Parameter uncertainties quantification for finite element based subspace fitting approaches

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Abstract

This paper addresses the issue of quantifying uncertainty bounds when updating the finite element model of a mechanical structure from measurement data. The problem arises as to assess the validity of the parameters identification and the accuracy of the results obtained. In this paper, a covariance estimation procedure is proposed about the updated parameters of a finite element model, which propagates the data-related covariance to the parameters by considering a first-order sensitivity analysis. In particular, this propagation is performed through each iteration step of the updating minimization problem, by taking into account the covariance between the updated parameters and the data-related quantities. Numerical simulations on a beam show the feasibility and the effectiveness of the method.

1. INTRODUCTION

The use of state-space model identificationin the analysis of vibrating structures constitutes an interesting research topic in mechanical engineering [1], known as experimental modal analysis. In many cases, the modal parameters of a structure — i.e., eigenfrequencies, damping ratios and observed mode shapes — are sought from vibration data. The stochastic subspace identification method (SSI), as presented by Van Overschee and De Moor [2], can be considered to address this task. It identifies a stochastic state-space model from output-only measurements, where it is assumed that the input is a realization of a white noise stochastic process. Two implementations of the SSI method of equal precision have been proposed in [3], namely, the covariance-driven (SSI-cov) and data-driven (SSI-data) implementations.

In the field of subspace identification methods, a finite element (FE) based subspace fitting (SF) approach has been recently proposed [4] which makes use of a coarse FE mesh of a structure. This leads to a minimization problem that consists in correlating a FE-based extended observability matrix with an experimental one [5]. A model reduction technique which uses the concept of mode basis truncation is considered to speed up the computation of the SF minimization problem. The FE-based SF approach can be used to identify either the structural or modal parameters of vibrating structures.

For any system identification method, the estimated parameters are afflicted with uncertainties [6]. In a broad sense, uncertainties can be classified into two categories [7] of aleatory (irreducible) and epistemic (reducible) uncertainties, and in many cases there is no strict distinction between these two categories. Aleatory uncertainty may result from geometric dimension variability due to manufacturing tolerances or inherent variability of materials such as concrete, while epistemic uncertainty is caused by lack of knowledge (e.g. due to finite number of data samples, undefined measurement noises, non-stationary excitations, and so on).

The statistical nature of subspace identification methods underlines the need of providing statistical evaluation of the estimated parameters. Two commonly used approaches in uncertainty propagation are the Monte Carlo (MC) method [8] and the perturbation method [9]. The MC method has been widely used in the resolution of mathematical and statistical problems [10]. The key idea consists in estimating the occurrence of the statistical expectation of a certain variable by means of stochastic sampling experiments [11]. The advantages of the MC method especially lies in its ease of implementation and good accuracy. However, its major drawback is that it requires huge computational costs for tackling engineering problems.

Several studies made on the estimation of uncertainties in the identification of modal parameters, from SSI, can be found in the literature [6, 9, 12]. These techniques invoke perturbation analyzes and appear to be interesting to estimate the variance of identified eigenfrequencies and modal damping coefficients, as well as the covariance of identified mode shapes. In [9], covariance matrices of estimates obtained from Maximum Likelihood, and prediction error based parameter estimation methods, have been used to obtain uncertainty bounds. In [12], a similar approach has been applied within the framework of operational modal analysis to obtain uncertainty bounds about the modal parameters estimated from SSI. The methodology described in [12] has been developed for output-only data acquired in a single experiment. In [6], an efficient algorithmic scheme has been proposed to speed up the computation of uncertainties.

The motivation behind the present work is to propose a variance analysis of the structural parameters obtained from the FE-based SF method. The proposed approach involves propagating, through sensitivity analysis, first-order perturbations from the data to the identified parameters [9, 13]. The derivations of the covariances of the FE model parameters constitute an original contribution of the paper.

2. FE-BASED SF METHOD AND UNCERTAINTIES QUANTIFICATION

2.1 Framework

The vibration behavior of a structure can be described through a discrete-time linear time-invariant (LTI) state-space model, as follows:

$$\begin{cases} \mathbf{x}_{k+1} = \mathbf{A}\mathbf{x}_k + \mathbf{v}_k, \\ \mathbf{y}_k = \mathbf{C}\mathbf{x}_k + \mathbf{w}_k, \end{cases} \tag{1}$$

where $\mathbf{x}_k \in \mathbb{R}^{2n}$ is the state vector, $\mathbf{y}_k \in \mathbb{R}^r$ is the vector of measurement outputs. Also, $\mathbf{A} \in \mathbb{R}^{2n\times 2n}$ and $\mathbf{C} \in \mathbb{R}^{r\times 2n}$ are state transition and output matrices, respectively, expressed by

