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# Convergence rates for the iteratively regularized Landweber iteration in Banach space

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**Abstract.** In this paper we provide a convergence rates result for a modified version of Landweber iteration with a priori regularization parameter choice in a Banach space setting.

**Keywords:** regularization, nonlinear inverse problems, Banach space, Landweber iteration

An increasing number of inverse problems is nowadays posed in a Banach space rather than a Hilbert space setting, cf., e.g., [2, 6, 13] and the references therein.

An Example of a model problem, where the use of non-Hilbert Banach spaces is useful, is the identification of the space-dependent coefficient function  $c$  in the elliptic boundary value problem

$$-\Delta u + cu = f \quad \text{in } \Omega \tag{1}$$

$$u = 0 \quad \text{on } \partial\Omega \tag{2}$$

from measurements of  $u$  in  $\Omega \subseteq \mathbb{R}^d$ ,  $d \in \{1, 2, 3\}$ , where  $f$  is assumed to be known. Here e.g., the choices  $p = 1$  for recovering sparse solutions,  $q = \infty$  for modelling uniformly bounded noise, or  $q = 1$  for dealing with impulsive noise are particularly promising, see, e.g., [3] and the numerical experiments in Section 7.3.3 of [13].

Motivated by this fact we consider nonlinear ill-posed operator equations

$$F(x) = y \tag{3}$$

where  $F$  maps between Banach spaces  $X$  and  $Y$ .

In the example above, the forward operator  $F$  maps the coefficient function  $c$  to the solution of the boundary value problem (1), (2), and is well-defined as an operator

$$F : \mathcal{D}(F) \subseteq L^p(\Omega) \rightarrow L^q(\Omega),$$

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where  $\mathcal{D}(F) = \{c \in X \mid \exists \hat{c} \in L^\infty(\Omega), \hat{c} \geq 0 \text{ a.e.} : \|c - \hat{c}\|_X \leq r\}$ ,  $r$  sufficiently small, for any

$$\begin{aligned} p, q \in [1, \infty], \quad f \in L^1(\Omega) & \quad \text{if } d \in \{1, 2\} \\ p \in [1, \infty], \quad q \in (\frac{d}{2}, \infty], \quad f \in L^s(\Omega), \quad s > \frac{d}{2} & \quad \text{if } d \geq 3, \end{aligned}$$

see Section 1.3 in [13].

Since the given data  $y^\delta$  are typically contaminated by noise, regularization has to be applied. We are going to assume that the noise level  $\delta$  in

$$\|y - y^\delta\| \leq \delta \tag{4}$$

is known and provide convergence results in the sense of regularization methods, i.e., as  $\delta$  tends to zero. In the following,  $x_0$  is some initial guess and we will assume that a solution  $x^\dagger$  to (3) exists.

Variational methods in Banach space have been extensively studied in the literature, see, e.g., [2, 10, 6] and the references therein.

Since these generalizations of Tikhonov regularization require computation of a global minimizer, iterative methods are an attractive alternative especially for large scale problems. After convergence results on iterative methods for nonlinear ill-posed operator equations in Banach spaces had already been obtained in the 1990's (cf. the references in [1]) in the special case  $X = Y$ , the general case  $X \neq Y$  has only been treated quite recently, see e.g. [5], [7], and [9] for an analysis of gradient and Newton type iterations. While convergence rates have already been established for the iteratively regularized Gauss-Newton iteration in [7], the question of convergence rates is still open for gradient type, i.e. Landweber methods. It is the aim of this paper to provide such a result.

In order to formulate and later on analyze the method, we have to introduce some basic notations and concepts.

