *coincides with the set of points where the matrix $\Sigma_i(g, a)$ has rank $i + 1$.*

Proof. $S_i(g,a)$ is the variety of $(x,y) \in \mathbb{C}^{n+1} \times \mathbb{C}^{n+1}$ where the evaluation of $\Sigma_i(g,a)$ is rank defective. There exists a Zariski dense subset $\mathscr{O}_1 \subset \mathbb{C}^{(n+1)i}$ such that for all $a \in \mathscr{O}_1$, the top-left $i \times i$ submatrix of A is invertible, where A is the matrix with rows $a_1, \dots, a_i$. For $a \in \mathscr{O}_1$, let $B = (b_{i,j})$ be an invertible $(n+1) \times (n+1)$ matrix such that $A \cdot B = [\mathbf{0} \mid I_i]$. The rank condition on $A \cdot B$ shows that $S_i(g,a)$ is the set of points $(x,y) \in \mathbb{C}^{n+1} \times \mathbb{C}^{n+1}$ where the rank of

$$M = \begin{bmatrix} \sum_{j=1}^{n+1} b_{j,1} Y_j & \cdots & \sum_{j=1}^{n+1} b_{j,n+1-i} Y_j \\ \sum_{j=1}^{n+1} b_{j,1} \partial g / \partial X_j & \cdots & \sum_{j=1}^{n+1} b_{j,n+1-i} \partial g / \partial X_j \end{bmatrix}$$

is at most 1, where $Y_1, \dots, Y_{n+1-1}$ are new variables. Next, let $S_i' \subset \mathbb{P}^{n-i} \times \mathbb{P}^{n-i}$ denote the determinantal variety of rank-defective matrices

$$\begin{bmatrix} \mathfrak{u}_{1,0} & \cdots & \mathfrak{u}_{1,n-i} \\ \mathfrak{u}_{2,0} & \cdots & \mathfrak{u}_{2,n-i} \end{bmatrix},$$

together with the grading given by $\deg(\mathfrak{u}_{1,j}) = 1$, $\deg(\mathfrak{u}_{2,j}) = D - 1$ for all $j \in \{0, \dots, n-i\}$. Setting $\mathsf{s} = 0$, $\deg(t_1) = D-1$, $\deg(t_2) = 1$, in [25, Example 15.39], the multidegree of S_i' is $\sum_{k=0}^{n-i} (D-1)^k T^k U^{n-k-i}$, where T (resp. U) corresponds to the class of a hyperplane in the first (resp. second) operand in the product $\mathbb{P}^{n-1} \times \mathbb{P}^{n-1}$. Let $\mathbb{C}[X_0, \dots, X_n]_{D-1}$ denote the set of homogeneous polynomials of degree $D-1$. Since determinantal varieties are Cohen-Macaulay [27, Thm. 11], by the same argument as in [15, Sec. 4], there exists a Zariski dense subset $\mathcal{O}_2 \subset \mathbb{C}[X_0, \dots, X_n]_{D-1}^{n+1} \times \mathbb{C}^{(n+1)i}$ such that for any $(h_0, \dots, h_n, a) \in \mathscr{O}_2$, the bidegree of the variety defined by $\operatorname{rank}(M) \le 1$ also has bidegree $\sum_{k=0}^{n-i} (D-1)^k T^k U^{n-k-i}$. Note that the set of $(h_0, \dots, h_n)$ which are of the form $(\partial g / \partial X_0, \dots, \partial g / \partial X_n)$ is a linear subspace of $\mathbb{C}[X_0, \dots, X_n]$. It remains to prove that the restriction of $\mathscr{O}_2$ to this subspace is nonempty. This is done by considering $(h_0, \dots, h_n) = (X_0^{D-1}, \dots, X_n^{D-1})$ (which comes from the derivatives of $g = (X_0^D + \cdots + X_n^D)/D$) and $a_{i,j} = 1$ if $i = j$ and 0 otherwise. Direct computations show that the corresponding variety has the expected bidegree. Therefore the open set $\mathcal{O}'$ of pairs (g,a) such that $(\partial g / \partial X_0, \dots, \partial g / \partial X_n, a) \in \mathscr{O}_2$ satisfies the desired properties. Writing the equations defining the variety of $S_i(g,a)$ from the rank of the matrix M shows that $(x,y) \in \operatorname{sing}(S_i(g,a))$ iff the evaluation of the first row of M is zero, which is equivalent to saying that $(Y_0, \dots, Y_n)$ lies in $\operatorname{Span}(a)$. This implies that the regular locus of S_i is the set of points where $\Sigma_i(g,a)$ has rank $i+1$. $\square$

By Proposition 6, there exists a non-empty Zariski open set $\mathscr{O} \subset \mathbb{C}^N \times \mathbb{C}^{(n+1)(i+1)}$ such that for g, a' in $\mathscr{O}$, $\mathcal{N}_V$ and $S_{i+1}(g,a')$ meet generically transversely outside the set $\mathscr{X} \cup \mathscr{Y}$ introduced before Lemma 4. Consider the map proj introduced in Section 2 (paragraph on projective varieties). We deduce that for $(g,a') \in \mathscr{O}_2$, $\mathcal{N}_V' = \mathsf{proj}(\mathcal{N}_V)$ and $S_i'(g,a') = \mathsf{proj}(S_i(g,a'))$ meet generically transversely (Lemma 4). Below, we take $(g,a) \in \mathscr{O} \cap \mathscr{O}'$ (where $\mathscr{O}'$ is the non-empty Zariski open set defined in Lemma 7).

Intersection theory [13, Theorem Definition 1.7] states that if two subvarieties Z_1 and Z_2 of $\mathbb{P}^n \times \mathbb{P}^n$ intersect generically transversely, then

$$\operatorname{bideg}(Z_1 \cap Z_2) = \operatorname{bideg}(Z_1) \cdot \operatorname{bideg}(Z_2) \bmod \langle T^{n+1}, U^{n+1} \rangle.$$

We deduce that $\operatorname{bideg}(\mathcal{N}_V' \cap S_i'(g,a))$ equals

$$\left(\sum_{k=0}^{d} \delta_{k+1}(V) T^{n-k} U^{k+1} \right) \left(\sum_{k=0}^{n-i-1} (D-1)^k T^k U^{n-k-i-1} \right) \bmod \langle T^{n+1}, U^{n+1} \rangle.$$

Note that the degree of the image of $S'_{i+1}(g,a') \cap \mathcal{N}'_V$ by the projection $\pi_1 : (x,y) \mapsto x$ is the coefficient of $T^{n-i-1}U^n$ in its bidegree. Direct computations show that it equals

$$\begin{cases} \delta_{i+1}(V) \text{ if } d = 1 \\ \sum_{j=i}^{d} \delta_{j+1}(V)(D-1)^{j-i} \text{ otherwise.} \end{cases}$$

