Data center interconnection networks are not hyperbolic

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Abstract

Topologies for data center interconnection networks have been proposed in the literature through various graph classes and operations. A common trait to most existing designs is that they enhance the symmetric properties of the underlying graphs. Indeed, symmetry is a desirable property for interconnection networks because it minimizes congestion problems and it allows each entity to run the same routing protocol. However, despite sharing similarities these topologies all come with their own routing protocol. Recently, generic routing schemes have been introduced which can be implemented for any interconnection network. The performances of such universal routing schemes are intimately related to the *hyperbolicity* of the topology. Roughly, graph hyperbolicity is a metric parameter which measures how close is the shortest-path metric of a graph from a tree metric (the smaller the gap the better). Motivated by the good performances in practice of these new routing schemes, we propose the first general study of the hyperbolicity of data center interconnection networks. Our findings are disappointingly negative: we prove that the hyperbolicity of most data center interconnection topologies scales linearly with their diameter, that it the worst-case possible for hyperbolicity. To obtain these results, we introduce original connection between hyperbolicity and the properties of the endomorphism monoid of a graph. In particular, our results extend to all vertex and edge-transitive graphs. Additional results are obtained for de Bruijn and Kautz graphs, grid-like graphs and networks from the so-called Cayley model.

Keywords. greedy routing scheme; metric embedding; graph endomorphism; Gromov hyperbolicity; Cayley graph; data center; interconnection network

1 Introduction

The network topologies that are used to interconnect the computing units of large-scale facilities (e.g., super computers, data centers hosting cloud applications, etc.) are designed to optimize various constraints such as equipment cost, deployment time, capacity and bandwidth, routing functionalities, reliability to equipment failures, power consumption, etc. This large variety of (conflicting) criteria has yielded numerous proposals of interconnection networks. See for instance [11, 12, 20, 47, 81] and [36, 38–40, 59] for the most recent ones. A common feature of the

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proposed constructions is to design network topologies offering a high-level of *symmetries*. Indeed, it is easier to balance the traffic load, and hence to minimize the congestion, on network topologies with a high-level of symmetry. Furthermore, it simplifies the initial wiring of the physical infrastructure and it ensures that each router node can run the protocol.

However, despite sharing properties, interconnection networks rely on specific routing algorithms that are optimized for each topology. As a novel step toward efficient and topology agnostic routing schemes, the authors in [64–66] proposed to use greedy routing schemes based on an embedding of the topology into certain metric space such as the hyperbolic metric space, and more recently the word metric space. This approach has been shown particularly efficient for Internet-like graphs [33, 42] where routes with low stretch are obtained. One explanation of this good behavior is that Internet-like graphs have low *hyperbolicity* [10, 74], a graph parameter providing sharp bounds on the stretch (or distortion) of the distances in a graph when it is embedded into an edge-weighted tree.

In this paper, we characterize or give upper and lower-bounds on the hyperbolicity of a broad range of interconnection network topologies. These bounds can be used to analyze the worst-case behavior of greedy routing schemes in these topologies. Before we present our results, let us further put in context the role they play in routing and in other distance-related problems.

Related work. Greedy routing schemes based on an embedding into the hyperbolic space have been introduced by Kleinberg in [33]. Since then, various authors explored further this approach [42, 45, 48]). In particular, they showed that the graphs of the Autonomous Systems of the Internet embed better into a hyperbolic space than into an Euclidean space[1]. It was only recently in [70] that a formal relationship between the performances of hyperbolic embeddings and the hyperbolicity was proved. Namely, the authors proved that the over-delay for such routing schemes, or equivalently the stretch of the routing, depends on the hyperbolicity. In [73], the authors proved that similar results hold for greedy routing schemes based on an embedding of the topology into some word metric space (*e.g.*, see [26] for more information). More precisely, they use hyperbolicity to upper-bound the complexity of their routings, as well as to bound the size of the automata that are involved in their routing schemes.

Their results add up to prior worst-case analysis of graph heuristics that already pointed out the important role played by the hyperbolicity. For instance, there are approximation algorithms for problems related to distances in graphs —like diameter and radius computation [37], and minimum ball covering [32]— whose approximation constant depends on the hyperbolicity. Sometimes the approximation factor is a universal constant but the algorithm relies on a data-structure whose size is proportional to the hyperbolicity of the network topology [31]. Geometric routing schemes in [42, 45, 48] do not make exception and so have a stretch *lower-bounded* by the hyperbolicity (the bound is reached by some of them).

There have been measurements to confirm that complex networks such as the graphs of the Autonomous Systems of the Internet, social networks and phylogenetic networks all have a low hyperbolicity. We refer to [60, 62, 63, 71, 74, 77] for the most important studies in this area. Additional related work in [50, 57] shows that the low hyperbolicity of complex networks may be a consequence

[1] In fact, it follows from [23] that for any n-vertex graph G there is an embedding φ of G into the Euclidean space (with unbounded dimension) such that for every $u, v \in V(G)$ we have $\mathrm{d}(\varphi(u), \varphi(v)) \leq \mathcal{O}(\sqrt{\log\log n}) \cdot \mathrm{d}_G(u, v) + \hat{\mathcal{O}}(\delta(G) \cdot \log n)$, with $\delta(G)$ being the hyperbolicity (the $\hat{\mathcal{O}}$-notation suppresses the polyloglog factors). However, it does not seem that hyperbolicity is the most relevant parameter in the study of Euclidean embeddings.

of some preferential attachment mechanisms. However, we are not informed of any study on the hyperbolicity of data center interconnection networks. In this paper, we aim to fill in this gap through a theoretical study of their underlying graphs.

Our contributions. In an attempt to confront with the diversity of interconnection network topologies proposed in the literature, we relate hyperbolicity with a few graph properties that are frequently encountered in these topologies. Indeed, we do not aim to provide a —long and non-exhaustive— listing of unrelated results for each network, but rather to exhibit a small number of their characteristics that are strongly related with their metric invariants. In particular, we relate hyperbolicity with the symmetries of a graph.

- We prove in Section 3 that for graphs whose center is a k-distance dominating set for some small value of k, the hyperbolicity scales linearly with the diameter. This class of graphs strictly contains graphs whose diameter equals the radius, *a.k.a.* the *self-centered graphs* [9, 14]. In particular, it comprises all vertex-transitive graphs (a strict subclass of self-centered graphs), as well as edge-transitive graphs. A main consequence of our result is that every interconnection network whose topology is based on a *Cayley graph* has large hyperbolicity[2].

- In addition, we prove that similar results hold for graphs admitting an *endomorphism* such that the distance between any vertex and its symmetric image is large. On the way to prove these results, we define a new graph invariant which is called *weak mobility*, that generalizes the so-called graph mobility (*e.g.*, see [22, 34]). We use these new results to improve our lower-bounds on the hyperbolicity of several interconnection networks.

- For completeness, we also characterize the hyperbolicity of other "symmetric" networks such as de Bruijn, Kautz and grid-like graphs. More precisely, we apply different techniques that are based on their shortest-paths distribution so that we can prove in Section 4 that they also have a large hyperbolicity. The techniques that are involved in the proofs have been introduced in previous papers [16, 41], but to the best of our knowledge the way we use them in this work is new.

  All of the above results are summarized in Table 1.

- Last, we extend our results in Section 5 to *heterogenous* data center interconnection networks. That is, we relate hyperbolicity with several graph operations, most of them being introduced in the Cayley model of [59] in order to enhance some desirable properties of data center interconnection networks.

Our main message is that existing designs in the literature yield graphs with the highest possible value for the hyperbolicity —w.r.t. their diameter. On the negative side, it means that any greedy routing scheme whose stretch depends on the hyperbolicity is not scalable enough to cope with large data centers. But on a more positive side, it also implies that any routing scheme relying on

[2] Independently from this work, the authors in [54] proved that for any vertex-transitive graph, the hyperbolicity scales linearly with the diameter. However, their proof relies on another definition of hyperbolicity, and it is unclear whether the proof can be extended to other graph classes. By contrast, our proof yields a tighter lower-bound for hyperbolicity, and it relies on a much simpler and more general argument (*i.e.*, see Theorem 4). Especially, it also applies to edge-transitive graphs.

a data-structure with size proportional to the hyperbolicity solely requires *sublogarithmic* space in the number of servers. Indeed, it is well-known that the data center interconnection networks often have a diameter that is logarithmic or sublogarithmic in their size.

We start this paper providing useful notations and definitions in Section 2, and we conclude it in Section 6 with open questions. Especially, can we infer a formal relationship between network congestion and graph hyperbolicity ?

2 Preliminaries

A data center is a facility that is used to house resources such as computer systems, servers, etc. Data center resources are interconnected using communication networks, that are called data center interconnection networks. They are modeled as a graph where the vertices are the data center resources (*e.g.*, computing units) and there is an edge between two resources if they are directly connected in the network. Different graph classes have been proposed in order to design data center interconnection networks [11, 12, 20, 36, 38–40, 47, 59, 81]. In what follows, our results apply to general graphs, but they are aimed at providing good lower-bounds on the hyperbolicity for these specific topologies.

We refer to [80, 82] for the usual graph terminology. Graphs in this study are finite, simple (hence, without loop nor multiple edges), connected and unweighted.

2.1 Metric graph theory

Given a connected graph $G = (V, E)$, the *distance* between any two vertices $u, v \in V$ is defined as the minimum number of edges on a uv-path. We will denote it by $\mathrm{d}_G(u, v)$, or by $\mathrm{d}(u, v)$ whenever G is clear from the context. For any subset $S \subseteq V$, the *eccentricity* of vertex $v \in S$, denoted $\mathrm{ecc}_G(v, S)$, is defined as the maximum distance in G between v and any other vertex in S. The *radius* of S is defined as the least eccentricity of vertices in S and is denoted by $\mathrm{rad}_G(S)$, while the *diameter* of S is defined as the largest eccentricity of vertices in S and is denoted by $\mathrm{diam}_G(S)$. Observe that it always holds $\mathrm{rad}_G(S) \leq \mathrm{diam}_G(S) \leq 2 \cdot \mathrm{rad}_G(S)$. In particular, for any vertex $v \in V$, we denote by $\mathrm{ecc}(v) = \mathrm{ecc}_G(v, V)$, $\mathrm{rad}(G) = \mathrm{rad}_G(V)$ and $\mathrm{diam}(G) = \mathrm{diam}_G(V)$. The *center* $\mathcal{C}(G)$ of the graph is the subset of all vertices with minimum eccentricity $\mathrm{rad}(G)$. We call the graph G *self-centered* if it holds $\mathrm{diam}(G) = \mathrm{rad}(G)$ *i.e.*, every vertex of G is in the center.

Last, we define graph hyperbolicity as follows.

Definition 1 (4-points Condition, [10]). Let G be a connected graph.

For every 4-tuple u, v, x, y of G, we define $\delta(u, v, x, y)$ as half of the difference between the two largest sums amongst:

$$S_1 = \mathrm{d}(u, v) + \mathrm{d}(x, y),\ S_2 = \mathrm{d}(u, x) + \mathrm{d}(v, y), \text{ and } S_3 = \mathrm{d}(u, y) + \mathrm{d}(v, x).$$

The graph hyperbolicity, denoted by $\delta(G)$, is equal to $\max_{u,v,x,y} \delta(u, v, x, y)$.

Moreover, we say that G is *δ-hyperbolic*, for every $\delta \geq \delta(G)$.