Consider, for some  $q \in (1, \infty)$ , the duality mapping  $J_q^X(x) := \partial \left\{ \frac{1}{q} \|x\|^q \right\}$ , which maps from  $X$  to its dual  $X^*$ . To analyze convergence rates we employ the Bregman distance

$$D_{j_q}(\tilde{x}, x) = \frac{1}{q} \|\tilde{x}\|^q - \frac{1}{q} \|x\|^q - \langle j_q^X(x), \tilde{x} - x \rangle_{X^* \cdot X}$$

(where  $j_q^X(x)$  denotes a single valued selection of  $J_q^X(x)$ ) or its shifted version

$$D_q^{x_0}(\tilde{x}, x) := D_{j_q}(\tilde{x} - x_0, x - x_0).$$

Throughout this paper we will assume that  $X$  is smooth, which means that the duality mapping is single-valued, and moreover, that  $X$  is  $q$ -convex, i.e.,

$$D_{j_q}(x, y) \geq c_q \|x - y\|^q \tag{5}$$

for some constant  $c_q > 0$ . As a consequence,  $X$  is reflexive and we also have

$$D_{j_{q^*}}(x^*, y^*) \leq C_{q^*} \|x^* - y^*\|^{q^*}, \tag{6}$$

for some  $C_{q^*}$ . Here  $q^*$  denotes the dual index  $q^* = \frac{q}{q-1}$ . Moreover, the duality mapping is bijective and  $J_q^{-1} = J_{q^*}^{X^*}$ , the latter denoting the (by  $q$ -convexity also single-valued) duality mapping on  $X^*$ . We will also make use of the identities

$$D_{j_q}(x, y) = D_{j_q}(x, z) + D_{j_q}(z, y) + \langle J_q^X(z) - J_q^X(y), x - z \rangle_{X^*, X} \quad (7)$$

and

$$D_{j_q}(y, x) = D_{j_{q^*}}(J_q^X(x), J_q^X(y)). \quad (8)$$

For more details on the geometry of Banach spaces we refer, e.g., to [12] and the references therein.

We here consider the iteratively regularized Landweber iteration

$$\begin{aligned} J_q^X(x_{n+1}^\delta - x_0) &= (1 - \alpha_n)J_q^X(x_n^\delta - x_0) - \mu_n A_n^* j_p^Y(F(x_n^\delta) - y^\delta), \\ x_{n+1}^\delta &= x_0 + J_{q^*}^{X^*}(J_q^X(x_{n+1}^\delta - x_0)), \quad n = 0, 1, \dots \end{aligned} \quad (9)$$

where we abbreviate

$$A_n = F'(x_n^\delta),$$

which, for an appropriate choice of the sequence  $\{\alpha_n\}_{n \in \mathbb{N}} \in (0, 1]$ , has been shown to be convergent with rates under a source condition

$$x^\dagger - x_0 \in \mathcal{R}(F'(x^\dagger)^* F'(x^\dagger))^{\nu/2}, \quad (10)$$

with  $\nu = 1$  in a Hilbert space setting in [11]. Since the linearized forward operator  $F'(x)$  typically has some smoothing property (reflecting the ill-posedness of the inverse problems) condition (10) can often be interpreted as a regularity assumption on the initial error  $x^\dagger - x_0$ , which is stronger for larger  $\nu$ .

In the Hilbert space case the proof of convergence rates for the plain Landweber iteration (i.e., (9) with  $\alpha_n = 0$ ) under source conditions (10) relies on the fact that the iteration errors  $x_n^\delta - x^\dagger$  remain in the range of  $(F'(x^\dagger)^* F'(x^\dagger))^{\nu/2}$  and their preimages under  $(F'(x^\dagger)^* F'(x^\dagger))^{\nu/2}$  form a bounded sequence (cf., Proposition 2.11 in [8]). Since carrying over this approach to the Banach space setting would require more restrictive assumptions on the structure of the spaces even in the special case  $\nu = 1$ , we here consider the modified version with an appropriate choice of  $\{\alpha_n\}_{n \in \mathbb{N}} \in (0, 1]$ .

