

# Fisheye Consistency: Keeping Data in Synch in a Georeplicated World

R Friedman, M Raynal, François Taïani

► **To cite this version:**

R Friedman, M Raynal, François Taïani. Fisheye Consistency: Keeping Data in Synch in a Georeplicated World. International Conference on NETworked sYStems (NETYS'2015), May 2015, Agadir, Morocco. Springer International Publishing, 2015, Networked Systems : Third International Conference, NETYS 2015, Agadir, Morocco, May 13-15, 2015, Revised Selected Papers. <10.1007/978-3-319-26850-7\_17>. <hal-01326888>

**HAL Id: hal-01326888**

**<https://hal.inria.fr/hal-01326888>**

Submitted on 6 Jun 2016

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Fisheye Consistency: Keeping Data in Synchrony in a Georeplicated World

R. Friedman<sup>1</sup>, M. Raynal<sup>2,3</sup>, and F. Taïani<sup>3</sup>

<sup>1</sup> The Technion Haifa, Israel

<sup>2</sup> Institut Universitaire de France

<sup>3</sup> Université de Rennes 1, IRISA, Campus de Beaulieu, 35042 Rennes Cedex, France

**Abstract.** Over the last thirty years, numerous consistency conditions for replicated data have been proposed and implemented. Popular examples include linearizability (or atomicity), sequential consistency, causal consistency, and eventual consistency. These conditions are usually defined independently from the computing entities (nodes) that manipulate the replicated data; i.e., they do not take into account how computing entities might be linked to one another, or geographically distributed. To address this lack, as a first contribution, this paper introduces the notion of *proximity graph* between computing nodes. If two nodes are connected in this graph, their operations must satisfy a strong consistency condition, while the operations invoked by other nodes are allowed to satisfy a weaker condition. The second contribution exploits this graph to provide a generic approach to the hybridization of data consistency conditions within the same system. We illustrate this approach on sequential consistency and causal consistency, and present a model in which all data operations are causally consistent, while operations by neighboring processes in the proximity graph are sequentially consistent. The third contribution of the paper is the design and the proof of a distributed algorithm based on this proximity graph, which combines sequential consistency and causal consistency (the resulting condition is called *fisheye consistency*). In doing so the paper provides a generic provably correct solution of direct relevance to modern georeplicated systems.

**Keywords:** Asynchronous message-passing systems, Broadcast, Causal consistency, Data replication, Georeplication, Linearizability, Sequential consistency.

## 1 Introduction

As distributed computer systems continue to grow in size, they make it increasingly difficult to provide strong consistency guarantees (e.g., linearizability [20]), prompting the rise of weaker guarantees (e.g., causal consistency [2] or eventual consistency [39]). These weaker consistency conditions strike a compromise between consistency, performance, and availability [5,7,10,13,40]. They try in general to minimize the violations of strong consistency, as these create anomalies for programmers and users, and emphasize the low probability of such violations in their real deployments [15].

*Recent related works* For brevity, we cannot name all the many weak consistency conditions that have been proposed in the past. We focus instead on the most recent

works in this area. One of the main hurdles in building systems and applications based on weak consistency models is how to generate an eventually consistent and meaningful image of the shared memory or storage [39]. In particular, a paramount sticking point is how to handle conflicting concurrent write (or update) operations and merge their result in a way that suits the target application. To that end, various conditions that enables custom conflict resolution and a host of corresponding data-types have been proposed and implemented [3,4,9,14,26,30,36,35].

Another form of hybrid consistency conditions can be found in the seminal works on *release consistency* [18,21] and *hybrid consistency* [6,16], which distinguish between strong and weak operations such that strong operations enjoy stronger consistency guarantees than weak operations. This line of work has given rise to a number of contributions in the context of large scale and geo-replicated data centers [38,40].

*Motivation and problem statement* In spite of their benefits, the above consistency conditions generally ignore the relative “distance” between nodes in the underlying “infrastructure”, where the notions of “distance” and “infrastructure” may be logical or physical, depending on the application. This is unfortunate as distributed systems must scale out and geo-replication is becoming more common. In a geo-replicated system, the network latency and bandwidth connecting nearby servers is usually at least an order of magnitude better than what is obtained between remote servers. This means that the cost of maintaining strong consistency among nearby nodes becomes affordable compared to the overall network costs and latencies in the system.

Some production-grade systems acknowledge the importance of distance when enforcing consistency, and do propose consistency mechanisms based on node locations (e.g. whether nodes are located in the same or in different data-centers). Unfortunately these production-grade systems usually do not distinguish between semantics and implementation. Rather, their consistency model is defined in operational terms, whose full implications can be difficult to grasp. In Cassandra [22], for instance, the application can specify for each operation a consistency guarantee that is dependent on the location of replicas. More precisely, the constraints LOCAL\_QUORUM requires a quorum of replicas in the local data center, while EACH\_QUORUM requires a quorum in each data center. Yet, although these constraints take distance into account, they do not provide the programmer with a precise image of the consistency they deliver.

The need to consider “distance” when defining consistency models, and the current lack of any formal underpinning to do so are exactly what motivates the hybridization of consistency conditions that we propose in this paper (which we call *fish-eye consistency*). Fisheye consistency conditions provide strong guarantees only for operations issued at nearby servers. In particular, there are many applications where one can expect that concurrent operations on the same objects are likely to be generated by geographically nearby nodes, e.g., due to business hours in different time zones, or because these objects represent localized information, etc. In such situations, a fisheye consistency condition would in fact provide global strong consistency at the cost of maintaining only locally strong consistency.

Consider for instance a node  $A$  that is “close” to a node  $B$ , but “far” from a node  $C$ , a causally consistent read/write register will offer the same (weak) guarantees to  $A$  on the operations of  $B$ , as on the operations of  $C$ . This may be suboptimal, as many

applications could benefit from varying levels of consistency conditioned on “how far” nodes are from each other. Stated differently: a node can accept that “remote” changes only reach it with weak guarantees (e.g., because information takes time to travel), but it wants changes “close” to it to come with strong guarantees (as “local” changes might impact it more directly).

In this work, we propose to address this problem by integrating a notion of *node proximity* in the definition of *data consistency*. To that end, we formally define a new family of hybrid consistency models that links the strength of data consistency with the proximity of the participating nodes. In our approach, a particular hybrid model takes as input a proximity graph, and two consistency conditions (a weaker one and a stronger one), taken from a set of totally ordered consistency conditions (e.g. linearizability, sequential consistency, causal consistency, and PRAM-consistency [25]).

The philosophy we advocate is related to that of Parallel Snapshot Isolation (PSI) proposed in [37]. PSI combines strong consistency (Snapshot Isolation) for transactions started at nodes in the same site of a geo-replicated system, but only ensures causality among transactions started at different sites.

Although PSI and our work operate at different granularities (fisheye-consistency is expressed on individual operations, each accessing a single object, while PSI addresses general transactions), they both show the interest of consistency conditions in which nearby nodes enjoy stronger semantics than remote ones. In spite of this similitude, however, the family of consistency conditions we propose distinguishes itself from PSI in a number of key dimensions. First, PSI is a specific condition while fisheye-consistency offers a general framework for defining multiple such conditions. PSI only distinguishes between nodes at the same physical site and remote nodes, whereas fisheye-consistency accepts arbitrary proximity graphs, which can be physical or logical. Finally, the definition of PSI is given in [37] by a reference implementation, whereas fisheye-consistency is defined in functional terms as restrictions on the ordering of operations that can be seen by applications, independently of the implementation we propose. As a result, we believe that our formalism makes it easier for users to express and understand the semantics of a given consistency condition and to prove the correctness of a program written w.r.t. such a condition.

*Roadmap* The next section introduces the system model and the classical sequential consistency (SC) [24] and causal consistency (CC) [2]. Then, Section 3 defines the notion of *proximity graph* and the associated fisheye consistency condition, which considers SC as its strong condition and CC as its weak condition. Section 4 presents a broadcast abstraction, and Section 5 proposes an algorithm based on this broadcast abstraction that implements the fisheye consistency condition that combines SC and CC. These algorithms are generic, where the genericity parameter is the proximity graph. Interestingly, their two extreme instantiations provide natural implementations of SC and CC. Section 6 concludes.

## 2 System Model and Basic Consistency Conditions

The system consists of  $n$  processes denoted  $\Pi = \{p_1, \dots, p_n\}$ . Each process is sequential and asynchronous. “Asynchronous” means that each process proceeds at its own speed,

which is arbitrary, may vary with time, and remains always unknown to the other processes.

Processes communicate by passing messages through bi-directional channels. Channels are reliable (no loss, duplication, creation, or corruption), and asynchronous (transit times are arbitrary but finite, and remain unknown to the processes).

## 2.1 Basic notions and definitions underpinning consistency conditions

This section is a short reminder of the fundamental notions typically used to define the consistency guarantees of distributed objects [8,19,27,31].

*Concurrent objects with sequential specification* A concurrent object is an object that can be simultaneously accessed by different processes. At the application level the processes interact through concurrent objects [19,31]. Each object is defined by a sequential specification, which is a set including all the correct sequences of operations and their results that can be applied to and obtained from the object. These sequences are called *legal* sequences.

*Execution history* The execution of a set of processes interacting through objects is captured by a *history*  $\widehat{H} = (H, \rightarrow_H)$ , where  $\rightarrow_H$  is a partial order on the set  $H$  of the object operations invoked by the processes.

*Concurrency and sequential history* If two operations are not ordered in a history, they are said to be *concurrent*. A history is said to be *sequential* if it does not include any concurrent operations. In this case, the partial order  $\rightarrow_H$  is a total order.

*Equivalent history* Let  $\widehat{H}|p$  represent the projection of  $\widehat{H}$  onto the process  $p$ , i.e., the restriction of  $\widehat{H}$  to operations occurring at process  $p$ . Two histories  $\widehat{H}_1$  and  $\widehat{H}_2$  are *equivalent* if no process can distinguish them, i.e.,  $\forall p \in \Pi : \widehat{H}_1|p = \widehat{H}_2|p$ .

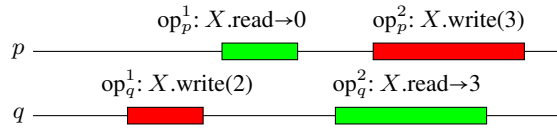
*Legal history*  $\widehat{H}$  being a sequential history, let  $\widehat{H}|X$  represent the projection of  $\widehat{H}$  onto the object  $X$ . A history  $\widehat{H}$  is *legal* if, for any object  $X$ , the sequence  $\widehat{H}|X$  belongs to the specification of  $X$ .

