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Abstract

In this paper, we extend the notion of diagonally strictly concave functions and use it to provide a sufficient condition for uniqueness of Nash equilibrium in some concave games. We then provide an alternative proof of the existence and uniqueness of Nash equilibrium for a network resource allocation game arising from the so-called Kelly mechanism by verifying the new sufficient condition. We then establish that the equilibrium resulting from the differential pricing in the Kelly mechanism is related to a normalised Nash equilibrium of a game with coupled strategy space.

Index Terms

concave games, diagonal strict concavity, differential pricing, Kelly mechanism, Nash equilibrium, network resource allocation, normalised Nash equilibrium.

I. Introduction

Consider a game played by N players where each player has to choose a portion of a pie, or some divisible good. Player i chooses actions $a_i \in [0, 1]$. The actions are constrained to satisfy

$$\sum_{i=1}^{N} a_i \leq 1. \tag{1}$$

Player i gets utility $U_i(a_i)$ for his action a_i, and acts to maximise his utility, subject to the constraint in (1).

The above abstract game is widely applicable. The divisible good could stand for

(a) amount of research grant money with a_i being the portion of the grant money claimed by the ith participant;

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(b) a communication resource like bandwidth with a_i being fraction of time player i uses the channel;

(c) net interference *temperature* [1] with a_i standing for the fraction player i wants as his in order to ensure that he transmits at a desired high enough rate.

The utility function is typically concave and increasing in the action variable. The key feature of this simultaneous action game is that the actions are coupled by the constraint that they should lie in the set given by (1). Write $a = (a_i, 1 \leq i \leq N)$ or more simply $a = (a_i)$ for the action profile.

Ideally, a social planner who works in the interest of greater social good may wish to pick an allocation vector

$$a^* \in \arg\max \left\{ \sum_{i=1}^{N} U_i(a_i) \mid a \text{ satisfies } (1) \right\}. \tag{2}$$

However, players can be strategic and can act to maximise their individual utilities. A Nash equilibrium (NE) for the above game is an action profile satisfying the constraint (1) and such that no player can strictly increase his utility by means of a unilateral deviation within the constraint set. The set of Nash equilibria for the action-constrained game is (quite straightforwardly) found to be the set of all action profiles a such that (1) is satisfied with equality.

When the system is decentralised, the social planner may not know the players' utilities or the worth of a portion of the resource for each player. In this case, Kelly [2] proposed a decentralised mechanism in which each player submits a 'bid' or *willingness-to-pay*. Let $b_i \geq 0$ denote this bid submitted by player i. The social planner then decides the unit price μ and assigns to each player a portion of the resource that is in proportion to his bid and inversely proportional to the unit price: $a_i = b_i/\mu$. The social planner then collects a payment that equals the bid. Kelly [2] showed that when each player chooses a bid that maximises his net utility given by

$$U_i\left(\frac{b_i}{\mu}\right) - b_i \tag{3}$$

there exists a good choice of the unit price μ^* that will enable player i to choose $b_i^* = \mu^* a_i^*$, so that the share of player i is a_i^*, the ith component of the system optimal vector in (2). In this mechanism, the social planner does not *price differentiate* the players, and the players are assumed to be price takers, i.e., players do not anticipate the effect of their bids on the unit price μ.

In a series of works, Hajek and Gopalakrishnan [3], Johari et al. ([4], [5], [6]) considered an alternative model where the players are price anticipating rather than price taking, and compete to maximise their utility. The social planner then implements a mechanism (henceforth *Kelly mechanism*) that apportions the pie in the fraction of the bids, i.e., with $b = (b_i, 1 \leq i \leq N)$, the 'proportional' allocation is as follows:

$$a_i(b) = \frac{b_i}{\sum_{j=1}^{N} b_j}.$$

This is then a new simultaneous action game where each player chooses a bid b_i. The net utility of player i is

$$V_i(b) := U_i(a_i(b)) - b_i = U_i\left(\frac{b_i}{\sum_{j=1}^{N} b_j}\right) - b_i. \tag{4}$$

Under the assumption that each U_i is concave, strictly increasing, and continuously differentiable over $\mathbb{R}_+$, and the right directional derivative at 0 is finite, the resulting game is known to have a unique NE. Further, the price anticipating nature of the players may result in a *suboptimal Nash equilibrium*, i.e., $\sum_i U_i(\cdot)$ at the NE can be lower than the value at the optimum profile of (2). Indeed, Johari and Tsitsiklis [4] showed that the proportional allocation mechanism leads to an efficiency loss of upto 25% of the social optimum value. To close this efficiency gap, a *price differentiation* scheme was proposed in [7]. Price differentiation is introduced by replacing the negative term in (4) by b_i/r_i, where $1/r_i$ is the price differentiation factor for player i. The resulting mechanism will be called the *Kelly mechanism with price differentiation*. The price differentiation results in a NE which is related to a special type of equilibrium called *normalised Nash equilibrium* as we show later in the paper.