Other definitions exist for the hyperbolicity, but they are pairwise equivalent up to a constant-factor (*e.g.*, see [10] for details). So far, the hyperbolicity of a few graph classes has been characterized such as: random graphs [56, 61, 69], chordal graphs [25], k-chordal graphs [53], outerplanar

Name	Degree max.	Diameter	Order	δ	Proof
de Bruijn graph, $UB(d,D)$	$2d$	D	d^D	$\frac{1}{2}\left\lfloor\frac{D}{2}\right\rfloor \le \delta \le \left\lfloor\frac{D}{2}\right\rfloor$	Prop. 37
Kautz graph, $UK(d,D)$	$2d$	D	$d^D(d+1)$	$\left\lfloor\frac{D}{4}\right\rfloor + \varepsilon \le \delta \le \left\lfloor\frac{D}{2}\right\rfloor, \varepsilon \in \{0,1\}$	Prop. 40
Shuffle exchange, $SE(n)$	3	$2n-1$	2^n	$\frac{1}{2}\left\lfloor\frac{n}{2}\right\rfloor \le \delta \le n-1$	Prop. 42
(n,m)-grid	4	$n+m-2$	nm	$\min\{n,m\}-1$	Cor. 48
d-dimensional grid of size s	$2d$	$d(s-1)$	s^d	$(s-1)\left\lfloor\frac{d}{2}\right\rfloor$	Cor. 49
Triangular (n,m)-grid	6	$n+m-2$	nm	$\frac{\min\{n,m\}-1}{2}$	Lem. 51
Hexagonal (n,m)-grid	6	$\begin{cases} n-1+\left\lceil\frac{m-1}{2}\right\rceil & \text{when } m \le 2n-1 \\ m-1 & \text{otherwise}\end{cases}$	nm	$\frac{\min\{n,m\}-1}{2}$	Lem. 53
Cylinder (n,m)-grid	4	$\left\lfloor\frac{n}{2}\right\rfloor + m - 1$	nm	$\min\left\{\left\lfloor\frac{n}{2}\right\rfloor, \frac{1}{2}\left(\left\lfloor\frac{n}{2}\right\rfloor + m\right) - \varepsilon\right\}, \varepsilon \in \left\{\frac{1}{2}, 1\right\}$	Lem. 55
Torus (n,m)-grid	4	$\left\lfloor\frac{n}{2}\right\rfloor + \left\lfloor\frac{m}{2}\right\rfloor$	nm	$\left\lfloor\frac{1}{2}\left(\left\lfloor\frac{n}{2}\right\rfloor + \left\lfloor\frac{m}{2}\right\rfloor\right)\right\rfloor - 1 \le \delta \le \left\lfloor\frac{1}{2}\left(\left\lfloor\frac{n}{2}\right\rfloor + \left\lfloor\frac{m}{2}\right\rfloor\right)\right\rfloor$	Lem. 11
Gen. hypercube, $G(m_1,\ldots,m_r)$	$\sum_{i=1}^r m_i - r$	r	$\prod_{i=1}^r m_i$	$\left\lfloor\frac{r}{2}\right\rfloor$	Lem. 13
Cube Connected Cycle, $CCC(n)$	3	$2n-2+\max\left\{2, \left\lfloor\frac{n}{2}\right\rfloor\right\}$	$n2^n$	$n \le \delta \le n-1+\left\lfloor\frac{\max\left\{2,\left\lfloor\frac{n}{2}\right\rfloor\right\}}{2}\right\rfloor$	Lem. 15
$\text{BCube}_k(n)$	$\max\{n,k+1\}$	$2(k+1)$	$2^k(n+k+1)$	$k+1$	Lem. 17
Fat-Tree_k	k	6	$\frac{k^2}{4}(k+5)$	2	Lem. 19
Butterfly graph, $BF(n)$	4	$2n$	$2^n(n+1)$	n	Lem. 21
Wrapped Butterfly graph, $WBF(n)$	4	$n+\left\lfloor\frac{n}{2}\right\rfloor$	$2^n(n+1)$	$\left\lfloor\frac{n}{2}\right\rfloor \le \delta \le \left\lfloor\frac{1}{2}\left(n+\left\lfloor\frac{n}{2}\right\rfloor\right)\right\rfloor$	Lem. 21
k-ary n-fly	$2k$	$2n$	$k^n(n+1)$	n	Lem. 23
k-ary n-tree	$3k$	$2n$	$k^{n-1}(n+k)$	$n-1$	Lem. 25
d-ary tree grid, $MT(d,h)$	$d+1$	$4h$	$d^h\left(d^h+2\frac{d^h-1}{d-1}\right)$	$2h$	Lem. 27
Bubble-sort graph, $BS(n)$	$n-1$	$\binom{n}{2}$	$n!$	$\left\lfloor\frac{n(n-1)}{4}\right\rfloor$	Lem. 30
Transposition graph, $T(n)$	$\binom{n}{2}$	$n-1$	$n!$	$\frac{1}{2}\left\lfloor\frac{n-1}{2}\right\rfloor \le \delta \le \left\lfloor\frac{n-1}{2}\right\rfloor$	Lem. 32
Star graph, $S(n)$	$n-1$	$\left\lfloor\frac{3(n-1)}{2}\right\rfloor$	$n!$	$\left\lfloor\frac{1}{2}\left\lfloor\frac{3(n-1)}{2}\right\rfloor - \frac{1}{2}\right\rfloor \le \delta \le \left\lfloor\frac{1}{2}\left\lfloor\frac{3(n-1)}{2}\right\rfloor\right\rfloor$	Lem. 34
Cayley graph, $G(\Gamma,S)$	$2\lvert S\rvert$	$\operatorname{diam}(G(\Gamma,S))$	$\lvert\Gamma\rvert$	$\frac{1}{2}\left\lfloor\frac{\operatorname{diam}(G(\Gamma,S))}{2}\right\rfloor \le \delta \le \left\lfloor\frac{\operatorname{diam}(G(\Gamma,S))}{2}\right\rfloor$	Cor. 5

Table 1: Summary of results

graphs [67] and other geometrical graph classes [37]. Lower and upper-bounds for the hyperbolicity are obtained in [58] using graph invariants, and also in [52, 67] using graph decompositions. We refer to [44] for a compelling of many well-known facts about hyperbolicity. In particular, we will make use of the following upper-bound for hyperbolicity:

Lemma 2 ([10, 44, 74]). *For every connected graph G, it holds that* $\delta(G) \leq \left\lfloor \frac{\mathrm{diam}(G)}{2} \right\rfloor$.

Based on Lemma 2, the authors in [71] have proposed the following classification of finite graphs. A graph G is *strongly hyperbolic* if $\delta(G) = \mathcal{O}(\log(\log(diam(G))))$, *hyperbolic* if $\delta(G) = \mathcal{O}(\log(diam(G)))$, and *non hyperbolic* otherwise. We follow their terminology and we aim at proving that some graph classes are non hyperbolic. This is in contrast with many graph classes in the literature that have a constant upper-bound on their hyperbolicity, and so, that are strongly hyperbolic [53]. By Lemma 2, in order to prove that a graph is non hyperbolic, and more precisely that its hyperbolicity scales linearly with its diameter, it suffices to prove that one can *lower-bound* the hyperbolicity with the diameter —up to a constant-factor. This line of work was followed in [21,76] to prove that expander graphs are non hyperbolic. Our proofs will make use of the notion of *isometric subgraphs*, the latter denoting a subgraph H of a graph G such that $\mathrm{d}_H(u, v) = \mathrm{d}_G(u, v)$ for any two vertices $u, v \in H$.

2.2 Algebraic graph theory

A graph *endomorphism* is a mapping σ from the vertex-set of a graph G to itself which preserves the adjacency relations, *i.e.*, for every $\{u, v\} \in E(G)$ we have that $\{\sigma(u), \sigma(v)\} \in E(G)$.

Definition 3. Let $G = (V, E)$ be a graph. Given an endomorphism σ of G, the *mobility* of σ is equal to $\min_{v \in V} \mathrm{d}(v, \sigma(v))$. The *weak mobility* of G is the largest integer l such that it admits an endomorphism with mobility l.

We note that a graph endomorphism might fail to preserve the *non-adjacency* relations, but it does so if it is a graph *automorphism*, *i.e.*, a one-to-one endomorphism. In particular a graph endomorphism σ is called *idempotent* if for every $v \in V(G)$ it holds that $\sigma^2(v) = v$, and in such a case it is an automorphism.

A graph is called *vertex-transitive* if for every $u, v \in V(G)$, there is an automorphism σ such that $\sigma(u) = v$. Similarly, we call a graph *edge-transitive* if for every $e = \{u, v\}, e' = \{u', v'\} \in E(G)$, there is an automorphism σ such that $\{\sigma(u), \sigma(v)\} = \{u', v'\}$. We emphasize that every vertex-transitive graph is self-centered. We will use this property in the following sections. Finally, let $(\Gamma, \cdot)$ be a group and let S be a generating set of Γ that is symmetric and that does not contain the neutral element of group Γ, *i.e.*, $S = S^{-1}$ and $S \cap S^{-1} = \emptyset$. The *Cayley graph* $G(\Gamma, S)$ of group Γ w.r.t. S has vertex-set Γ and edge-set $\{\{g, g \cdot s\} \mid g \in \Gamma, s \in S\}$. It is well-known that every Cayley graph is vertex-transitive [12].

3 The metric properties of the endomorphism monoid of a graph

Our belief is that any method to lower-bound the value of hyperbolicity needs to rely *as few as possible* on the shortest-path distribution of the graphs so as to be of practical use. Indeed, in most cases there is no good characterization of this distribution. There even exist interconnection networks topologies the diameter of which is still unknown [3,46]. In a need of more robust methods, we introduce new lower-bounds on the hyperbolicity that are based on non-trivial symmetries of the graphs. For clarity, our results are presented separately from their applications to interconnection networks topologies.

3.1 Main results

We first introduce a very generic argument to obtain lower-bounds on the hyperbolicity. In particular, we will show that it applies to highly symmetric graphs such as transitive graphs.

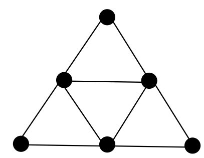


Figure 1: A self-centered graph G with $diam(G) = rad(G) = 2$, while $\delta(G) = 1/2 = diam(G)/4$.

Theorem 4. *Let G be a connected graph, and let $k \geq 0$ be such that all vertices are at distance at most k from the center of G. Then, $\delta(G) \geq \frac{1}{2} \cdot \left\lfloor \frac{\text{diam}(G)}{2} \right\rfloor - \frac{k}{2}$ and this bound is sharp.*

Proof. Let $\mathcal{C}(G)$ be the center of G. By the hypothesis every node in G is at distance at most k from $\mathcal{C}(G)$, therefore $\text{diam}_G(\mathcal{C}(G)) \geq \text{diam}(G) - 2k$. Moreover, by [37, Proposition 5] $\text{diam}_G(\mathcal{C}(G)) \leq 4\delta(G) + 1$. Consequently, it holds $\delta(G) \geq \lfloor \text{diam}_G(\mathcal{C}(G))/2 \rfloor /2 \geq \lfloor \text{diam}(G)/2 \rfloor /2 - k/2$.

The lower-bound is sharp, as shown with the example of Figure 1 where $diam(G) = rad(G) = 2$ while $\delta(G) = 1/2 = diam(G)/4$. □

Unlike all other techniques that we will discuss next, we can use the lower-bound of Theorem 4 to prove that all graphs studied in this work are non hyperbolic. However, the bounds obtained with this first method are usually loose, and they never outmatch the bounds obtained with the other techniques — when they apply. We will illustrate this point in what follows.

It is straightforward that Theorem 4 applies to self-centered graphs (with $k = 0$). Especially, it applies to vertex-transitive graphs.

Corollary 5. *Let G be a connected vertex-transitive graph. Then, $\delta(G) \geq \frac{1}{2} \cdot \left\lfloor \frac{\text{diam}(G)}{2} \right\rfloor$ and this bound is sharp.*

The lower-bound of Corollary 5 is sharp, as shown by any clique (that has diameter one and null hyperbolicity).

On the practical side, most of the interconnection networks topologies are based on vertex-transitive graphs. This comprises hypercube-based networks [8], generalized Petersen graphs [1,18], generalized Heawood graphs [30,68] and Cayley graphs [12]. For some of these topologies such as the Pancake graph [3], a well-known Cayley graph, Corollary 5 is the best lower-bound on the hyperbolicity we know so far.

Corollary 6. *Let G be a connected edge-transitive graph. Then, $\delta(G) \geq \frac{1}{2} \cdot \left\lfloor \frac{\text{diam}(G)}{2} \right\rfloor - \frac{1}{2}$ and this bound is sharp.*

Proof. We first claim that the center $\mathcal{C}(G)$ is a dominating set of G. Indeed, let $u \in V(G)$ and $v \in \mathcal{C}(G)$, and let $x \in N_G(u)$ and $y \in N_G(v)$. Since G is edge-symmetric by the hypothesis, there exists an automorphism σ such that $\{\sigma(v), \sigma(y)\} = \{u, x\}$. Furthermore $\sigma(v) \in \mathcal{C}(G)$ because σ is an automorphism and so, $\mathrm{d}_G(u, \mathcal{C}(G)) \leq \mathrm{d}_G(u, \sigma(v)) \leq 1$ which proves the claim. As a result, we can apply Theorem 4 by setting $k = 1$.