*Process Order* Notice that since we assumed that processes are sequential, we restrict the discussion in this paper to execution histories  $\widehat{H}$  for which for every process  $p$ ,  $\widehat{H}|p$  is sequential. This total order is also called the *process order* for  $p$ .

## 2.2 Sequential consistency

Intuitively, an execution is sequentially consistent if it could have been produced by executing (with the help of a scheduler) the processes on a multiprocessor. Formally, a history  $\widehat{H}$  is *sequentially consistent* (SC) if there exists a history  $\widehat{S}$  such that:

- $\widehat{S}$  is sequential,
- $\widehat{S}$  is legal (the specification of each object is respected),



**Fig. 1.** A sequentially consistent execution

- $\widehat{H}$  and  $\widehat{S}$  are equivalent (no process can distinguish  $\widehat{H}$ —what occurred—and  $\widehat{S}$ —what we would like to see, to be able to reason about).

One can notice that SC does not demand that the sequence  $\widehat{S}$  respects the real-time occurrence order on the operations. This is the fundamental difference between linearizability and SC.

Figure 1 shows an history  $\widehat{H}$  that is sequentially consistent. Let us observe that, although  $op_q^1$  occurs before  $op_p^1$  in physical time,  $op_p^1$  does not see the effect of the write operation  $op_q^1$ , and still returns 0. A legal sequential history  $\widehat{S}$ , equivalent to  $\widehat{H}$ , can be easily built, namely,  $X.read \rightarrow 0$ ,  $X.write(2)$ ,  $X.write(3)$ ,  $X.read \rightarrow 3$ .

### 2.3 Causal consistency

In a sequentially consistent execution, all processes perceive all operations in the same order, which is captured by the existence of a sequential and legal history  $\widehat{S}$ . Causal consistency [2] relaxes this constraint for read-write registers, and allows different processes to perceive different orders of operations, as long as causality is preserved.

Formally, a history  $\widehat{H}$  in which processes interact through concurrent read/write registers is causally consistent (CC) if:

- There is a causal order  $\rightsquigarrow_H$  on the operations of  $\widehat{H}$ , i.e., a partial order that links each read to at most one latest write (or otherwise to an initial value  $\perp$ ), so that the value returned by the read is the one written by this latest write and  $\rightsquigarrow_H$  respects the process order of all processes.
- For each process  $p_i$ , there is a sequential and legal history  $\widehat{S}_i$  that
  - is equivalent to  $\widehat{H}|(p_i + W)$ , where  $\widehat{H}|(p_i + W)$  is the sub-history of  $\widehat{H}$  that contains all operations of  $p_i$ , plus the writes of all the other processes,
  - respects  $\rightsquigarrow_H$  (i.e.,  $\rightsquigarrow_H \subseteq \rightarrow_{S_i}$ ).

Intuitively, this definition means that all processes see causally related writes in the same order, but can see writes that are not causally related in different orders.

An example of causally consistent execution is given in Figure 2. The processes  $r$  and  $s$  observe the write operations on  $X$  by  $p$  ( $op_p^1$ ) and  $q$  ( $op_q^1$ ) in two different orders. This is acceptable in a causally consistent history because  $op_p^1$  and  $op_q^1$  are not causally related. This would not be acceptable in a sequentially consistent history, where the same total order on operations must be observed by all the processes.

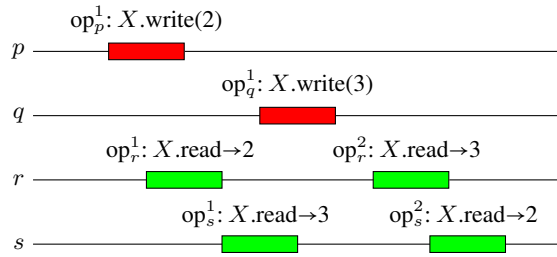


Fig. 2. An execution that is causally consistent (but not sequentially consistent)

<b>process <i>paris</i> is</b>	<b>process <i>berlin</i> is</b>	<b>process <i>new-york</i> is</b>
$X \leftarrow 1$	$X \leftarrow 2$	<b>repeat</b> $c \leftarrow R$ <b>until</b> $c = 1$
$R \leftarrow 1$	$S \leftarrow 1$	<b>repeat</b> $d \leftarrow S$ <b>until</b> $d = 1$
$a \leftarrow X$	$b \leftarrow X$	$X \leftarrow 3$
<b>end process</b>	<b>end process</b>	<b>end process</b>

Fig. 3. *new-york* does not need to be closely synchronized with *paris* and *berlin*, calling for a hybrid form of consistency

### 3 The Family of Fisheye Consistency Conditions

This section introduces a hybrid consistency model based on (a) two consistency conditions and (b) the notion of a proximity graph defined on the computing nodes (processes). The two consistency conditions must be totally ordered in the sense that any execution satisfying the stronger one also satisfies the weaker one.

#### 3.1 The notion of a proximity graph

Let us assume that for physical or logical reasons linked to the application, each process (node) can be considered either close to or remote from other processes. This notion of “closeness” can be captured through a *proximity graph* denoted  $\mathcal{G} = (\Pi, E_{\mathcal{G}} \subseteq \Pi \times \Pi)$ , whose vertices are the  $n$  processes of the system ( $\Pi$ ). The edges are undirected.  $N_{\mathcal{G}}(p_i)$  denotes the neighbors of  $p_i$  in  $\mathcal{G}$ .  $\mathcal{G}$  captures the level of consistency imposed on processes: processes connected in  $\mathcal{G}$  must respect a stronger data consistency than unconnected processes.

*Example* To illustrate the semantic of  $\mathcal{G}$ , we extend the original scenario that Ahamad, Niger *et al* use to motivate causal consistency in [2]. Consider the three processes of Figure 3, *paris*, *berlin*, and *new-york*. Processes *paris* and *berlin* interact closely with one another and behave symmetrically : they concurrently write the shared variable  $X$ , then set the flags  $R$  and  $S$  respectively to 1, and finally read  $X$ . By contrast, process *new-york* behaves sequentially w.r.t. *paris* and *berlin*: *new-york* waits for *paris* and *berlin* to write on  $X$ , using the flags  $R$  and  $S$ , and then writes  $X$ .

If we assume a model that provides causal consistency at a minimum, the write of  $X$  by *new-york* is guaranteed to be seen after the writes of *paris* and *berlin* by all

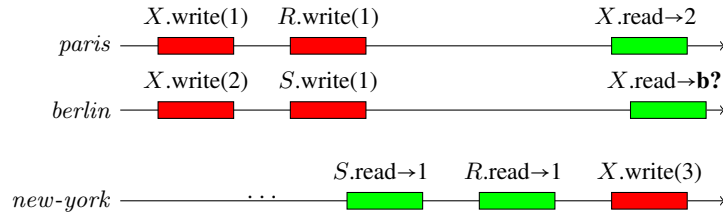


Fig. 4. Executing the program of Figure 3.



Fig. 5. Capturing the synchronization needs of Fig. 3 with a proximity graph  $\mathcal{G}$

processes (because *new-york* waits on *R* and *S* to be set to 1). Causal consistency however does not impose any consistent order on the writes of *paris* and *berlin* on *X*. In the execution shown on Figure 4, this means that although *paris* reads 2 in *X* (and thus sees the write of *berlin* after its own write), *berlin* might still read 1 in *b* (thus perceiving ‘*X.write(1)*’ and ‘*X.write(2)*’ in the opposite order to that of *paris*).

Sequential consistency removes this ambiguity: in this case, in Figure 4, *berlin* can only read 2 (the value it wrote) or 3 (written by *new-york*), but not 1. Sequential consistency is however too strong here: because the write operation of *new-york* is already causally ordered with those of *paris* and *berlin*, this operation does not need any additional synchronization effort. This situation can be seen as an extension of the *write concurrency freedom* condition introduced in [2]: *new-york* is here free of concurrent write w.r.t. *paris* and *berlin*, making causal consistency equivalent to sequential consistency for *new-york*. *paris* and *berlin* however write to *X* concurrently, in which case causal consistency is not enough to ensure strongly consistent results.

If we assume *paris* and *berlin* execute in the same data center, while *new-york* is located on a distant site, this example illustrates a more general case in which, because of a program’s logic or activity patterns, no operations at one site ever conflict with those at another. In such a situation, rather than enforce a strong (and costly) consistency in the whole system, we propose a form of consistency that is strong for processes within the same site (here *paris* and *berlin*), but weak between sites (here between *paris*, *berlin* on one hand and *new-york* on the other).

In our model, the synchronization needs of individual processes are captured by the *proximity graph*  $\mathcal{G}$  introduced at the start of this section and shown in Figure 5: *paris* and *berlin* are connected, meaning the operations they execute should be perceived as strongly consistent w.r.t. one another ; *new-york* is neither connected to *paris* nor *berlin*, meaning a weaker consistency is allowed between the operations executed at *new-york* and those of *paris* and *berlin*.



### 3.2 Fisheye consistency for the pair (sequential consistency, causal consistency)

When applied to the scenario of Figure 4, fisheye consistency combines two consistency conditions (a weak and a stronger one, here causal and sequential consistency) and a proximity graph to form an hybrid distance-based consistency condition, which we call  $\mathcal{G}$ -fisheye (SC,CC)-consistency.

The intuition in combining SC and CC is to require that write operations be observed in the same order by all processes if:

- They are causally related (as in causal consistency),
- Or they occur on “close” nodes (as defined by  $\mathcal{G}$ ).