For $j \in \{1,\ldots,i+1\}$, let ν_j be the first coefficient of a'_j and let $\mathcal{U}$ be the set of $a' \in \mathbb{C}^{(n+1)(i+1)}$ such that $\nu_1 \neq 0$. Set $\tilde{\mathscr{O}} = \{(g,a') \in \mathscr{O} \cap \mathscr{O}' \mid a' \in \mathcal{U}\}$. For $a' \in \mathcal{U}$, let χ be the map sending a' to $(a'_2 - \nu_2 a'_1/\nu_1, \ldots, a'_i - \nu_i a'_1/\nu_1)$. The image of $\mathcal{U}$ by χ is a dense open subset $\mathcal{U}' \subset \mathbb{C}^{ni}$. Finally, we write Ω for the set $(g_{|X_0=1}, a) \in \mathbb{C}[X_1,\ldots,X_n]_{\leq D} \times \mathbb{C}^{ni}$ such that there exists $(g,a') \in \tilde{\mathscr{O}}$ with $\chi(a') = a$. For $(g_{|X_0=1}, a) \in \Omega$, $S'_{i+1}(g,a')$ and $\mathcal{N}'_V$ intersect generically transversely. Moreover, its image by the projection Π (see (3.1)) restricted to the chart $x_0 = 1$, $y_0 = 1$ is $W(g_{|X_0=1}, a)$. Consequently,

$$\begin{aligned} \deg(W(g_{|X_0=1}, a)) &\leq \deg(\Pi(S'_{i+1}(g^h, a') \cap \mathcal{N}'_V)) \\ &= \begin{cases} \delta_{i+1}(V) & \text{if } d = 1 \\ \sum_{j=i}^{d} \delta_{j+1}(V)(D-1)^{j-i} & \text{otherwise.} \end{cases} \end{aligned}$$

4 Proof of Proposition 6

Our proof relies on applying Lemma 3 with $V_1 = \mathcal{N}_V$ and $V_2 = S_i(g,a)$ for a generic choice of (a,g). It simply consists in proving that properties T_1 and T_2 defined in Lemma 3 hold. This leads us to define atlases and local equations for $\mathcal{N}_V$. Next, we define an atlas (and hence local equations) for a set related to $S_i(g,a)$. We will apply an algebraic version of Thom's weak transversality Theorem to a well chosen map constructed using these local equations, establishing that this map is regular at the origin. Finally, we will use these results in the last paragraph of this Section to prove properties T_1 and T_2 under some genericity assumption on (g,a).

4.1 Local equations for $\mathcal{N}_V$

By assumption, V is d-equidimensional and smooth as is its projective closure; we denote by c its codimension. This implies that the affine cone $\mathsf{aff}(V)$ of the projective closure of V is also equidimensional of codimension c. Besides, if $(x,y) \in \mathsf{aff}(V)$ with $x \neq 0$ then x is a regular point of $\mathsf{aff}(V)$. By [33, Lemma 5.2.4], there exists an atlas $\boldsymbol{\psi} = ((\mathbf{h}_j, m_j))_{1 \leq j \leq J}$ for $(\mathsf{aff}(V), \mathrm{sing}(\mathsf{aff}(V)))$ (see Subsection 2.2). This leads us to define the set

$$U_j = \{(x,y) \mid x \in \mathsf{aff}(V) \cap \mathcal{O}(m_j), y \perp T_x\mathsf{aff}(V) \setminus \{\mathbf{0}\}\}.$$

Since the open sets $\mathcal{O}(m_j)$ cover $\mathsf{aff}(V) \setminus \mathrm{sing}(\mathsf{aff}(V))$ (property P_4), the sets U_j cover $\mathcal{N}_V \setminus \mathscr{X} \cup \mathscr{Y}$. Let $m'_{j,1}, \ldots, m'_{j,L_j}$ be the $c \times c$ minors of $\mathrm{jac}(\mathbf{h}_j)$ such that $\mathcal{O}(m_j m'_{j,k}) \cap V \neq \emptyset$ for $1 \leq k \leq L_j$. For $1 \leq r \leq n-c$, we denote by $M_{r,k}(m'_{j,k})$ the minor of the $(c+1, c+1)$ minors of the $(c+1, c+1)$ submatrix of

$$J = \begin{bmatrix} \mathrm{jac}(\mathbf{h}_j) \\ Y_0 \cdots Y_n \end{bmatrix}$$

whose upper left $(c \times c)$ minor is $m'_{j,k}$ and adding the missing row and column. In the sequel, we denote by $\mathbf{H}_{j,k}$ the sequence $\mathbf{h}_j, M_{1,k}(m'_{j,k}), \ldots, M_{n-c,k}(m'_{j,k})$.

Lemma 8. *Under the above notation and assumptions, the sequence of couples* $(\mathbf{H}_{j,k}, m_j m'_{j,k})$ *for* $1 \leq j \leq J$ *and* $1 \leq k \leq L_j$ *is an atlas for* $(\mathscr{N}_V, \mathrm{sing}(\mathscr{N}_V) \cup \mathscr{X} \cup \mathscr{Y})$.

