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1 Introduction

Folkerts, Launert and Thom have recently announced a very interesting discovery about Jost Bürgi (1552–1632) [6, 7]. Bürgi was a remarkable mechanician, clockmaker and instrument maker, and also the author of a table of progressions which can be viewed as a table of logarithms. In this note, we focus on some specific points concerning Bürgi's newly discovered algorithms. In earlier notes, we have given a critical analysis of Bürgi's work on progressions [12],[1] we gave a possible source for Bürgi's iterative algorithm for the computation of sines [14] and we considered the complexity and accuracy of that iterative algorithm [15].

In this note, we want to concentrate on the computation of the first sines using the method of differences. It appears that Bürgi worked out quite elaborate and interesting methods.

2 Bürgi's algorithms

Before we enter into the details, we should clarify what are Bürgi's algorithms, because there is some confusion in the literature, including in our previous articles. There has been much talk about Bürgi's *Kunstweg* (artistic way), and we consider that this *Kunstweg* is the iterative algorithm discussed

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[1] A translation of Bürgi's introduction to the tables was recently published by Kathleen Clark [4].

in our earlier articles [14, 15]. Others authors may call *Kunstweg* the set of all of Bürgi's algorithms, and the reader is therefore advised to be careful at what is exactly meant. In any case, we briefly recall that *Kunstweg*.

3 The so-called *Kunstweg*

Folkerts *et al.* [6] give the following table adapted from Bürgi's manuscript dated from the 1580s. Bürgi used instead sexagesimal values, but we can safely convert them to base 10.

	c_5		c_4		c_3		c_2		c_1
0	0		0		0		0		0
		2,235,060		67,912		2,064		63	
10	2,235,060		67,912		2,064		63		2
		2,167,148		67,848		2,001		61	
20	4,402,208		133,760		4,065		124		4
		2,033,388		61,783		1,877		57	
30	6,435,596		195,543		5,942		181		6
		1,837,845		55,841		1,696		51	
40	8,273,441		251,384		7,638		232		7
		1,586,461		48,203		1,464		44	
50	9,859,902		299,587		9,102		276		8
		1,286,874		39,101		1,188		36	
60	11,146,776		338,688		10,290		312		9
		948,186		28,811		876		27	
70	12,094,962		367,499		11,166		339		10
		580,687		17,645		537		17	
80	12,675,649		385,144		11,703		356		11
		195,543		5,942		181		6	
90	12,871,192		391,086		11,884		362		12

The purpose of this table is to compute the values of $\sin 10°$, $\sin 20°$, ..., $\sin 90°$, to any desired accuracy. Bürgi's algorithm is deceptively simple. There is no bisection, no roots, only two basic operations: many additions, and a few divisions by 2. The computations can be done with integers, or with sexagesimal values, or in any other base.

In order to compute the sines, Bürgi starts with an arbitrary list of values, which can be considered as first approximations of the sines, but need not be. These values are given in column c_1. In all these columns, c_1 to c_5, the last value is always the *sinus totus*. That is, the first approximation starts with $\sin 90° = 12$. This gives in modern terms for $\sin 60°$ the value $\frac{9}{12} = 0.75$, an approximation of the actual sine which is $0.866\ldots$. It basically matters very little with what values one starts. It is even possible to take all initial values equal to 1, for instance, or in decreasing order, or at random, but they can't all be taken equal to 0, except if the last one is equal to 2 (when working with integers). If real values are used, it would work even if all values are equal to 0, and the last one is equal to 1. Negative values do also work, but some distributions will fail, for instance with values that cancel each other, such as -1, followed by 1.

Column c_5 shows the result of that algorithm, here conducted to four steps, but the table could have been extended at will towards the left. Now, in column c_5, there is a new value for the *sinus totus*, namely 12871192, and therefore we have as a new approximation of $\sin 60°$ the fraction $\frac{11146776}{12871192} = 0.86602515136$, the exact value being $\frac{\sqrt{3}}{2} = 0.866025403\ldots$.