*Formal definition* Formally, we say that a history  $\widehat{H}$  is  $\mathcal{G}$ -fisheye (SC,CC)-consistent if:

- There is a causal order  $\sim_H$  induced by  $\widehat{H}$  (as in causal consistency); and
- $\sim_H$  can be extended to a subsuming order  $\overset{\star}{\sim}_{H,\mathcal{G}}$  (i.e.  $\sim_H \subseteq \overset{\star}{\sim}_{H,\mathcal{G}}$ ) so that

$$\forall (p, q) \in E_{\mathcal{G}} : (\overset{\star}{\sim}_{H,\mathcal{G}})|(\{p, q\} \cap W) \text{ is a total order}$$

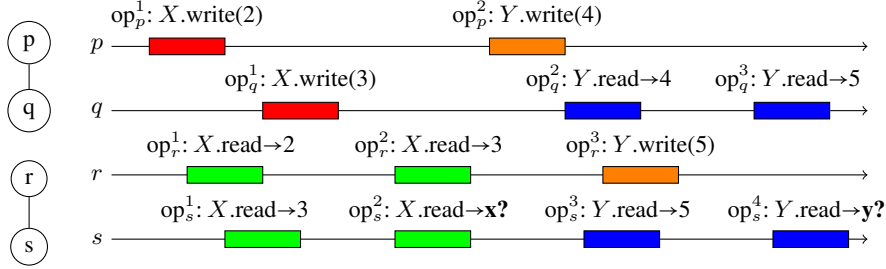
where  $(\overset{\star}{\sim}_{H,\mathcal{G}})|(\{p, q\} \cap W)$  is the restriction of  $\overset{\star}{\sim}_{H,\mathcal{G}}$  to the write operations of  $p$  and  $q$ ; and

- for each process  $p_i$  there is a history  $\widehat{S}_i$  that
  - (a) is sequential and legal;
  - (b) is equivalent to  $\widehat{H}|(p_i + W)$ ; and
  - (c) respects  $\overset{\star}{\sim}_{H,\mathcal{G}}$ , i.e.,  $(\overset{\star}{\sim}_{H,\mathcal{G}})|(p_i + W) \subseteq (\rightarrow_{S_i})$ .

If we apply this definition to the example of Figure 4 with the proximity graph proposed in Figure 5 we obtain the following: because *paris* and *berlin* are connected in  $\mathcal{G}$ ,  $X.write(1)$  by *paris* and  $X.write(2)$  by *berlin* must be totally ordered in  $\overset{\star}{\sim}_{H,\mathcal{G}}$  (and hence in any sequential history  $\widehat{S}_i$  perceived by any process  $p_i$ ).  $X.write(3)$  by *new-york* must be ordered after the writes on  $X$  by *paris* and *berlin* because of the causality imposed by  $\sim_H$ . As a result, if the system is  $\mathcal{G}$ -fisheye (SC,CC)-consistent, **b?** can be equal to 2 or 3, but not to 1. This set of possible values is as in sequential consistency, with the difference that  $\mathcal{G}$ -fisheye (SC,CC)-consistency does not impose any total order on the operation of *new-york*.

Given a system of  $n$  processes, let  $\emptyset_{\Pi} = (\Pi, \emptyset)$  denote the graph with no edges, and  $K_{\Pi}$  denote the complete graph  $(\Pi, \Pi \times \Pi)$ . It is easy to see that CC is  $\emptyset_{\Pi}$ -fisheye (SC,CC)-consistency. Similarly SC is  $K_{\Pi}$ -fisheye (SC,CC)-consistency.

*A larger example* Figure 6 and Table 1 illustrate the semantic of  $\mathcal{G}$ -fisheye (SC,CC) consistency on a second, larger, example. In this example, the processes  $p$  and  $q$  on one hand, and  $r$  and  $s$  on the other hand, are neighbors in the proximity graph  $\mathcal{G}$  (shown on the left). There are two pairs of write operations:  $op_p^1$  and  $op_q^1$  on the register  $X$ , and  $op_p^2$  and  $op_r^3$  on the register  $Y$ . In a sequentially consistency history, both pairs of writes must be seen in the same order by all processes. As a consequence, if  $r$  sees the value 2 and then 3 for  $X$ ,  $s$  must do the same, and only 3 can be returned by **x?**. For the same reason, only 3 can be returned by **y?**, as shown in the first line of Table 1.



**Fig. 6.** Illustrating  $\mathcal{G}$ -fisheye (SC,CC)-consistency

**Table 1.** Possible executions for the history of Figure 6

Consistency	$x?$	$y?$
Sequential Consistency	3	5
Causal Consistency	{2,3}	{4,5}
$\mathcal{G}$ -fisheye (SC,CC)-consistency	3	{4,5}

In a causally consistent history, however, both pairs of writes ( $\{op_p^1, op_q^1\}$  and  $\{op_p^2, op_r^3\}$ ) are causally independent. As a result, any two processes can see each pair in different orders.  $x?$  may return 2 or 3, and  $y?$  4 or 5 (second line of Table 1).

$\mathcal{G}$ -fisheye (SC,CC)-consistency provides intermediate guarantees: because  $p$  and  $q$  are neighbors in  $\mathcal{G}$ ,  $op_p^1$  and  $op_q^1$  must be observed in the same order by all processes.  $x?$  must return 3, as in a sequentially consistent history. However, because  $p$  and  $r$  are not connected in  $\mathcal{G}$ ,  $op_p^2$  and  $op_r^3$  may be seen in different orders by different processes (as in a causally consistent history), and  $y?$  may return 4 or 5 (last line of Table 1).

## 4 Construction of an Underlying (SC,CC)-Broadcast Operation

Our implementation of  $\mathcal{G}$ -fisheye (SC,CC)-consistency uses a broadcast operation with hybrid ordering guarantees. We present here this hybrid broadcast, before moving on to the actual implementation of  $\mathcal{G}$ -fisheye (SC,CC)-consistency in Section 5.

### 4.1 $\mathcal{G}$ -fisheye (SC,CC)-broadcast: definition

The hybrid broadcast we proposed, denoted  $\mathcal{G}$ -(SC,CC)-broadcast, is parametrized by a proximity graph  $\mathcal{G}$  which determines which kind of delivery order should be applied to which messages, according to the position of the sender in the graph  $\mathcal{G}$ . Messages (SC,CC)-broadcast by neighbors in  $\mathcal{G}$  must be delivered in the same order at all the processes, while the delivery of the other messages only need to respect causal order.

The (SC,CC)-broadcast abstraction provides the processes with two operations, denoted `TOCO_broadcast()` and `TOCO_deliver()`. We say that messages are toco-broadcast and toco-delivered.

*Causal message order* Let  $M$  be the set of messages that are toco-broadcast. The causal message delivery order, denoted  $\rightsquigarrow_M$ , is defined as follows [11,34]. Let  $m_1, m_2 \in M$ ;  $m_1 \rightsquigarrow_M m_2$ , iff one of the following conditions holds:

- $m_1$  and  $m_2$  have been toco-broadcast by the same process, with  $m_1$  first;
- $m_1$  was toco-delivered by a process  $p_i$  before this process toco-broadcast  $m_2$ ;
- There exists a message  $m$  such that  $(m_1 \rightsquigarrow_M m) \wedge (m \rightsquigarrow_M m_2)$ .

*Definition of the  $\mathcal{G}$ -fisheye (SC,CC)-broadcast* The (SC,CC)-broadcast abstraction is defined by the following properties.

**Validity.** If a process toco-delivers a message  $m$ , this message was toco-broadcast by some process. (No spurious message.)

**Integrity.** A message is toco-delivered at most once. (No duplication.)

**$\mathcal{G}$ -delivery order.** For all the processes  $p$  and  $q$  such that  $(p, q)$  is an edge of  $\mathcal{G}$ , and for all the messages  $m_p$  and  $m_q$  such that  $m_p$  was toco-broadcast by  $p$  and  $m_q$  was toco-broadcast by  $q$ , if a process toco-delivers  $m_p$  before  $m_q$ , no process toco-delivers  $m_q$  before  $m_p$ .

**Causal order.** If  $m_1 \rightsquigarrow_M m_2$ , no process toco-delivers  $m_2$  before  $m_1$ .

**Termination.** If a process toco-broadcasts a message  $m$ , this message is toco-delivered by all processes.

It is easy to see that if  $\mathcal{G}$  has no edges, this definition boils down to causal delivery, and if  $\mathcal{G}$  is fully connected (clique), this definition specifies total order delivery respecting causal order. Finally, if  $\mathcal{G}$  is fully connected and we suppress the “causal order” property, the definition boils down to total order delivery.

## 4.2 $\mathcal{G}$ -fisheye (SC,CC)-broadcast: algorithm

*Local variables* To implement the  $\mathcal{G}$ -fisheye (SC,CC)-broadcast abstraction, each process  $p_i$  manages three local variables.

- $causal_i[1..n]$  is a local vector clock used to ensure a causal delivery order of the messages;  $causal_i[j]$  is the sequence number of the next message that  $p_i$  will toco-deliver from  $p_j$ .
- $total_i[1..n]$  is a vector of logical clocks such that  $total_i[i]$  is the local logical clock of  $p_i$  (Lamport’s clock), and  $total_i[j]$  is the value of  $total_j[j]$  as known by  $p_i$ .
- $pending_i$  is a set of messages received but not yet toco-delivered by  $p_i$ .

*Description of the algorithm* Let us remind that for simplicity, we assume that the channels are FIFO. Algorithm 1 describes the behavior of a process  $p_i$ . This behavior is decomposed into four parts.

The first part (lines 1-6) is the code of the operation `TOCO_broadcast( $m$ )`. Process  $p_i$  first increases its local clock  $total_i[i]$  and sends the protocol message `TOCOBC( $m, \langle causal_i[\cdot], total_i[i], i \rangle$ )` to each other process. In addition to the application message  $m$ , this protocol message carries the control information needed to ensure the correct toco-delivery of  $m$ , namely, the local causality vector ( $causal_i[1..n]$ ), and