$$\mathbf{A} = \exp\left(\begin{bmatrix} \mathbf{0} & \mathbf{I}_n \\ -\mathbf{M}^{-1}\mathbf{K} & -\mathbf{M}^{-1}\gamma \end{bmatrix}\tau\right) \quad , \quad \mathbf{C} = \begin{bmatrix}\mathbf{H}_d - \mathbf{H}_a\mathbf{M}^{-1}\mathbf{K} & | & \mathbf{H}_v - \mathbf{H}_a\mathbf{M}^{-1}\gamma\end{bmatrix}. \tag{2}$$

where $\mathbf{M} \in \mathbb{R}^{n\times n}$, $\mathbf{K} \in \mathbb{R}^{n\times n}$ and $\gamma \in \mathbb{R}^{n\times n}$ are, respectively, the mass, stiffness and viscous damping matrices and τ is the sampling rate. Also, $\mathbf{H}_d$, $\mathbf{H}_v$ and $\mathbf{H}_a \in \mathbb{R}^{r\times n}$ are Boolean matrices to localize the degrees of freedom (DOFs) at which displacements, velocities and accelerations are measured in the vector $\mathbf{y} \in \mathbb{R}^r$. Finally, $\mathbf{v} \in \mathbb{R}^n$ and $\mathbf{w} \in \mathbb{R}^r$ are white noise vectors of unmeasured excitation forces and measurement noise, respectively. Furthermore, the observability matrix associated with the state-space representation (1) can be expressed as

$$\mathscr{O} = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \vdots \\ \mathbf{C}(\mathbf{A})^p \end{bmatrix}. \tag{3}$$

Model updating has the purpose to calibrate the parameters of a FE model such that some model properties are close to the truly observed structural properties. In the proposed SF method, the considered quantity for calibration is the extended observability matrix $\mathscr{O}$. The main idea is to correlate – in a least squares sense – the matrix $\hat{\mathscr{O}}$ obtained from experimental data with a reduced matrix $\tilde{\mathscr{O}}^h(\boldsymbol{\theta}^h) \in$

$\mathbb{R}^{(p+1)r\times n_r}$, where $n_r \ll n$. The extended observability matrix $\tilde{\mathscr{O}}^h(\boldsymbol{\theta}^h)$ is issued from a FE model of the structure and a mode-based model reduction technique, which consists in projecting the FE matrices (mass, stiffness and damping) on a reduced basis of mode shapes (see [4] for further details). Hence, a SF minimization problem can be proposed as follows:

$$\boldsymbol{\theta}^h = \text{argmin}\,||\mathbf{r}||_2^2, \quad \mathbf{r} = \left[\mathbf{I}_{2n_r} \otimes (\mathbf{I}_{(p+1)r} - \hat{\mathscr{O}}\hat{\mathscr{O}}^{\dagger})\right] \text{vec}\{\tilde{\mathscr{O}}^h(\boldsymbol{\theta}^h)\}, \tag{4}$$

where $\boldsymbol{\theta}^h \in \mathbb{R}^{n_h}$ is the vector of structural parameters of the FE model to be updated. One of the most popular and effective algorithms for solving least squares problems like (4) is the Gauss-Newton method. The flowchart of the related approach is reported in Figure 1. Here, the matrices $\hat{\Sigma}_{\mathscr{H}}$, $\hat{\Sigma}_{\mathscr{O}}$ and $\hat{\Sigma}_{\boldsymbol{\theta}}$ denote the covariances of matrices related to the SF approach and are obtained through a sensitivity analysis.