The lower-bound is sharp, as shown by any star (that has diameter two and null hyperbolicity). □

3.1.1 Improved lower-bounds using graph endomorphisms

However, despite its wide applicability to interconnection networks, the above Corollaries 5 and 6 require graphs to have an automorphism group with constrained properties. A natural question is whether we can weaken the requirements by considering endomorphisms instead of automorphisms. To answer this question, we use weakly vertex-transitive graphs that have been defined in [29] in a similar fashion to vertex-transitive graphs. Namely, a graph G is weakly vertex-transitive if, for any two vertices $u, v \in V(G)$ there exists a graph *endomorphism* σ satisfying $\sigma(u) = v$. Unlike vertex-transitive graphs, the gap between hyperbolicity and diameter may be arbitrarily large for weakly vertex-transitive graphs. Indeed, on the one hand it was proved in [29] that bipartite graphs are weakly vertex-transitive. On the other hand, trees are bipartite 0-hyperbolic graphs, whereas they may have a diameter that is arbitrarily large. We now show that surprisingly, some lower-bounds on the hyperbolicity can still be deduced from graph endomorphisms.

Theorem 7. *Let G be a connected graph of weak mobility $l \geq 2$. Then it holds $\delta(G) \geq \frac{1}{2} \cdot \left\lceil \frac{l}{2} \right\rceil$.*

Proof. We will consider a graph game which is a slight variation of the well-known 'Cop and Robber' game (*e.g.* see [5–7]). There are two players in this game that are playing alternatively on a (connected) graph, by moving along a path of length at most s, for some positive integer s. The first player to position herself on the graph is the Cop, and the second player is called the Robber. Last a graph is said *Cop-win* for this game if the Cop always has a winning-strategy *i.e.*, she can always reach the position of the Robber in a finite number of moves, and hence eventually catch the Robber. In [49] the authors proved that every connected graph G is Cop-win whenever $s \geq 4\delta(G)$. So, to prove the theorem we claim that it suffices to show that G is not Cop-win if $s \leq l - 1$. Indeed, in such a case it holds $4\delta(G) \geq l$, hence $2\delta(G) \geq l/2$ that implies $2\delta(G) \geq \lceil l/2 \rceil$ and so, $\delta(G) \geq \lceil l/2 \rceil / 2$. Equivalently, we will exhibit a winning-strategy for the Robber in such a case.

Let σ be an endomorphism of G with mobility l, that exists by the hypothesis. One can observe that if at each turn of the Cop the Robber is onto the image by σ of her current position, then it is a winning strategy for the Robber because by the hypothesis, both vertices are at distance at least l, and the maximum speed of the Cop is $l - 1$. To achieve the result, let us proceed as follows. First if the Cop picks vertex u as her initial position then the Robber starts the game at vertex $\sigma(u)$. Then, if the Cop moves along a path $(u = x_0, x_1, \ldots, x_i, \ldots, x_k = v)$, $k \leq l - 1$, then the Robber moves along the path $(\sigma(u), \sigma(x_1), \ldots, \sigma(x_i), \ldots, \sigma(v))$ which exists because σ is a graph endomorphism. Such a move for the Robber is valid as long as $v \notin \{\sigma(u), \sigma(x_1), \ldots, \sigma(x_i), \ldots, \sigma(v)\}$, and that is always the case since $\sigma(x_i) = v$ would imply $\mathrm{d}(x_i, \sigma(x_i)) \leq l - 1$. □

We are particularly interested in the special case of the graphs G with weak mobility equal to their diameter $diam(G)$. These graphs are self-centered, and so, their hyperbolicity is at least

$\lfloor diam(G)/2 \rfloor/2$ by Theorem 4. The lower-bound is slightly improved by Theorem 7 in this situation. However, not all self-centered graphs have their weak mobility equal to the diameter [34].

In what follows, we will mostly rely upon the below refinement of Theorem 7 in our proofs. This way, we will obtain almost tight bounds on the hyperbolicity of data center interconnection networks. However, note that the following results require stronger constrictions on the endomorphism monoid than Theorem 7.

Theorem 8. *Let G be a connected graph, and l, l' be two non-negative integers. Suppose there exists an endomorphism σ of G with mobility l and such that for every $v \in V(G)$, $\mathrm{d}(v, \sigma^2(v)) \leq l'$. Then, it holds $\delta(G) \geq \left\lfloor \frac{l}{2} \right\rfloor - \frac{l'}{2}$.*

Proof. Clearly, if $l \leq l'$ then $\delta(G) \geq 0 \geq \lfloor l/2 \rfloor - l'/2$. Therefore, we will assume w.l.o.g. that $l \geq l' + 1$. Let $u \in V(G)$ minimizing $\mathrm{d}_G(u, \sigma(u))$ and let v be on a $u\sigma(u)$-shortest-path such that $\mathrm{d}_G(u, v) = \lfloor \mathrm{d}_G(u, \sigma(u))/2 \rfloor$. Then, we deduce from the endomorphism σ the following inequalities:

$$\begin{aligned} S_1 &= \mathrm{d}(u, \sigma(u)) + \mathrm{d}(v, \sigma(v)) \geq 2 \cdot \mathrm{d}(u, \sigma(u)) \geq 2l; \\ S_2 &= \mathrm{d}(u, v) + \mathrm{d}(\sigma(u), \sigma(v)) \leq 2 \cdot \mathrm{d}(u, v) \leq 2 \lfloor \mathrm{d}(u, \sigma(u))/2 \rfloor; \\ S_3 &= \mathrm{d}(u, \sigma(v)) + \mathrm{d}(v, \sigma(u)) \leq \mathrm{d}(u, \sigma^2(u)) + \mathrm{d}(\sigma^2(u), \sigma(v)) + \mathrm{d}(v, \sigma(u)) \leq l' + 2 \cdot \mathrm{d}(v, \sigma(u)) \\ &\quad \leq 2 \lceil \mathrm{d}(u, \sigma(u))/2 \rceil + l' \leq \mathrm{d}(u, \sigma(u)) + 1 + l'. \end{aligned}$$

In such a case, $S_1 \geq \max\{S_2, S_3\}$ and as a result:

$$\delta(G) \geq \delta(u, v, \sigma(u), \sigma(v)) \geq \min\left(\left\lceil \frac{\mathrm{d}(u, \sigma(u))}{2} \right\rceil, \left\lfloor \frac{\mathrm{d}(u, \sigma(u))}{2} \right\rfloor - \frac{l'}{2} \right) \geq \left\lfloor \frac{l}{2} \right\rfloor - \frac{l'}{2}.$$

□

The lower-bound of Theorem 8 outmatches the one of Theorem 7 when $l' \leq \lfloor l/2 \rfloor - 1$. Furthermore, in practice, we will use Theorem 8 with $l = diam(G)$ and $l' \in \{0, 1\}$. This way, we will improve by a factor two all previous lower-bounds.

It can be noticed that the lower-bound of Theorem 8 is sharp for almost every cycle. Indeed, let $\mathbb{Z}_n$ be the vertex set of the n-cycle C_n, and let σ be the automorphism mapping any vertex i to the vertex $i + \lfloor n/2 \rfloor \pmod n$. Applying Theorem 8 to σ, we obtain a lower-bound $\lfloor n/4 \rfloor$ for the hyperbolicity of even-length cycles, which is exact, and a lower-bound $\lfloor n/4 \rfloor - 1/2$ for odd-length cycles, which is exact when $n \equiv 1 \pmod 4$ and below $1/2$ of the true hyperbolicity when $n \equiv 3 \pmod 4$ [28, 53].

We emphasize on the following consequence of Theorem 8.

Corollary 9. *Let G be a connected graph and σ be an idempotent endomorphism with mobility l. Then, it holds $\delta(G) \geq \left\lfloor \frac{l}{2} \right\rfloor$.*

Proof. By the hypothesis, the endomorphism σ is idempotent and so, we can apply Theorem 8 by setting $l' = 0$. □

In the special case when $l = diam(G)$, the lower-bound of Corollary 9 is best possible. Indeed, it coincides with the upper-bound of Lemma 2, thereby giving the exact value for hyperbolicity.

It is natural to ask whether Theorems 7 and 8 can be further improved by using bounds on the distances $\mathrm{d}(v, \sigma^3(v)), \mathrm{d}(v, \sigma^4(v))$ and so on. However, answering this question is nontrivial since the techniques used for Theorems 7 and 8 are already quite different. We leave it as an interesting open question.

3.2 Applications

Equipped with Theorems 7, 8 and Corollary 9, we subsequently apply them on a broad range of topologies studied in the literature. We will combine the lower-bounds that we obtain with a slight variation of the well-known upper-bound of Lemma 2. Indeed it is folklore that the hyperbolicity of a graph is the maximum hyperbolicity taken over all of its biconnected components. So, $\delta(G) \leq \lfloor \mathrm{effdiam}(G)/2 \rfloor$, where the so-called *efficient diameter* $\mathrm{effdiam}(G)$ denotes the largest diameter amongst the biconnected components of the graph. This way, we will show that for most graphs found in the literature, their hyperbolicity scales linearly with the efficient diameter —that is the worst-case possible for hyperbolicity.

3.2.1 Torus

Let us first consider the torus, a well-known grid-like graph which is highly symmetrical. Other grid-like graphs will be considered in Section 4.2 using a different approach.

Definition 10. The *torus* (n,m)-grid has vertex-set $\mathbb{Z}_n \times \mathbb{Z}_m$; any two vertices $(i,j),(i',j')$ are adjacent if either $i' = i, j' \equiv j+1 \pmod{m}$, or $i' \equiv i+1 \pmod{n}, j' = j$.

Lemma 11. *Let $n = 2p+r,\ m = 2q+s$, with $r,s \in \{0,1\}$. Then, the hyperbolicity $\delta_{n,m}$ of the torus (n,m)-grid satisfies:*

$$\left\lfloor \frac{1}{2} \cdot \left(\left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{m}{2} \right\rfloor \right) \right\rfloor - \frac{r+s}{2} \leq \delta_{n,m} \leq \left\lfloor \frac{1}{2} \cdot \left(\left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{m}{2} \right\rfloor \right) \right\rfloor.$$

Proof. For any two vertices $u = (i_u, j_u), v = (i_v, j_v)$:

$$\mathrm{d}(u,v) = \min\{|i_u - i_v|, n - |i_u - i_v|\} + \min\{|j_u - j_v|, m - |j_u - j_v|\}.$$

It implies that the diameter of the torus grid is $\lfloor n/2 \rfloor + \lfloor m/2 \rfloor$ and so, $\delta_{n,m} \leq \lfloor (\lfloor n/2 \rfloor + \lfloor m/2 \rfloor)/2 \rfloor$ by Lemma 2. Finally, let σ be the automorphism of the torus grid which maps any vertex (i,j) to the vertex $(i + \lfloor n/2 \rfloor \pmod{n}, j + \lfloor m/2 \rfloor \pmod{m})$. Since for any vertex v, $\mathrm{d}(v, \sigma(v)) = \lfloor n/2 \rfloor + \lfloor m/2 \rfloor$ and $\mathrm{d}(v, \sigma^2(v)) = r+s$, then it follows from Theorem 8 that $\delta_{n,m} \geq \lfloor (\lfloor n/2 \rfloor + \lfloor m/2 \rfloor)/2 \rfloor - \frac{r+s}{2} \geq \lfloor (\lfloor n/2 \rfloor + \lfloor m/2 \rfloor)/2 \rfloor - 1$. □

3.2.2 Hybercube-like networks

Definition 12 ([8, 11])**.** Let $m_1, m_2, \ldots, m_r$ be positive integers with for every $i,\ m_i \geq 2$ and $r \geq 1$. The generalized hypercube $G(m_1, m_2, \ldots, m_r)$ has vertex-set $\{(x_1, x_2, \ldots, x_r) \mid \forall i, 0 \leq x_i \leq m_i - 1\}$, and two vertices $(x_1, x_2, \ldots, x_r), (y_1, y_2, \ldots, y_r)$ are adjacent in the graph if and only if their Hamming distance $\sum_i \mathbb{I}_{\{x_i \neq y_i\}}$ is equal to 1.

In particular, the k-ary hypercube $H_k(n)$ is the generalized hypercube $G(m_1, m_2, \ldots, m_n)$ with for every $i,\ m_i = k$.

Lemma 13. $\delta\left(G(m_1, m_2, \ldots, m_r)\right) = \left\lfloor \frac{r}{2} \right\rfloor$.