Let us return to the Kelly mechanism defined by utilities (4). Notice that the bids (or actions) in the decentralised mechanism are no longer coupled, but the utilities of the players are coupled. This is reminiscent of the special class of games with coupled utilities and decoupled actions sets dealt with in [8].

In another class of resource allocation problems called *routing games* players share a communication network to ship their demand (or traffic) from a source to a destination. The communication network consists of several interconnected links which are capacity constrained, and cost on each link depends on the total traffic on that link. As higher congestion implies higher delay or higher loss rate, the players prefer to use a link that is less congested. The action space of each player is constrained in these games as sum of flows across the links must equal its total demand. In [9], the authors studied the amount of traffic sent by each player on each link at equilibrium assuming that the players aim to minimise their total cost. They establish existence and uniqueness of NE in routing games under the assumption that the cost function of each player is convex in its flow and satisfies certain monotonicity properties. Noting that this game can be studied as a game where each player aims to maximise the negative of its cost function, this is again reminiscent of the special class of games with coupled utilities and decoupled actions sets dealt with in Rosen's work [8].

In [8], Rosen provided a general framework to study games where utility of each player is concave and the action (strategy) space is convex and compact. His framework includes competitions where not only utilities of the players are coupled, but also the action space of the players can be coupled, hence covering a rich class of concave games. When the action space of the players are coupled, a player is restricted to take only certain actions (a strict subset of his action space), given action profile of his opponents. To study the equilibrium behaviour of games in such generality, Rosen introduced the concept of normalised Nash equilibrium (NNE). He established the existence of NNE in these games, and further provided a sufficient condition called *diagonal strict concavity* for uniqueness of NNE. In Rosen's setting, NNE is same as the NE when the utilities of players are coupled but the strategy spaces are independent of each other, i.e., each player can take any action independent of his opponents. The problems studied by Hajek and Gopalakrishnan [3] and Johari et al. [4] fall within the setting considered by Rosen in [8].

Our work was motivated by the following question. Could one apply Rosen's result, with a suitable modification to handle noncompactness of the action spaces, and prove the uniqueness of the NE obtained by Hajek and Gopalakrishnan [3]? Could one provide a unified approach to establish uniqueness of NE in network games, in particular, resource allocation and routing games?

Study of uniqueness of NE is important in network games. Besides its theoretical interest, uniqueness of NE is of obvious importance in predicting network behaviour in equilibrium. Uniqueness of NE is also of particular importance for network management, where regulating player behaviour in a single equilibrium (using pricing, for example) is usually much easier than for several equilibria simultaneously. For a survey on network games with unique NE see [10].

Though unique NE in a game is favourable, in many games the NE may not be unique.

Indeed, there are games that possess infinitely many NE points; see [11]. This difficulty is overcome by requiring that payoff functions satisfy some properties. In [12], it is shown that if the game admits a *potential function*, then the Nash equilbria are given by the local optima of the potential function. Naturally, if the potential function has a unique optimum, it corresponds to the unique NE of the game. Another requirement that ensures uniqueness of NE is the *dominant diagonal property* in supermodular games [13]. Yet another test for uniqueness of NE in a game where action space of the players is bounded and continuous is to verify that the best response of each players is a *standard function* (see [14] and [15, Th. 2]). In this paper, we provide another sufficient condition for the existence of unique NE in concave games.

Our contributions are as follows:

- We provide a generalisation of *diagonal strict concavity* (DSC) property, and show that when it is satisfied, uniqueness of NE is guaranteed in concave games.
- We provide an example network resource allocation game where the proposed sufficient condition holds, but the DSC property is difficult to verify.
- We show that the NE in the Kelly mechanism with differential pricing is related to the NNE of another game with coupled action space.
- We provide a unified approach to establishing uniqueness of NE in resource allocation games using the Rosen's framework of concave games.

The paper is organised as follows. In Section II, we briefly introduce $N-$person concave games studied by Rosen [8] and discuss the DSC property. In Section III, we motivate the need to extend the definition the DSC property to establish uniqueness of NE by providing examples where DSC property is difficult to verify. In Section IV, we generalise the Rosen's work by providing a new sufficient condition (based on DSC) to establish uniqueness of NE, and show its application to the study of resource allocation problems. In Section VI, we make the connection between the Nash equilibrium in the Kelly mechanism having differential pricing with the normalised Nash equilibrium of another game with related but uncoupled utility functions and a coupled action set. We end the paper with a concise summary and a brief discussion of future work in Section VII.