In place of the Hilbert space source condition (10), we consider variational inequalities

$$\begin{aligned} \exists \beta > 0 \quad \forall x \in \mathcal{B}_\rho^D(x^\dagger) : \\ |\langle J_q^X(x^\dagger - x_0), x - x^\dagger \rangle_{X^* \times X}| \leq \beta D_q^{x_0}(x^\dagger, x)^{\frac{1-\nu}{2}} \|F'(x^\dagger)(x - x^\dagger)\|^\nu, \end{aligned} \quad (11)$$

cf., e.g., [4], where

$$\mathcal{B}_\rho^D(x^\dagger) = \{x \in X \mid D_q^{x_0}(x^\dagger, x) \leq \rho^2\}$$

with  $\rho > 0$  such that  $x_0 \in \mathcal{B}_\rho^D(x^\dagger)$ . Using the interpolation and the Cauchy-Schwarz inequality, it is readily checked that in the Hilbert space case (10)

implies (11). For more details on such variational inequalities we refer to Section 3.2.3 in [13] and the references therein.

The assumptions on the forward operator besides a condition on the domain

$$\mathcal{B}_\rho^D(x^\dagger) \subseteq \mathcal{D}(F) \quad (12)$$

include a structural condition on its degree of nonlinearity (cf. [4])

$$\begin{aligned} \|(F'(x^\dagger + v) - F'(x^\dagger))v\| &\leq K \|F'(x^\dagger)v\|^{c_1} D_q^{x_0}(x^\dagger, v + x^\dagger)^{c_2}, \\ v \in X, x^\dagger + v \in \mathcal{B}_\rho^D(x^\dagger), \end{aligned} \quad (13)$$

whose strength depends on the smoothness index in (11). Namely, we assume that

$$c_1 = 1 \text{ or } c_1 + c_2 p > 1 \text{ or } (c_1 + c_2 p \geq 1 \text{ and } K \text{ is sufficiently small}) \quad (14)$$

$$c_1 + c_2 \frac{2\nu}{\nu+1} \geq 1, \quad (15)$$

so that in case  $\nu = 1$ , a Lipschitz condition on  $F'$ , corresponding to  $(c_1, c_2) = (0, 1)$  is sufficient.

Here  $F'$  denotes the Gateaux derivative of  $F$ , hence a Taylor remainder estimate

$$\begin{aligned} &\|F(x_n^\delta) - F(x^\dagger) - F'(x^\dagger)(x_n^\delta - x^\dagger)\| \quad (16) \\ &= \|g(1) - g(0) - F'(x^\dagger)(x_n^\delta - x^\dagger)\| \\ &= \left\| \int_0^1 g'(t) dt - F'(x^\dagger)(x_n^\delta - x^\dagger) \right\| \\ &= \left\| \int_0^1 F'(x^\dagger + t(x_n^\delta - x^\dagger))(x_n^\delta - x^\dagger) dt - F'(x^\dagger)(x_n^\delta - x^\dagger) \right\| \\ &\leq K \|F'(x^\dagger)(x_n^\delta - x^\dagger)\|^{c_1} D_q^{x_0}(x^\dagger, x_n^\delta)^{c_2} \end{aligned} \quad (17)$$

where  $g : t \mapsto F(x^\dagger + t(x_n^\delta - x^\dagger))$ , follows from (13).

We will assume that in each step the step size  $\mu_n > 0$  in (9) is chosen such that

$$\mu_n \frac{1 - 3C(c_1)K}{3(1 - C(c_1)K)} \|F(x_n^\delta) - y^\delta\|^p - 2^{q^* + q - 2} C_{q^*} \mu_n^{q^*} \|A_n^* j_p^Y(F(x_n^\delta) - y^\delta)\|^{q^*} \geq 0 \quad (18)$$

where  $C(c_1) = c_1^{c_1} (1 - c_1)^{1 - c_1}$ , and  $c_1, K$  are as in (13), which is possible, e.g.,