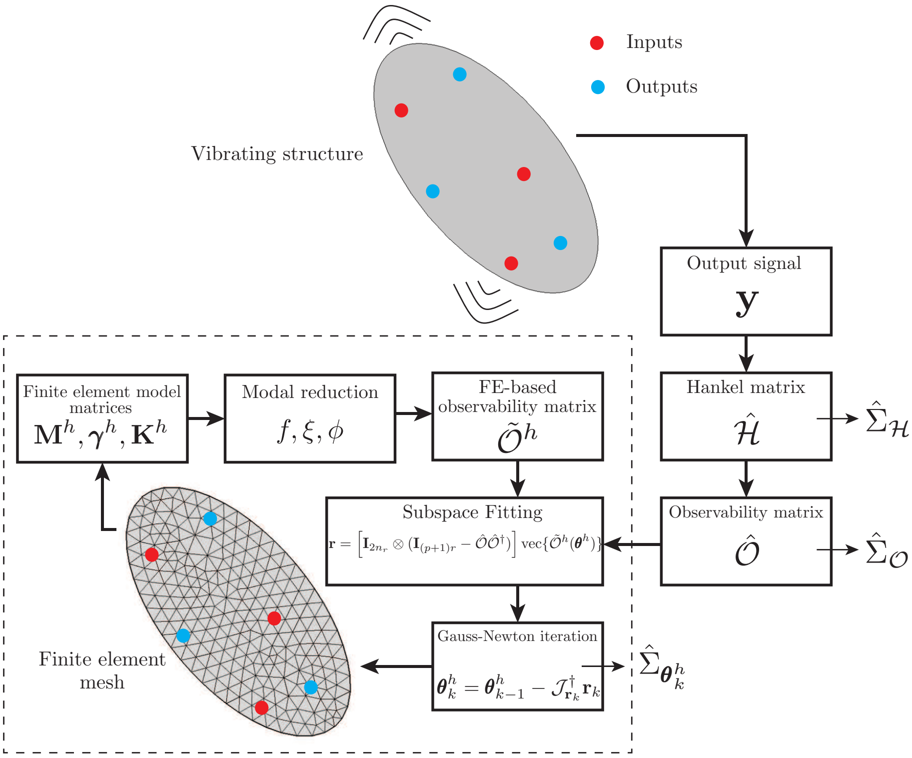

Figure 1 : Flowchart of the SF based approach

Let Y be a matrix-valued function of $\hat{X}$. Using delta notation, theoretical first-order perturbations are defined, yielding:

$$\text{vec}\{\Delta Y\} = \mathscr{J}_{Y,X}\text{vec}\{\Delta X\}. \tag{5}$$

The covariance of Y is defined as:

$$\text{cov}(Y) = \text{vec}\{\Delta Y\}(\text{vec}\{\Delta Y\})^T \tag{6}$$

and follows from a Taylor approximation, i.e.:

$$\text{vec}\{Y(\hat{X})\} \approx \text{vec}\{Y(X)\} + \mathscr{J}_{Y,X}\text{vec}\{\hat{X} - X\} \quad \Rightarrow \quad \text{cov}(Y(\hat{X})) \approx \mathscr{J}_{Y,X}\hat{\Sigma}_X \mathscr{J}_{Y,X}^T, \tag{7}$$

where $\mathscr{J}_{Y,X}$ is the sensitivity matrix, defined by $\mathscr{J}_{Y,X} = \partial\text{vec}\{Y(X)\}/\partial\text{vec}\{X\}$. A consistent estimate is obtained by replacing, in the sensitivity matrix, the theoretical variables X with consistent estimates $\hat{X}$ issued from data.

Using this relationship, a perturbation in measurements is propagated towards the observability matrix $\hat{\mathscr{O}}$ and ultimately the structural parameters through the Gauss-Newton algorithm to obtain the related sensitivity matrices.

2.2 Extended observability matrix estimation and uncertainties quantification

Define the output covariance matrix $\mathscr{R}_i \in \mathbb{R}^{r\times r}$ as

$$\mathscr{R}_i = \mathbf{E}[\mathbf{y}_{k+i}\mathbf{y}_k^T], \tag{8}$$

and the state-output covariance matrix $\mathbf{G} \in \mathbb{R}^{2n\times r}$ as

$$\mathbf{G} = \mathbf{E}[\mathbf{x}_{k+1}\mathbf{y}_k^T]. \tag{9}$$

The output covariance matrices $\mathscr{R}_i$ can be stacked into a block Hankel matrix $\mathscr{H} \in \mathbb{R}^{(p+1)r\times qr}$, where p and q are chosen such that $\mathtt{min}(pr, qr) \geq 2n$, as follows:

$$\mathscr{H} = \begin{bmatrix} \mathscr{R}_1 & \mathscr{R}_2 & \cdots & \mathscr{R}_q \\ \mathscr{R}_2 & \mathscr{R}_3 & \cdots & \mathscr{R}_{q+1} \\ \vdots & \vdots & \ddots & \vdots \\ \mathscr{R}_{p+1} & \mathscr{R}_{p+2} & \cdots & \mathscr{R}_{p+q} \end{bmatrix}. \tag{10}$$

An interesting feature of the Hankel matrix $\mathscr{H}$ is that it can be factorized, as follows [14]:

$$\mathscr{H} = \mathscr{O}\mathscr{C}, \tag{11}$$

where $\mathscr{O} \in \mathbb{R}^{(p+1)r\times 2n}$ is the extended observability matrix (see Eq. (3)), and $\mathscr{C} \in \mathbb{R}^{2n\times qr}$ is the controllability matrix, given by

$$\mathscr{C} = \begin{bmatrix} \mathbf{G} & \mathbf{AG} & \ldots & \mathbf{A}^{q-1}\mathbf{G} \end{bmatrix}. \tag{12}$$