Proof. The diameter of $G(m_1, m_2, \ldots, m_r)$ is r and so, $\delta\left(G(m_1, m_2, \ldots, m_r)\right) \leq \lfloor r/2 \rfloor$ by Lemma 2. To prove the lower-bound, we first make the observation that the binary hypercube $H_2(r)$ is an isometric subgraph of $G(m_1, m_2, \ldots, m_r)$. Let σ be the automorphism mapping any vertex

$(x_1, x_2, \ldots, x_r) \in V(H_2(r))$ to its *complementary* vertex $(1-x_1, 1-x_2, \ldots, 1-x_r)$. Note that σ has mobility r and it is idempotent. As a result, we conclude by Corollary 9 that $\delta\left(G(m_1, m_2, \ldots, m_r)\right) \geq \delta\left(H_2(r)\right) \geq \lfloor r/2 \rfloor$. $\square$

As we will show later, Lemma 13 also follows from Corollary 49 and the fact that the n-dimensional grid of size 2 is exactly the hypercube $H_2(n)$.

Definition 14 ([4]). The cube-connected-cycle $CCC(n)$ has vertex-set the pairs $\langle i, w \rangle$, for $0 \leq i \leq n-1$ and for w any binary word of length n; two vertices $\langle i, x_1 x_2 \ldots x_n \rangle$ and $\langle j, y_1 y_2 \ldots y_n \rangle$ are adjacent in the graph if and only if either $i = j$, $x_i = 1 - y_i$ and for every $k \neq i$, $x_k = y_k$; or $i \equiv j + 1 \pmod n$ and for every k, $x_k = y_k$.

Lemma 15. $n \leq \delta\left(CCC(n)\right) \leq n - 1 + \left\lfloor \max\left\{1, \frac{1}{2} \cdot \left\lfloor \frac{n}{2} \right\rfloor\right\} \right\rfloor$.

Proof. By [19], $\operatorname{diam}\left(CCC(n)\right) = 2n-2+\max\left\{2, \lfloor n/2 \rfloor\right\}$ and so, $\delta\left(CCC(n)\right) \leq n-1+\left\lfloor \left(\max\left\{2, \lfloor n/2 \rfloor\right\}\right)/2 \right\rfloor$ by Lemma 2. Furthermore, the mapping $\sigma : \langle i, w \rangle \to \langle i, \bar{w} \rangle$ is an idempotent endomorphism and it has mobility $2n$ by [19]. We conclude by Corollary 9 that $\delta\left(CCC(n)\right) \geq n$. $\square$

Definition 16 ([39]). Let $\mathbb{Z}_n^l$ be the set of words of length l over the alphabet $\{0, 1, \ldots, n-1\}$. The graph $\text{BCube}_k(n)$ has vertex-set $\mathbb{Z}_n^{k+1} \cup \left(\{0, 1, \ldots, k\} \times \mathbb{Z}_n^k\right)$ and edge-set $\left\{\left\{\langle l, s_k s_{k-1} \ldots s_{l+1} s_{l-1} \ldots s_0 \rangle, s_k s_{k-1} \ldots s_{l+1} s_l s_{l-1} \ldots s_0\right\} \mid 0 \leq l \leq k \text{ and for every } i, 0 \leq s_i \leq n-1\right\}$.

Lemma 17. $\delta\left(\text{BCube}_k(n)\right) = k + 1$.

Proof. By [43] $\operatorname{diam}\left(\text{BCube}_k(n)\right) = 2(k+1)$ and so, $\delta\left(\text{BCube}_k(n)\right) \leq k+1$ by Lemma 2. Then, let us assume that $n = 2$ because we have by [43] that $\text{BCube}_k(2)$ is an isometric subgraph of $\text{BCube}_k(n)$. We define the automorphism σ satisfying that for all binary word $w \in \mathbb{Z}_2^{k+1}$, $\sigma(w) = \bar{w}$, and for every pair $< l, w > \in \{0, 1, \ldots, k\} \times \mathbb{Z}_2^k$, $\sigma(< l, w >) = < l, \bar{w} >$. By [39, 43] σ has mobility $2(k+1)$ and so, by noticing that σ is idempotent we can conclude by Corollary 9 that $\delta\left(\text{BCube}_k(n)\right) \geq \delta\left(\text{BCube}_k(2)\right) \geq k+1$. $\square$

3.2.3 Tree-like networks

Definition 18 ([36]). Let $k \geq 4$ be even. The Fat-Tree_k is a graph with vertex-set that is partitioned into four layers:

1. a *core layer*, labeled with $\{0\} \times \mathbb{Z}_{(k/2)^2}$;
2. an *aggregation layer*, labeled with $\{1\} \times \mathbb{Z}_k \times \mathbb{Z}_{k/2}$. For every $0 \leq i \leq (k/2)^2 - 1$ the vertex labeled $\langle 0, i \rangle$ in the core layer is adjacent to all the vertices labeled $\langle 1, j, i \pmod k / 2 \rangle$ in the aggregation layer, with $0 \leq j \leq k-1$;
3. an *edge layer*, labeled with $\{2\} \times \mathbb{Z}_k \times \mathbb{Z}_{k/2}$. For every $0 \leq i \leq k-1$ there is a complete join between the subsets of vertices $\{\langle 1, i, j \rangle \mid 0 \leq j \leq k/2 - 1\}$ and $\{\langle 2, i, j \rangle \mid 0 \leq j \leq k/2 - 1\}$;
4. finally, a *server layer* labeled with $\{3\} \times \mathbb{Z}_k \times \mathbb{Z}_{(k/2)^2}$. For any $0 \leq q, r < k/2$ the vertex labeled $\langle 3, k, (k/2)q + r \rangle$ in the server layer is adjacent to the vertex labeled $\langle 2, k, q \rangle$ in the edge layer.

An example of a Fat-Tree_4 is given in Figure 2.

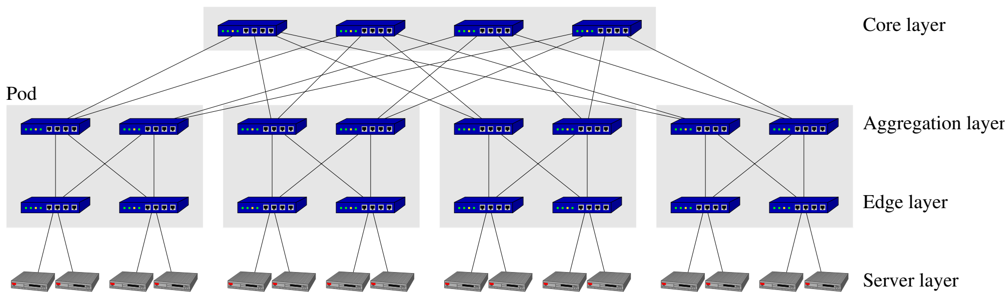


Figure 2: The graph Fat-Tree$_4$.

Lemma 19. $\delta\left(\text{Fat-Tree}_k\right) = 2$.

Proof. By construction, every vertex in the server layer is a pending vertex, that is a vertex of degree one. As a result, it can be ignored for the computation of hyperbolicity because the hyperbolicity of a graph is equal to the maximum hyperbolicity taken over all its biconnected components. It follows that the efficient diameter of Fat-Tree$_k$ is 4, hence $\delta\left(\text{Fat-Tree}_k\right) \leq 2$.

Furthermore, by construction Fat-Tree$_4$ is an isometric subgraph of Fat-Tree$_k$. So, let σ be the idempotent endomorphism of Fat-Tree$_4$ mapping: any vertex $\langle 0, i \rangle$ to the vertex $\langle 0, 3-i \rangle$ in the core layer; any vertex $\langle 1, i, j \rangle$ to the vertex $\langle 1, 3-i, 1-j \rangle$ in the aggregation layer, and in the same way any vertex $\langle 2, i, j \rangle$ to the vertex $\langle 2, 3-i, 1-j \rangle$ in the edge layer; last, any vertex $\langle 3, i, j \rangle$ to the vertex $\langle 3, 3-i, 3-j \rangle$ in the server layer. It can be hand-checked that σ has mobility 4 and so, by Corollary 9 $\delta\left(\text{Fat-Tree}_k\right) \geq \delta\left(\text{Fat-Tree}_4\right) \geq 2$. $\square$

Definition 20 ([47]). The Butterfly graph $BF(n)$ has vertex-set $\{0, 1, \ldots, n\} \times \mathbb{Z}_2^n$; two vertices $\langle i, w \rangle, \langle i', w' \rangle$ are adjacent if $i' = i + 1$ and for every $j \neq i,\ w_j = w'_j$.

Lemma 21. $\delta\left(BF(n)\right) = n$.

Proof. Let w and w' be two binary words of length n and let i_1 and i_l be respectively the least and the largest index in which they differ. Then, it can be checked that for every integer i, $\mathrm{d}_{BF(n)}(\langle i, w \rangle, \langle i, w' \rangle) = 2(i_l - i_1)$. As a result, the endomorphism σ mapping any vertex $\langle i, w \rangle$ to the vertex $\langle i, \bar{w} \rangle$ has mobility $2n$. Since σ is idempotent then it follows from Corollary 9 that $\delta\left(BF(n)\right) \geq n$. Last, we also have that $\mathrm{diam}\left(BF(n)\right) = 2n$, hence $\delta\left(BF(n)\right) \leq n$ by Lemma 2. $\square$

In the literature, the edge-set of the Butterfly network is sometimes defined as $\{\{\langle i, w \rangle, \langle i+1 \pmod{n}, w' \rangle\} \mid 0 \leq i \leq n \text{ and for every } j \neq i, w_j = w'_j\}$ [81], and this definition is also known as the wrapped Butterfly network. It modifies the diameter of the topology from $2n$ to $n + \lfloor n/2 \rfloor$, and the distance between any two vertices $\langle i, w \rangle, \langle i, \bar{w} \rangle$ from $2n$ to n. As a result, using the same arguments as for Lemma 21 one obtains that the hyperbolicity of the wrapped Butterfly graph is comprised between $\lfloor n/2 \rfloor$ and $\lfloor \left(n + \lfloor n/2 \rfloor\right)/2 \rfloor$.

Definition 22 ([20]). The k-ary n-fly has vertex-set $\{0, 1, \ldots, n\} \times \mathbb{Z}_k^n$; two vertices $\langle i, w \rangle, \langle i', w' \rangle$ are adjacent if $i' = i + 1$ and for every $j \neq i,\ w_j = w'_j$.

Observe that the Butterfly graph $BF(n)$ is isomorphic to the 2-ary n-fly.

Lemma 23. *The k-ary n-fly is n-hyperbolic.*

Proof. By [20], the diameter of the k-ary n-fly is $2n$ and so, it has hyperbolicity bounded from above by n by Lemma 2. Moreover, by construction it contains the Butterfly graph $BF(n)$ as an isometric subgraph and so, it has hyperbolicity at least n by Lemma 21. $\square$

Definition 24 ([20]). The k-ary n-tree is the graph with vertex-set $\mathbb{Z}_k^n \cup \left(\{0,1,\ldots,n-1\} \times \mathbb{Z}_k^{n-1}\right)$ such that any two vertices $\langle i, w \rangle, \langle i', w' \rangle$ are adjacent if $i' = i+1$ and for every $j \neq i,\ w_j = w'_j$; any two vertices $\langle i, w \rangle, w'$ are adjacent if $i = n-1$ and $w' = w \cdot b$ for some $b \in \mathbb{Z}_k$.

Lemma 25. *The k-ary n-tree is $(n-1)$-hyperbolic.*

Proof. By construction, the biconnected components of the k-ary n-tree are composed of one single-vertex graph for each vertex $w \in \mathbb{Z}_k^n$, and of the k-ary $(n-1)$-fly. Since the hyperbolicity of the graph is equal to the maximum hyperbolicity taken over its biconnected components, then it follows from Lemma 23 that the k-ary n-tree is $(n-1)$-hyperbolic. $\square$

Definition 26 ([81]). The d-ary tree grid $MT(d,h)$ is a graph whose vertices are labeled with the pairs of words $< u, v >$ over an alphabet of size d and such that $\max\{|u|,|v|\} = h$. Any two vertices $\langle u, v \rangle$ and $\langle u', v' \rangle$ are adjacent in $MT(d,h)$ if and only if there is some letter λ such that: either $|u| = h$, $u = u'$ and $v = v' \cdot \lambda$; or $|v| = h$, $v = v'$ and $u' = u \cdot \lambda$.