II. Rosen's Uniqueness Theorem

In this section, we describe Rosen's result on the sufficiency of diagonal strict concavity for uniqueness of NE in a game with vector strategies. We first discuss the game where only the utilities are coupled and then discuss the case where the strategy spaces of the players are also coupled. We first set up some notation and state the assumptions.

Consider the following N-person concave game. We describe the constraint set in the next paragraph and the utility functions in the following paragraph.

Let $A_i = \{b_i \in \mathbb{R}^{m_i}, h_{ik}(b_i) \geq 0, k = 1, 2, \cdots, K_i\}$ denote bounded action set of player i, where $\mathbb{R}^{m_i}$ denotes the Euclidian plane of dimension m_i, and for $k = 1, 2, \cdots, K_i$, $h_{ik} : \mathbb{R}^{m_i} \to \mathbb{R}$ is a concave and continuously differentiable function on $\mathbb{R}^{m_i}$. Write, as before, $b := (b_i)$ for the action profile, where $b \in A = S := \prod_{i=1}^{N} A_i \subset \mathbb{R}^m$ with $m = \sum_{i=1}^{N} m_i$. In this case, the action space A is the rectangle S, and we say action set of players are *orthogonal*. The game is said to be *decoupled in the action set*. We denote the jth component of the action b_i as b_{ij}. More generally, we will also consider a *coupled constraint set* $A = \{b \in \mathbb{R}^m, h_j(x) \geq 0 \text{ for } j = 1, \ldots, K\}$ where K is a natural number and $h_j, j = 1, \ldots, K$ are concave and continuously differentiable functions on $\mathbb{R}^m$. Whether the action set is orthogonal or coupled, we will assume that there exists a $\overline{b} \in A$ that is strictly interior to every nonlinear constraint. This is a sufficient condition for the Kuhn-Tucker constraint qualification.

Consider a family of N coupled utility functions, where the ith utility function is $V_i : A \to \mathbb{R}$. $V_i(b)$ is assumed to be continuous in b and concave and continuously differentiable in b_i for a given $b_{-i} = (b_1, b_2, \cdots, b_{i-1}, b_{i+1}, \cdots, b_N)$. Write $V := (V_i)$ for the family of utility functions.

For any scalar function $\alpha(b)$ we denote the gradient with respect to b_i as $\nabla_i \alpha(b)$. Define a mapping $\sigma : \mathbb{R}^m \times \mathbb{R}_+^N \to \mathbb{R}$, where $\mathbb{R}_+$ denotes the set of positive real numbers, as the weighted sum of functions V as follows:

$$\sigma(b, r) = \sum_{i=1}^{N} r_i V_i(b), \quad r_i \geq 0 \quad \forall\ i. \tag{5}$$

The *pseudogradient* of $\sigma(b, r)$ for any given nonnegative r is defined as

$$g(b, r) = \begin{bmatrix} r_1 \nabla_1 V_1(b) \\ r_2 \nabla_2 V_2(b) \\ \vdots \\ r_N \nabla_N V_N(b) \end{bmatrix}. \tag{6}$$

Let $G(b, r)$ denote the Jacobian with respect to b of $g(b, r)$. Note that $G(b, r)$ is a matrix of dimension $m \times m$. We use the notation M^t to denote transpose of matrix M. For a vector

r, we say $r \geq 0$, respectively $r > 0$, if each component is nonnegative, respectively strictly positive.

Definition 1 (Rosen [8](p. 524)): The function $\sigma(\cdot, r)$ is called *diagonally strictly concave* (DSC) for a given $r \geq 0$ if for every distinct pair $b^0, b^1 \in A$ we have

$$\left(b^1 - b^0\right)^t \left(g(b^1, r) - g(b^0, r)\right) < 0. \tag{7}$$

When the players are interested in *minimising* their respective convex *cost* functions, the corresponding $\sigma(\cdot, r)$ is *diagonally strictly convex* if (7) holds with the opposite inequality.

A sufficient condition for the family V to be diagonally strictly concave (convex) for a given $r \geq 0$ is that the symmetric matrix

$$[G(b, r) + G^t(b, r)] \tag{8}$$

is negative (positive) definite over the domain A [8, Th. 6].