Notice that the matrix $\mathscr{O}$ is full column rank [2,4]. Within the SSI framework [3], an estimate $\hat{\mathscr{H}}$ of the output Hankel matrix $\mathscr{H}$ can be built from $N+p+q$ measurements, i.e.:

$$\hat{\mathscr{H}} = \frac{1}{N}\mathscr{Y}^+(\mathscr{Y}^-)^T, \tag{13}$$

where

$$\mathscr{Y}^- = \begin{bmatrix} \mathbf{y}_q & \mathbf{y}_{q+1} & \cdots & \mathbf{y}_{N+q-1} \\ \mathbf{y}_{q-1} & \mathbf{y}_q & \cdots & \mathbf{y}_{N+q-2} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{y}_1 & \mathbf{y}_2 & \cdots & \mathbf{y}_N \end{bmatrix}, \quad \mathscr{Y}^+ = \begin{bmatrix} \mathbf{y}_{q+1} & \mathbf{y}_{q+2} & \cdots & \mathbf{y}_{N+q} \\ \mathbf{y}_{q+2} & \mathbf{y}_{q+3} & \cdots & \mathbf{y}_{N+q+1} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{y}_{q+p+1} & \mathbf{y}_{q+p+2} & \cdots & \mathbf{y}_{N+p+q} \end{bmatrix}. \tag{14}$$

The derivation of an estimate of the covariance of the Hankel matrix, namely $\hat{\Sigma}_{\mathscr{H}}$, is achieved by splitting the data matrices $\mathscr{Y}^+$ and $\mathscr{Y}^-$ in (14) into n_b blocks, as follows:

$$\mathscr{Y}^+ = \begin{bmatrix} \mathscr{Y}_1^+ & \mathscr{Y}_2^+ & \ldots & \mathscr{Y}_{n_b}^+ \end{bmatrix}, \quad \mathscr{Y}^- = \begin{bmatrix} \mathscr{Y}_1^- & \mathscr{Y}_2^- & \ldots & \mathscr{Y}_{n_b}^- \end{bmatrix}. \tag{15}$$

Hence, for each pair of sub-matrices $(\mathscr{Y}_j^+, \mathscr{Y}_j^-)$, the corresponding Hankel matrix $\hat{\mathscr{H}}_j$ can be estimated as in Eq. (13):

$$\hat{\mathscr{H}}_j = \frac{1}{N_b}\mathscr{Y}_j^+(\mathscr{Y}_j^-)^T, \tag{16}$$

where $N_b = N/n_b$. As a result, one obtains $\hat{\mathscr{H}} = (1/n_b)\sum_{j=1}^{n_b}\hat{\mathscr{H}}_j$. Also, the estimate $\hat{\Sigma}_{\mathscr{H}}$ can be written as follows:

$$\hat{\Sigma}_{\mathscr{H}} = \frac{1}{n_b(n_b-1)}\sum_{j=1}^{n_b}\left(\mathrm{vec}\{\hat{\mathscr{H}}_j\} - \mathrm{vec}\{\hat{\mathscr{H}}\}\right)\left(\mathrm{vec}\{\hat{\mathscr{H}}_j\} - \mathrm{vec}\{\hat{\mathscr{H}}\}\right)^T. \tag{17}$$

From (11), it can be readily proved that $\mathscr{H}$ and $\mathscr{O}$ share the same column space. Hence, an estimate $\hat{\mathscr{O}}$ of the extended observability matrix can be obtained from a Singular Value Decomposition (SVD) of $\hat{\mathscr{H}}$ and its truncation at the order $2n$, i.e.:

$$\hat{\mathscr{H}} = \mathbf{U}\Delta\mathbf{V}^T = \begin{bmatrix}\mathbf{U}_1 & \mathbf{U}_0\end{bmatrix}\begin{bmatrix}\Delta_1 & \mathbf{0} \\ \mathbf{0} & \Delta_0\end{bmatrix}\begin{bmatrix}\mathbf{V}_1^T \\ \mathbf{V}_0^T\end{bmatrix} \approx \mathbf{U}_1\Delta_1\mathbf{V}_1^T, \tag{18}$$

and thus

$$\hat{\mathscr{O}} = \mathbf{U}_1, \tag{19}$$

where Δ_1 is the diagonal matrix of the $2n$ dominant singular values of $\hat{\mathscr{H}}$, and $\mathbf{U}_1 \in \mathbb{R}^{(p+1)r\times 2n}$ is the related matrix of left singular vectors.