Lemma 27. $\delta\left(MT(d,h)\right) = 2h$.

Proof. By [81] $\mathrm{diam}\left(MT(d,h)\right) = 4h$ and so, $\delta\left(MT(d,h)\right) \leq 2h$. Furthermore, $MT(2,h)$ is an isometric subgraph of $MT(d,h)$ by construction. Let σ be the idempotent endomorphism of $MT(2,h)$ mapping any vertex $\langle u, v \rangle$ to the vertex $\langle \bar{u}, \bar{v} \rangle$. By construction σ has mobility $4h$ and so, we conclude by Corollary 9 that $\delta\left(MT(d,h)\right) \geq \delta\left(MT(2,h)\right) \geq 2h$. $\square$

3.2.4 Symmetric networks and Cayley graphs

Let $(\Gamma, \cdot)$ be a group and let S be a generating set of Γ that is symmetric and that does not contain the neutral element of Γ. We remind that the *Cayley graph* $G(\Gamma, S)$ —of group Γ w.r.t. S— has vertex-set Γ and edge-set $\{\{g, g \cdot s\} \mid g \in \Gamma, s \in S\}$. It is well-known that every Cayley graph is vertex-transitive [12]. Furthermore, it has been shown (see for instance Exercise 2.4.14 in [81]) that the cube connected cycle $CCC(n)$ and the Butterfly graph $BF(n)$ are Cayley graphs.

Lemma 28. *Let $(\Gamma, \cdot)$ be a commutative group and S be a symmetric generating set that does not contain the neutral element of Γ. If $G(\Gamma, S)$ is not a clique, then $\delta\left(G(\Gamma, S)\right) \geq \frac{1}{2}\left\lceil \frac{\mathrm{diam}(G(\Gamma,S))}{2} \right\rceil$.*

Proof. Let $id_\Gamma, g \in \Gamma$ be such that id_Γ is the neutral element of group Γ and $\mathrm{d}(id_\Gamma, g) = \mathrm{diam}\left(G(\Gamma, S)\right) = D > 1$. The mapping $\sigma : v \to g \cdot v$ is an automorphism satisfying that for every $v \in \Gamma,\ \mathrm{d}(v, \sigma(v)) = \mathrm{d}(id_\Gamma, v^{-1} \cdot g \cdot v) = \mathrm{d}(id_\Gamma, g) = D$. Therefore, we can conclude by Theorem 7 that $\delta\left(G(\Gamma, S)\right) \geq \lceil D/2 \rceil / 2$. $\square$

Definition 29 ([12]). The Bubble-sort graph $BS(n)$ has vertex-set the n-element permutations, that is $\{\phi_1\phi_2\ldots\phi_i\ldots\phi_n \mid \{\phi_1,\ldots,\phi_n\} = \{1,\ldots,n\}\}$. Any two vertices ϕ, ψ are adjacent if and only if there is some index $i < n$ such that $\phi_i = \psi_{i+1}, \phi_{i+1} = \psi_i$ and for every $j \notin \{i, i+1\},\ \phi_j = \psi_j$.

Lemma 30. $\delta(BS(n)) = \left\lfloor \frac{n(n-1)}{4} \right\rfloor$.

Proof. By [12] $\mathrm{diam}(BS(n)) = \binom{n}{2}$, hence $\delta(BS(n)) \leq \lfloor \mathrm{diam}(BS(n))/2 \rfloor$ by Lemma 2. Now, let σ be the idempotent endomorphism mapping any vertex $\phi_1\phi_2\ldots\phi_i\ldots\phi_n$ to $\phi_n\ldots\phi_{n-i+1}\ldots\phi_2\phi_1$. By [12] all pairs $(u, \sigma(u))$ are diametral pairs and so, we can conclude by Corollary 9 that $\delta(BS(n)) \geq \lfloor \mathrm{diam}(BS(n))/2 \rfloor$. □

Definition 31 ([17]). The Transposition graph $T(n)$ has vertex-set the n-element permutations. Any two vertices ϕ, ψ are adjacent if and only if there are $i, j,\ i \neq j$ such that $\phi_i = \psi_j, \phi_j = \psi_i$ and for every $k \notin \{i, j\},\ \phi_k = \psi_k$.

Lemma 32. $\frac{1}{2}\left\lceil \frac{n-1}{2} \right\rceil \leq \delta\left(T(n)\right) \leq \left\lfloor \frac{n-1}{2} \right\rfloor$.

Proof. By [17] the diameter of $T(n)$ is $n-1$ and so, by Lemma 2 $\delta\left(T(n)\right) \leq \lfloor (n-1)/2 \rfloor$. Moreover, let σ be the endomorphism mapping any vertex $\phi_1\phi_2\ldots\phi_i\ldots\phi_{n-1}\phi_n$ to $\phi_2\phi_3\ldots\phi_{i+1}\ldots\phi_n\phi_1$. Again by [17] all pairs $(u, \sigma(u))$ are diametral pairs and so, we can conclude by Theorem 7 that $\delta\left(S(n)\right) \geq \lceil n - 1/2 \rceil / 2$. □

Definition 33 ([12]). The star graph $S(n)$ has vertex-set the n-element permutations and edge-set $\{\{\phi_1\ldots\phi_{i-1}\phi_i\phi_{i+1}\ldots\phi_n, \phi_i\ldots\phi_{i-1}\phi_1\phi_{i+1}\ldots\phi_n\} \mid 2 \leq i \leq n\}$.

Lemma 34. $\left\lfloor \frac{1}{2}\left\lfloor \frac{3(n-1)}{2} \right\rfloor - \frac{1}{2} \right\rfloor \leq \delta\left(S(n)\right) \leq \left\lfloor \frac{1}{2}\left\lfloor \frac{3(n-1)}{2} \right\rfloor \right\rfloor$.

Proof. By [12] the diameter of $S(n)$ is $\lfloor 3(n-1)/2 \rfloor$ and so, $\delta\left(S(n)\right) \leq \lfloor \lfloor 3(n-1)/2 \rfloor / 2 \rfloor$ by Lemma 2. Then, given $\phi = \phi_1\phi_2\ldots\phi_i\ldots\phi_{n-1}\phi_n$, let ψ be the unique n-element permutation satisfying that $\psi_{n-2j} = \phi_{n-2j-1}, \psi_{n-2j-1} = \phi_{n-2j}$, for every $0 \leq j \leq \lfloor (n-1)/2 \rfloor - 1$. Again by [12], $\mathrm{d}(\psi, \phi) \geq \lfloor 3(n-1)/2 \rfloor - \varepsilon \geq \lfloor 3(n-1)/2 \rfloor - 1$, with $\varepsilon = n + 1 \pmod 2$. Moreover it can be checked that the mapping $\sigma : \psi \to \phi$ is an idempotent endomorphism of $S(n)$. Therefore, by Corollary 9 $\delta\left(S(n)\right) \geq \lfloor \lfloor 3(n-1)/2 \rfloor / 2 - 1/2 \rfloor$. □

4 Using the shortest-path distribution

It turns out that for "simple" topologies that are commonly found in the literature, desirable symmetries such as those in use in Section 3 might fail to exist. For instance, the infinite rectangular grid is vertex-symmetric, but finite rectangular grids are not. As we will show next, the more generic Theorem 4 could still be applied in order to obtain loose lower-bounds in these situations. However, since the shortest-path distributions of the "simplest" topologies are well-known and characterized, that allows us to lower-bound their hyperbolicity using more involved techniques. In particular, our proofs for grid-like graphs introduce a novel way to make use of the maximal shortest-paths in the study of graph hyperbolicity.

4.1 The fellow traveler property for graphs defined on an alphabet

As a warm up, we will lower-bound the hyperbolicity of some graph classes defined on alphabets, starting with the undirected de Bruijn graph.

Definition 35 ([13]). The undirected de Bruijn graph $UB(d, D)$ has vertex-set the words of length D taken over an alphabet Σ of size d. The 2-set $\{u, v\}$ is an edge of $UB(d, D)$ if and only if $u = u_{d-1}u_{d-2}\ldots u_1u_0$ and $v = u_{d-2}\ldots u_1u_0v_0$ for some letters $u_{d-1}, u_{d-2}, \ldots, u_1, u_0, v_0 \in \Sigma$.

De Bruijn graphs have been extensively studied in the literature [15, 24, 27, 81]. In particular, $UB(d,D)$ has diameter D, maximum degree $2d$, and d^D vertices. Shortest-path routing and shortest-path distances in $UB(d,D)$ are characterized as follows.

Lemma 36 ([24]). *Let u, v be two words of length D taken over some alphabet Σ of size d, and write $u = u_L \cdot x \cdot u_R$ and $v = v_L \cdot x \cdot v_R$ so that $D - |x| + \min\{|u_L| + |v_R|, |v_L| + |u_R|\}$ is minimized. Then it holds $\mathrm{d}_{UB(d,D)}(u,v) = D - |x| + \min\{|u_L| + |v_R|, |v_L| + |u_R|\}$.*

We say that a graph G falsifies the *k-fellow traveler property* if there are two shortest-paths $\mathcal{P}_1, \mathcal{P}_2$ with same endpoints $u, v \in V(G)$, and there are two vertices $x \in \mathcal{P}_1, y \in \mathcal{P}_2$ such that $\mathrm{d}_G(u,x) = \mathrm{d}_G(u,y)$ and $\mathrm{d}_G(x,y) > k$. By a straightforward calculation we obtain that in such a case $\delta(u,v,x,y) = \mathrm{d}_G(x,y)/2 > k/2$. So, we can lower-bound the hyperbolicity of G with the least k such that it *satisfies* the $2k$-fellow traveler property. This standard argument will be the one in use throughout the remaining of Section 4.1.

Proposition 37. *For any positive integers d and D, $\delta(UB(d,D)) \geq \frac{1}{2} \cdot \left\lfloor \frac{D}{2} \right\rfloor$.*

Proof. We prove that $UB(d,D)$ cannot satisfy the k-fellow traveler property for some range of k. W.l.o.g. the vertices of $UB(d,D)$ are labeled with the words of length D taken over the alphabet $\Sigma = \{0, 1, \ldots, d-1\}$. Let $u = 0^D$, $v = 1^D$, $x = 0^{\lfloor D/2 \rfloor} \cdot 1^{\lceil D/2 \rceil}$, and $y = 1^{\lceil D/2 \rceil} \cdot 0^{\lfloor D/2 \rfloor}$. By Lemma 36 it comes that $\mathrm{d}(u,v) = D = \lceil D/2 \rceil + \lfloor D/2 \rfloor = \mathrm{d}(u,x) + \mathrm{d}(x,v) = \mathrm{d}(u,y) + \mathrm{d}(y,v)$. As a result, the graph $UB(d,D)$ cannot satisfy the k-fellow traveler property for $k < \mathrm{d}(x,y) = \lfloor D/2 \rfloor$ and so, $\delta(UB(d,D)) \geq \lfloor D/2 \rfloor / 2$. □

To compare the bounds of Theorem 4 and Proposition 37, we note that it has been proved in [79] that de Bruijn graphs with maximum degree $d \geq 3$ are self-centered. Therefore, if $d \geq 3$ then Proposition 37 follows from Theorem 4 (with $k = 0$), but it is not the case if $d = 2$. Furthermore, the lower-bound of Proposition 37 is reached for $d = D = 2$, *a.k.a.* the diamond graph. It can be computer-checked that is also holds for $d = 2, D = 4$. However, $\delta(UB(2,D)) = \left\lfloor \frac{D}{2} \right\rfloor$ for every odd $D \leq 11$. Based on computer experiments (for $d = 2, D \leq 12$), we made the following stronger conjecture:

Conjecture 38. *For every $D \geq 7, \delta(UB(d,D)) = \left\lfloor \frac{D}{2} \right\rfloor$.*

A closely related graph class that has been extensively studied in the literature is the class of undirected Kautz graphs $UK(d,D)$ [2, 13]. The graph $UK(d,D)$ has diameter D, maximum degree $2d$, and $d^D(d+1)$ vertices. Furthermore, it can be checked that the Kautz graph $UK(d,D)$ is an induced subgraph of the de Bruijn graph $UB(d+1,D)$.

Definition 39 ([2, 13]). The undirected Kautz graph $UK(d,D)$ has vertex-set the words of length D taken over an alphabet Σ of size $d+1$ and satisfying that no two adjacent letters are equal. The 2-set $\{u,v\}$ is an edge of $UK(d,D)$ if and only if $u = u_{d-1}u_{d-2}\ldots u_1u_0$ and $v = u_{d-2}\ldots u_1u_0v_0$ for some letters $u_{d-1}, u_{d-2}, \ldots, u_1, u_0, v_0 \in \Sigma$.