by a choice  $0 < \mu_n \leq C_\mu \frac{\|F(x_n^\delta) - y^\delta\|^{\frac{q-p}{q-1}}}{\|A_n\|^{q^*}} =: \bar{\mu}_n$  with  $C_\mu := \frac{2^{2-q^*-q}}{3} \frac{1 - 3C(c_1)K}{(1 - C(c_1)K)C_{q^*}}$ . If

$$p \geq q \quad (19)$$

and  $F, F'$  are bounded on  $\mathcal{B}_\rho^D(x^\dagger)$ , it is possible to bound  $\bar{\mu}_n$  away from zero

$$\bar{\mu}_n \geq C_\mu \left( \sup_{x \in \mathcal{B}_\rho^D(x^\dagger)} (\|F(x) - y\| + \bar{\delta})^{p-q} \|F'(x)\|^q \right)^{-1/(q-1)} =: \underline{\mu} \quad (20)$$

for  $\delta \in [0, \bar{\delta}]$ , provided the iterates remain in  $\mathcal{B}_\rho^D(x^\dagger)$  (which we will show by induction in the proof of Theorem 1). Hence, there exist  $\underline{\mu}, \bar{\mu} > 0$  independent of  $n$  and  $\delta$  such that we can choose

$$0 < \underline{\mu} \leq \mu_n \leq \bar{\mu}, \quad (21)$$

(e.g., by simply setting  $\mu_n \equiv \underline{\mu}$ ).

Moreover, we will use an a priori choice of the stopping index  $n_*$  according to

$$n_*(\delta) = \min\{n \in \mathbb{N} : \alpha_n^{\frac{\nu+1}{p(\nu+1)-2\nu}} \leq \tau\delta\}, \quad (22)$$

and of  $\{\alpha_n\}_{n \in \mathbb{N}}$  such that

$$\left(\frac{\alpha_{n+1}}{\alpha_n}\right)^{\frac{2\nu}{p(\nu+1)-2\nu}} + \frac{1}{3}\alpha_n - 1 \geq c\alpha_n \quad (23)$$

for some  $c \in (0, \frac{1}{3})$  independent of  $n$ , where  $\nu \in [0, 1]$  is the exponent in the variational inequality (11).

*Remark 1.* A possible choice of  $\{\alpha_n\}_{n \in \mathbb{N}}$  satisfying (23) and smallness of  $\alpha_{\max}$  is given by

$$\alpha_n = \frac{\alpha_0}{(n+1)^x}$$

with  $x \in (0, 1]$  such that  $3x\theta < \alpha_0$  sufficiently small, since then with  $c := \frac{1}{3} - \frac{x\theta}{\alpha_0} > 0$ , using the abbreviation  $\theta = \frac{2\nu}{p(\nu+1)-2\nu} \in [0, \frac{1}{p-1}]$  we get by the Mean Value Theorem

$$\begin{aligned} & \left(\frac{\alpha_{n+1}}{\alpha_n}\right)^\theta + \left(\frac{1}{3} - c\right)\alpha_n - 1 \\ &= \frac{\alpha_n}{\alpha_0} \left\{ \alpha_0 \left(\frac{1}{3} - c\right) - \frac{(n+2)^{x\theta} - (n+1)^{x\theta}}{(n+2)^{x\theta}} (n+1)^x \right\} \\ &= \frac{\alpha_n}{\alpha_0} \left\{ \alpha_0 \left(\frac{1}{3} - c\right) - \frac{x\theta(n+1+t)^{x\theta-1}}{(n+2)^{x\theta}} (n+1)^x \right\} \\ &\geq \frac{\alpha_n}{\alpha_0} \left\{ \alpha_0 \left(\frac{1}{3} - c\right) - x\theta \frac{(n+1)^x}{n+1+t} \right\} \geq 0, \end{aligned}$$

for some  $t \in [0, 1]$ .