A perturbation $\Delta\mathscr{H}$ of the Hankel matrix is propagated towards the estimate of the extended observability matrix $\hat{\mathscr{O}}$, see Eq. (19). In order to express the perturbation $\Delta\hat{\mathscr{O}}$, the sensitivity of the matrix of left singular vectors $\mathbf{U}_1$ (see Eq. (19)) is to be derived, as follows [6]:

$$\operatorname{vec}\{\Delta\mathbf{U}_1\} = \mathscr{J}_{\mathbf{U}_1,\mathscr{H}}\operatorname{vec}\{\Delta\hat{\mathscr{H}}\} = \begin{bmatrix}\mathscr{B}_1\mathscr{C}_1 \\ \vdots \\ \mathscr{B}_{2n}\mathscr{C}_{2n}\end{bmatrix}\operatorname{vec}\{\Delta\hat{\mathscr{H}}\}, \tag{20}$$

where

$$\mathscr{B}_i = \left[\mathbf{I}_{(p+1)r} + \frac{\hat{\mathscr{H}}}{\sigma_i}\mathscr{D}_i\left(\frac{\hat{\mathscr{H}}^T}{\sigma_i} - \begin{bmatrix}\mathbf{0}_{qr-1,(p+1)r} \\ \mathbf{u}_i^T\end{bmatrix}\right) \;\middle|\; \frac{\hat{\mathscr{H}}}{\sigma_i}\mathscr{D}_i\right], \quad \mathscr{D}_i = \left(\mathbf{I}_{qr} + \begin{bmatrix}\mathbf{0}_{qr-1,(p+1)r} \\ 2\mathbf{v}_i^T\end{bmatrix} - \frac{\hat{\mathscr{H}}^T\hat{\mathscr{H}}}{\sigma_i^2}\right)^{-1} \tag{21}$$

and

$$\mathscr{C}_i = \frac{1}{\sigma_i}\begin{bmatrix}(\mathbf{I}_{(p+1)r} - \mathbf{u}_i\mathbf{u}_i^T)(\mathbf{v}_i^T \otimes \mathbf{I}_{(p+1)r}) \\ (\mathbf{I}_{qr} - \mathbf{v}_i\mathbf{v}_i^T)(\mathbf{I}_{qr} \otimes \mathbf{u}_i^T)\end{bmatrix}, \tag{22}$$

where $\mathbf{u}_i$ and $\mathbf{v}_i$ stand for the $i-$th left and right singular vectors, respectively. Then, an estimate of the covariance matrix of $\hat{\mathscr{O}}$ can be derived from (7) as

$$\hat{\Sigma}_{\mathscr{O}} = \mathscr{J}_{\mathbf{U}_1,\mathscr{H}}\hat{\Sigma}_{\mathscr{H}}\mathscr{J}_{\mathbf{U}_1,\mathscr{H}}^T. \tag{23}$$

2.3 Structural parameters updating and uncertainties quantification

The Gauss-Newton method [15] is based on a second-order expansion of the objective function $||\mathbf{r}||_2^2$ about some approximated values of the parameters $\boldsymbol{\theta}^h$, and iteration steps so as to find the local extrema of $||\mathbf{r}||_2^2$. Assuming an initial deterministic parameter value $\boldsymbol{\theta}_0^h$, the $k-$th iteration ($k \geq 1$) of the Gauss-Newton algorithm can be written as

$$\boldsymbol{\theta}_k^h = \boldsymbol{\theta}_{k-1}^h - \mathscr{J}_{\mathbf{r}_k}^\dagger\mathbf{r}_k, \tag{24}$$

where $\boldsymbol{\theta}_k^h = [\theta_{1,k}^h \;\cdots\; \theta_{n_h,k}^h]^T$ is the vector of structural parameters identified at iteration k. Also, $\mathbf{r}_k$ is the residual vector and $\mathscr{J}_{\mathbf{r}_k}^\dagger$ is the Moore-Penrose pseudoinverse of the Jacobian matrix $\mathscr{J}_{\mathbf{r}_k} \in \mathbb{R}^{2(p+1)rn\times n_h}$, at iteration k, defined as:

$$\mathbf{r}_k = \left[\mathbf{I}_{2n} \otimes (\mathbf{I}_{(p+1)r} - \hat{\mathscr{O}}\hat{\mathscr{O}}^\dagger)\right]\operatorname{vec}\left\{\mathscr{O}^h(\boldsymbol{\theta}_{k-1}^h)\right\}, \tag{25}$$

and

$$\mathscr{J}_{\mathbf{r}_k} = \left[\mathbf{I}_{2n} \otimes (\mathbf{I}_{(p+1)r} - \hat{\mathscr{O}}\hat{\mathscr{O}}^\dagger)\right]\mathscr{J}_{\mathscr{O}^h,\boldsymbol{\theta}_{k-1}^h}. \tag{26}$$