---

**Algorithm 1** The  $\mathcal{G}$ -fisheye (SC,CC)-broadcast algorithm executed by  $p_i$

---

```

1: operation TOCO_broadcast( $m$ )
2:    $total_i[i] \leftarrow total_i[i] + 1$ 
3:   for all  $p_j \in II \setminus \{p_i\}$  do send TOCOBC( $m, \langle causal_i[\cdot], total_i[i], i \rangle$ ) to  $p_j$ 
4:    $pending_i \leftarrow pending_i \cup \langle m, \langle causal_i[\cdot], total_i[i], i \rangle \rangle$ 
5:    $causal_i[i] \leftarrow causal_i[i] + 1$ 
6: end operation
7: on receiving TOCOBC( $m, \langle s\_caus_j^m[\cdot], s\_tot_j^m, j \rangle$ )
8:    $pending_i \leftarrow pending_i \cup \langle m, \langle s\_caus_j^m[\cdot], s\_tot_j^m, j \rangle \rangle$ 
9:    $total_i[j] \leftarrow s\_tot_j^m$   $\triangleright$  Last message from  $p_j$  had timestamp  $s\_tot_j^m$ 
10:  if  $total_i[i] \leq s\_tot_j^m$  then
11:     $total_i[i] \leftarrow s\_tot_j^m + 1$   $\triangleright$  Ensuring global logical clocks
12:    for all  $p_k \in II \setminus \{p_i\}$  do send CATCH_UP( $total_i[i], i$ ) to  $p_k$ 
13:  end if
14: end on receiving
15: on receiving CATCH_UP( $last\_date_j, j$ )
16:    $total_i[j] \leftarrow last\_date_j$ 
17: end on receiving
18: background task  $T$  is
19:  loop forever
20:    wait until  $C \neq \emptyset$  where
21:     $C \equiv \{ \langle m, \langle s\_caus_j^m[\cdot], s\_tot_j^m, j \rangle \rangle \in pending_i \mid s\_caus_j^m[\cdot] \leq causal_i[\cdot] \}$ 
22:    wait until  $T_1 \neq \emptyset$  where
23:     $T_1 \equiv \{ \langle m, \langle s\_caus_j^m[\cdot], s\_tot_j^m, j \rangle \rangle \in C \mid \forall p_k \in N_G(p_j) : \langle total_i[k], k \rangle > \langle s\_tot_j^m, j \rangle \}$ 
24:    wait until  $T_2 \neq \emptyset$  where
25:     $T_2 \equiv \left\{ \langle m, \langle s\_caus_j^m[\cdot], s\_tot_j^m, j \rangle \rangle \in T_1 \mid \begin{array}{l} \forall p_k \in N_G(p_j), \\ \forall \langle m_k, \langle s\_caus_k^{m_k}[\cdot], s\_tot_k^{m_k}, k \rangle \rangle \\ \in pending_i : \\ \langle s\_tot_k^{m_k}, k \rangle > \langle s\_tot_j^m, j \rangle \end{array} \right\}$ 
26:     $\langle m_0, \langle s\_caus_{j_0}^{m_0}[\cdot], s\_tot_{j_0}^{m_0}, j_0 \rangle \rangle \leftarrow \arg \min_{\langle m, \langle s\_caus_j^m[\cdot], s\_tot_j^m, j \rangle \rangle \in T_2} \{ \langle s\_tot_j^m, j \rangle \}$ 
27:     $pending_i \leftarrow pending_i \setminus \langle m_0, \langle s\_caus_{j_0}^{m_0}[\cdot], s\_tot_{j_0}^{m_0}, j_0 \rangle \rangle$ 
28:    TOCO_deliver( $m_0$ ) to application layer
29:    if  $j_0 \neq i$  then  $causal_i[j_0] \leftarrow causal_i[j_0] + 1$  end if  $\triangleright$  for  $causal_i[i]$  see line 5
30:  end loop forever
31: end background task

```

---

the value of the local clock ( $total_i[i]$ ). Then, this protocol message is added to the set  $pending_i$  and  $causal_i[i]$  is increased by 1 (this captures the fact that the future application messages toco-broadcast by  $p_i$  will causally depend on  $m$ ).

The second part (lines 7-14) is the code executed by  $p_i$  when it receives a protocol message TOCOBC( $m, \langle s\_caus_j^m[\cdot], s\_tot_j^m, j \rangle$ ) from  $p_j$ . When this occurs  $p_i$  adds first this protocol message to  $pending_i$ , and updates its view of the local clock of  $p_j$  ( $total_i[j]$ ) to the sending date of the protocol message (namely,  $s\_tot_j^m$ ). Then, if the local clock of  $p_i$  is late ( $total_i[i] \leq s\_tot_j^m$ ),  $p_i$  catches up (line 11), and informs the

other processes of it (line 12).

The third part (lines 15-17) is the processing of a catch up message from a process  $p_j$ . In this case,  $p_i$  updates its view of  $p_j$ 's local clock to the date carried by the catch up message. Let us notice that, as channels are FIFO, a view  $stotal_i[j]$  can only increase.

The final part (lines 18-31) is a background task executed by  $p_i$ , where the application messages are toco-delivered. The set  $C$  contains the protocol messages that were received, have not yet been toco-delivered, and are “minimal” with respect to the causality relation  $\sim_M$ . This minimality is determined from the vector clock  $s\_caus_j^m[1..n]$ , and the current value of  $p_i$ 's vector clock ( $causal_i[1..n]$ ). If only causal consistency was considered, the messages in  $C$  could be delivered.

Then,  $p_i$  extracts from  $C$  the messages that can be toco-delivered. Those are usually called *stable* messages. The notion of stability refers here to the delivery constraint imposed by the proximity graph  $\mathcal{G}$ . More precisely, a set  $T_1$  is first computed, which contains the messages of  $C$  that (thanks to the FIFO channels and the catch up messages) cannot be made unstable (with respect to the total delivery order defined by  $\mathcal{G}$ ) by messages that  $p_i$  will receive in the future. Then the set  $T_2$  is computed, which is the subset of  $T_1$  such that no message received, and not yet toco-delivered, could make incorrect – w.r.t.  $\mathcal{G}$  – the toco-delivery of a message of  $T_2$ .

Once a non-empty set  $T_2$  has been computed,  $p_i$  extracts the message  $m$  whose timestamp  $\langle s\_tot_j^m[j], j \rangle$  is “minimal” with respect to the timestamp-based total order ( $p_j$  is the sender of  $m$ ). This message is then removed from  $pending_i$  and toco-delivered. Finally, if  $j \neq i$ ,  $causal_i[j]$  is increased to take into account this toco-delivery (all the messages  $m'$  toco-broadcast by  $p_i$  in the future will be such that  $m \sim m'$ , and this is encoded in  $causal_i[j]$ ). If  $j = i$ , this causality update was done at line 5.

**Theorem 1.** *Algorithm 1 implements a  $\mathcal{G}$ -fisheye (SC,CC)-broadcast.*

The proof, provided in the appendix relies on the monotonicity of the clocks  $causal_i[1..n]$  and  $total_i[1..n]$ , and the reliability and FIFO properties of the underlying communication channels [7,12,23,34].

## 5 An Algorithm Implementing $\mathcal{G}$ -Fisheye (SC,CC)-Consistency

### 5.1 The high level object operations read and write

Algorithm 2 uses the  $\mathcal{G}$ -fisheye (SC,CC)-broadcast we have just presented to realize  $\mathcal{G}$ -fisheye (SC,CC)-consistency using a fast-read strategy. This algorithm is derived from the fast-read algorithm for sequential consistency proposed by Attiya and Welch [7], in which the total order broadcast has been replaced by our  $\mathcal{G}$ -fisheye (SC,CC)-broadcast.

The  $write(X, v)$  operation uses the  $\mathcal{G}$ -fisheye (SC,CC)-broadcast to propagate the new value of the variable  $X$ . To insure any other write operations that must be seen *before*  $write(X, v)$  by  $p_i$  are properly processed,  $p_i$  enters a waiting loop (line 4), which ends after the message  $WRITE(X, v, i)$  that has been toco-broadcast at line 2 is toco-delivered at line 11.

The  $read(X)$  operation simply returns the local copy  $v_x$  of  $X$ . These local copies are updated in the background when  $WRITE(X, v, j)$  messages are toco-delivered.

---

**Algorithm 2** Implementing  $\mathcal{G}$ -fisheye (SC,CC)-consistency, executed by  $p_i$ 

---

```
1: operation  $X.write(v)$ 
2:   TOCO_broadcast(WRITE( $X, v, i$ ))
3:    $delivered_i \leftarrow false$ ;
4:   wait until  $delivered_i = true$ 
5: end operation
6: operation  $X.read()$ 
7:   return  $v_x$ 
8: end operation
9: on toco_deliver WRITE( $X, v, j$ )
10:   $v_x \leftarrow v$ ;
11:  if ( $i = j$ ) then  $delivered_i \leftarrow true$  endif
12: end on toco_deliver
```

---

**Theorem 2.** *Algorithm 2 implements  $\mathcal{G}$ -fisheye (SC,CC)-consistency.*

The proof (cf. appendix) uses the causal order on messages  $\sim_M$  provided by the  $\mathcal{G}$ -fisheye (SC,CC)-broadcast to construct the causal order on operations  $\sim_H$ . It then gradually extends  $\sim_H$  to obtain  $\overset{\star}{\sim}_{H,G}$  by adapting the technique used in [28,32].

## 6 Conclusion

This work was motivated by the increasing popularity of geographically distributed systems. We have presented a framework that enables to formally define and reason about mixed consistency conditions in which the operations invoked by nearby processes obey stronger consistency requirements than operations invoked by remote ones. The framework is based on the concept of a proximity graph, which captures the “closeness” relationship between processes. As an example, we have formally defined  $\mathcal{G}$ -fisheye (SC,CC)-consistency, which combines sequential consistency for operations by close processes with causal consistency among all operations. We have also provided a formally proven protocol for implementing  $\mathcal{G}$ -fisheye (SC,CC)-consistency.

Another natural example that has been omitted from this paper for brevity is  $\mathcal{G}$ -fisheye (LIN,SC)-consistency, which combines linearizability for operations by nearby nodes with an overall sequential consistency guarantee.

The significance of our approach is that the definitions of consistency conditions are functional rather than operational. That is, they are independent of a specific implementation, and provide a clear rigorous understanding of the provided semantics. This formal underpinning comes with improved complexity and performance, as illustrated in our implementation of  $\mathcal{G}$ -fisheye (SC,CC)-consistency, in which operations can terminate without waiting to synchronize with remote parts of the system.

More generally, we expect the general philosophy we have presented to extend to Convergent Replicated Datatypes (CRDT) in which not all operations are commutative [29]. These CRDTs usually require at a minimum causal communications to implement eventual consistency. The hybridization we have proposed opens up the path of CRDTs

which are globally eventually consistent, and locally sequentially consistent, a route we plan to explore in future work.

## Acknowledgments

This work was partially funded by the French ANR project SocioPlug (ANR-13-INFR-0003), and by the DeScENt project (Labex CominLabs excellence laboratory ANR-10-LABX-07-01).