**Theorem 1.** *Assume that  $X$  is smooth and  $q$ -convex, that  $x_0$  is sufficiently close to  $x^\dagger$ , i.e.,  $x_0 \in \mathcal{B}_\rho^D(x^\dagger)$ , (which by (5) implies that  $\|x^\dagger - x_0\|$  is also small), that a variational inequality (11) with  $\nu \in (0, 1]$  and  $\beta$  sufficiently small is satisfied, that  $F$  satisfies (13) with (14), (15), that  $F$  and  $F'$  are continuous and uniformly bounded in  $\mathcal{B}_\rho^D(x^\dagger)$ , that (12) holds and that*

$$q^* \geq \frac{2\nu}{p(\nu+1) - 2\nu} + 1. \quad (24)$$

Let  $n_*(\delta)$  be chosen according to (22) with  $\tau$  sufficiently large. Moreover assume that (19) holds and the sequence  $\{\mu_n\}_{n \in \mathbb{N}}$  is chosen such that (21) holds for  $0 < \underline{\mu} < \bar{\mu}$  according to (20), and let the sequence  $\{\alpha_n\}_{n \in \mathbb{N}} \subseteq [0, 1]$  be chosen such that (23) holds, and  $\alpha_{\max} = \max_{n \in \mathbb{N}} \alpha_n$  is sufficiently small.

Then, the iterates  $x_{n+1}^\delta$  remain in  $\mathcal{B}_\rho^D(x^\dagger)$  for all  $n \leq n_*(\delta) - 1$  with  $n_*$  according to (22). Moreover, we obtain optimal convergence rates

$$D_q^{x_0}(x^\dagger, x_{n_*}) = O(\delta^{\frac{2\nu}{\nu+1}}), \quad \text{as } \delta \rightarrow 0 \quad (25)$$

as well as in the noise free case  $\delta = 0$

$$D_q^{x_0}(x^\dagger, x_n) = O\left(\alpha_n^{\frac{2\nu}{p(\nu+1)-2\nu}}\right) \quad (26)$$

for all  $n \in \mathbb{N}$ .

*Remark 2.* Note that the rate exponent in (26)  $\frac{2\nu}{p(\nu+1)-2\nu} = \frac{2\nu}{\nu+1}(p - \frac{2\nu}{\nu+1})^{-1}$ , always lies in the interval  $[0, \frac{1}{p-1}]$ , since  $\frac{2\nu}{\nu+1} \in [0, 1]$ .

Moreover, note that Theorem 1 provides a results on rates only, but no convergence result without variational inequality. This corresponds to the situation from [11] in a Hilbert space setting.

*Proof.* First of all, for  $x_n^\delta \in \mathcal{B}_\rho^D(x^\dagger)$ , (13) allows us to estimate as follows (see also (16)) in case  $c_1 \in [0, 1]$ :

$$\begin{aligned} & \|F(x_n^\delta) - F(x^\dagger) - A(x_n^\delta - x^\dagger)\| \\ & \leq K \|A(x_n^\delta - x^\dagger)\|^{c_1} D_q^{x_0}(x^\dagger, x_n^\delta)^{c_2} \\ & \leq C(c_1)K \left( \|A(x_n^\delta - x^\dagger)\| + D_q^{x_0}(x^\dagger, x_n^\delta)^{\frac{c_2}{1-c_1}} \right), \end{aligned} \quad (27)$$

where we have used the abbreviation  $A = F'(x^\dagger)$  and the elementary estimate

$$a^{1-\lambda}b^\lambda \leq C(\lambda)(a+b) \text{ with } C(\lambda) = \lambda^\lambda(1-\lambda)^{1-\lambda} \text{ for } a, b \geq 0, \lambda \in (0, 1), \quad (28)$$

and therewith, by the second triangle inequality,

$$\|A(x_n^\delta - x^\dagger)\| \leq \frac{1}{1 - C(c_1)K} \left( \|F(x_n^\delta) - F(x^\dagger)\| + C(c_1)K D_q^{x_0}(x^\dagger, x_n^\delta)^{\frac{c_2}{1-c_1}} \right) \quad (29)$$

as well as analogously

$$\begin{aligned} & \|F(x_n^\delta) - F(x^\dagger) - A_n(x_n^\delta - x^\dagger)\| \\ & \leq 2C(c_1)K \left( \|A(x_n^\delta - x^\dagger)\| + D_q^{x_0}(x^\dagger, x_n^\delta)^{\frac{c_2}{1-c_1}} \right) \\ & \leq \frac{2C(c_1)K}{1 - C(c_1)K} \left( \|F(x_n^\delta) - F(x^\dagger)\| + D_q^{x_0}(x^\dagger, x_n^\delta)^{\frac{c_2}{1-c_1}} \right). \end{aligned} \quad (30)$$