A. Equilibrium in Games with Decoupled Action Set

In this subsection we consider games with decoupled or orthogonal action sets, i.e., $A = S$. In a concave N-person game with decoupled action set, a point $b^0 \in A$ is said to be a Nash equilibrium (NE) if for every $i = 1, 2, \cdots, N$, we have

$$V_i(b^0) = \max_{b_i \in A_i} V_i(b_i, b^0_{-i}).$$

Rosen established the following result.

Theorem 1 (Rosen [8], Th.2): Assume that the constraint set is orthogonal, and that $\sigma(\cdot, r)$ is DSC for some $r > 0$. If an equilibrium point exists, then it is unique.

Rosen's uniqueness theorem [8, Th. 2] was stated for compact domains, for which he also established existence of a Nash equilibrium (Rosen [8, Th. 1]). But the proof works for any unbounded domain, provided an equilibrium point exists. Rosen's concept of equilibrium with coupled constraints is also known as variational equilibrium (see [16] and references therein). Rosen studies this concept under the assumption that all players have common constraints. (A more general framework for games with constraints that are not necessarily common is known as the generalised Nash equilibrium - GNE, see [17], [16], and references therein).

B. Equilibrium in Games with Coupled Constraint Set

Now consider the coupled constraint set $A = \{b \in \mathbb{R}^m, h_j(b) \geq 0, j = 1, 2, \cdots, K\}$. To study equilibrium in such concave games where both utilities and actions are coupled, Rosen introduced the concept of *normalised Nash equilibrium* (NNE). The NNE is a special kind of NE where the Lagrange multipliers[1] across players are interrelated. Formally, it is defined as follows.

Definition 2: Let b be a NE and let $(u_i^0) \geq 0$ be the associated Lagrange multipliers given by the Kuhn-Tucker conditions at the NE. If the (u_i^0) satisfy $u_i^0 = \lambda / r_i$ for some $(r_i) > 0$ and $\lambda \geq 0$, then b is called a normalised Nash equilibrium (NNE) for r.

Rosen established existence of NNE for every specified $r = (r_i) > 0$ in concave games with compact constraint sets. He also established a uniqueness result for the NNE, which we now state.

Theorem 2 (Rosen [8], Th.4): Let $r > 0$ and let $\sigma(\cdot, r)$ be diagonally strictly concave. If a NNE for r exists, then it is unique.

We will return to use of NNE later in subsection VI where we study NE in the Kelly mechanism with differential pricing. We first discuss the need for extending the DSC property.

We close this section with the reiteration that the DSC property is a sufficient condition to establish uniqueness of NE in a concave N-person game. Many problems in network games, such as Kelly's resource allocation problem [2], Tullock's rent seeking problem [18], and routing games [9] are concave games. However, the DSC property is either difficult to verify [19] or can be established only in some special cases [9, Sec. III.B]. We will see an example in the next section of a situation where the DSC property is yet to be verified, and yet, the NE is known to be unique. In Section V, we will show how the extension of the DSC property in Section IV applies to the example of Section III.

III. An example where DSC is unverified

In this section, we provide an example of a concave game for which the DSC property is not yet verified. We also demonstrate that Rosen's sufficient condition for the DSC property fails. This will then set the stage for an extension of the DSC property.

[1]The Lagrange multipliers are those associated the equilibrium point and satisfy the Kuhn-Tucker conditions.

Consider the resource allocation problem defined by utility (4). We first observe that $b_i \in \mathbb{R}_+$ for all i. By setting $K_i = 1, h_{i1}(b_i) = b_i, i = 1, \cdots, N$, this problem fall within the framework of concave games. Also note that the linear terms in the utility function do not affect the DSC property. So, we may focus on the modified family $V = (V_i)$ where

$$V_i(b) := U_i\left(\frac{b_i}{\sum_{j=1}^N b_j}\right) \tag{9}$$

without the linear term. Observe that

$$\frac{\partial V_i(b)}{\partial b_i} = U_i'\left(\frac{b_i}{\sum_j b_j}\right)\left(\frac{1 - b_i/\sum_j b_j}{\sum_j b_j}\right). \tag{10}$$

Consider the setting of two players; $N = 2$. For an $r > 0$, with V as in (9), $\sigma(\cdot, r)$ is DSC if for any pair $(b_1^1, b_2^1) \in \mathbb{R}_+^2$ and $(b_1^0, b_2^0) \in \mathbb{R}_+^2$, we have

$$\sum_i r_i\left(\beta_i^1\left(\sum_j b_j^1\right) - \beta_i^0\left(\sum_j b_j^0\right)\right)\left(\frac{(1-\beta_i^1)U_i'(\beta_i^1)}{\sum_j b_j^1} - \frac{(1-\beta_i^0)U_i'(\beta_i^0)}{\sum_j b_j^0}\right) < 0, \tag{11}$$

where $\beta_i^0 = b_i^0/(b_1^0 + b_2^0)$ and $\beta_i^1 = b_i^1/(b_1^1 + b_2^1)$ for $i = 1, 2$.