## References

1. Adve S. and Gharachorloo K., Shared memory consistency models: a tutorial. *IEEE Comp. Mag.*, 29(12):66-76, 1996.
2. Ahamad M., Niger G., Burns J.E., Hut to P.W., and Kohl P., Causal memory: definitions, implementation and programming. *Dist. Computing*, 9:37-49, 1995.
3. Almeida S., Leitaõ J., Rodrigues L., ChainReaction: a Causal+ Consistent Datastore based on Chain Replication. *8th ACM Europ. Conf. on Comp. Sys. (EuroSys'13)*, pp. 85-98, 2013.
4. Alvaro P., Bailis P., Conway N., and Hellerstein J. M., Consistency without borders *4th ACM Symp. on Cloud Computing (SOCC '13)*, 2013, 23
5. Attiya H. and Friedman R., A correctness condition for high-performance multiprocessors. *SIAM Journal on Computing*, 27(6):1637-1670, 1998.
6. Attiya H. and Friedman R., Limitations of Fast Consistency Conditions for Distributed Shared Memories. *Information Processing Letters*, 57(5):243-248, 1996.
7. Attiya H. and Welch J.L., Sequential consistency versus linearizability. *ACM Trans. on Comp. Sys.*, 12(2):91-12, 1994.
8. Attiya H. and Welch J.L., *Distributed computing: fundamentals, simulations and advanced topics, (2nd Edition)*, Wiley-Inter science, 414 pages, 2004 (ISBN 0-471-45324-2).
9. Bailis P., Ghodsi A., Hellerstein J. M., and Stoica I., Bolt-on Causal Consistency *2013 ACM SIGMOD Int. Conf. on Management of Data (SIGMOD'13)*, pp. 761-772, 2013.
10. Birman K.P. and Friedman R., Trading Consistency for Availability in Distributed Systems. *Technical Report #TR96-1579*, Computer Science Department, Cornell University, April 2016.
11. Birman K.P. and Joseph T.A., Reliable communication in presence of failures. *ACM Trans. on Comp. Sys.*, 5(1):47-76, 1987.
12. Birman K., Schiper A., and Stephenson P., Lightweight Causal and Atomic Group Multicast *ACM Trans. Comput. Syst.*, vol. 9, pp. 272-314, 1991.
13. Brewer E., Towards Robust Towards Robust Distributed Systems *19th ACM Symposium on Principles of Distributed Computing (PODC)*, Invited talk, 2000.
14. Burckhardt S., Gotsman A., Yang H., and Zawirski M., Replicated Data Types: Specification, Verification, Optimality *41<sup>st</sup> ACM Symp. Princ. of Prog. Lang. (POPL)*, pp. 271-284, 2014.
15. DeCandia G., Hastorun D., Jampani M., Kakulapati G., Lakshman A., Pilchin A., Sivasubramanian S., Vosshall P., and Vogels W., Dynamo: amazon's highly available key-value store *21st ACM Symp. on Op. Sys. Principles (SOSP'07)*, pp. 205-220, 2007
16. Friedman R., Implementing Hybrid Consistency with High-Level Synchronization Operations. *Distributed Computing*, 9(3):119-129, 1995.
17. Garg V.K. and Raynal M., Normality: a consistency condition for concurrent objects. *Parallel Processing Letters*, 9(1):123-134, 1999.
18. Gharachorloo K., Lenoski D., Laudon J., Gibbons P., Gupta A., and Hennessy J., Memory consistency and event ordering in scalable shared-memory multiprocessors. *17th ACM Annual International Symp. on Comp. Arch. (ISCA)*, pp. 15-26, 1990.

19. Herlihy M. and Shavit N., *The Art of Multiprocessor Programming*, Morgan Kaufmann Publishers Inc., 508 pages, 2008 (ISBN 978-0-12-370591-4).
20. Herlihy M. and Wing J., Linearizability: a correctness condition for concurrent objects. *ACM Transactions on Programming Languages and Systems*, 12(3):463–492, 1990.
21. Keleher P. Cox A.L., and Zwaenepoel W., Lazy release consistency for software distributed shared memory. *Proc. 19th ACM Int'l Symp. on Comp. Arch. (ISCA '92)*, pages 13–21, 1992.
22. Lakshman A., and Malik P., Cassandra: a decentralized structured storage system. *SIGOPS Oper. Syst. Rev.*, volume 44, pp. 35-40, 201.
23. Lamport L., Time, Clocks and the Ordering of Events in a Distributed System *Comm. of the ACM*, vol. 21, pp. 558-565, 1978
24. Lamport L., How to make a multiprocessor computer that correctly executes multiprocess programs. *IEEE Trans. on Comp.*, C28(9):690–691, 1979.
25. PRAM: A Scalable Shared Memory. *Technical Report CS-TR-180-88*, Princeton University, September 1988.
26. Lloyd W., Freedman M. J., Kaminsky M., and Andersen D. G., Don't Settle for Eventual: Scalable Causal Consistency for Wide-area Storage with COPS. *23rd ACM Symp. on Op. Sys. Principles*, pp. 401-416, 2011.
27. Lynch N.A., *Distributed Algorithms*. Morgan Kaufman, San Francisco, 872 pages, 1996.
28. Mizuno M., Raynal M., and Zhou J. Z., Sequential Consistency in Distributed Systems. *Selec.Papers from the Int. W. on Theory & Prac. in Dist. Sys.*, Springer, pp. 224-241, 1995
29. Oster G., Urso P., Molli P., and Imine A., Data consistency for P2P collaborative editing. *20th Anniv. Conf. on Comp. Supported Cooperative Work*, ACM, pp. 259-268, 2006
30. Preguiça N.M., Marquês J.M., Shapiro M., and Letia M., A Commutative Replicated Data Type for Cooperative Editing. *Proc. 29th IEEE Int'l Conf. on Dist. Comp. Sys. (ICDCS'09)*, pp. 395–403, 2009.
31. Raynal M., *Concurrent Programming: Algorithms, Principles, and Foundations*, Springer, 515 pages, 2013, ISBN 978-3-642-32026-2.
32. Raynal M., *Distributed Algorithms for Message-passing Systems*, Springer, 500 pages, 2013, ISBN 978-3-642-38122-5.
33. Raynal M. and Schiper A., A suite of formal definitions for consistency criteria in distributed shared Memories. *9th Int'l IEEE Conf. on Parallel and Dist. Comp. Sys. (PDCS'96)*, pp. 125-131, 1996.
34. Raynal M., Schiper A., and Toueg S., The Causal Ordering Abstraction and a Simple Way to Implement. *Information Processing Letters*, 39(6):343-350, 1991.
35. Saito Y. and Shapiro M., Optimistic Replication. *it ACM Computing Survey*, 37(1):42-81, March 2005.
36. Shapiro M., Preguiça N.M., Baquero C., and Zawirski M., Convergent and Commutative Replicated Data Types. *Bulletin of the EATCS*, 104:67-88, 2011.
37. Sovran Y., Power R., Aguilera M. K., and Li J., Transactional Storage for Geo-Replicated Systems. *23rd ACM Symp. on Operating Sys. Princ. (SOSP)*, pp. 385-400, 2011.
38. Terry D. B., Prabhakaran V., Kotla R., Balakrishnan M., Aguilera M. K., and Abu-Libdeh H., Consistency-based Service Level Agreements for Cloud Storage, *24th ACM Symp. on Op. Sys. Principles (SOSP'13)*, pp. 309-324, 2013.
39. Terry D. B., Theimer M. M., Petersen K., Demers A. J., Spreitzer M. J., and Hauser C. H., Managing Update Conflicts in Bayou, a Weakly Connected Replicated Storage System *15th ACM Symp. on Op. Sys. Principles (SOSP'95)*, pp. 172-182, 1995.
40. Xie C., Su C., Kapritsos M., Wang Y., Yaghmazadeh N., Alvisi L., and Mahajan P., SALT: Combining ACID and BASE in a Distributed Database. *USENIX Op. Sys. Design & Imp. (OSDI)*, 2014.



## Appendix: Detail of Proofs

### Proof that Algorithm 1 implements a $\mathcal{G}$ -fisheye (SC,CC)-broadcast

We use the usual partial order on vector clocks:

$$C_1[\cdot] \leq C_2[\cdot] \text{ iff } \forall p_i \in \Pi : C_1[i] \leq C_2[i]$$

with its accompanying strict partial order:

$$C_1[\cdot] < C_2[\cdot] \text{ iff } C_1[\cdot] \leq C_2[\cdot] \wedge C_1[\cdot] \neq C_2[\cdot]$$

We use the lexicographic order on the scalar clocks  $\langle s_{tot_j}, j \rangle$ :

$$\langle s_{tot_j}, j \rangle < \langle s_{tot_i}, i \rangle \text{ iff } (s_{tot_j} < s_{tot_i}) \vee (s_{tot_j} = s_{tot_i} \wedge i < j)$$

We start by three useful lemmata on  $causal_i[\cdot]$  and  $total_i[\cdot]$ . These lemmata establish the traditional properties expected of logical and vector clocks.

**Lemma 1.** *The following holds on the clock values taken by  $causal_i[\cdot]$ :*

1. *The successive values taken by  $causal_i[\cdot]$  in Process  $p_i$  are monotonically increasing.*
2. *The sequence of  $causal_i[\cdot]$  values attached to TOCOBC messages sent out by Process  $p_i$  are strictly increasing.*

**Proof** Proposition 1 is derived from the fact that the two lines that modify  $causal_i[\cdot]$  (lines 5, and 29) only increase its value. Proposition 2 follows from Proposition 1 and the fact that line 5 insures successive TOCOBC messages cannot include identical  $causal_i[i]$  values.  $\square$ *Lemma 1*

**Lemma 2.** *The following holds on the clock values taken by  $total_i[\cdot]$ :*

1. *The successive values taken by  $total_i[i]$  in Process  $p_i$  are monotonically increasing.*
2. *The sequence of  $total_i[i]$  values included in TOCOBC and CATCH\_UP messages sent out by Process  $p_i$  are strictly increasing.*
3. *The successive values taken by  $total_i[\cdot]$  in Process  $p_i$  are monotonically increasing.*

**Proof** Proposition 1 is derived from the fact that the lines that modify  $total_i[i]$  (lines 2 and 11) only increase its value (in the case of line 11 because of the condition at line 10). Proposition 2 follows from Proposition 1, and the fact that lines 2 and 11 insures successive TOBOBC and CATCH\_UP messages cannot include identical  $total_i[i]$  values.

To prove Proposition 3, we first show that:

$$\forall j \neq i : \text{ the successive values taken by } total_i[j] \text{ in } p_i \text{ are monotonically increasing.} \quad (1)$$

For  $j \neq i$ ,  $total_i[j]$  can only be modified at lines 9 and 16, by values included in TOBOBC and CATCH\_UP messages, when these messages are received. Because the underlying channels are FIFO and reliable, Proposition 2 implies that the sequence of  $last\_date_j$  and  $s\_tot_j^m$  values received by  $p_i$  from  $p_j$  is also strictly increasing, which shows Equation (1).