Bürgi's algorithm is an iterative procedure for computing all the values of column c_{i+1} from those of column c_i. The computations use an intermediate column whose last value is half of the previous *sinus totus*. This intermediate column actually provides the values of the cosines, but for intermediate angles, here 5°, 15°, ..., 85°. In the above example, the *sinus totus* in column c_1 is 12, and the last value of the column between c_2 and c_1 is 6. The last value of the column between c_3 and c_2 is 181, half of 362. If the *sinus totus* is odd, one might take the exact half, but it actually does not matter, as this algorithm leads to increasingly larger numbers, and ignoring a half integer only has marginal consequences on the convergence.

Once the last value of an intermediate column has been obtained, all other values of that column are obtained by adding the values in the previous column, as if the previous column were differences. So, we have $6 + 11 = 17$, $17 + 10 = 27$, and so on.

When the intermediate column has been filled, the new column c_{i+1} is constructed by starting with 0, and adding the values of the intermediate column.

And that's all!

4 The use of differences

As we have shown in our recent note [15], the previous algorithm, although very interesting, can hardly be used to compute a table of sines for every minute, let alone for every $2''$, as Bürgi is supposed to have done. Bürgi therefore introduced other methods, which are in fact also very novel and ingenious.

Bürgi may have used his *Kunstweg* to compute accurate values of $\sin 1°$, $\sin 2°$, etc., but next he needed to compute the values of $\sin 1'$, $\sin 2'$, etc. Instead of using his *Kunstweg*, Bürgi instead first obtained an accurate value of $\sin 1'$, then worked his way up using well-known formulæ or the cumulation of differences.

One of the most interesting features in Bürgi's approach is how he analyzed the difference between $\sin(1°)/60$, which is an approximation of $\sin 1'$, and the correct value of $\sin 1'$. The way he did this rests on an analysis of the cumulation of smaller differences, and the details are in fact not yet totally understood.

In his edition of Bürgi's *Fundamentum Astronomiæ* [7], Launert sheds new light on Bürgi's methods, but both Bürgi's and Launert's descriptions are far from clear and at times opaque, and even wrong.

What Bürgi must have done, and of which he only gives the result, is a reconstruction of how the higher differences compose the value of $\sin 1^\circ$. This is one thing, but Bürgi went beyond, and sought to use this analysis in order to find a better value of $\sin 1'$ from that of $\sin 1^\circ$. This is really clever, although it is not clear, as we will see, how this was really used.

In order to explain what Bürgi did as clearly as possible, we take the notations used by Legendre in 1815 [8]. We had already done so when we expounded Briggs' methods [11]. y, y', y'', etc., are the values of the sines for $0'$, $1'$, $2'$, etc. The various differences are δy, $\delta y'$, …, $\delta^2 y$, …

$$
\begin{array}{lllll}
\sin 0' & y & & & \\
 & & \delta y & & \\
\sin 1' & y' & & \delta^2 y & \\
 & & \delta y' & & \delta^3 y \\
\sin 2' & y'' & & \delta^2 y' & \\
 & & \delta y'' & & \delta^3 y' \\
\sin 3' & y''' & & \delta^2 y'' & \\
 & & \delta y''' & & \delta^3 y'' \\
\sin 4' & y^{\text{IV}} & & \delta^2 y''' & \\
 & & \delta y^{\text{IV}} & & \delta^3 y''' \\
\sin 5' & y^{\text{V}} & & \delta^2 y^{\text{IV}} & \\
 & & \delta y^{\text{V}} & & \delta^3 y^{\text{IV}} \\
\sin 6' & y^{\text{VI}} & & \delta^2 y^{\text{V}} & \\
 & & & & \delta^3 y^{\text{V}} \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
\sin 1^\circ & y^{\text{LX}} & & &
\end{array}
$$

Bürgi did only consider what we now call the third differences, but without naming them. He called the second differences δ^2 the "differences of the differences." Basically, Bürgi noticed that the second differences are practically linear, and therefore he implicitely assumed that the third differences δ^3 are constant, or could be considered so. This is his assumption. Moreover, in the above scheme, $\delta y = \sin 1'$ and $\delta^2 y \approx \delta^3 y$.