Proposition 40. *For any positive integers d and D, $\delta(UK(d,D)) \geq \left\lfloor \frac{D}{4} \right\rfloor + \left\lfloor \frac{D \pmod 4}{3} \right\rfloor$.*

Proof. As for the proof of Proposition 37, we prove that $UK(d,D)$ cannot satisfy the k-fellow traveler property for some range of k. W.l.o.g. the vertices of $UK(d,D)$ are labeled with the words of length D taken over the alphabet $\{0, 1, 2, \ldots, d\}$. Let $D = 2D' + r$, $r \in \{0, 1\}$, let

$u = (01)^{D'} \cdot 0^r, v = (21)^{D'} \cdot 2^r$. Note that 0^r (resp. 2^r) is either the empty word or it is equal to 0 (resp. to 2). By Lemma 36 $\mathrm{d}_{UK(d,D)}(u,v) \geq \mathrm{d}_{UB(d+1,D)}(u,v) = D$ and so, $\mathrm{d}_{UK(d,D)}(u,v) = D$ because $\mathrm{diam}\,(UK(d,D)) = D$. In particular, let $\mathcal{P}_1$ be the uv-shortest-path in $UK(d,D)$ that one obtains by applying "right shiftings" on u until one obtains vertex v *i.e.*,

$$\mathcal{P}_1 = (01)^{D'} \cdot 0^r \to 1 \cdot (01)^{D'-1} \cdot 0^r \cdot 2 \to (01)^{D'-1} \cdot 0^r \cdot 21 \to \cdots \to (21)^{D'} \cdot 2^r$$

Similarly, let $\mathcal{P}_2$ be the vu-shortest-path in $UK(d,D)$ that one obtains by applying "right shiftings" on v until one obtains vertex u. That is,

$$\mathcal{P}_2 = (21)^{D'} \cdot 2^r \to 1 \cdot (21)^{D'-1} \cdot 2^r \cdot 0 \to (21)^{D'-1} \cdot 2^r \cdot 01 \to \ldots \to (01)^{D'} \cdot 0^r$$

Let now

$$x = (01)^{\lfloor D'/2 \rfloor} \cdot 0^r \cdot (21)^{\lceil D'/2 \rceil} \in \mathcal{P}_1$$
$$\text{and } y = 1^r \cdot (21)^{\lceil D'/2 \rceil - r} \cdot 2^r \cdot (01)^{\lfloor D'/2 \rfloor} \cdot 0^r \in \mathcal{P}_2$$

be such that $\mathrm{d}(u,x) = \mathrm{d}(u,y)$.

The graph $UK(d,D)$ falsifies the k-fellow traveler property for all $k < \mathrm{d}_{UK(d,D)}(x,y)$, and we have by Lemma 36 that $\mathrm{d}_{UK(d,D)}(x,y) \geq \mathrm{d}_{UB(d+1,D)}(x,y) \geq 2\left(\lfloor D/4 \rfloor + \lfloor (D \pmod 4)/3 \rfloor\right)$.

As a result, it holds $\delta\,(UK(d,D)) \geq \lfloor D/4 \rfloor + \lfloor (D \pmod 4)/3 \rfloor$. $\square$

The lower-bound of Proposition 40 is reached for $d = 2, D = 3$. Again to compare with Theorem 4, we note that it was also proved in [79] that Kautz graphs are self-centered, for every $d \geq 2$. Therefore, applying Theorem 4 (with $k = 0$) gives us a lower-bound $\lfloor D/2 \rfloor/2$ for the hyperbolicity of $UK(d,D)$, that is of the same order of magnitude as the one of Proposition 40 (Proposition 40 is slightly better if $D \equiv 3 \pmod 4$, and slightly worse if $D \equiv 2 \pmod 4$). We last define another topology that is related to the de Bruijn graph:

Definition 41 ([81]). The shuffle-exchange graph $SE(n)$ has vertex-set the binary words of length n. The 2-set $\{u,v\}$ is an edge of $SE(n)$ if and only if $u = u_{n-1}u_{n-2}\ldots u_1u_0$ and: either $v = u_0u_{n-1}u_{n-2}\ldots u_1$, or $v = u_{n-2}\ldots u_1u_0u_{n-1}$, or $v = u_{n-1}u_{n-2}\ldots u_1\bar{u_0}$, for some booleans $u_{n-1}, u_{n-2}, \ldots, u_1, u_0$.

It was proved in [81] that the diameter of $SE(n)$ is $2n-1$, and that the pair of vertices $(0^n, 1^n)$ is a diametral pair. Furthermore, it can be checked that one can obtain the de Bruijn graph $UB(2, n-1)$ from $SE(n)$ as follows: for each edge $\{u_{n-1}u_{n-2}\ldots u_1u_0, u_{n-1}u_{n-2}\ldots u_1\bar{u_0}\}$, we contract the edge and we label $u_{n-1}u_{n-2}\ldots u_1$ the resulting vertex. This defines a *contraction mapping* σ, mapping any vertex $u_{n-1}u_{n-2}\ldots u_1u_0$ of $SE(n)$ to the vertex $u_{n-1}u_{n-2}\ldots u_1$ of $UB(2, n-1)$. In the following, it will be useful to observe that by construction, for every two vertices u, v of $SE(n)$ it holds $\mathrm{d}_{SE(n)}(u,v) \geq \mathrm{d}_{UB(2,n-1)}(\sigma(u), \sigma(v))$.

Proposition 42. *For any positive integer n, $\delta\,(SE(n)) \geq \frac{1}{2} \cdot \left\lfloor \frac{n}{2} \right\rfloor$.*

Proof. As for the proof of Proposition 37, we prove that $SE(n)$ cannot satisfy the k-fellow traveler property for some range of k. Let $u = 0^n$, $v = 1^n$ be a diametral pair of $SE(n)$, with $\mathrm{d}(u,v) = 2n-1$. Let $\mathcal{P}_1$ be the uv-shortest-path:

$$0^n \to 0^{n-1} \cdot 1 \to 1 \cdot 0^{n-1} \to 1 \cdot 0^{n-2} \cdot 1 \to 11 \cdot 0^{n-2} \to \ldots \to 1^{n-1} \cdot 0 \to 1^n$$

Similarly, let $\mathcal{P}_2$ be the vu-shortest-path:

$$1^n \to 1^{n-1} \cdot 0 \to 0 \cdot 1^{n-1} \to 0 \cdot 1^{n-2} \cdot 0 \to 00 \cdot 1^{n-2} \to \ldots \to 0^{n-1} \cdot 1 \to 0^n.$$

Finally, let $x = 1^{\lfloor n/2 \rfloor} \cdot 0^{\lceil n/2 \rceil} \in \mathcal{P}_1$, $y = 0^{\lceil n/2 \rceil - 1} \cdot 1^{\lfloor n/2 \rfloor} \cdot 0 \in \mathcal{P}_2$ be such that $\mathrm{d}(u, x) = \mathrm{d}(u, y)$. By using the above contraction mapping from $SE(n)$ to $UB(2, n-1)$ one obtains $\mathrm{d}_{UB(2,n-1)}(x', y') \leq \mathrm{d}_{SE(n)}(x, y)$ with $x' = 1^{\lfloor n/2 \rfloor} \cdot 0^{\lceil n/2 \rceil - 1}$, $y' = 0^{\lceil n/2 \rceil - 1} \cdot 1^{\lfloor n/2 \rfloor}$. As a result, we have by Lemma 36 that the shuffle-exchange graph falsifies the k-fellow traveler property for every $k < \mathrm{d}_{UB(2,n-1)}(x', y') = \left\lfloor \frac{n}{2} \right\rfloor$ and so, it holds $\delta\left(SE(n)\right) \geq \frac{1}{2} \cdot \left\lfloor \frac{n}{2} \right\rfloor$. □

4.2 The maximal shortest-paths in grid-like topologies

In this section, we name grid-like graphs some slight variations of the 2-dimensional grid. As a reminder, an (n, m)-grid is the Cartesian product of the path P_n, with n vertices, with the path P_m, with m vertices. That is, the vertex-set is $\{0, \ldots, n-1\} \times \{0, \ldots, m-1\}$, and the edge-set is $\{\{(i, j), (i', j')\} \mid |i - i'| + |j - j'| = 1\}$. Grid-like networks are used for modeling interconnection networks and other computational applications. We now propose to compute their hyperbolicity. Our main tool in this section is the notion of *far-apart pairs*, first introduced in [41, 52]:

Definition 43 (Far-apart pair [41, 52])**.** Given $G = (V, E)$, the pair (u, v) is far-apart if for every $w \in V \setminus \{u, v\}$, $\mathrm{d}(w, u) + \mathrm{d}(u, v) > \mathrm{d}(w, v)$ and $\mathrm{d}(w, v) + \mathrm{d}(u, v) > \mathrm{d}(w, u)$.

Said differently, far-apart pairs are the endpoints of *maximal* shortest-paths in the graph. The main motivation for introducing far-apart pairs was to speed-up the computation of hyperbolicity, via the following pre-processing method.

Lemma 44 ([41, 52])**.** *Let G be a connected graph. There exist two far-apart pairs (u, v) and (x, y) satisfying:*

- $\mathrm{d}_G(u, v) + \mathrm{d}_G(x, y) \geq \max\{\mathrm{d}_G(u, x) + \mathrm{d}_G(v, y), \mathrm{d}_G(u, y) + \mathrm{d}_G(v, x)\}$;
- $\delta(u, v, x, y) = \delta(G)$.

We here propose a novel application of this result in order to simplify proofs for the hyperbolicity of grid-like topologies.

Definition 45. The $(s_1, s_2, \ldots, s_d)$-grid is a graph with vertex set $\Pi_{i=1}^{d}\{0, \ldots, s_i - 1\}$ such that any two vertices $\langle u_1, u_2, \ldots, u_d \rangle, \langle v_1, v_2, \ldots, v_d \rangle$ are adjacent only if $\sum_{i=1}^{d} |u_i - v_i| = 1$.

Definition 46. The d-dimensional grid of size s is the $(s_1, s_2, \ldots, s_d)$-grid with for every i, $s_i = s$.

Let us determine the hyperbolicity of the above graphs. By doing so, we answer an open question of the literature [44, Remark 7].

Proposition 47. *The $(s_1, s_2, \ldots, s_d)$-grid has hyperbolicity:*

$$h_d(s_1, s_2, \ldots, s_d) = \max_{\mathcal{E} \subseteq \{1, \ldots, d\}} \min \left\{ \sum_{i \in \mathcal{E}} s_i - 1, \sum_{i \notin \mathcal{E}} s_i - 1 \right\}.$$

Proof. The 2^{d-1} far-apart pairs of the grid are the diametral pairs $\{(\langle u_1,\ldots,u_d\rangle, \langle v_1,\ldots,v_d\rangle) \mid \forall i, \{u_i, v_i\} = \{0, s_i - 1\}\}$. Let $(\langle u_1,\ldots,u_d\rangle, \langle v_1,\ldots,v_d\rangle)$ and $(\langle x_1,\ldots,x_d\rangle, \langle y_1,\ldots,y_d\rangle)$ be two such pairs, denoted with $(\overrightarrow{u}, \overrightarrow{v})$ and $(\overrightarrow{x}, \overrightarrow{y})$ for short. Finally, let $D = \sum_i s_i - 1$ be the diameter of the grid and let $l = \sum_{i|u_i \neq x_i} s_i - 1$. Then it comes:

$$\begin{aligned}
S_1 &= \mathrm{d}(\overrightarrow{u}, \overrightarrow{v}) + \mathrm{d}(\overrightarrow{x}, \overrightarrow{y}) = 2D \\
S_2 &= \mathrm{d}(\overrightarrow{u}, \overrightarrow{x}) + \mathrm{d}(\overrightarrow{v}, \overrightarrow{y}) = 2l \\
S_3 &= \mathrm{d}(\overrightarrow{u}, \overrightarrow{y}) + \mathrm{d}(\overrightarrow{v}, \overrightarrow{x}) = 2(D-l).
\end{aligned}$$

As a result, $\delta(\overrightarrow{x}, \overrightarrow{y}, \overrightarrow{u}, \overrightarrow{v}) = \min\{l, D-l\}$ which is maximum for $l = h_d(s_1, s_2, \ldots, s_d)$. We conclude that $h_d(s_1, s_2, \ldots, s_d)$ is the hyperbolicity by Lemma 44. □

We highlight two particular cases of Proposition 47 that were already known in the literature.