Proof. Recall that we are given an atlas $\boldsymbol{\psi} = ((\mathbf{h}_j, m_j))_{1 \leq j \leq J}$ for $(\mathsf{aff}(V), \mathrm{sing}(\mathsf{aff}(V)))$. Let $(x, y) \in \mathscr{N}_V \setminus (\mathrm{sing}(\mathscr{N}_V) \cup \mathscr{X} \cup \mathscr{Y})$. Then, $x \in \mathrm{reg}(\mathsf{aff}(V))$ (because $x \neq \mathbf{0}$ and the projective closure of V is assumed to be smooth) and there exists $1 \leq j \leq J$ such that $x \in \mathsf{aff}(V) \cap \mathcal{O}(m_j)$. Besides note that $\mathsf{aff}(V) \cap \mathcal{O}(m_j)$ coincides with $Z(\mathbf{h}_j) \cap \mathcal{O}(m_j)$ (property P_1) and that $\mathrm{jac}(\mathbf{h}_j)$ has maximal rank at x (property P_3). We let $m'_{j,k}$ be a $(c \times c)$-minor of $\mathrm{jac}(\mathbf{h}_j)$ which does not vanish at x.
Since $(x, y) \in \mathscr{N}_V$, we have $y \perp T_x\mathsf{aff}(V)$. Using property P_1 and P_3, we deduce that $T_x\mathsf{aff}(V)$ is the kernel of $\mathrm{jac}(\mathbf{h}_j)$. We deduce by elementary linear algebra that the matrix J introduced above is rank defective at (x, y). Besides, elementary linear algebra (e.g. using a Schur complement) shows that over $\mathcal{O}(m_j m'_{j,k})$, the variety defined by $\mathbf{h}_j(x) = 0$ and $\mathsf{rank}(J(x, y)) \leq c$ is defined by $\mathbf{H}_{j,k}$. We have established properties P_1 and P_2. Establishing the fact that the sets $\mathcal{O}(m_j m'_k)$ cover $\mathscr{N}_V \setminus (\mathrm{sing}(\mathscr{N}_V) \cup \mathscr{X} \cup \mathscr{Y})$ (property P_4) is immediate from the above discussion. It remains to prove that $\mathrm{jac}(\mathbf{H}_{j,k})$ has maximal rank at (x, y) (property P_3). Without loss of generality, assume that $m'_{j,k}$ is the upper left minor of $\mathrm{jac}(\mathbf{h}_j)$. Observe that the minors $M_{1,k}(m'_{j,k}), \ldots, M_{n-c,k}(m'_{j,k})$ can be written as $Y_{c+\ell} m'_{j,k} + \rho_\ell$ where $\rho_\ell \subset \mathbb{Q}[X_1, \ldots, X_n, Y_1, \ldots, Y_c]$. Extracting from $\mathrm{jac}(\mathbf{H}_{j,k})$ the columns of $\mathrm{jac}(\mathbf{h}_j)$ corresponding to $m'_{j,k}$ and those corresponding to the partial derivatives w.r.t $Y_{c+\ell}$ for $1 \leq \ell \leq n - c$ yields a submatrix which is not rank defective over $Z(\mathbf{h}_j) \cap \mathcal{O}(m_j m'_{j,k})$ which ends the proof. ☐

4.2 Local equations for $S_i(g, a)$

In this section, we build an atlas for $S_i(g, a)$ for generic (g, a). To do that, we see (g, a) as in point in the space $\mathbb{C}^N \times \mathbb{C}^{(n+1)i}$ (recall that N is the dimension of the vector space of homogeneous polynomials in $\mathbb{C}[X_0, \ldots, X_n]$) and see the entries of a and the coefficients of g as variables.
Formally, for $1 \leq r \leq i$, let $A_r = (A_{0,r}, \ldots, A_{n,r})$ be a vector of indeterminates. Let also $\mathcal{M} = \{(\alpha_0, \ldots, \alpha_n) \in \mathbb{N}^{n+1} \mid \sum_{j=0}^n \alpha_j = D\}$ and $G = (G_\alpha, \alpha \in \mathcal{M})$ be a vector of indeterminates. By abuse of notation, we also denote by G the polynomial $\sum_{\alpha \in \mathcal{M}} G_\alpha X^\alpha$; it lies in $\mathbb{Q}(G)[X_0, \ldots, X_n]$.
We consider now the matrix

$$\Sigma_i = \begin{bmatrix} Y_0 & \cdots & Y_n \\ \partial G/\partial X_0 & \cdots & \partial G/\partial X_n \\ \text{———} & A_1 & \text{———} \\ & \vdots & \\ \text{———} & A_i & \text{———} \end{bmatrix}$$

and the algebraic set $\mathscr{S}_i \subset \mathbb{C}^{n+1} \times \mathbb{C}^{n+1} \times \mathbb{C}^N \times \mathbb{C}^{(n+1)i}$ defined by $\mathrm{rank}(\Sigma_i) \leq i + 1$.
Let $\sigma_1, \ldots, \sigma_L$ be the sequence of $(i + 1, i + 1)$-minors of the submatrix Σ_i obtained by removing the line containing partial derivatives of G or the line A_j for $1 \leq j \leq L$ such that $\mathscr{S}_i \cap \mathcal{O}(\sigma_\ell) \neq \emptyset$.
For $1 \leq \ell \leq L$, we denote by $S_{1,\ell}, \ldots, S_{n-i-1,\ell}$ the $(i + 2, i + 2)$-minors of Σ_i obtained by selecting the rows and columns used to compute σ_ℓ and adding the missing row and column from Σ_i. We denote by $\mathbf{S}_\ell$ the sequence $S_{1,\ell}, \ldots, S_{n-i-1,\ell}$.
Finally, we define the set $\mathscr{T} \subset \mathbb{C}^N \times \mathbb{C}^{(n+1)i}$ as the complementary of the set of points $(g, a = (a_1, \ldots, a_i)) \in \mathbb{C}^N \times \mathbb{C}^{(n+1)i}$ such that

- the coefficients of $X_r X_s^{D-1}$ in G for $1 \leq r, s \leq n$ with $r \neq s$ are not zero;

- (g, a) lies in the non-empty open set $\mathscr{O}$ defined in Lemma 7;
- $\mathrm{Span}(a_1, \dots, a_i)$ has dimension i and none of the entries of A_r is 0 (for $1 \leq r \leq i$).

Note that $\mathscr{T}$ is Zariski closed in $\mathbb{C}^N \times \mathbb{C}^{(n+1)i}$. Finally, we denote by $\mathscr{S}'$ the union of $\mathrm{sing}(\mathscr{S}_i)$, the set $\mathbb{C}^{n+1} \times \mathbb{C}^{n+1} \times \mathscr{T}$ and the subset of points $\mathscr{S}_i$ such that their Y-coordinates are all 0.
Up to renumbering the sequence of couples $(\mathbf{S}_\ell, \sigma_\ell)_{1 \leq \ell \leq L}$ we assume that the set of indices ℓ such that $(\mathscr{S}_i \setminus \mathscr{S}') \cap \mathcal{O}(\sigma_\ell) \neq \emptyset$ is $\{1, \dots, L'\}$ (for $L' \leq L$).