From Equation (1) and Proposition 1, we conclude that the successive values taken by the vector  $total_i[\cdot]$  in  $p_i$  are monotonically increasing (Proposition 3).  $\square_{Lemma 2}$

**Lemma 3.** *Consider an execution of the protocol. The following invariant holds: for  $i \neq j$ , if  $m$  is a message sent from  $p_j$  to  $p_i$ , then at any point of  $p_i$ 's execution outside of lines 28-29,  $s\_caus_j^m[j] < causal_i[j]$  iff that  $m$  has been toco-delivered by  $p_i$ .*

**Proof** We first show that if  $m$  has been toco-delivered by  $p_i$ , then  $s\_caus_j^m[j] < causal_i[j]$ , outside of lines 28-29. This implication follows from the condition  $s\_caus_j^m[\cdot] \leq causal_i[\cdot]$  at line 21, and the increment at line 29.

We prove the reverse implication by induction on the protocol's execution by process  $p_i$ . When  $p_i$  is initialized  $causal_i[\cdot]$  is null:

$$causal_i^0[\cdot] = [0 \dots 0] \quad (2)$$

because the above is true of any process, with Lemma 2, we also have

$$s\_caus_j^m[\cdot] \geq [0 \dots 0] \quad (3)$$

for all message  $m$  that is toco-broadcast by Process  $p_j$ .

(2) and (3) imply that there are no messages sent by  $p_j$  so that  $s\_caus_j^m[j] < causal_i^0[j]$ , and the Lemma is thus true when  $p_i$  starts.

Let us now assume that the invariant holds at some point of the execution of  $p_i$ . The only step at which the invariant might become violated is when  $causal_i[j_0]$  is modified for  $j_0 \neq i$  at line 29. When this increment occurs, the condition  $s\_caus_{j_0}^m[j_0] < causal_i[j_0]$  of the lemma potentially becomes true for additional messages. We want to show that there is only one single additional message, and that this message is  $m_0$ , the message that has just been delivered at line 28, thus completing the induction, and proving the lemma.

For clarity's sake, let us denote  $causal_i^{\circ}[j_0]$  the value of  $causal_i[j_0]$  just before line 29, and  $causal_i^{\bullet}[j_0]$  the value just after. We have  $causal_i^{\bullet}[j_0] = causal_i^{\circ}[j_0] + 1$ .

We show that  $s\_caus_{j_0}^{m_0}[j_0] = causal_i^{\circ}[j_0]$ , where  $s\_caus_{j_0}^{m_0}[\cdot]$  is the causal timestamp of the message  $m_0$  delivered at line 28. Because  $m_0$  is selected at line 26, this implies that  $m_0 \in T_2 \subseteq T_1 \subseteq C$ . Because  $m_0 \in C$ , we have

$$s\_caus_{j_0}^{m_0}[\cdot] \leq causal_i^{\circ}[\cdot] \quad (4)$$

at line 21, and hence

$$s\_caus_{j_0}^{m_0}[j_0] \leq causal_i^{\circ}[j_0] \quad (5)$$

At line 21,  $m_0$  has not been yet delivered (otherwise it would not be in  $pending_i$ ). Using the contrapositive of our induction hypothesis, we have

$$s\_caus_{j_0}^{m_0}[j_0] \geq causal_i^{\circ}[j_0] \quad (6)$$

(5) and (6) yield

$$s\_caus_{j_0}^{m_0}[j_0] = causal_i^o[j_0] \quad (7)$$

Because of line 5,  $m_0$  is the only message toco-broadcast by  $P_{j_0}$  whose causal timestamp verifies (7). From this unicity and (7), we conclude that after  $causal_i[j_0]$  has been incremented at line 29, if a message  $m$  sent by  $P_{j_0}$  verifies  $s\_caus_{j_0}^m[j_0] < causal_i^o[j_0]$ , then

- either  $s\_caus_{j_0}^m[j_0] < causal_i^o[j_0] - 1 = causal_i^o[j_0]$ , and by induction assumption,  $m$  has already been delivered;
- or  $s\_caus_{j_0}^m[j_0] = causal_i^o[j_0] - 1 < causal_i^o[j_0]$ , and  $m = m_0$ , and  $m$  has just been delivered at line 28.

□ *Lemma 3*

### Termination

**Theorem 3.** *All messages toco-broadcast using Algorithm 1 are eventually toco-delivered by all processes in the system.*

**Proof** We show Termination by contradiction. Assume a process  $p_i$  toco-broadcasts a message  $m_i$  with timestamp  $\langle s\_caus_i^{m_i}[\cdot], s\_tot_i^{m_i}, i \rangle$ , and that  $m_i$  is never toco-delivered by  $p_j$ .

If  $i \neq j$ , because the underlying communication channels are reliable,  $p_j$  receives at some point the TOCOBC message containing  $m_i$  (line 7), after which we have

$$\langle m_i, \langle s\_caus_i^{m_i}[\cdot], s\_tot_i^{m_i}, i \rangle \rangle \in pending_j \quad (8)$$

If  $i = j$ ,  $m_i$  is inserted into  $pending_i$  immediately after being toco-broadcast (line 4), and (8) also holds.

$m_i$  might never be toco-delivered by  $p_j$  because it never meets the condition to be selected into the set  $C$  of  $p_j$  (noted  $C_j$  below) at line 21. We show by contradiction that this is not the case. First, and without loss of generality, we can choose  $m_i$  so that it has a minimal causal timestamp  $s\_caus_i^{m_i}[\cdot]$  among all the messages that  $j$  never toco-delivers (be it from  $p_i$  or from any other process). Minimality means here that

$$\forall m_x, p_j \text{ never delivers } m_x \Rightarrow \neg(s\_caus_x^{m_x} < s\_caus_i^{m_i}) \quad (9)$$

Let us now assume  $m_i$  is never selected into  $C_j$ , i.e., we always have

$$\neg(s\_caus_i^{m_i}[\cdot] \leq causal_j[\cdot]) \quad (10)$$

This means there is a process  $p_k$  so that

$$s\_caus_i^{m_i}[k] > causal_j[k] \quad (11)$$

If  $i = k$ , we can consider the message  $m'_i$  sent by  $i$  just before  $m_i$  (which exists since the above implies  $s\_caus_i^{m_i}[i] > 0$ ). We have  $s\_caus_i^{m'_i}[i] = s\_caus_i^{m_i}[i] - 1$ , and hence from (11) we have

$$s\_caus_i^{m'_i}[i] \geq causal_j[k] \quad (12)$$

Applying Lemma 3 to (12) implies that  $p_j$  never toco-delivers  $m'_i$  either, with  $s\_caus_i^{m'_i}[i] < s\_caus_i^{m_i}[i]$  (by way of Proposition 2 of Lemma 1), which contradicts (9).

If  $i \neq k$ , applying Lemma 3 to  $causal_i[\cdot]$  when  $p_i$  toco-broadcasts  $m_i$  at line 3, we find a message  $m_k$  sent by  $p_k$  with  $s\_caus_k^{m_k}[k] = s\_caus_i^{m_i}[k] - 1$  such that  $m_k$  was received by  $p_i$  before  $p_i$  toco-broadcast  $m_i$ . In other words,  $m_k$  belongs to the causal past of  $m_i$ , and because of the condition on  $C$  (line 21) and the increment at line 29, we have

$$s\_caus_k^{m_k}[\cdot] < s\_caus_i^{m_i}[\cdot] \quad (13)$$

As for the case  $i = k$ , (11) also implies

$$s\_caus_k^{m_k}[k] \geq causal_j[k] \quad (14)$$

which with Lemma 3 implies that  $p_j$  never delivers the message  $m_k$  from  $p_k$ , and with (13) contradicts  $m_i$ 's minimality (9).

We conclude that if a message  $m_i$  from  $p_i$  is never toco-delivered by  $p_j$ , after some point  $m_i$  remains indefinitely in  $C_j$

$$m_i \in C_j \quad (15)$$

Without loss of generality, we can now choose  $m_i$  with the smallest total order timestamp  $\langle s\_tot_i^{m_i}, i \rangle$  among all the messages never delivered by  $p_j$ . Since these timestamps are totally ordered, and no timestamp is allocated twice, there is only one unique such message.

We first note that because channels are reliable, all processes  $p_k \in N_G(p_i)$  eventually receive the TOCOBC protocol message of  $p_i$  that contains  $m_i$  (line 7 and following). Lines 10-11 together with the monotonicity of  $total_k[k]$  (Proposition 1 of Lemma 2), insure that at some point all processes  $p_k$  have a timestamp  $total_k[k]$  strictly larger than  $s\_tot_i^{m_i}$ :

$$\forall p_k \in N_G(p_i) : total_k[k] > s\_tot_i^{m_i} \quad (16)$$

Since all changes to  $total_k[k]$  are systematically rebroadcast to the rest of the system using TOCOBC or CATCHUP protocol messages (lines 2 and 11),  $p_j$  will eventually update  $total_j[k]$  with a value strictly higher than  $s\_tot_i^{m_i}$ . This update, together with the monotonicity of  $total_j[\cdot]$  (Proposition 3 of Lemma 2), implies that after some point:

$$\forall p_k \in N_G(p_i) : total_j[k] > s\_tot_i^{m_i} \quad (17)$$

and that  $m_i$  is selected in  $T_1^j$ . We now show by contradiction that  $m_i$  eventually progresses to  $T_2^j$ . Let us assume  $m_i$  never meets  $T_2^j$ 's condition. This means that every time  $T_2^j$  is evaluated we have:

$$\exists p_k \in N_G(p_i), \exists \langle m_k, \langle s\_caus_k^{m_k}[\cdot], s\_tot_k^{m_k}, k \rangle \rangle \in pending_j : \quad (18)$$

$$\langle s\_tot_k^{m_k}, k \rangle \leq \langle s\_tot_i^m, i \rangle$$

Note that there could be different  $p_k$  and  $m_k$  satisfying (18) in each loop of Task  $T$ . However, because  $N_G(p_i)$  is finite, the number of timestamps  $\langle s\_tot_k^{m_k}, k \rangle$  such that  $\langle s\_tot_k^{m_k}, k \rangle \leq \langle s\_tot_i^m, i \rangle$  is also finite. There is therefore one process  $p_{k_0}$  and one message  $m_{k_0}$  that appear infinitely often in the sequence of  $(p_k, m_k)$  that satisfy (18). Since  $m_{k_0}$  can only be inserted once into  $pending_j$ , this means  $m_{k_0}$  remains indefinitely into  $T_2^j$ , and hence  $pending_j$ , and is never delivered. (18) and the fact that  $i \neq k_0$  (because  $p_i \notin N_G(p_i)$ ) yields

$$\langle s\_tot_k^{m_{k_0}}, k_0 \rangle < \langle s\_tot_i^m, i \rangle \quad (19)$$

which contradicts our assumption that  $m_i$  has the smallest total order timestamps  $\langle s\_tot_i^m, i \rangle$  among all messages never delivered to  $p_j$ . We conclude that after some point  $m_i$  remains indefinitely into  $T_2^j$ .

$$m_i \in T_2^j \quad (20)$$

If we now assume  $m_i$  is never returned by  $\text{argmin}$  at line 26, we can repeat a similar argument on the finite number of timestamps smaller than  $\langle s\_tot_i^m, i \rangle$ , and the fact that once they have been removed from  $pending_j$  (line 27), messages are never inserted back, and find another message  $m_k$  with a strictly smaller timestamp than  $p_j$  that is never delivered. The existence of  $m_k$  contradicts again our assumption on the minimality of  $m_i$ 's timestamp  $\langle s\_tot_i^m, i \rangle$  among undelivered messages.