Consequently, Bürgi could make his way up and recompute $\sin 1^\circ$ from $\sin 1'$ and δ^3:

$$\sin 1^\circ = \sum_{i=0}^{59} \delta y^i = 60\delta y + \left(\delta^2 y + (\delta^2 y' + \delta^2 y'') + \cdots + (\delta^2 y' + \delta^2 y'' + \cdots + \delta^2 y^{59})\right) \tag{1}$$

$$\approx 60\delta y + \left(\sum_{i=1}^{59} i\right)\delta^2 y + \left(\sum_{i=1}^{58}\sum_{j=1}^{i} j\right)\delta^3 y \tag{2}$$

This expression is in fact a particular case of Newton's forward difference formula [11].
Now since $\delta^2 y \approx \delta^3 y$, we have

$$\approx 60\delta y + \left(\sum_{i=1}^{59} i + \sum_{i=1}^{58}\sum_{j=1}^{i} j\right)\delta^3 y \tag{3}$$

But we know that $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$, and also that $\sum_{i=1}^{n}\sum_{j=1}^{i} j = \frac{n(n+1)(n+2)}{6}$. Therefore

$$\sin 1^\circ \approx 60\delta y + \left(\frac{59 \times 60}{2} + \frac{58 \times 59 \times 60}{6}\right)\delta^3 y \tag{4}$$

$$\approx 60\delta y + \left(\frac{59 \times 60}{2} + 34220\right)\delta^3 y \tag{5}$$

Hence

$$\sin 1' = \delta y \approx \frac{\sin 1^\circ}{60} - \left(\frac{59}{2} + \frac{34220}{60}\right)\delta^3 y \tag{6}$$

and since $\delta^3 y < 0$

$$\sin 1' = \delta y \approx \frac{\sin 1^\circ}{60} + \left(\frac{59}{2} + \frac{34220}{60}\right)|\delta^3 y| \tag{7}$$

So, $\sin 1'$ can be computed from the approximation $\frac{\sin 1^\circ}{60}$ and an approximation of the third difference, that is also from an approximation of the first second difference. But this is a somewhat circular reasoning, since computing an accurate value of $\delta^2 y = \sin 2' - 2\sin 1'$ may require knowing $\sin 1'$ and $\sin 2'$. At this point, we are therefore not clear as to how Bürgi obtained the value of the first second difference. If he obtained the first second difference ($\delta^2 y$) from other means, then the corrections applied to $\frac{\sin 1^\circ}{60}$ merely help to check the value of $\sin 1'$, and are not really used to compute an accurate one.

Before we go on with our analysis, it is necessary to explain the sexagesimal notation. Bürgi expresses the angles, but also the numbers in sexagesimal notation. The conventions are however not exactly the same. Angles are expressed in the usual way. But for a numerical value, such as the value of a sine, the whole length of the sine (the *sinus totus*) is taken equal to 60. Consequently, $\sin 90^\circ$ is equal to 60°, or better, 60 parts.

We can therefore have such values as $\sin 1^\circ = 0.01745240643728\ldots = 1^\circ 2^{\mathrm{I}} 49^{\mathrm{II}}\ldots$, because $0.01745240643728\ldots = (1 + (2 + (49 + \ldots)/60)/60)/60$

Returning to Bürgi's algorithm, Bürgi does not seem to give a value of $\delta^2 y$ [7, p. 59], but Launert, in his description of the algorithm, gives the following value [7, p. 63]:

$$\delta^3 y = 1^{\mathrm{IV}} 8^{\mathrm{V}} 45^{\mathrm{VI}} 38^{\mathrm{VII}} 27^{\mathrm{VIII}}$$

However ... this is incorrect. If these figures are correct, it should be

$$\delta^3 y = 1^{\mathrm{V}} 8^{\mathrm{VI}} 45^{\mathrm{VII}} 38^{\mathrm{VIII}} 27^{\mathrm{IX}}$$

which is about

$$(1 + (8 + (45 + (38 + 27/60)/60)/60)/60)/60^6 = 0.00000000002456300\ldots$$

The exact values of $\delta^2 y$ and $\delta^3 y$ are in fact

$$\delta^2 y = 1^{\mathrm{V}} 8^{\mathrm{VI}} 54^{\mathrm{VII}} 10^{\mathrm{VIII}} 12^{\mathrm{IX}} \ldots \tag{8}$$
$$\delta^3 y = 1^{\mathrm{V}} 8^{\mathrm{VI}} 54^{\mathrm{VII}} 10^{\mathrm{VIII}} 11^{\mathrm{IX}} \ldots \tag{9}$$

It is possible that Launert derived this value, by computing it from accurate values of $\sin 1^\circ$ and $\sin 1'$, but he doesn't say so, and we have been unable to find it in Bürgi's text.