Lemma 9. *The sequence $(\mathbf{S}_\ell, \sigma_\ell)_{1 \leq \ell \leq L'}$ is an atlas for the couple $(\mathscr{S}_i, \mathscr{S}')$. Besides, the truncated Jacobian matrix of $\mathbf{S}_\ell$ obtained by considering the partial derivatives w.r.t the entries of $A_1, \dots, A_i$ and the coefficients of G has full rank over $\mathcal{O}(\sigma_\ell)$. Moreover, there exists a non-empty Zariski open set $\mathscr{O}''$ such that for all $(g, a) \in \mathscr{O}''$, $(\mathbf{S}_\ell, \sigma_\ell)_{1 \leq \ell \leq L'}$ is an atlas of the couple $(S_i(g, a), \mathrm{sing}(S_i(g, a))$.*

Proof. Take (x, y, g, a) in $\mathscr{S}_i \setminus \mathscr{S}'$. Since $(g, a) \notin \mathscr{S}$, $(g, a) \notin \mathscr{O}$ and $(x, y) \notin \mathrm{sing}(S_i(g, a))$. We deduce that Σ_i has rank $i + 1$ at (x, y, g, a). Then, either $\dim(\mathrm{Span}(a_1, \dots, a_i, y)) = i + 1$ or $y \in \mathrm{Span}(a_1, \dots, a_i)$ while $\nabla_{x,y,a,g}(G) \notin \mathrm{Span}(a_1, \dots, a_i)$ (because Σ_i has rank $i + 1$ at (x, y, g, a)). Since $(y_0, \dots, y_n) \neq 0$ (because $(x, y, g, a) \notin \mathscr{S}$), we deduce that there exists $1 \leq r \leq i$ such that $\mathrm{Span}(a_1, \dots, a_{r-1}, a_{r+1}, \dots, a_i, y) = \mathrm{Span}(a_1, \dots, a_i)$ and we deduce that

$$\dim(\mathrm{Span}(a_1, \dots, a_{r-1}, a_{r+1}, \dots, a_i, \nabla_x(g), y)) = i + 1.$$

This implies that one of the $(i + 1, i + 1)$-minor σ_ℓ of Σ_i does not vanish at (x, y, g, a). Elementary linear algebra shows that $\mathscr{S}_i \cap \mathcal{O}(\sigma_\ell) \setminus \mathscr{S}'$ coincides with $Z(\mathbf{S}_\ell)$ over $\mathcal{O}(\sigma_\ell) \setminus \mathscr{S}'$. Thus, we have established properties P_1 and P_2. The covering property P_4 is immediate and follows also from the above discussion.
It remains to prove property P_3, i.e. $\mathrm{jac}(\mathbf{S}_\ell)$ has maximal rank at any point of $\mathscr{S}_i \cap \mathcal{O}(\sigma_\ell) \setminus \mathscr{S}'$. Assume first that σ_ℓ is a $(i + 1, i + 1)$-minor obtained from removing the partial derivatives of G from Σ_i. Without loss of generality, we may also assume that it is obtained by selecting the first $i + 1$ columns of Σ_i. Then, polynomials in $\mathbf{S}_\ell$ can be written as $\sigma_\ell A_{r,i+1} + \rho_{r,\ell}$ for $i + 1 \leq r \leq n$ where $\rho_{r,\ell}$ has degree 0 in A_r. That implies that one can extract a diagonal matrix with σ_ℓ on the diagonal from $\mathrm{jac}(\mathbf{S}_\ell)$ which, of course, has maximal rank over $\mathcal{O}(\sigma_\ell)$.
When σ_ℓ is obtained by removing one of the line A_r (e.g. A_i) a more involved but similar conclusion can be made. Since we work over the complementary of $\mathscr{S}'$, there exists $0 \leq r \leq n$ such that the X_r-coordinate of x is not 0. Extracting the submatrix of $\mathrm{jac}(\mathbf{S}_\ell)$ corresponding to the partial derivatives with respect to the coefficients of G of the monomials X_r^D and $X_s X_r^{D-1}$ yields a diagonal matrix with a power of the X_r-coordinate of x multiplied by σ_ℓ on the diagonal. These are non-zero over $\mathcal{O}(\sigma_\ell) \setminus \mathscr{S}'$.
The rank property of the truncated Jacobian matrix of $\mathbf{S}_\ell$ is an immediate consequence of the above discussion. Details on the proof of the specialization property of the atlas $(\mathbf{S}_\ell, \sigma_\ell)$ are left to the reader; we mention that it is a direct consequence of specialization properties of minors with polynomial entries and Lemma 7. ☐

4.3 A map and its regularity at the origin

Let m_j, m'_k and $\mathbf{H}_{j,k}$ be the polynomials introduced in the paragraph on local equations for $\mathcal{N}_V$ and σ_ℓ, $\mathbf{S}_\ell$ be the $(i + 1, i + 1)$-minor and $(i + 2, i + 2)$-minors of Σ introduced in the paragraph on local equations of $S_i(g, a)$. Consider the Zariski open set $\mathscr{U}_{j,k,\ell} \subset \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C}^N \times \mathbb{C}^{ni}$ defined by

$$m_j m'_k \neq 0, \quad (X_0, \dots, X_n) \neq \mathbf{0} \quad (Y_0, \dots, Y_n) \neq \mathbf{0}, \quad \sigma_\ell \neq 0$$

and the inequations defining the complement $\mathscr{S}'$. We define now the following map:

$$\boldsymbol{\phi}_{j,k,\ell} : z \in \mathscr{U}_{j,k,\ell} \to (\mathbf{H}_{j,k}(z), \mathbf{S}_\ell(z)).$$

Observe that $\boldsymbol{\phi}_{j,k,\ell}^{-1}(\mathbf{0}) \subset \mathcal{N}_V \cap \mathscr{S}_i$.

Lemma 10. *The map $\boldsymbol{\phi}_{j,k,\ell}$ is regular at $\mathbf{0}$.*

Proof. Since j, k and ℓ are fixed in the sequel, we omit them as subscripts. Observe that $\mathrm{jac}(\mathbf{H}, \mathbf{S})$ has the following shape

$$\mathbf{J}_{\boldsymbol{\phi}} = \begin{bmatrix} \mathrm{jac}_{\mathbf{X}}(\mathbf{H}) & \mathbf{0} \\ \mathrm{jac}_{\mathbf{X}}(\mathbf{S}) & \Delta \end{bmatrix}$$

where the last columns correspond to the partial derivates with respect to the entries of $A_1, \ldots, A_i$ and G. By Lemma 8, $(\mathbf{H}_{j,k}, m_j m'_k)$ satisfies properties $\mathsf{P}_1, \mathsf{P}_2$ and P_3. This implies that it has maximal rank at any point in $\boldsymbol{\phi}^{-1}(\mathbf{0}) \subset \mathscr{U}$. By Lemma 9, Δ has maximal rank at any point of $\boldsymbol{\phi}^{-1}(\mathbf{0})$. We deduce that $\mathbf{J}_{\boldsymbol{\phi}}$ has maximal rank at any point of $\boldsymbol{\phi}^{-1}(\mathbf{0})$ and our conclusion follows. ☐

In the sequel, for $(g, a) \in \mathbb{C}^N \times \mathbb{C}^{(n+1)i}$, we denote by $\boldsymbol{\phi}_{j,k,\ell}^{(g,a)}$ the restricted map $(x, y) \to \boldsymbol{\phi}_{j,k,\ell}(x, y, g, a)$. Applying Thom's weak transversality Theorem (see [33, Sec 4.2]) to $\boldsymbol{\phi}_{j,k,\ell}$ shows that there exists a non-empty Zariski open set $\mathscr{O}'''_{j,k,\ell} \subset \mathbb{C}^N \times \mathbb{C}^{(n+1)i}$ such that for all $(g, a) \in \mathscr{O}'''_{j,k,\ell}$, the restricted map $\boldsymbol{\phi}_{j,k,\ell}^{(g,a)}$ is regular at $\mathbf{0}$. Letting $\mathscr{O}'''$ be the intersection of all these non-empty Zariski open sets $\mathscr{O}'''_{j,k,\ell}$ leads to the following result.