We have not been able to prove that there is an $r > 0$ such that (11) holds for distinct b^1 and b^0, even for the simple case when (U_i)s are identity maps, i.e.,

$$U_i\left(\frac{b_i}{\sum_j b_j}\right) = \frac{b_i}{\sum_j b_j}. \tag{12}$$

Further, we have not been able to prove the negation of the above statement holds, which would establish that $\sigma(\cdot, r)$ is not DSC for any $r > 0$.

The above are interesting open questions because, on the one hand, we know that there exists a unique (normalised) NE (for $r_i \equiv 1$), and yet the Jacobian-based sufficient condition for DSC fails, as we show next.

Proposition 1: Let $N = 2$ and consider the family $V = (V_i)$ with (U_i) as in (12). Then, for any $r > 0$, $[G(b, r) + G^t(b, r)]$ for the family V is not negative definite.

Proof: It suffices to consider $r \neq 0$. Fix such an r and, without loss of generality, assume $r_2 > 0$. The second order derivatives of V_1 and V_2 are

$$\begin{aligned}
\frac{\partial^2 V_1(b_1, b_2)}{\partial b_1^2} &= \frac{-2b_2}{(b_1+b_2)^3} \\
\frac{\partial^2 V_2(b_1, b_2)}{\partial b_2^2} &= \frac{-2b_1}{(b_1+b_2)^3} \\
\frac{\partial^2 V_1(b_1, b_2)}{\partial b_1 \partial b_2} &= \frac{b_1 - b_2}{(b_1+b_2)^3} = -\frac{\partial V_2(b_1, b_2)}{\partial b_2 \partial b_1},
\end{aligned}$$

and the symmetric matrix in (8) is given by

$$[G(b,r)+G^t(b,r)] = \frac{1}{(b_1+b_2)^3}\begin{bmatrix} -2r_1b_2 & (b_1-b_2)(r_1-r_2) \\ (b_1-b_2)(r_1-r_2) & -2r_2b_1 \end{bmatrix}.$$

Now consider a bid $b_1 > 0$ and $b_2 = 0$. We then have

$$[G(b,r)+G^t(b,r)] = \frac{r_2}{(b_1)^2}\begin{bmatrix} 0 & (a-1) \\ (a-1) & -2 \end{bmatrix},$$

where $a = r_1/r_2 \geq 0$. The eigenvalues of this matrix are $-1+\sqrt{1+(a-1)^2}$ and $-1-\sqrt{1+(a-1)^2}$. Clearly the first of these is nonnegative for all $a \geq 0$, and so $G[(b,r)+G^t(b,r)]$ is not negative definite on $\mathbb{R}^2_+$. ■

We next provide a generalisation of DSC property in the following section and show that uniqueness of NE is guaranteed if the new property holds. We verify the validity of the new sufficient condition in the resource allocation problems of (4).

IV. Generalised Diagonal Strict Concavity

In this section we generalise the notion of diagonally strictly concave functions. Its usefulness arises from its application to the resource allocation problem with utilities as in (4).

We assume that the action space of each player has the same dimension, i.e., $m_1 = m_2 = \cdots = m_N := \overline{m}$, and that each player's actions are nonnegative. The generalisation of the DSC property is as follows.

Definition 3: Let $T_j : \mathbb{R}_+^{\overline{m}N} \to \mathbb{R}_+$ be a nonnegative function for $j = 1,2,\cdots,\overline{m}$. The function $\sigma(\cdot, r)$ is *generalised diagonally strictly concave* (GDSC) for a given $r \geq 0$ if for every pair $b^0, b^1 \in \mathbb{R}_+^{\overline{m}N}$ such that

$$\left(b^0_{ij}/T_j(b^0), 1 \leq i \leq N, 1 \leq j \leq \overline{m}\right) \neq \left(b^1_{ij}/T_j(b^1), 1 \leq i \leq N, 1 \leq j \leq \overline{m}\right),$$

we have

$$\sum_i r_i \sum_j \left(\frac{b^1_{ij}}{T_j(b^1)} - \frac{b^0_{ij}}{T_j(b^0)}\right)\left(T_j(b^1)\frac{\partial V_i}{\partial b_{ij}}(b^1) - T_j(b^0)\frac{\partial V_i}{\partial b_{ij}}(b^0)\right) < 0. \tag{13}$$

For the case of $\overline{m} = 1$ when each player's action is a nonnegative scalar, writing T for T_1, the above condition simplifies to

$$\sum_i r_i \left(\frac{b^1_i}{T(b^1)} - \frac{b^0_i}{T(b^0)}\right)\left(T(b^1)\frac{\partial V_i}{\partial b_i}(b^1) - T(b^0)\frac{\partial V_i}{\partial b_i}(b^0)\right) < 0.$$

If all T_j were identically 1, we get back Rosen's definition of diagonally strictly concave functions given in (7). The generalisation here allows the functions T_j to be more general than the "identically 1" function. The generalisation is useful because we can leverage it to extract the following theorem.