Corollary 48 ([44,55]). *The (n,m)-grid is $(\min\{n,m\}-1)$-hyperbolic.*

Corollary 49 ([55]). *The d-dimensional grid of size s is $(s-1)\cdot\left\lfloor \frac{d}{2} \right\rfloor$-hyperbolic.*

Similar results can be obtained for other grid-like graphs which can be found in the literature. We prove some of these results before concluding this section.

Definition 50. The triangular (n,m)-grid is a supergraph of the (n,m)-grid with same vertex-set and with additional edges $\{(i,j),(i+1,j+1)\}$ for every $0 \leq i \leq n-2$ and $0 \leq j \leq m-2$.

An example of a triangular $(6,7)$-grid is given in Figure 3a.

Lemma 51. *The triangular (n,m)-grid is $\frac{\min\{n,m\}-1}{2}$-hyperbolic.*

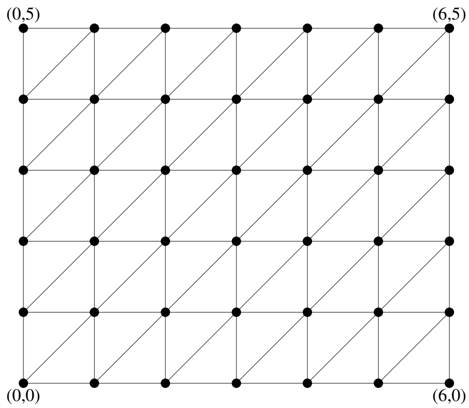
Proof. Let $u = (i_u, j_u)$ and $v = (i_v, j_v)$ be two vertices of the grid. We can assume w.l.o.g. that $i_u \geq i_v$. In such a case, either $j_u \geq j_v$ and so, $\mathrm{d}(u,v) = \max\{i_u - i_v, j_u - j_v\}$; or $j_u < j_v$ and so, $\mathrm{d}(u,v) = (i_u - i_v) + (j_v - j_u)$. We deduce from the above characterization that there is only one far-apart pair (u,v) such that $\mathrm{d}(u,v) \neq \max\{|i_u - i_v|, |j_u - j_v|\}$ namely, $u = (n-1, 0)$ and $v = (0, m-1)$ for which $\mathrm{d}(u,v) = n+m-2$. Furthermore, for any other far-apart pair (x,y) either $\mathrm{d}(x,y) = n-1$ or $\mathrm{d}(x,y) = m-1$.

Let (u,v) and (x,y) be two far-apart pairs satisfying the conditions of the above Lemma 44. We assume w.l.o.g. that $\mathrm{d}(u,v) \geq \mathrm{d}(x,y)$, and we claim that $2\delta(u,v,x,y) \leq \min\{n,m\} - 1$. First, by [74] $2\delta(u,v,x,y) \leq \min\{\mathrm{d}(u,v), \mathrm{d}(x,y)\} \leq \mathrm{d}(x,y)$. Note that $\mathrm{d}(x,y) = k \in \{n-1, m-1\}$ by the above characterization of the far-apart pairs in the grid. As a result, if $n = m$ then we are done because $\mathrm{d}(x,y) = \min\{n,m\} - 1$.

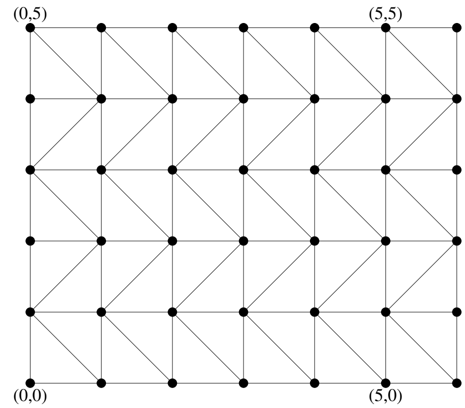
For the remaining of the proof, we will suppose that $n \neq m$ and $\mathrm{d}(x,y) = \max\{n,m\} - 1 = k$ (else we are done because $\mathrm{d}(x,y) = \min\{n,m\} - 1$). If $k = n-1$, it implies that $\mathrm{d}(u,v) \geq |i_u - i_v| = |i_x - i_y| = \mathrm{d}(x,y) = n-1$; else, it implies $\mathrm{d}(u,v) \geq |j_u - j_v| = |j_x - j_y| = \mathrm{d}(x,y) = m-1$. Therefore, we always have that $\max\{\mathrm{d}(u,x) + \mathrm{d}(v,y), \mathrm{d}(u,y) + \mathrm{d}(v,x)\} \geq 2k$. It follows by Lemma 44 that the hyperbolicity of the triangular grid is:

$$\begin{aligned}
2\delta(u,v,x,y) &= \mathrm{d}(u,v) + \mathrm{d}(x,y) - \max\{\mathrm{d}(u,x) + \mathrm{d}(v,y), \mathrm{d}(u,y) + \mathrm{d}(v,x)\} \\
&\leq n+m-2+k-2k = n+m-2-\max\{n-1,m-1\} = \min\{n,m\} - 1
\end{aligned}$$

The bound is reached by setting $u = (n-1,0)$, $v = (0,m-1)$, $x = (0,0)$, $y = (n-1,m-1)$. □



(a) The triangular $(7,6)$-grid has hyperbolicity $\delta = \frac{5}{2} = \delta(u,v,x,y)$ with $u = (6,0)$, $v = (0,5)$, $x = (0,0)$, $y = (6,5)$.



(b) The hexagonal $(7,6)$-grid has hyperbolicity $\delta = \frac{5}{2} = \delta(u,v,x,y)$ with $u = (0,5)$, $v = (5,0)$, $x = (0,0)$, $y = (5,5)$.

Figure 3: Examples of grid-like graphs.

In the example of Figure 3a, the hyperbolicity of the graph is given by the 4-tuple $u = (6,0)$, $v = (0,5)$, $x = (0,0)$, $y = (6,5)$.

Definition 52. The hexagonal (n,m)-grid is a supergraph of the (n,m)-grid with same vertex-set and with additional edges $\{\{(i, m-2j-1), (i+1, m-2j-2)\} \mid 0 \leq i \leq n-2 \text{ and } 0 \leq j \leq \left\lfloor \frac{m}{2} \right\rfloor - 1\} \cup \{\{(i, m-2j-3), (i+1, m-2j-2)\} \mid 0 \leq i \leq n-2 \text{ and } 0 \leq j \leq \left\lfloor \frac{m-1}{2} \right\rfloor - 1\}$.
The additional edges are called *diagonal* edges.

Informally, the difference between the triangular grid and the hexagonal grid is that in the hexagonal grid, the direction of diagonal edges alternate at each row. We refer to Figure 3b for an illustration. The hyperbolicity of hexagonal grids has already received some attention in [28]. In fact, they showed using the hexagonal grid that the gap between hyperbolicity of a graph and the length of its longest isometric cycle can be arbitrarily large (see also [78] for more explanations). However, to the best of our knowledge there was no formal bound so far established for the hyperbolicity of hexagonal grids.

Lemma 53. *The hexagonal (n,m)-grid is $\frac{\min\{n,m\}-1}{2}$-hyperbolic.*

Proof. We will first characterize the distances in the grid. Let $u = (i_u, j_u)$, $v = (i_v, j_v)$ be two vertices of the hexagonal grid. W.l.o.g., $i_u \geq i_v$. Let us observe that in order to obtain an uv-shortest-path, it suffices to maximize the number of *diagonal* edges used in the path, that is $\min\{k, |i_u - i_v|\}$ with:

- $k = \lfloor |j_u - j_v|/2 \rfloor$ if both $j_u - j_v$ and $2\,[m - j_v \pmod 2] - 1$ have the same sign;
- $k = \lceil |j_u - j_v|/2 \rceil$ otherwise.

As a result $\mathrm{d}(u,v) = |i_u - i_v| + |j_u - j_v| - \min\{k, |i_u - i_v|\}$ for some k depending on j_u and j_v, $k \in \{\lfloor |j_u - j_v|/2 \rfloor, \lceil |j_u - j_v|/2 \rceil\}$.

Suppose in addition that (u,v) is a far-apart pair. There are two cases. If $\mathrm{d}(u,v) = |j_u - j_v|$ then it is monotonically increasing with $|j_u - j_v|$ and so, $|j_u - j_v| = m - 1$. Else, $\mathrm{d}(u,v) =$

$|i_u - i_v| + |j_u - j_v| - k$ for some k *only* depending on j_u and j_v, that is monotonically increasing with $|i_u - i_v|$ and so, $|i_u - i_v| = n - 1$.

Finally, let (u, v), (x, y) be two far-apart pairs satisfying the conditions of the above Lemma 44. We will prove that $2\delta(u, v, x, y) \leq \min\{n, m\} - 1$.

Case $\mathbf{m \leq n}$. If $\min\{\mathrm{d}(u, v), \mathrm{d}(x, y)\} \leq m - 1$ then we are done because by [74] we have that $\delta(u, v, x, y) \leq \min\{\mathrm{d}(u, v), \mathrm{d}(x, y)\}/2 \leq (m-1)/2$. Else, we must have that $|i_u - i_v| = |i_x - i_y| = n - 1$ and so, $\max\{\mathrm{d}(u, x) + \mathrm{d}(v, y), \mathrm{d}(u, y) + \mathrm{d}(v, x)\} \geq 2(n-1)$. Since in such a case $\mathrm{d}(u, v) + \mathrm{d}(x, y) \leq (n - 1 + \lceil (m-1)/2 \rceil) + (n - 1 + \lfloor (m-1)/2 \rfloor) = 2(n-1) + m - 1$ then it follows once again that $\delta(u, v, x, y) \leq (m-1)/2$.

Case $\mathbf{m > n}$. There are three subcases to be considered.

- Suppose $\mathrm{d}(u, v) = |j_u - j_v| = m - 1$, $\mathrm{d}(x, y) = |j_x - j_y| = m - 1$. Then it comes that $\max\{\mathrm{d}(u, x) + \mathrm{d}(v, y), \mathrm{d}(u, y) + \mathrm{d}(v, x)\} \geq 2(m-1)$ and so, $\delta(u, v, x, y) = 0$.
- Suppose $\mathrm{d}(u, v) = j_u - j_v = m - 1$ and $n - 1 + \lfloor (j_x - j_y)/2 \rfloor \leq \mathrm{d}(x, y) \leq n - 1 + \lceil (j_x - j_y)/2 \rceil$. Then it holds that $\mathrm{d}(u, y) + \mathrm{d}(v, x) \geq (j_u - j_y) + (j_x - j_v) = (j_u - j_x) + (j_x - j_y) + (j_x - j_v) = m - 1 + (j_x - j_y)$. As a result,

  $$2\delta(u, v, x, y) \leq (n - 1 + \lceil (j_x - j_y)/2 \rceil + m - 1) - (m - 1 + j_x - j_y) = n - 1 - \lfloor (j_x - j_y)/2 \rfloor \leq n - 1.$$

- Else, we consider the smallest hexagonal grid of dimensions (n', m') for which there exists two far-apart pairs (u', v') and (x', y') that satisfy the conditions of the above Lemma 44 and such that $\delta(u', v', x', y') \geq \delta(u, v, x, y)$. We assume w.l.o.g. that $n' < m'$ and $\mathrm{d}(u', v') \neq |j_{u'} - j_{v'}|$, $\mathrm{d}(x', y') \neq |j_{x'} - j_{y'}|$ (otherwise we fall in one of the above cases). Note that it implies that $|i_{u'} - i_{v'}| = |i_{x'} - i_{y'}| = n' - 1$ by our above characterization of the far-apart pairs.

  If the two far-apart pairs are $((0,0), (n'-1, m'-1))$ and $((0, m'-1), (n'-1, 0))$, then we obtain by the computation that $2\delta(u', v', x', y') = n' - 1 + (n' - m') < n' - 1 \leq n - 1$.