Lemma 11. *There exists a non-empty Zariski open set $\mathscr{O}''' \subset \mathbb{C}^N \times \mathbb{C}^{(n+1)i}$ such that for any (j, k, ℓ) and $(g, a) \in \mathscr{O}'$, the restricted map $\boldsymbol{\phi}_{j,k,\ell}^{(g,a)}$ is regular at $\mathbf{0}$.*

4.4 Transversality of the intersection

Let $\mathscr{O}$ be the intersection of the non-empty Zariski open sets $\mathscr{O}''$ and $\mathscr{O}'''$ defined in Lemma 9 and Lemma 11. Take $(g, a) \in \mathscr{O}$ and $Z_{(g,a)}$ be the Zariski closure of $\bigcup_{j,k,\ell} {\boldsymbol{\phi}_{j,k,\ell}^{(g,a)}}^{-1}(\mathbf{0})$. Recall that we need to prove the transversality of $\mathcal{N}_V \cap S_i(g, a)$ at any point outside $\mathscr{X} \cup \mathscr{Y}$. Let $\boldsymbol{\alpha}_1 = (\mathbf{H}_{j,k}, m_j m'_k)$ be the atlas of $(\mathcal{N}_V, \mathrm{sing}(\mathcal{N}_V))$ defined in Lemma 8 and $\boldsymbol{\alpha}_2 = (\mathbf{S}_\ell, \sigma_\ell)$ be the atlas of $(S_i(g, a), \mathrm{sing}(S_i(g, a)))$ defined in Lemma 9. We start by proving that the Zariski closure of $\mathcal{N}_V \cap S_i(g, a) \setminus (\mathscr{X} \cup \mathscr{Y})$ equals $Z_{g,a}$. The inclusion $Z_{(g,a)} \subset \mathcal{N}_V \cap S_i(g, a) \setminus (\mathscr{X} \cup \mathscr{Y})$ is immediate since all points in ${\boldsymbol{\phi}_{j,k,\ell}^{(g,a)}}^{-1}(\mathbf{0}) \subset Z(\mathbf{H}_{j,k}, \mathbf{S}_{k,\ell}) \cap \mathcal{O}(m_j m'_k \sigma_\ell)$ and $Z(\mathbf{H}_{j,k}, \mathbf{S}_{k,\ell}) \cap \mathcal{O}(m_j m'_k \sigma_\ell) = \mathcal{N}_V \cap S_i(g, a) \cap \mathcal{O}(m_j m'_k \sigma_\ell)$ (property P_2). We prove now the reverse inclusion. It is sufficient to prove that for any irreducible component Z of the Zariski closure of $\mathcal{N}_V \cap S_i(g, a) \setminus (\mathscr{X} \cup \mathscr{Y})$, there exists a triple (j, k, ℓ) and a Zariski closed subset $F \subsetneq Z$ such that $Z \setminus F \subset {\boldsymbol{\phi}_{j,k,\ell}^{(g,a)}}^{-1}(\mathbf{0})$. Since Z is an irreducible component of the Zariski closure of $\mathcal{N}_V \cap S_i(g, a) \setminus (\mathscr{X} \cup \mathscr{Y})$, there exists $(x, y) \in Z$ such that $(x, y) \notin \mathscr{X} \cup \mathscr{Y}$. Let $F = Z \cap (\mathscr{X} \cup \mathscr{Y})$. Now, take $(x, y) \in Z \setminus F$. By property P_4 applied to $\boldsymbol{\alpha}_1$, that implies that there exists j and k such that $x \in Z(\mathbf{H}_{j,k}) \cap \mathcal{O}(m_j m'_k)$. Besides, $(y_0, \ldots, y_n) \neq \mathbf{0}$ since $(x, y) \notin F$. This latter property implies that there exists ℓ such that $\sigma_\ell(x, y) \neq 0$. Finally, we have established that $(Z \setminus F) \cap \mathcal{O}(m_j m'_k \sigma_\ell)$ is not empty for some (j, k, ℓ). Property P_2 applied to $\boldsymbol{\alpha}_1$ and $\boldsymbol{\alpha}_2$ imply that (x, y) lies in $Z(\mathbf{H}_{j,k})$ and $Z(\mathbf{S}_\ell)$. We deduce that $(x, y) \in {\boldsymbol{\phi}_{j,k,\ell}^{(g,a)}}^{-1}(\mathbf{0})$ which implies that $Z \setminus F \subset {\boldsymbol{\phi}_{j,k,\ell}^{(g,a)}}^{-1}(\mathbf{0})$ as requested.

Property (T_1)**.** Consider an irreducible component Z of the Zariski closure of $\mathcal{N}_V \cap S_i(g,a) \setminus (\mathscr{X} \cup \mathscr{Y})$. The above discussion implies that Z is an irreducible component of $Z_{(g,a)}$ and that there exists j,k,ℓ such that $Z \cap \mathcal{O}(m_j m'_k \sigma_\ell)$ is not empty.
Property (T_2)**.** Recall that $(g,a) \in \mathscr{O}'$ and let Z be an irreducible component of $\mathcal{N}_V \cap S_i(g,a)$. We already proved that Z there exists (j,k,ℓ) such that $Z \cap \mathcal{O}(m_j m'_k \sigma_\ell)$ is not empty and meets ${\boldsymbol{\phi}_{j,k,\ell}^{(g,a)}}^{-1}(\mathbf{0})$. By Lemma 11, the restricted map $\boldsymbol{\phi}_{j,k,\ell}^{(g,a)}$ is regular at $\mathbf{0}$. Then, the jacobian matrix associated to $\mathbf{H}_{j,k}, \mathbf{S}_\ell$ has maximal rank at any point of $Z \cap {\boldsymbol{\phi}_{j,k,\ell}^{(g,a)}}^{-1}(\mathbf{0})$, which concludes the proof.