The setting of the generalisation is one where the action sets are decoupled, $K_i = \overline{m}$ for each i, and $h_{ik}(b_i) = b_{ik}$, for all $k = 1, 2, \cdots, K_i$. Note that this setting covers the game with utilities (4).

Theorem 3: Assume that the family $\sigma(\cdot, r)$ is GDSC for some $r > 0$. Assume further that if $T_j(b) = 0$ for some j, then b is not a NE for the game with utility functions (V_i). If a NE exists for this game then it is unique up to scaling of the action components by (T_j).

Proof: The proof is a simple extension of Rosen's proof of [8, Th. 2]. Let $b^0, b^1 \in \mathbb{R}_+^{\overline{m}N}$ be two equilibrium points. Then for each $i = 1, \ldots, N$, we have

$$b_i^l \in \arg\max_{b_i} \left\{ V_i(b_i, b_{-i}^l) \mid b_i \in \mathbb{R}_+^{\overline{m}} \right\}, \quad l = 0, 1.$$

By the Kuhn-Tucker necessary conditions, for each $i = 1, \ldots, N$, and for $l = 0, 1$, there exist Lagrange multipliers $u_i^l \in \mathbb{R}_+^{\overline{m}}$ so that

$$b_i^l \geq 0 \tag{14}$$

$$u_i^l \geq 0 \tag{15}$$

$$(u_i^l)^t b_i^l = 0 \tag{16}$$

$$\frac{\partial V_i}{\partial b_{ij}}(b^l) + u_{ij}^l = 0 \quad j = 1, 2, \cdots, \overline{m}. \tag{17}$$

By assumption, $T_j(b) = 0$ for some j implies that b is not an equilibrium point, and hence $T_j(b^l) > 0$ for all j and $l = 0, 1$. Multiply the last equation above by $T_j(b^l)$ and subtract the equation for $l = 0$ from the equation for $l = 1$ to get

$$T_j(b^1)\left(\frac{\partial V_i}{\partial b_{ij}}(b^1) + u_{ij}^1\right) - T_j(b^0)\left(\frac{\partial V_i}{\partial b_{ij}}(b^0) + u_{ij}^0\right) = 0.$$

Now multiply by $r_i \left(b_{ij}^1 / T_j(b^1) - b_{ij}^0 / T_j(b^0) \right)$, sum over j, and then sum over i, to get

$$\begin{aligned} &\sum_i r_i \sum_j \left(\frac{b_{ij}^1}{T_j(b^1)} - \frac{b_{ij}^0}{T_j(b^0)} \right) \left(T_j(b^1) \frac{\partial V_i}{\partial b_{ij}}(b^1) - T_j(b^0) \frac{\partial V_i}{\partial b_{ij}}(b^0) \right) \\ &+ \sum_i r_i \sum_j \left(\frac{b_{ij}^1}{T_j(b^1)} - \frac{b_{ij}^0}{T_j(b^0)} \right) \left(T_j(b^1) u_{ij}^1 - T_j(b^0) u_{ij}^0 \right) = 0. \end{aligned} \tag{18}$$

The second term, using $u_{ij}^l b_{ij}^l = 0$ for each i, is

$$-\sum_j \left(\frac{T_j(b^1)}{T_j(b^0)} b_{ij}^0 u_{ij}^1 + \frac{T_j(b^0)}{T_j(b^1)} b_{ij}^1 u_{ij}^0 \right) \leq 0$$

for each i, and as a consequence the first term in (18) must be nonnegative to make the sum zero. But, by the assumption of GDSC, this is possible only when $b_{ij}^0/T_j(b^0) = b_{ij}^1/T_j(b^1)$ for all i, j. This establishes uniqueness up to scaling of the action components by (T_j). ■

V. Application to resource allocation games

We now go back to the study of resource allocation problems of Hajek and Gopalakrishnan [3] and Johari and Tsitsiklis [4]. The action variable of each player is a scalar, i.e., $\overline{m} = 1$. We consider a generalisation of the utility function V_i of (4). For a fixed $r = (r_i)$ with $r_i > 0$ for every i, define

$$V_i(b) = U_i \left(\frac{b_i}{\sum_{j=1}^N b_j} \right) - \frac{b_i}{r_i}. \tag{19}$$

Recall that $1/r_i$ is the price differentiation factor for player i in the Kelly mechanism with price differentiation. We verify that the GDSC property holds for the associated $\sigma(\cdot, r)$ under a suitable choice of T.