  Else, by minimality of the subgrid there is some vertex in the 4-tuple, say u', which is contained amongst $\{(0,0), (n'-1, m'-1), (n'-1, 0), (0, m'-1)\}$ and such that no other vertex $z \in \{v', x', y'\}$ satisfies that $j_{u'} = j_z$. By symmetry, we will assume that $u' \in \{(0, m'-1), (n'-1, m'-1)\}$. Then, using the above characterization of the distances in the hexagonal grid, it can be checked that for any $0 \leq i \leq n' - 1$ and for any $0 \leq j \leq m' - 2$:

  $$\begin{aligned} \mathrm{d}\left((n'-1, m'-2), (i, j)\right) &= \mathrm{d}\left((n'-1, m'-1), (i, j)\right) - 1 \\ \text{and } \mathrm{d}\left((1, m'-2), (i, j)\right) &= \mathrm{d}\left((0, m'-1), (i, j)\right) - 1 \text{ unless } (i, j) = (0, m'-2) \end{aligned}$$

  Therefore, by the 4-point condition $\delta(u', v', x', y') = \delta\left((n'-1, m'-2), v', x', y'\right)$ when $u' = (n'-1, m'-1)$; $\delta(u', v', x', y') \leq \max\left\{\mathrm{d}\left((0, m'-1), (0, m'-2)\right), \delta\left((n'-1, m'-2), v', x', y'\right)\right\} \leq \max\left\{1, \delta\left((n'-1, m'-2), v', x', y'\right)\right\}$ when $u' = (0, m'-1)$. In both cases, it contradicts the minimality of (n', m').

To conclude, let $l = \min\{n, m\} - 1$. The upper-bound $l/2$ for the hyperbolicity is reached by setting $u = (0, m-1), v = (l, m-1-l), x = (0, m-1-l), y = (l, m-1)$. □

In the example of Figure 3b for an illustration, the hyperbolicity of the graph is given by the 4-tuple $u = (0, 5)$, $v = (5, 0)$, $x = (0, 0)$, $y = (5, 5)$.

Definition 54. The cylinder (n, m)-grid is the supergraph of the (n, m)-grid with the same vertex-set and with additional edge-set $\{\{(0, j), (n-1, j)\} \mid 0 \leq j \leq m-1\}$.

In particular, when $m = 1$, then the cylinder (n, m)-grid is the n-cycle C_n. More generally, each row induces a cycle instead of inducing a path.

Lemma 55. *The cylinder (n, m)-grid is*

$$\begin{cases} \left\lfloor \frac{n}{2} \right\rfloor \text{-hyperbolic} & \text{when } m > \left\lfloor \frac{n}{2} \right\rfloor \\ \left(\frac{\lfloor \frac{n}{2} \rfloor + m}{2} - 1 \right) \text{-hyperbolic} & \text{when } m \leq \left\lfloor \frac{n}{2} \right\rfloor \text{ and } (n \text{ is odd or } \left\lceil \frac{n}{2} \right\rceil - m + 1 \text{ is odd}) \\ \left(\frac{\lfloor \frac{n}{2} \rfloor + m}{2} - \frac{1}{2} \right) \text{-hyperbolic} & \text{otherwise.} \end{cases}$$

Proof. Let $u = (i_u, j_u), v = (i_v, j_v)$ be two vertices of the grid. We have:

$$\mathrm{d}(u, v) = \min\{|i_u - i_v|, n - |i_u - i_v|\} + |j_u - j_v|.$$

As a result, the far-apart pairs of the cylinder (n, m)-grid are exactly the pairs $\{(i, 0), (i + \lfloor n/2 \rfloor \pmod n), m-1)\}$, and the pairs $\{(i, 0), (i + \lceil n/2 \rceil \pmod n), m-1)\}$, with $0 \leq i \leq n-1$. Equivalently, these are the pairs $\{(u', 0), (v', m-1)\}$ with (u', v') an arbitrary far-apart pair of the n-cycle C_n.

Let (u, v) and (x, y) be two far-apart pairs of the cylinder (n, m)-grid satisfying the conditions of the above Lemma 44. Write $u = (u', 0)$, $v = (v', m-1)$, $x = (x', 0)$, $y = (y', m-1)$. Furthermore, let $S_1 = \mathrm{d}(u, v) + \mathrm{d}(x, y)$, $S_2 = \mathrm{d}(u, x) + \mathrm{d}(v, y)$, and $S_3 = \mathrm{d}(u, y) + \mathrm{d}(v, x)$. Similarly, let $S_1' = \mathrm{d}_{C_n}(u', v') + \mathrm{d}_{C_n}(x', y')$, $S_2' = \mathrm{d}_{C_n}(u', x') + \mathrm{d}_{C_n}(v', y')$, and $S_3' = \mathrm{d}_{C_n}(u', y') + \mathrm{d}_{C_n}(v', x')$. Note that it holds: $S_1' = 2\lfloor n/2 \rfloor = \max\{S_1', S_2', S_3'\}$; $S_1 = S_1' + 2(m-1) = 2(\lfloor n/2 \rfloor + m - 1)$, $S_2' = S_2$, and $S_3 = S_3' + 2(m-1)$.

There are two cases to be considered.

- Suppose that $m > \lfloor n/2 \rfloor$. We have that $\delta(u, v, x, y) \leq (S_1 - S_3)/2 \leq (S_1' - S_3')/2 \leq S_1'/2 \leq \lfloor n/2 \rfloor$. The bound is reached by setting $u' = y'$ and $v' = x'$.
- Suppose that $m \leq \lfloor n/2 \rfloor$. If $(u', v') = (y', x')$ then we obtain by the calculation that $\delta(u, v, x, y) = (m-1)/2$. Otherwise, $S_2' + S_3' = n$ and hence

  $$2\delta(u, v, x, y) = S_1' - \max\{S_3', S_2' - 2(m-1)\} = S_1' - \max\{S_3', (n - 2(m-1)) - S_3'\},$$

  this is maximum when $\lfloor n/2 \rfloor - (m-1) \leq S_3' \leq \lceil n/2 \rceil - (m-1)$. In the following, let $\lceil n/2 \rceil - (m-1) = 2q + r$ with $0 \leq r \leq 1$. There are two subcases to be considered.

  (i) Assume that n is odd and let us set $u' = 0$, $v' = \lfloor n/2 \rfloor$, $x' = \lfloor n/2 \rfloor - q$, and $y' = n - q - r$. In such a case, $S_3' = (q + r) + q = 2q + r$. As a result, $\delta(u, v, x, y) = (\lfloor n/2 \rfloor + m)/2 - 1$ and so, the above upper-bound is always reached when n is odd.

(ii) Assume that n is even. Then, $S'_3 = 2\,\mathrm{d}(u', y')$ cannot be odd. It implies that the hyperbolicity is bounded from above by $n/4 + (m-1-r)/2$. We set $u' = 0$, $v' = n/2$, $x' = n/2 - q$, and $y' = n - q$. In such a case, $S'_3 = 2q$, hence $(n - 2(m-1)) - S'_3 = 4q + 2r - 2q = 2q + 2r$ and so, $\delta(u, v, x, y) = n/4 + (m-1-r)/2$ that is maximum.

□

Before concluding this section, let us compare the techniques employed for grids with Theorem 4. It is easy to see that for all grid-like graphs considered (cf. Definitions 10, 50, 52 and 54), there is either a row or a column contained in the center. Therefore, the diameter of the center is at least $\min\{n, m\} - 1$, and so, by [37, Proposition 5] one obtains the lower-bound $\lfloor (\min\{n, m\} - 1)/2 \rfloor / 2$ on the hyperbolicity for all these graphs. In this section, we have conducted an in-depth analysis of their shortest-path distribution in order to establish the exact hyperbolicity of these grid-like graphs.

5 Relations between hyperbolicity and some graph operations

Our results so far are heavily focused on the so-called *homogeneous* data center interconnection networks. By contrast, heterogeneous data centers are based on the composition of homogeneous data center interconnection topologies through graph operations. We survey a few of these operations so that we can study the impact that they may have on the hyperbolicity of the network.

5.1 Biswap operation and biswapped networks

Definition 56 ([35]). Let G be a graph. The biswapped graph $Bsw(G)$ has vertex set $\{0, 1\} \times V(G) \times V(G)$. Two vertices (b, u, v) and (b', u', v') are adjacent if, and only if either $b = b'$, $u = u'$ and $\{v, v'\} \in E(G)$, or $b = \bar{b}' = 1 - b'$, $u = v'$, and $u' = v$.

Lemma 57. *For any connected graph G, $\delta(Bsw(G)) = \mathrm{diam}(G) + 1$.*

Proof. By [35] $\mathrm{diam}(Bsw(G)) = 2 \cdot \mathrm{diam}(G) + 2$ and so, by Lemma 2 $\delta(Bsw(G)) \leq \mathrm{diam}(G) + 1$. To prove the lower-bound, let $u, v \in V(G)$ be such that $\mathrm{diam}(G) = \mathrm{d}_G(u, v)$. We define $\overrightarrow{x_1} = (0, u, v)$, $\overrightarrow{x_2} = (0, v, u)$, $\overrightarrow{x_3} = (1, u, u)$ and $\overrightarrow{x_4} = (1, v, v)$. We deduce from [35] that:

$$
\begin{aligned}
S_1 &= \mathrm{d}(\overrightarrow{x_1}, \overrightarrow{x_2}) + \mathrm{d}(\overrightarrow{x_3}, \overrightarrow{x_4}) = 2(2\,\mathrm{d}_G(u, v) + 2) \\
S_2 &= \mathrm{d}(\overrightarrow{x_1}, \overrightarrow{x_3}) + \mathrm{d}(\overrightarrow{x_2}, \overrightarrow{x_4}) = 2(\mathrm{d}_G(u, v) + 1) \\
S_3 &= \mathrm{d}(\overrightarrow{x_1}, \overrightarrow{x_4}) + \mathrm{d}(\overrightarrow{x_2}, \overrightarrow{x_3}) = 2(\mathrm{d}_G(u, v) + 1)
\end{aligned}
$$

As a result, $\delta(Bsw(G)) \geq \delta(\overrightarrow{x_1}, \overrightarrow{x_2}, \overrightarrow{x_3}, \overrightarrow{x_4}) = \mathrm{d}_G(u, v) + 1 = \mathrm{diam}(G) + 1$. □

It follows from Lemma 57 that the hyperbolicity of a biswap network always scales with its diameter, *regardless of the topology that is used for the operation.*

5.2 Generic Cayley construction

Let us finally consider the following transformation of a Hamiltonian graph, and the consequences of it on the hyperbolicity.

Lemma 58. *Let G be a Hamiltonian graph and c be a positive integer. We construct a graph G' from G by replacing every edge in some Hamilton cycle of G with a path of length c. Then, it holds $\delta(G') \geq \frac{1}{2}\left\lceil\frac{c-1}{2}\right\rceil$.*

Proof. Let P be a path of length c added by the construction, let x and y be the endpoints of P, and let P' be a xy-shortest-path in $G' \setminus (P \setminus \{x, y\})$. The union of P with P' is an isometric cycle and so, it has length upper-bounded by $4 \cdot \delta(G') + 3$ by [78]. Moreover, the length of P' is at least 2 because $\{x, y\}$ is an edge of G by the hypothesis. Thus it comes that the length of the cycle is at least $c + 2$ and so, $c \leq 4 \cdot \delta(G') + 1$. □

The Cayley model in [59] aims to apply the construction defined in Lemma 58 to some Hamiltonian graph G of order N, with $c = \Omega(\log N)$ and so that the diameter of the resulting graph G' is $O(\log N)$. Summarizing, we get.

Theorem 59. *Graphs in the Cayley model have hyperbolicity $\Theta(\log N)$, which scales linearly with their diameter.*

6 Conclusion

We proved in this work that the topologies of various interconnection networks have their hyperbolicity that scales linearly with their diameter. This property is inherent to any graph having desired properties for data centers such as a high-level of symmetry. Interestingly, symmetries are a common way to minimize network congestion whereas it was shown in [51], using a simplified model, that a bounded hyperbolicity might explain the congestion phenomenon observed in some real-life networks. This was formally proved in [75] for shortest-path routing, but to the best of our knowledge no relation is known between hyperbolicity and congestion in *general*. Therefore, we let open whether a more general relationship between congestion and hyperbolicity can be determined.

Finally, our results imply that in any greedy routing scheme based on an embedding into the hyperbolic space —and in some cases, on an embedding into some word metric space— there is a *linear* number of routing paths for which the stretch is arbitrarily bad. However, this does not preclude the possibility that for most other routing paths, the stretch is bounded by a small constant. We thus believe that it might be of interest to compute the *average hyperbolicity* [63, 72] of the data center interconnection topologies so as to verify whether it is the case.

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