This shows that  $m_i$  is eventually delivered, and ends our proof by contradiction.

□*Theorem 3*

**Causal Order** We prove the causal order property by induction on the causal order relation  $\rightsquigarrow_M$ .

**Lemma 4.** Consider  $m_1$  and  $m_2$ , two messages toco-broadcast by Process  $p_i$ , with  $m_1$  toco-broadcast before  $m_2$ . If a process  $p_j$  toco-delivers  $m_2$ , then it must have toco-delivered  $m_1$  before  $m_2$ .

**Proof** We first consider the order in which the messages were inserted into  $pending_j$  (along with their causal timestamps  $s\_caus_i^{m_1, m_2}$ ). For  $i = j$ ,  $m_1$  was inserted before  $m_2$  at line 4 by assumption. For  $i \neq j$ , we note that if  $p_j$  delivers  $m_2$  at line 28, then  $m_2$  was received from  $p_i$  at line 7 at some earlier point. Because channels are FIFO, this also means

$$m_1 \text{ was received and added to } pending_j \text{ before } m_2 \text{ was.} \quad (21)$$

We now want to show that when  $m_2$  is delivered by  $p_j$ ,  $m_1$  is no longer in  $pending_j$ , which will show that  $m_1$  has been delivered before  $m_2$ . We use an argument by contradiction. Let us assume that

$$\langle m_1, \langle s\_caus_i^{m_1}, s\_tot_i^{m_1}, i \rangle \rangle \in pending_j \quad (22)$$

at the start of the iteration of Task  $T$  which delivers  $m_2$  to  $p_j$ . From Proposition 2 of Lemma 1, we have

$$s\_caus_i^{m_1} < s\_caus_i^{m_2} \quad (23)$$

which implies that  $m_1$  is selected into  $C$  along with  $m_2$  (line 21):

$$\langle m_1, \langle s\_caus_i^{m_1}, s\_tot_i^{m_1}, i \rangle \rangle \in C$$

Similarly, from Proposition 2 of Lemma 2 we have:

$$s\_tot_i^{m_1} < s\_tot_i^{m_2} \quad (24)$$

which implies that  $m_1$  must also belong to  $T_1$  and  $T_2$  (lines 23 and 25). (24) further implies that  $\langle s\_tot_i^{m_2}, i \rangle$  is not the minimal  $s\_tot$  timestamp of  $T_2$ , and therefore  $m_0 \neq m_2$  in this iteration of Task  $T$ . This contradicts our assumption that  $m_2$  was delivered in this iteration; shows that (22) must be false; and therefore with (21) that  $m_1$  was delivered before  $m_2$ .  $\square_{Lemma 4}$

**Lemma 5.** *Consider  $m_1$  and  $m_2$  so that  $m_1$  was toco-delivered by a process  $p_i$  before  $p_i$  toco-broadcasts  $m_2$ . If a process  $p_j$  toco-delivers  $m_2$ , then it must have toco-delivered  $m_1$  before  $m_2$ .*

**Proof** Let us note  $p_k$  the process that has toco-broadcast  $m_1$ . Because  $m_2$  is toco-broadcast by  $p_i$  after  $p_i$  toco-delivers  $m_1$  and increments  $causal_i[k]$  at line 29, we have, using Lemma 3 and Proposition 1 of Lemma 1:

$$s\_caus_k^{m_1}[k] < s\_caus_i^{m_2}[k] \quad (25)$$

Because of the condition on set  $C$  at line 21, when  $p_j$  toco-delivers  $m_2$  at line 28, we further have

$$s\_caus_i^{m_2}[\cdot] \leq causal_j[\cdot] \quad (26)$$

and hence using (25)

$$s\_caus_k^{m_1}[k] < s\_caus_i^{m_2}[k] \leq causal_j[k] \quad (27)$$

Applying Lemma 3 to (27), we conclude that  $p_j$  must have toco-delivered  $m_1$  when it delivers  $m_2$ .  $\square_{Lemma 5}$

**Theorem 4.** *Algorithm 1 respects causal order.*

**Proof** We finish the proof by induction on  $\sim_M$ . Let's consider three messages  $m_1, m_2, m_3$  such that

$$m_1 \sim_M m_3 \sim_M m_2 \quad (28)$$

and such that:

- if a process toco-delivers  $m_3$ , it must have toco-delivered  $m_1$ ;
- if a process toco-delivers  $m_2$ , it must have toco-delivered  $m_3$ ;

We want to show that if a process toco-delivers  $m_2$ , it must have toco-delivered  $m_1$ . This follows from the transitivity of temporal order. This result together with Lemmas 4 and 5 concludes the proof.  $\square_{Theorem 4}$

### **$\mathcal{G}$ -delivery order**

**Theorem 5.** *Algorithm 1 respects  $\mathcal{G}$ -delivery order.*

**Proof** Let's consider four processes  $p_l, p_h, p_i,$  and  $p_j$ .  $p_l$  and  $p_h$  are connected in  $\mathcal{G}$ .  $p_l$  has toco-broadcast a message  $m_l$ , and  $p_h$  has toco-broadcast a message  $m_h$ .  $p_i$  has toco-delivered  $m_l$  before  $m_h$ .  $p_j$  has toco-delivered  $m_h$ . We want to show that  $p_j$  has toco-delivered  $m_l$  before  $m_h$ .

We first show that:

$$\langle s\_tot_h^{m_h}, h \rangle > \langle s\_tot_l^{m_l}, l \rangle \quad (29)$$

We do so by considering the iteration of the background task  $T$  (lines 18-31) of  $p_i$  that toco-delivers  $m_l$ . Because  $p_h \in N_{\mathcal{G}}(p_l)$ , we have

$$\langle total_i[h], h \rangle > \langle s\_tot_l^{m_l}, l \rangle \quad (30)$$

at line 23.

If  $m_h$  has not been received by  $p_i$  yet, then because of Lemma 3.2, and because communication channels are FIFO and reliable, we have:

$$\langle s\_tot_h^{m_h}, l \rangle > \langle total_i[h], h \rangle \quad (31)$$

which with (30) yields (29).

If  $m_h$  has already been received by  $p_i$ , by assumption it has not been toco-delivered yet, and is therefore in  $pending_i$ . More precisely we have:

$$\langle m_h, \langle s\_caus_h^{m_h}[\cdot], s\_tot_h^{m_h}, h \rangle \rangle \in pending_i \quad (32)$$

which, with  $p_h \in N_{\mathcal{G}}(p_l)$ , and the fact that  $m_l$  is selected in  $T_2^i$  at line 25 also gives us (29).

We now want to show that  $p_j$  must have toco-delivered  $m_l$  before  $m_h$ . The reasoning is somewhat the symmetric of what we have done. We consider the iteration of the background task  $T$  of  $p_j$  that toco-delivers  $m_h$ . By the same reasoning as above we have

$$\langle total_j[l], l \rangle > \langle s\_tot_h^{m_h}, h \rangle \quad (33)$$

at line 23.

Because of Lemma 3.2, and because communication channels are FIFO and reliable, (33) and (29) imply that  $m_l$  has already been received by  $p_j$ . Because  $m_h$  is selected in  $T_2^j$  at line 25, (29) implies that  $m_h$  is no longer in  $pending_j$ , and so must have been toco-delivered by  $p_j$  earlier, which concludes the proof.  $\square_{Theorem 5}$

**Theorem 1.** *Algorithm 1 implements a  $\mathcal{G}$ -fisheye (SC,CC)-broadcast.*

### **Proof**

- Validity and Integrity follow from the validity and integrity of the underlying communication channels, and from how a message  $m_j$  is only inserted once into  $pending_i$  (at line 4 if  $i = j$ , at line 8 otherwise) and always removed from  $pending_i$  at line 27 before it is toco-delivered by  $p_i$  at line 28;

- $\mathcal{G}$ -delivery order follows from Theorem 5;
- Causal order follows from Theorem 4;
- Termination follows from Theorem 3.

□*Theorem 1*

### Proof that Algorithm 2 implements $\mathcal{G}$ -fisheye (SC,CC)-consistency

For readability, we denote in the following  $r_p(X, v)$  the read operation invoked by process  $p$  on object  $X$  that returns a value  $v$  ( $X.\text{read} \rightarrow v$ ), and  $w_p(X, v)$  the write operation of value  $v$  on object  $X$  invoked by process  $p$  ( $X.\text{write}(v)$ ). We may omit the name of the process when not needed.

Let us consider a history  $\widehat{H} = (H, \xrightarrow{po}_H)$  that captures an execution of Algorithm 2, i.e.,  $\xrightarrow{po}_H$  captures the sequence of operations in each process (process order,  $po$  for short). We construct the causal order  $\rightsquigarrow_H$  required by the definition of Section 3.2 in the following, classical, manner:

- We connect each read operation  $r_p(X, v) = X.\text{read} \rightarrow v$  invoked by process  $p$  (with  $v \neq \perp$ , the initial value) to the write operation  $w(X, v) = X.\text{write}(v)$  that generated the  $\text{WRITE}(X, v)$  message carrying the value  $v$  to  $p$  (line 10 in Algorithm 2). In other words, we add an edge  $\langle w(X, v) \xrightarrow{rf} r_p(X, v) \rangle$  to  $\xrightarrow{po}_H$  (with  $w$  and  $r_p$  as described above) for each read operation  $r_p(X, v) \in H$  that does not return the initial value  $\perp$ . We connect initial read operations  $r(X, \perp)$  to an  $\perp$  element that we add to  $H$ .

We call these additional relations *read-from links* (noted  $\xrightarrow{rf}$ ).

- We take  $\rightsquigarrow_H$  to be the transitive closure of the resulting relation.