Theorem 4: Let $r > 0$ and consider the (V_i) of (19). Let $T(b) = \sum_i b_i$. Then $\sigma(\cdot, r)$ is GDSC.

Proof: Observe that $\partial V_i / \partial b_i$ is

$$\frac{\partial V_i}{\partial b_i}(b) = U_i' \left(\frac{b_i}{\sum_j b_j} \right) \left(\frac{1 - b_i/\sum_j b_j}{\sum_j b_j} \right) - \frac{1}{r_i}, \tag{20}$$

and so, with $\beta = b/T(b)$, we get

$$r_i T(b) \frac{\partial V_i}{\partial b_i}(b) = r_i U_i' \left(\beta_i \right) \left(1 - \beta_i \right) - T(b).$$

We now verify the GDSC property of $\sigma(\cdot, r)$. Take any $b^0, b^1 \in \mathbb{R}_+^N$ and form $\beta^l = b^l / T(b^l)$ for $l = 0, 1$. The left-hand side of (13) is then

$$\sum_i r_i(\beta_i^1 - \beta_i^0)(U_i' \left(\beta_i^1\right) (1 - \beta_i^1) - U_i' \left(\beta_i^0\right) (1 - \beta_i^0)) - (T(b^1) - T(b^0)) \sum_i (\beta_i^1 - \beta_i^0).$$

The second term is zero since (β_i^l) sums to 1 for both $l = 0, 1$. The first term is strictly negative for distinct b^0, b^1. To see this observe that $(1 - x)U_i'(x) = W_i'(x)$, where $W_i(x) := (1 - x)U_i(x) + \int_0^x U_i(z)dz$. Since W_i is obviously increasing, concave, and a continuously differentiable function, the family $W = (W_i)$ yield $\sigma_W(\cdot, r)$, which is (5) with W_i in place of V_i, that is DSC for r. This proves the claim that the first term is negative, and proof of the theorem is complete. ■

Let us now leverage this result to get a proof of uniqueness of NE for the Kelly mechanism with price differentiation.

Corollary 1: Let $r > 0$. The game defined by decoupled action sets $\mathbb{R}_+^N$ and utility functions V_i given by (19) has a unique NE. In particular, with $r_i = 1$ for all i, the game with utility functions V_i given by (4) has a unique NE.

Proof: We prove the result in the following sequence of simple steps.

Step 1 (Compact action spaces): The action space of each player can be restricted to a compact rectangular set. Fix player i. The net utility of player i is $U_i(a_i) - b_i/r_i$. Since U_i is strictly increasing, we have $U_i(a_i) \leq U_i(1)$ for any allocation to player i. If player i places a bid strictly larger than $b_i^{max} := r_i U_i(1)$, his aggregate utility is strictly negative, regardless of the allocation. Hence therefore he has no incentive to place a bid larger than b_i^{max}, and his action set is effectively reduced to the bounded and closed interval $[0,\ b_i^{max}]$.

Step 2 (Existence of equilibrium): By Step 1, the action space of each player is a compact and convex subset of $\mathbb{R}_+$. The action sets are decoupled. Existence of a NE follows by Rosen's [8, Th. 1].

Step 3 ($T(b) > 0$): A b with $T(b) = 0$ cannot be an equilibrium. Indeed, $T(b) = \sum_i b_i = 0$ implies that $b_i = 0$ for all i. But then player 1 can increase his bid to a small value $b_1' \in (0,\ b_1^{max}]$, get the entire good, and pay a negligible amount of b_1'/r_1, and strictly improve his net utility.

Step 4 (Uniqueness up to scaling): In Step 2, we verified that a NE exists. In Step 3, we verified that a b with $T(b) = 0$ cannot be an equilibrium. In Theorem 4, we verified that for every $r > 0$, $\sigma(\cdot, r)$ is GDSC. By Theorem 3 the NE is unique up to scaling by $T(b)$.