Here, the sensitivity matrix $\mathscr{J}_{\mathscr{O}^h,\boldsymbol{\theta}_{k-1}^h}$ is expressed by:

$$\mathscr{J}_{\mathscr{O}^h,\boldsymbol{\theta}_{k-1}^h} = \left[\frac{\partial\operatorname{vec}\{\mathscr{O}^h(\boldsymbol{\theta}^h)\}}{\partial\theta_1^h} \quad \cdots \quad \frac{\partial\operatorname{vec}\{\mathscr{O}^h(\boldsymbol{\theta}^h)\}}{\partial\theta_{n_h}^h}\right]\Bigg|_{\boldsymbol{\theta}^h=\boldsymbol{\theta}_{k-1}^h}, \tag{27}$$

where $\frac{\partial \mathrm{vec}\{\mathscr{O}^h(\boldsymbol{\theta}^h)\}}{\partial \theta_j^h}$ is obtained by numerical differentiation.

Separating zeroth- and first-order terms and using the fact that $\mathrm{vec}\{\mathbf{AB}\} = (\mathbf{B}^T \otimes \mathbf{I}_a)\mathrm{vec}\{\mathbf{A}\}$ for any matrices $\mathbf{A} \in \mathbb{R}^{a\times b}$ and $\mathbf{B} \in \mathbb{R}^{b\times c}$ [16], this yields

$$\Delta\boldsymbol{\theta}_k^h = \Delta\boldsymbol{\theta}_{k-1}^h - (\mathbf{r}_k^T \otimes \mathbf{I}_{n_h})\mathrm{vec}\{\Delta\mathscr{J}_{\mathbf{r}_k}^{\dagger}\} - \mathscr{J}_{\mathbf{r}_k}^{\dagger}\Delta\mathbf{r}_k. \tag{28}$$

Assume that the perturbation $\Delta\boldsymbol{\theta}_k^h$ is linked to the perturbation of the observability matrix as follows:

$$\Delta\boldsymbol{\theta}_k^h = \mathscr{M}_k\mathrm{vec}\{\Delta\hat{\mathscr{O}}\} \qquad \forall k \geq 1. \tag{29}$$

Then, by introducing Eq. (29) in Eq. (28), it comes:

$$\mathscr{M}_k = \mathscr{M}_{k-1} - (\mathbf{r}_k^T \otimes \mathbf{I}_{n_h})\mathscr{L}_k\mathscr{N}_k - \mathscr{J}_{\mathbf{r}_k}^{\dagger}\mathscr{Q}_k \tag{30}$$

where $\mathscr{M}_0 = \mathbf{0}_{n_h,2n(p+1)r}$ due to the deterministic nature of the initial value. In (30), $\mathscr{L}_k$, $\mathscr{N}_k$ and $\mathscr{Q}_k$ are matrices which link $\mathrm{vec}\{\Delta\mathscr{J}_{\mathbf{r}_k}^{\dagger}\}$ and $\Delta\mathbf{r}_k$ to $\mathrm{vec}\{\Delta\hat{\mathscr{O}}\}$. They are expressed as follows:

$$\begin{aligned}
\mathscr{L}_k &= \{-[(\mathscr{J}_{\mathbf{r}_k}\mathscr{J}_{\mathbf{r}_k}^{\dagger})^T \otimes (\mathscr{J}_{\mathbf{r}_k}^T\mathscr{J}_{\mathbf{r}_k})^{-1}] + [\mathbf{I}_{2(p+1)rn_r} \otimes (\mathscr{J}_{\mathbf{r}_k}^T\mathscr{J}_{\mathbf{r}_k})^{-1}]\}\mathscr{P}_{2(p+1)rn_r,n_h} - [(\mathscr{J}_{\mathbf{r}_k}^{\dagger})^T \otimes \mathscr{J}_{\mathbf{r}_k}^{\dagger}], && (31)\\
\mathscr{N}_k &= \begin{bmatrix} \mathscr{N}_{1,k} \\ \mathscr{N}_{2,k} \\ \vdots \\ \mathscr{N}_{n_h,k} \end{bmatrix}, && (32)\\
\mathscr{N}_{j,k} &= -\left[\left(\hat{\mathscr{O}}^{\dagger}\mathscr{J}^{*}_{\tilde{\mathscr{O}}^h,\theta^h_{j,k-1}}\right)^T \otimes \mathbf{I}_{(p+1)r}\right] + \left[\mathbf{I}_{2n_r} \otimes (\mathbf{I}_{(p+1)r} - \hat{\mathscr{O}}\hat{\mathscr{O}}^{\dagger})\right]\mathscr{J}_{\mathscr{J}^{*}_{\tilde{\mathscr{O}}^h,\theta^h_{j,k-1}},\boldsymbol{\theta}^h_{k-1}}\mathscr{M}_{k-1} && (33)\\
&\quad - \left[\left(\mathscr{J}^{*}_{\tilde{\mathscr{O}}^h,\theta^h_{j,k-1}}\right)^T \otimes \hat{\mathscr{O}}\right]\{-[(\hat{\mathscr{O}}\hat{\mathscr{O}}^{\dagger})^T \otimes (\hat{\mathscr{O}}^T\hat{\mathscr{O}})^{-1}] + [\mathbf{I}_{(p+1)r} \otimes (\hat{\mathscr{O}}^T\hat{\mathscr{O}})^{-1}]\}\mathscr{P}_{(p+1)r,2n} - [(\hat{\mathscr{O}}^{\dagger})^T \otimes \hat{\mathscr{O}}^{\dagger}], &&\\
\mathscr{Q}_k &= -\left[\left(\hat{\mathscr{O}}^{\dagger}\tilde{\mathscr{O}}^h(\boldsymbol{\theta}^h_{k-1})\right)^T \otimes \mathbf{I}_{(p+1)r}\right] - [(\hat{\mathscr{O}}^{\dagger})^T \otimes \hat{\mathscr{O}}^{\dagger}] + \left[\mathbf{I}_{2n_r} \otimes (\mathbf{I}_{(p+1)r} - \hat{\mathscr{O}}\hat{\mathscr{O}}^{\dagger})\right]\mathscr{J}_{\tilde{\mathscr{O}}^h,\boldsymbol{\theta}^h_{k-1}}\mathscr{M}_{k-1} && (34)\\
&\quad - \left[\left(\tilde{\mathscr{O}}^h(\boldsymbol{\theta}^h_{k-1})\right)^T \otimes \hat{\mathscr{O}}\right]\{-[(\hat{\mathscr{O}}\hat{\mathscr{O}}^{\dagger})^T \otimes (\hat{\mathscr{O}}^T\hat{\mathscr{O}})^{-1}] + [\mathbf{I}_{(p+1)r} \otimes (\hat{\mathscr{O}}^T\hat{\mathscr{O}})^{-1}]\}\mathscr{P}_{(p+1)r,2n}, &&
\end{aligned}$$

where $\mathscr{P}_{a,b}$ are permutation matrices defined such that $\mathrm{vec}\{\mathbf{A}^T\} = \mathscr{P}_{a,b}\mathrm{vec}\{\mathbf{A}\}$ [6, 12]; also, $\mathscr{J}^{*}_{\tilde{\mathscr{O}}^h,\theta^h_{j,k-1}} \in \mathbb{R}^{(p+1)r\times 2n_r}$ is defined such that $\mathrm{vec}\{\mathscr{J}^{*}_{\tilde{\mathscr{O}}^h,\theta^h_{j,k-1}}\} = \mathscr{J}_{\tilde{\mathscr{O}}^h,\theta^h_{j,k-1}}$, while $\mathscr{J}_{\mathscr{J}^{*}_{\tilde{\mathscr{O}}^h,\theta^h_{j,k-1}},\boldsymbol{\theta}^h_{k-1}}$ is defined such that $\mathrm{vec}\{\Delta\mathscr{J}^{*}_{\tilde{\mathscr{O}}^h,\theta^h_{j,k-1}}\} = \mathscr{J}_{\mathscr{J}^{*}_{\tilde{\mathscr{O}}^h,\theta^h_{j,k-1}},\boldsymbol{\theta}^h_{k-1}}\mathrm{vec}\{\Delta\boldsymbol{\theta}^h_{k-1}\}$ and is obtained by numerical differentiation.

Hence, an estimate of the covariance matrix of the identified vector of structural parameters $\boldsymbol{\theta}_k^h$, at iteration k of the SF minimization procedure (see Eq. (4)), is given by:

$$\hat{\Sigma}_{\boldsymbol{\theta}_k^h} = \mathscr{M}_k\hat{\Sigma}_{\mathscr{O}}\mathscr{M}_k^T. \tag{35}$$

3. NUMERICAL APPLICATION

The theory described in Section 2 is applied through the analysis of a 3D FE model of a beam in operational modal analysis. Here, a clamped-free beam is considered with dimensions $1m \times 0.1m \times 0.1m$ as shown in Figure 2.