$\rightsquigarrow_H$  is acyclic, as assuming otherwise would imply at least one of the  $\text{WRITE}(X, v)$  messages was received before it was sent.  $\rightsquigarrow_H$  is therefore an order. We now need to show  $\rightsquigarrow_H$  is a causal order in the sense of the definition of Section 2.3, i.e., that the result of each read operation  $r(X, v)$  is the value of the latest write  $w(X, v)$  that occurred before  $r(X, v)$  in  $\rightsquigarrow_H$  (said differently, that no read returns an overwritten value).

**Lemma 6.**  $\rightsquigarrow_H$  is a causal order.

**Proof** We show this by contradiction. We assume without loss of generality that all values written are distinct. Let us consider  $w_p(X, v)$  and  $r_q(X, v)$  so that  $w_p(X, v) \xrightarrow{rf} r_q(X, v)$ , which implies  $w_p(X, v) \rightsquigarrow_H r_q(X, v)$ . Let us assume there exists a second write operation  $w_r(X, v') \neq w_p(X, v)$  on the same object, so that

$$w_p(X, v) \rightsquigarrow_H w_r(X, v') \rightsquigarrow_H r_q(X, v) \quad (34)$$

(illustrated in Figure 7).  $w_p(X, v) \rightsquigarrow_H w_r(X, v')$  means we can find a sequence of operations  $op_i \in H$  so that

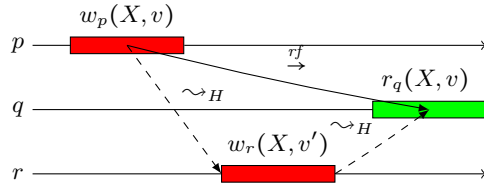
$$w_p(X, v) \rightarrow_0 op_0 \dots \rightarrow_i op_i \rightarrow_{i+1} \dots \rightarrow_k w_r(X, v') \quad (35)$$



with  $\rightarrow_i \in \{\overset{po}{\rightarrow}_H, \overset{rf}{\rightarrow}\}$ ,  $\forall i \in [1, k]$ . The semantics of  $\overset{po}{\rightarrow}_H$  and  $\overset{rf}{\rightarrow}$  means we can construct a sequence of causally related (SC,CC)-broadcast messages  $m_i \in M$  between the messages that are toco-broadcast by the operations  $w_p(X, v)$  and  $w_r(X, v')$ , which we note  $\text{WRITE}_p(X, v)$  and  $\text{WRITE}_r(X, v')$  respectively:

$$\text{WRITE}_p(X, v) = m_0 \rightsquigarrow_M m_1 \dots \rightsquigarrow_M m_i \rightsquigarrow_M \dots \rightsquigarrow_M m_{k'} = \text{WRITE}_r(X, v') \quad (36)$$

where  $\rightsquigarrow_M$  is the message causal order introduced in Section 4.1. We conclude that  $\text{WRITE}_p(X, v) \rightsquigarrow_M \text{WRITE}_r(X, v')$ , i.e., that  $\text{WRITE}_p(X, v)$  belongs to the causal past of  $\text{WRITE}_r(X, v')$ , and hence that  $q$  in Figure 7 toco-delivers  $\text{WRITE}_r(X, v')$  after  $\text{WRITE}_p(X, v)$ .



**Fig. 7.** Proving that  $\rightsquigarrow_H$  is causal by contradiction

We now want to show that  $\text{WRITE}_r(X, v')$  is toco-delivered by  $q$  before  $q$  executes  $r_q(X, v)$ . We can apply the same reasoning as above to  $w_r(X, v') \rightsquigarrow_H r_q(X, v)$ , yielding another sequence of operations  $op'_i \in H$ :

$$w_r(X, v') \rightarrow'_0 op'_0 \dots \rightarrow'_i op'_i \rightarrow'_{i+1} \dots \rightarrow'_{k''} r_q(X, v) \quad (37)$$

with  $\rightarrow'_i \in \{\overset{po}{\rightarrow}_H, \overset{rf}{\rightarrow}\}$ . Because  $r_q(X, v)$  does not generate any (SC,CC)-broadcast message, we need to distinguish the case where all  $op'_i$  relations correspond to the process order  $\overset{po}{\rightarrow}_H$  (i.e.,  $op'_i = \overset{po}{\rightarrow}_H, \forall i$ ). In this case,  $r = q$ , and the blocking behavior of  $X.\text{write}()$  (line 4 of Algorithm 2), insures that  $\text{WRITE}_r(X, v')$  is toco-delivered by  $q$  before executing  $r_q(X, v)$ . If at least one  $op'_i$  corresponds to the read-from relation, we can consider the latest one in the sequence, which will denote the toco-delivery of a  $\text{WRITE}_z(Y, w)$  message by  $q$ , with  $\text{WRITE}_r(X, v') \rightsquigarrow_M \text{WRITE}_z(Y, w)$ . From the causality of the (SC,CC)-broadcast, we also conclude that  $\text{WRITE}_r(X, v')$  is toco-delivered by  $q$  before executing  $r_q(X, v)$ .

Because  $q$  toco-delivers  $\text{WRITE}_p(X, v)$  before  $\text{WRITE}_r(X, v')$ , and toco-delivers  $\text{WRITE}_r(X, v')$  before it executes  $r_q(X, v)$ , we conclude that the value  $v$  of  $v_x$  is overwritten by  $v'$  at line 10 of Algorithm 2, and that  $r_q(X, v)$  does not return  $v$ , contradicting our assumption that  $w_p(X, v) \overset{rf}{\rightarrow} r_q(X, v)$ , and concluding our proof that  $\rightsquigarrow_H$  is a causal order.  $\square_{\text{Lemma 6}}$

To construct  $\overset{\star}{\rightsquigarrow}_{H, \mathcal{G}}$ , as required by the definition of (SC,CC)-consistency (Section 3.2), we need to order the write operations of neighboring processes in the proximity graph  $\mathcal{G}$ . We do so as follows:

- We add an edge  $w_p(X, v) \xrightarrow{ww} w_q(Y, w)$  to  $\sim_H$  for each pair of write operations  $w_p(X, v)$  and  $w_q(Y, w)$  in  $H$  such that:
  - $(p, q) \in E_G$  (i.e.,  $p$  and  $q$  are connected in  $\mathcal{G}$ );
  - $w_p(X, v)$  and  $w_q(Y, w)$  are not ordered in  $\sim_H$ ;
  - The broadcast message  $\text{WRITE}_p(X, v)$  of  $w_p(X, v)$  has been toco-delivered before the broadcast message  $\text{WRITE}_p(Y, w)$  of  $w_q(Y, w)$  by all processes.
 We call these additional edges *ww links* (noted  $\xrightarrow{ww}$ ).
- We take  $\overset{\star}{\sim}_{H, \mathcal{G}}$  to be the recursive closure of the relation we obtain.

$\overset{\star}{\sim}_{H, \mathcal{G}}$  is acyclic, as assuming otherwise would imply that the underlying (SC,CC)-broadcast violates causality. Because of the  $\mathcal{G}$ -delivery order and termination of the toco-broadcast (Section 4.1), we know all pairs of  $\text{WRITE}_p(X, v)$  and  $\text{WRITE}_p(Y, w)$  messages with  $(p, q) \in E_G$  as defined above are toco-delivered in the same order by all processes. This insures that all write operations of neighboring processes in  $\mathcal{G}$  are ordered in  $\overset{\star}{\sim}_{H, \mathcal{G}}$ .

We need to show that  $\overset{\star}{\sim}_{H, \mathcal{G}}$  remains a causal order, i.e., that no read in  $\overset{\star}{\sim}_{H, \mathcal{G}}$  returns an overwritten value.

**Lemma 7.**  $\overset{\star}{\sim}_{H, \mathcal{G}}$  is a causal order.

**Proof** We extend the original causal order  $\sim_M$  on the messages of an (SC,CC)-broadcast execution with the following order  $\overset{\mathcal{G}}{\sim}_M$ :

$m_1 \overset{\mathcal{G}}{\sim}_M m_2$  if

- $m_1 \sim_M m_2$ ; or
- $m_1$  was sent by  $p$ ,  $m_2$  by  $q$ ,  $(p, q) \in E_G$ , and  $m_1$  is toco-delivered before  $m_2$  by all processes; or
- there exists a message  $m_3$  so that  $m_1 \overset{\mathcal{G}}{\sim}_M m_3$  and  $m_3 \overset{\mathcal{G}}{\sim}_M m_2$ .

$\overset{\mathcal{G}}{\sim}_M$  captures the order imposed by an execution of an (SC,CC)-broadcast on its messages. The proof is then identical to that of Lemma 6, except that we use the order  $\overset{\mathcal{G}}{\sim}_M$ , instead of  $\sim_M$ .  $\square_{\text{Lemma 7}}$

**Theorem 2.** Algorithm 2 implements  $\mathcal{G}$ -fisheye (SC,CC)-consistency.

**Proof** The order  $\overset{\star}{\sim}_{H, \mathcal{G}}$  we have just constructed fulfills the conditions required by the definition of  $\mathcal{G}$ -fisheye (SC,CC)-consistency (Section 3.2):

- by construction  $\overset{\star}{\sim}_{H, \mathcal{G}}$  subsumes  $\sim_H$  ( $\sim_H \subseteq \overset{\star}{\sim}_{H, \mathcal{G}}$ );
- also by construction  $\overset{\star}{\sim}_{H, \mathcal{G}}$ , any pair of write operations invoked by processes  $p, q$  that are neighbors in  $\mathcal{G}$  are ordered in  $\overset{\star}{\sim}_{H, \mathcal{G}}$ ; i.e.,  $(\overset{\star}{\sim}_{H, \mathcal{G}})|(\{p, q\} \cap W)$  is a total order.

To finish the proof, we choose, for each process  $p_i$ ,  $\widehat{S}_i$  as one of the topological sorts of  $(\overset{\star}{\sim}_{H, \mathcal{G}})|(\{p_i\} \cap W)$ , following the approach of [28,32].  $\widehat{S}_i$  is sequential by construction. Because  $\overset{\star}{\sim}_{H, \mathcal{G}}$  is causal,  $\widehat{S}_i$  is legal. Because  $\overset{\star}{\sim}_{H, \mathcal{G}}$  respects  $\xrightarrow{p_o}_H$ ,  $\widehat{S}_i$  is equivalent to  $\widehat{H}|(\{p_i\} \cap W)$ . Finally,  $\widehat{S}_i$  respects  $(\overset{\star}{\sim}_{H, \mathcal{G}})|(\{p_i\} \cap W)$  by construction.  $\square_{\text{Theorem 2}}$