Step 5 (Uniqueness): The NE is unique if $T(b)$ is unique for the equilibrium. By Step 3, there is a player who places a positive bid. Without loss of generality, let this player be 1. Then $b_1 > 0$. With $\beta = b/T(b)$, we must then have

$$0 = \frac{\partial V_1(b)}{\partial b_1} = U_1'(\beta)(1 - \beta_1)/T(b) - 1/r_1.$$

It follows that $T(b)$ is unique and so, by Step 4, the equilibrium is unique.

This completes the proof of the corollary. ■

VI. NORMALISED NASH AND DIFFERENTIAL PRICING

Consider the game defined by utility in (19). The vector (r_i) is price differentiation vector in the Kelly mechanism [7]. In this section, we show that equilibrium in the game with price differentiation can be interpreted as the normalised Nash equilibrium of another game with coupled constraints, the game with which we began this paper, for weights (r_i).

Recall the coupled action set given by

$$A = \left\{ a = (a_i) : \sum_i a_i = 1 \right\}$$

introduced at the beginning of this paper. Consider the game defined by the family of utility functions $W_i : A \to \mathbb{R}_+$

$$W_i(a) = U_i(a_i)(1 - a_i) + \int_0^{a_i} U_i(z)dz, \tag{21}$$

where $U_i(\cdot)$ is concave, continuously differentiable and strictly increasing function. Note that the above utility does not depend on the actions of the other players. The interaction in the game is only through the constraint set. In the proof of Theorem 4 we argued that $W_i(\cdot)$ is a concave, increasing and continuously differentiable function on $[0,\ 1]$, and the corresponding $\sigma_W(\cdot, r)$ formed with (W_i) satisfies the DSC property for any $r > 0$. Then, by Theorem 2, the family W_i has a unique normalised Nash equilibrium for each $r > 0$. Let (a_i^*) denote the normalised Nash equilibrium for a given $r > 0$. By the Kuhn-Tucker conditions, there exists a $\lambda \geq 0$ such that

$$r_i \frac{\partial W_i(a^*)}{\partial a_i} - \lambda = 0$$

for each $i = 1, \ldots, N$, or, equivalently,

$$(1 - a_i^*) \frac{\partial U_i(a_i^*)}{\partial a_i} - \frac{\lambda}{r_i} = 0 \tag{22}$$

for each $i = 1, \ldots, N$.

Now, consider the coupled utility decoupled action set game defined by (19). By Theorem 4 and Corollary 1, this game has a unique Nash equilibrium. Let (x_i^*) denote the unique Nash equilibrium and define $\mu = \sum_j x_j^* > 0$. By the optimality conditions, for each $i = 1, 2, \cdots N$, there exists $\gamma_i > 0$ such that

$$U_i' \left(\frac{x_i^*}{\sum_j x_j^*} \right) \left(\frac{1}{\sum_j x_j^*} - \frac{x_i^*}{(\sum_j x_j^*)^2} \right) - \frac{1}{r_i} + \gamma_i = 0 \tag{23}$$

Assume (r_i) are such that $x_i^* > 0$ for each $i = 1, 2, \cdots, N$. Then, $\gamma_i = 0$ for each $i = 1, 2, \cdots, N$. Multiplying both sides by μ, we have

$$U_i'\left(\frac{x_i^*}{\sum_j x_j^*}\right)\left(1 - \frac{x_i^*}{\sum_j x_j^*}\right) - \frac{\mu}{r_i} = 0 \tag{24}$$

If we define $a_i^* = x_i^*/\sum_j x_j^*$, we see that both (24) and (22) are identical with $\lambda = \mu$. Thus, the normalised Nash equilibrium corresponding to r in the game defined by utilities (21) maximises the equilibrium utilities in the Kelly mechanism with price differentiation vector r.

VII. Concluding Remarks

In this paper we applied Rosen's framework of concave games to establish uniqueness of Nash equilibrium in resource allocation games. First, we provided the example of the Kelly mechanism where diagonal strict concavity (DSC), which is a sufficient condition for uniqueness of Nash equilibrium in Rosen's framework, is not yet verified, and yet the game is known to have unique Nash equilibrium. We then provided a sufficient condition, as a generalisation of the DSC property, to establish the uniqueness of Nash equilibrium. Our generalisation exploits the structure of utilities to establish uniqueness of Nash equilibrium.

Further, applying Rosen's framework to study Kelly mechanism with differential pricing, we showed that the resulting Nash equilibrium of the game is the normalised Nash equilibrium of another game where strategy space is coupled.

Rosen developed a dynamic model to study stability in concave games. He showed that when DSC property holds the system is globally asymptotically stable, and starting from any point the system converges to the unique Nash equilibrium. It would be of interest to see if a similar stability results hold under the new GDSC property.

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