Putting this (important) problem aside, and assuming we have a value of $\delta^3 y$, we can use it to correct the value of $\frac{\sin 1^\circ}{60}$ to obtain a better approximation of $\sin 1'$.

However, Bürgi's explanations are very general, and do not use actual values for the differences. Launert is more explicit and gives the following steps for Bürgi's corrections [7, p. 63]:

1. first, one computes $\frac{\sin 1^\circ}{60}$;
2. then, one defines the difference of the differences, and Launert exhibits the above value which he calls $dd = 1^{\mathrm{IV}} 8^{\mathrm{V}} \ldots$ (instead of the more correct $1^{\mathrm{V}} 8^{\mathrm{VI}} \ldots$)

3. then Launert computes the first correction $\frac{dd\times 30}{4}$, but this should give $0.00000000018422\ldots = 8^{\mathrm{V}}35^{\mathrm{VI}}\ldots$, whereas Launert gives $8^{\mathrm{IV}}35^{\mathrm{V}}\ldots$;

4. Bürgi then makes us compute $1+2\cdots+29 = 435$, which we multiply by $(1+30)/3$, and this number, 4495 is again multiplied by 2 and by dd to produce the second correction $2^{\mathrm{IV}}51^{\mathrm{V}}\ldots$; but, in fact neither $1^{\mathrm{IV}}8^{\mathrm{V}}\ldots$, nor $1^{\mathrm{V}}8^{\mathrm{VI}}\ldots$, when multiplied by 4495×2, give $2^{\mathrm{IV}}51^{\mathrm{V}}\ldots$;

5. the addition of these two corrections to the value of $\frac{\sin 1^\circ}{60}$ would give a better approximation of $\sin 1'$, if the two corrections were correct...

This procedure will certainly appear confusing to all those who will try to apply it, but there are good reasons for this confusion. Bürgi himself adds to the confusion, for instance when he computes $1+2\cdots+29 = 435$, and also asks us to compute the sum of the squares, although he doesn't seem to use the result.

In order to get the things right, we have to clarify the above steps. First, it appears that the value of dd given by Launert was probably modified by him by a factor 60 so that the first corrective term becomes $8^{\mathrm{IV}}35^{\mathrm{V}}\ldots$ instead of $8^{\mathrm{V}}35^{\mathrm{VI}}\ldots$... That is, as we have said earlier, the correct value of dd should be about $1^{\mathrm{V}}8^{\mathrm{VI}}$, not the one used by Launert. But if Launert uses the correct value for dd, then Bürgi's steps do no longer work. So, what is the solution?

In fact, Bürgi asks not only to multiply the difference of differences by 30 and divide it by 4, but in addition to *shift* it, or to magnify it by one unit. So, we have to compute $\frac{30\times 60}{4}dd$.

The step for the second correction has also been misunderstood. Perhaps there is a missing word in Bürgi's original text, but what Bürgi really wanted us to do is to multiply the difference of differences by 4495, multiply it by 2, *and make the result one unit smaller*, that is, divide the result by 60. This is why Launert's computations are incorrect.

With these modifications, Bürgi's steps do provide the right corrections. However... Bürgi ends up with the expression

$$\sin 1' = \delta y \approx \frac{\sin 1^\circ}{60} + \left(\frac{30\times 60}{4} + \frac{4495\times 2}{60}\right)|\delta^3 y| \tag{10}$$

But earlier, we had found

$$\sin 1' = \delta y \approx \frac{\sin 1^\circ}{60} + \left(\frac{59}{2} + \frac{34220}{60}\right)|\delta^3 y| \tag{11}$$

The reader can check that

$$\frac{30\times 60}{4} + \frac{4495\times 2}{60} = \frac{59}{2} + \frac{34220}{60} \tag{12}$$