5 Non-generic function

We show in this section that the bounds in Theorem 1 hold under milder conditions than the genericity of the coefficients of g. Consider a d-equidimensional algebraic set $V \subset \mathbb{C}^n$ whose projective closure is smooth, a set of generators $f_1, \ldots, f_p$ of $I(V)$, and $g \in \mathbb{Q}[X_1, \ldots, X_n]$ of degree D. Let $a \in \mathbb{C}^{ni}$, $g \in \mathbb{Q}[X_1, \ldots, X_n]$ and $I_{\mathsf{crit}}(g,a)$ be the ideal generated by $f_1, \ldots, f_p$ and the $(n-d+i+1)$-minors of the matrix

$$\begin{bmatrix} \rule{1cm}{0.4pt} & \mathrm{jac}(\mathbf{f}) & \rule{1cm}{0.4pt} \\ \partial g/\partial X_1 & \cdots & \partial g/\partial X_n \\ \rule{1cm}{0.4pt} & a_1 & \rule{1cm}{0.4pt} \\ & \vdots & \\ \rule{1cm}{0.4pt} & a_i & \rule{1cm}{0.4pt} \end{bmatrix}.$$

Proposition 12. *Let $i \in \{0, \ldots, d\}$ and $a \in \mathbb{C}^{ni}$. Assume that the ideal $I_{\mathsf{crit}}(g,a)$ is radical and $W(g,a)$ is empty or $(i-1)$ equidimensional. Then there exists a non-empty Zariski open subset $\mathscr{O} \subset \mathbb{C}^{in}$ such that the following holds. For any $a = (a_1, \ldots, a_i) \in \mathscr{O}$, the degree of $W((g,\mathbf{a}),V)$ is bounded above by the bounds in Theorem 1.*

Lemma 13. *Let $Q \in \mathbb{C}[T_1, \ldots, T_\ell]$ be a nonzero multivariate polynomial, and $(t_1, \ldots, t_\ell) \in \mathbb{C}^\ell$ be such that $Q(t_1, \ldots, t_\ell) = 0$. Then there exist univariate polynomials $u_1, \ldots, u_\ell \in \mathbb{C}[\mathfrak{e}]$ such that for all $i \in \{1, \ldots, \ell\}$, $u_i(0) = t_i$ and $Q(u_1(\mathfrak{e}), \ldots, u_\ell(\mathfrak{e})) \in \mathbb{C}[\mathfrak{e}]$ is not identically zero.*

Proof. We prove the existence of $u_1, \ldots, u_\ell$ of the form $u_i(\mathfrak{e}) = t_i + s_i\mathfrak{e}$, where $s_i \in \mathbb{C}$ for all $i \in \{1, \ldots, \ell\}$. Let $\mathbf{t}$ and $\mathbf{s}$ be shorthands for $(t_1, \ldots, t_\ell)$ and $(s_1, \ldots, s_\ell)$. Using Taylor's expansion, we write $Q(\mathbf{t} + \mathfrak{e}\mathbf{s}) = \mathfrak{e}\partial Q(\mathbf{t})(\mathbf{s}) + \mathfrak{e}^2 \partial^2 Q(\mathbf{t})(\mathbf{s},\mathbf{s})/2 + \ldots + \mathfrak{e}^{\deg(Q)} \partial^{\deg(Q)} Q(\mathbf{t})(\mathbf{s}, \ldots, \mathbf{s})/\deg(Q)!$. Since $Q \neq 0$, at least one of its derivatives is not zero at $\mathbf{t}$. Let k be the smallest integer such that $\mathbf{u} \mapsto \partial^k Q(\mathbf{t})(\mathbf{u}, \ldots, \mathbf{u})$ is not the zero map. Finally, let $\mathbf{s}$ be such that $\partial^k Q(\mathbf{t})(\mathbf{s}, \ldots, \mathbf{s}) \neq 0$. Hence, we have $Q(\mathbf{t} + \mathfrak{e}\mathbf{s}) - \mathfrak{e}^k \partial^k Q(\mathbf{t})(\mathbf{s}, \ldots, \mathbf{s})/k! = O(|\mathfrak{e}^{k+1}|)$. Consequently, $Q(\mathbf{t} + \mathfrak{e}\mathbf{s})$ cannot be identically zero, as this would imply $\mathfrak{e}^k = O(|\mathfrak{e}^{k+1}|)$. ☐

of Proposition 12. The proof is a classical deformation argument similar to the one used in [26]. Further we assume that $W(g,a)$ is not empty (else the result is immediate). By Theorem 1, there exists a polynomial Q in $N + ni = \binom{n+D}{n} + ni$ variables whose zero-set encode the pairs (g,a) for which the bounds are not satisfied. By Lemma 13, there exists $(\mathfrak{g}, \mathfrak{a}) \in \mathbb{Q}[\mathfrak{e}][X_1, \ldots, X_n] \times \mathbb{Q}[\mathfrak{e}]^{ni}$ such that their evaluation at $\mathfrak{e} = 0$ is (g,a) and the evaluation of Q at $(\mathfrak{g}, \mathfrak{a})$ (seen as an element in $\mathbb{Q}[\mathfrak{e}]^{N+ni}$) is nonzero. For $\varepsilon \in \mathbb{C}$, we let $(g_\varepsilon, a_\varepsilon)$ denote the evaluation of $\mathfrak{g}$ and $\mathfrak{a}$ at $\mathfrak{e} = \varepsilon$. For $i \in \{1, \ldots, d\}$, the set of affine spaces in $\mathbb{C}^n$ of codimension $i-1$ can be identified with a dense open subset of the Grassmaniann of $(n-i+2)$-dimensional vector spaces in $\mathbb{C}^{n+1}$. Since $W((g,a),V)$ is $(i-1)$-equidimensional, there exists a dense open subset $\mathscr{O}$ of this Grassmanian such that for any E in $\mathscr{O}$, the intersection $W((g,a),V)\cap E$ is transverse, finite and its cardinality equals the degree of $W((g,\mathbf{a}_i),V)$. Let $\mathbf{x}\in\mathbb{C}^n$ be a point in this intersection. Let $v_1,\ldots,v_n\in\mathbb{C}[X_1,\ldots,X_n,\mathfrak{e}]$ be polynomials satisfying the following assumptions: $v_1,\ldots,v_n$ is a regular sequence, for every $\varepsilon\in\mathbb{C}$ their evaluations at $\mathfrak{e}=\varepsilon$ vanish on $W((g_\varepsilon,a_\varepsilon),V)\cap E$, and the jacobian matrix $\mathrm{jac}(v_1(X_1,\ldots,X_n,0),\ldots,v_n(X_1,\ldots,X_n,0))$ is invertible at $\mathbf{x}$ (since $I_{\mathsf{crit}}(g,a)$ is radical). In order to obtain such polynomials, we consider n generic linear combinations of the equations defining $W((\mathfrak{g},\mathfrak{a}),V)\cap E$. Then the holomorphic implicit mapping theorem [24, Thm. 8.6] states that for $\mathbf{x}_0\in W((g,a),V)\cap E$ there exist open neighborhoods (for the Euclidean topology) $0\in U_1\subset\mathbb{C}$, $\mathbf{x}_0\in U_2$ such that there is a holomorphic map $\varepsilon\mapsto\{\mathbf{x}\in U_2\mid v_1(\mathbf{x},\varepsilon)=\cdots=v_n(\mathbf{x},\varepsilon)=0\}$ on U_1. In particular this map is continuous, which implies that for $\varepsilon\in\mathbb{C}$ with sufficiently small complex modulus, the cardinality of $W((g_\varepsilon,a_\varepsilon),V)\cap E$ is bounded below by the degree of $W((g,a),V)$. Since this is true for any E in the Zariski dense open subset $\mathscr{O}$ of affine subsets, the cardinality of $W((g_\varepsilon,a_\varepsilon),V)\cap E$ equals its degree. Finally, as Q is not identically zero on the coefficients of $(\mathfrak{g},\mathfrak{a})$, for ε_0 with sufficiently small modulus, the evaluation of Q at the coefficients of $(g_{\varepsilon_0},a_{\varepsilon_0})$ is nonzero. Consequently, the bounds in Theorem 1 hold for $W((g_{\varepsilon_0},a_{\varepsilon_0}),V)$ and hence they also hold for $W((g,a),V)$. ☐