The vibrating behavior of the beam is analyzed with the FE method in order to assess its time response. The output data refer to the transverse displacement of the structure which is recorded by means of one displacement sensor located at the free end of the beam, as shown in Figure 2. Numerical

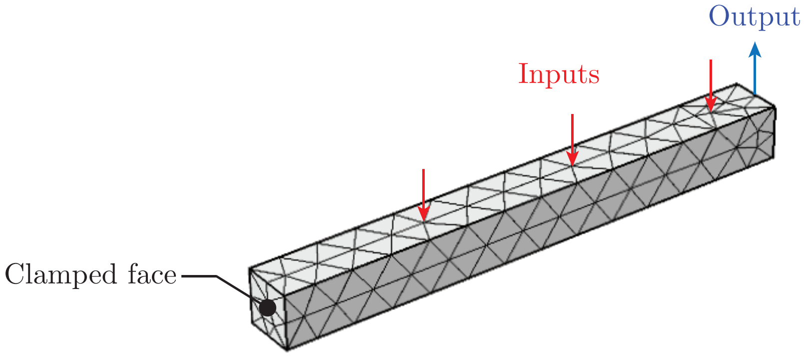

Figure 2 : Illustration of the 3D FE mesh of the beam.

vibration data are thus considered, which are sampled at a frequency of 1280Hz. In addition, different white noise rates are added to the data at a Signal to Noise Ratio (SNR) of 40 to 10dB.
The FE-based SF approach and uncertainties quantification are investigated. In this framework, a coarse FE mesh based on the Euler-Bernoulli beam theory is considered, as shown in Figure 3. This mesh is made up of 10 Euler-Bernoulli beam elements of same length with six DOFs per node. The issue here

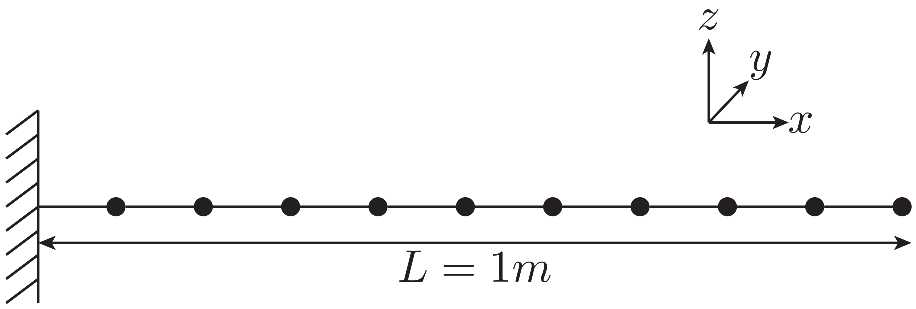

Figure 3 : Coarse FE mesh of the beam.

consists in updating the Young's modulus – namely, E – of the coarse 1D FE model, which is assumed to be unknown. This yields a single-parameter SF minimization problem, as follows:

$$E = \text{argmin}||\mathbf{r}||_2^2 \qquad \text{where} \qquad \mathbf{r} = \left[\mathbf{I}_{2n_r} \otimes (\mathbf{I}_{(p+1)r} - \hat{\mathscr{O}}\hat{\mathscr{O}}^\dagger)\right] \text{vec}\left\{\tilde{\mathscr{O}}^h(E)\right\}, \tag{36}$$

where $\tilde{\mathscr{O}}^h$ is the observability matrix issued from the 1D FE model of the beam (Figure 3), with $p = 50$ and $2n_r = 14$. The minimization procedure is repeated for different levels of noise, hence providing different estimates of the updated Young's modulus, with different estimates of the standard deviations $\sigma_E = \sqrt{\hat{\Sigma}_E}$ (see Table 1). The results show clearly that the dispersion of the Young's modulus increases as the noise level growths, as expected. Below 20dB of the SNR, the estimated standard deviation is higher than 25%, i.e., the estimated Young's modulus is inaccurate.

4. CONCLUSION

A SF strategy has been proposed to calculate covariance estimates when updating the parameters of a FE model from measurement data of a vibrating structure. The procedure involves propagating first-order perturbations at each iteration step of the Gauss-Newton algorithm which is involved in the resolution of the SF minimization problem. The relevance of the proposed approach has been highlighted for identifying the Young's modulus of a numerical beam, subject to various levels of noise, and related uncertainty bounds.

Table 1 : Updated Young's modulus and related standard deviations.

SNR [dB]	**Updated Young modulus** E [GPa]	**Standard deviation** σ_E [GPa]
40	2.03	0.02
35	2.02	0.02
30	2.02	0.05
25	2.03	0.15
20	2.02	0.56
15	2.09	3.43

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