At this point, what is still not totally clear is how Bürgi came to his expression, and why his and ours are identical. We have also not yet explained why Bürgi mentions the sum of the squares, without using it. We do not have complete answers to these questions, but we can still go a little further, by relating a number of notions. Recall that $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ and $\sum_{i=1}^{n} \sum_{j=1}^{i} j = \frac{n(n+1)(n+2)}{6}$. Then, the above identity is a particular case of

$$\sum_{i=1}^{2n+1} i + \sum_{i=1}^{2n} \sum_{j=1}^{i} j = (n+1)^3 + \frac{\sum_{i=1}^{n} i}{3} \times (1+n+1) \times 2 \tag{13}$$

For instance, if $n = 29$, we have

$$(1+2+\cdots+59) + 34220 = (29+1)^3 + \frac{435}{3} \times (1+29+1) \times 2 \tag{14}$$

$$= \frac{30 \times 60 \times 60}{4} + 4495 \times 2 \tag{15}$$

The connection with the sum of the squares can be made if we notice that

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}. \tag{16}$$

That sum could then be used in an alternate expression of the second correction.

Once Bürgi has $\sin 1'$, he can compute $\sin 2'$ and other values by the well-known formulæ, but he can also use the accumulation of differences [7, p. 65]. One therefore presumes that Bürgi had pivot points, namely values with accurate expressions of the sine and the differences. For instance, for $1'$, Bürgi had $\sin 1'$, as well as the first difference which is also $\sin 1'$, the second difference $\delta^2 y$, and the third one which he could take equal to the second one. By taking the third difference constant, he could just add up the values until $\sin 1°$. In order to get five correct sexagesimal places, Bürgi would in fact have needed to use fourth differences, but the technique is the same. We believe that Bürgi computed the sines of the degrees with his first algorithm (*Kunstweg*), and probably also the sines of the half degrees which can be obtained from the two surrounding sines by solving a quadratic equation, and then computed the first, second, third and fourth differences for all these ranges. Once Bürgi had for instance an accurate value of $\sin 4°$, for instance, it was very easy for him to obtain the first, second, third and fourth differences, because he could compute $\sin(4° + 1')$, $\sin(4° + 2')$, $\sin(4° + 3')$, $\sin(4° + 4')$, and merely do the subtractions.

Bürgi's table giving the sines for every minute [7, pp. 72–73][2] appears quite accurate, but it is easy to see that this table was copied from another one and truncated.[3] For instance, the use of differences on this table will not reveal its underlying structure, and practically all the values seem to be correctly rounded. We have only checked some of the values, but we trust that Bürgi had more digits, and that he could at the same time check an interpolation by comparing the end of an interpolation with the next pivot.

This method could be applied for computing a table of sines with a $2''$ interval, merely by using a table to $1'$ with 8 sexagesimal places as pivots and computing $\sin 2''$. It is therefore possible that Bürgi did compute such a table.

It should be noted that the above sketched procedure is exactly what we suggested Bürgi did in our earlier analysis [12, § 3.3], prior to the (re)discovery of the *Fundamentum Astronomiæ*. In other words, Bürgi appears to be a forerunner of the application of differences to construct a table, instead of merely using differences to check tabular values. His approach is in fact closer to that of Prony [13] than to that of Briggs. Briggs did not calculate pivots and apply the method of differences, but he found intermediate values by the application of subtabulation methods [11]. In other words, Briggs used larger differences in order to find intermediate values, whereas Bürgi would use known differences to build up. Of course, both Briggs and Bürgi *analyzed* the structure of the differences, but the choices made were different. Had Briggs known of Bürgi's ideas, he might have considered computing his tables differently, and this suggests that he had no detailed knowledge of Bürgi's work.

5 Conclusion

This brief note was only meant to clarify the computation of the sine of the first minute, and we hope that it did so. But in fact, what has appeared is that Bürgi developed a very ingenious collection of algorithms and obviously had a very deep sense of numbers. The use of differences, not for checking tabular values, but in order to compute new ones, is a very modern approach. It anticipates by 200 years (!) the work of Prony [13], and even Babbage (!). and this is quite meritory and should deserve our admiration.

[2] The entire table was put on `http://locomat.loria.fr/buergi/fundamentum.html`, with permission.

[3] On the excerpt reproduced by Launert, we can notice a typo of the copier, namely that the value of $\sin 4°$ is given as $4°11'17''23'''54''''$, but it should be $4°11'7''23'''54''''$. The difference given below that value is correct, as well as the following value for $\sin(4°1')$.

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