6 Algorithms

Terminology and computational model. In this section, we consider *bounded error probabilistic algorithms*. These algorithms are probabilistic random-access stored-program machines whose probability of success is bounded from above by an *a priori* bound. It is the same computational model as in [17]. Complexity bounds count the number of arithmetic operations $(+,-,\times,/)$ in $\mathbb{Q}$. A lifting fiber is a data structure giving an exact representation of an equidimensional algebraic set. We recall below its definition and we refer to [17, Sec. 3.4] for more details.

Definition 14. *[17, Def. 4] Let $V\subset\mathbb{C}^n$ be a d-equidimensional variety defined over $\mathbb{Q}$ (*i.e.* $Z_{\mathbb{C}}(I_{\mathbb{Q}}(V))=V$). A* lifting fiber *for V is a tuple $\mathscr{L}=(\mathbf{G},M,\mathbf{z},u,Q,\mathbf{v})$:*

- *A* lifting system *$\mathbf{H}=(h_1,\ldots,h_{n-d})\in\mathbb{Q}[X_1,\ldots,X_n]$, such that $h_1,\ldots,h_{n-d}$ is a reduced regular sequence and $V\subset Z(\mathbf{H})$.*
- *A $n\times n$ invertible matrix M with entries in $\mathbb{Q}$ such that the coordinates $Y=M^{-1}X$ are in* Noether position *w.r.t. V;*
- *A* rational lifting point *$\mathbf{z}=(z_1,\ldots,z_d)\in\mathbb{Q}^d$;*
- *A* primitive element *$u:\mathbb{C}^n\to\mathbb{C}$, which is a linear form with rational coefficients having distinct values at all points of the finite set $V^{(\mathbf{z})}=V\cap\{Y_1-z_1=\cdots=Y_d-z_d=0\}\subset\mathbb{C}^n$;*
- *A* polynomial *$Q\in\mathbb{Q}[T]$ of minimal degree vanishing at all points of $u(V^{(\mathbf{z})})$;*
- univariate polynomials *$\mathbf{v}=(v_{d+1},\ldots,v_n)\in\mathbb{Q}[T]^{n-d}$ of degree less than* $\deg(Q)$ *such that*

$$Y_1-z_1=\cdots=Y_d-z_d=0$$
$$Y_{d+1}-v_{d+1}(T)=\cdots=Y_n-v_n(T)=0, Q(T)=0$$

is a rational parametrization of $V^{(\mathbf{z})}$ by the roots of Q.

The sequence $(M, u, Q, \mathbf{v})$ *is called a* geometric resolution *of* V.

Computing a lifting fiber can be achieved in a probabilistic way with the Kronecker solver [11]. We assume that we know a probabilistic algorithm PolarVar which takes as input $d \in \mathbb{N}$, a lifting fiber of a d-equidimensional variety $V \subset \mathbb{C}^n$ and a sequence $\mathbf{a} = (a_1, \ldots, a_d) \in \mathbb{Q}^{d\times n}$; it returns a geometric resolution of the 0-dimensional polar variety $W(\mathbf{a}_1, V)$ or "fail". We use also the routine ChangePrimitiveElement [17, Algo. 6].
In [1], the authors propose an algorithm which takes as input a reduced regular sequence $f_1, \ldots, f_{n-d}$ defining a d-equidimensional algebraic set $V \subset \mathbb{C}^n$, a matrix $\mathbf{F}$ whose entries are multivariate polynomials, and a sequence $a = (a_1 \ldots, a_d)$ of vectors in $\mathbb{Q}^n$. It returns lifting fibers for the associated *degeneracy loci.* If $\mathbf{F}$ turns out to be the jacobian matrix of the regular sequence defining the variety, then these degeneracy loci are the classical polar varieties $W(a, V)$, see [1, Section 5.1]. This algorithm works in two steps: it computes first a lifting fiber for V; then, from this lifting fiber and from the matrix a, it computes lifting fibers for the degeneracy loci. In the case of polar varieties, the complexity of the second step is bounded by $L(nD_{\max})^{O(1)}\delta^2$, where δ is the maximum of the degrees of the polar varieties $W(\mathbf{a}_i, V)$ (where $\mathbf{a}_i = (a_1, \ldots, a_i)$), $D_{\max}$ is the maximum of the degrees of $f_1, \ldots, f_{n-d}$, and L is the size of an essentially division-free straight line program for evaluating $f_1, \ldots, f_{n-d}$.
Let $a = (a_1, \ldots, a_d)$ be a sequence of d vectors in $\mathbb{Q}^n$. We construct another sequence $a' = (e_{n+1}, a'_1, \ldots, a'_d)$ of vectors in $\mathbb{Q}^{n+1}$ defined by the $(d+1) \times (n+1)$ coefficient matrix

$$A' = \left[\begin{array}{ccc|c} & \mathbf{0} & & 1 \\ \hline \text{---} & a_1 & \text{---} & \\ \vdots & \vdots & \vdots & \mathbf{0} \\ \text{---} & a_d & \text{---} & \end{array}\right].$$

Lemma 15. *Let* $\Pi_n : \mathbb{C}^{n+1} \to \mathbb{C}^n$ *be the projection on the* n *first coordinates, and* $f_1, \ldots, f_p \in \mathbb{Q}[X_1, \ldots, X_n]$ *be polynomials defining a reduced smooth* d*-equidimensional variety and* $g \in \mathbb{Q}[X_1, \ldots, X_n]$ *be a polynomial. Then for any* $\mathbf{a} \in \mathbb{Q}^{d\times n}$ *and for* $i \in \{0, \ldots, d\}$*, the modified polar variety* $W((g, a), Z(f_1, \ldots, f_p))$ *equals* $\Pi_n(W(a'_{i+1}, Z(f_1, \ldots, f_p, g - X_{n+1})))$.

Proof. Set $V = Z(f_1, \ldots, f_p) \subset \mathbb{C}^n$ and $V' = Z(f_1, \ldots, f_p, g - X_{n+1}) \subset \mathbb{C}^{n+1}$. Direct computations show that if V is smooth, then so is V'. The modified polar variety $W((g, \mathbf{a}_i), V)$ is defined by the set of points in V at which

$$\operatorname{rank}\begin{bmatrix} \text{---} & \mathrm{jac}(\mathbf{f}) & \text{---} \\ \text{---} & \nabla g & \text{---} \\ \text{---} & a_1 & \text{---} \\ \vdots & \vdots & \vdots \\ \text{---} & a_1 & \text{---} \end{bmatrix} \leq n - d + i.$$

Direct computations show that the corresponding matrix for $W(\mathbf{a}'_i, V')$ has the same rank at any point $(x, g(x))$ where $x \in V$. ∎

Algorithm 1: CRITPOINTS

Input :

- A lifting fiber $(\mathbf{H}, M, \mathbf{z}, u, Q, \mathbf{v})$ for a smooth d-equidimensional variety $V \subset \mathbb{C}^n$
- $g \in \mathbb{Q}[X_1, \dots, X_n]$ and $a = (a_1, \dots, a_d) \in \mathbb{Q}^{d \times n}$
- A primitive element u_{crit} for $W(\mathbf{a}_1, V)$

$M' \leftarrow \left[\begin{array}{c|c} M & \mathbf{0} \\ \hline \mathbf{0} & 1 \end{array}\right]$;

$\mathscr{L}' \leftarrow (\mathbf{H} \cup \{g - X_{n+1}\}, M', (z_1, \dots, z_n, g(z_1, \dots, z_n)),$
$u, Q, (v_1, \dots, v_n, g \circ (v_1(T), \dots, v_n(T)) \bmod Q(T)))$;

$a' \leftarrow$ sequence of rows of $\left[\begin{array}{ccc|c} & \mathbf{0} & & 1 \\ \hline \text{---} & a_1 & \text{---} & \\ \vdots & \vdots & \vdots & \mathbf{0} \\ \text{---} & a_d & \text{---} & \end{array}\right]$;

$\mathscr{L}^{(2)} \leftarrow$ POLARVAR$(d, \mathscr{L}', a')$ or return “fail”;

$(\mathbf{H}', M^{(2)}, \mathbf{z}', u_{\mathsf{crit}}, Q', \mathbf{v}') \leftarrow$ CHANGEPRIMITIVEELEMENT$(\mathscr{L}^{(2)}, u_{\mathsf{crit}} \circ M^{(2)})$;

$\mathbf{v}^{(2)} \leftarrow (M^{(2)})^{-1} \cdot (v'_1, \dots, v'_{n+1})^T$;

return $(\mathsf{Id}_n, u_{\mathsf{crit}} \circ (M^{(2)})^{-1}, Q', (v'_1, \dots, v'_n))$;

Theorem 16. *Let $\mathscr{L} = (\mathbf{H}, M, \mathbf{z}, u, Q, \mathbf{v})$ be a lifting fiber for a d-equidimensional algebraic set V, $g \in \mathbb{Q}[X_1, \dots, X_n]$ be a polynomial of degree $D \geq 2$ and $a \in \mathbb{C}^{ni}$. Assume that V, g and a satisfy the same assumptions as in Proposition 12. Let $D_{\max}$ be the maximum of the degrees of $h_1, \dots, h_{n-d}, g$, (where $\mathbf{H} = (h_1, \dots, h_{n-d})$) and u be a primitive element for $W(g, V)$. Assume that the evaluation map $x \mapsto (h_1(x), \dots, h_{n-d}(x), g(x))$ is represented by an essentially division-free straight-line program of size L. Algorithm 1 with input $(\mathscr{L}, \mathbf{a}, u)$ computes a geometric resolution of the set $W(g, V)$ or it returns "fail". Using the algorithm in [1] for* PolarVar, *it requires at most $(nD_{\max})^{O(1)}\widetilde{O}(L\Delta^2)$ operations in $\mathbb{Q}$, where $\Delta = \sum_{j=0}^{d} \delta_{j+1}(V)(D-1)^{j-i}$.*

Proof. We prove first the correctness of the algorithm. Note that $\mathscr{L}'$ computed during Algorithm 1 is a lifting fiber for $V' = \{(x, g(x)) \mid x \in V\}$. Assuming that PolarVar returns a lifting fiber for $W(\mathbf{a}'_1, V')$, Lemma 15 shows that the output of PolarVar is a lifting fiber of the pairs $(x, g(x))$ for $x \in W(g, V)$. The last steps compute a geometric resolution of the projection on the n first coordinates, which is $W(g, V)$. We prove now the complexity statement. The first step of Algorithm 1 does not cost any arithmetic operations. The second step requires $\widetilde{O}(L \deg(V))$ operations in $\mathbb{Q}$ for the modular composition using quasi-linear algorithms for multiplication and reduction. The evaluation of g costs L operations. The cost of the computation of $\mathbf{a}'$ is negligible. By [1, Thm. 18], the call to PolarVar requires $L(pnd)^{O(1)}\delta'^2$, where δ' is the maximum of the degrees of the polar varieties of V'. By Lemma 15, the projection of $W(\mathbf{a}'_1, V')$ on the n first coordinates is $W(g, V)$. By Theorem 1, Proposition 12 and since $\deg(g) \geq 2$, we have $\deg(W(g, V)) \leq \Delta$. Changing the primitive element costs $\widetilde{O}(n\Delta^2)$ by [17, Lemma 6]. Finally, the linear algebra computations in the last step cost $O(n^2\Delta)$ operations in $\mathbb{Q}$. Summing all these complexities proves the complexity statement. □

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