For any  $n \leq n_*$  according to (22), by (7) we have

$$\begin{aligned}
 & D_q^{x_0}(x^\dagger, x_{n+1}^\delta) - D_q^{x_0}(x^\dagger, x_n^\delta) \\
 &= D_q^{x_0}(x_n^\delta, x_{n+1}^\delta) + \langle J_q^X(x_n^\delta - x_0) - J_q^X(x_{n+1}^\delta - x_0), x^\dagger - x_n^\delta \rangle_{X^* \times X} \\
 &= D_q^{x_0}(x_n^\delta, x_{n+1}^\delta) - \mu_n \langle j_p^Y(F(x_n^\delta) - y^\delta), A_n(x_n^\delta - x^\dagger) \rangle_{Y^* \times Y} \\
 &\quad + \alpha_n \langle J_q^X(x^\dagger - x_0), x^\dagger - x_n^\delta \rangle_{X^* \times X} \\
 &\quad - \alpha_n \langle J_q^X(x^\dagger - x_0) - J_q^X(x_n^\delta - x_0), x^\dagger - x_n^\delta \rangle_{X^* \times X}
 \end{aligned} \tag{31}$$

where the terms on the right hand side can be estimated as follows.

By (6) and (8) we have

$$\begin{aligned}
 & D_q^{x_0}(x_n^\delta, x_{n+1}^\delta) \\
 &\leq C_{q^*} \|J_q^X(x_{n+1}^\delta - x_0) - J_q^X(x_n^\delta - x_0)\|^{q^*} \\
 &= C_{q^*} \|\alpha_n J_q^X(x_n^\delta - x_0) + \mu_n A_n^* j_p^Y(F(x_n^\delta) - y^\delta)\|^{q^*} \\
 &\leq 2^{q^*-1} C_{q^*} \left( \alpha_n^{q^*} \|x_n^\delta - x_0\|^q + \mu_n^{q^*} \|A_n^* j_p^Y(F(x_n^\delta) - y^\delta)\|^{q^*} \right) \\
 &\leq 2^{q^*-1} C_{q^*} \left( \alpha_n^{q^*} (2^{q-1} (\|x^\dagger - x_0\|^q + \frac{1}{c_q} D_q^{x_0}(x^\dagger, x_n^\delta)) + \mu_n^{q^*} \|A_n^* j_p^Y(F(x_n^\delta) - y^\delta)\|^{q^*}) \right)
 \end{aligned} \tag{32}$$

where we have used the triangle inequality in  $X^*$  and  $X$ , the inequality

$$(a + b)^\lambda \leq 2^{\lambda-1}(a^\lambda + b^\lambda) \quad \text{for } a, b \geq 0, \lambda \geq 1, \tag{34}$$

and (5).

For the second term on the right hand side of (31) we get, using (30), (28), (34),

$$\begin{aligned}
 & \langle j_p^Y(F(x_n^\delta) - y^\delta), A_n(x_n^\delta - x^\dagger) \rangle_{Y^* \times Y} \\
 &= \langle j_p^Y(F(x_n^\delta) - y^\delta), F(x_n^\delta) - y^\delta \rangle_{Y^* \times Y} \\
 &\quad - \langle j_p^Y(F(x_n^\delta) - y^\delta), F(x_n^\delta) - y^\delta - A_n(x_n^\delta - x^\dagger) \rangle_{Y^* \times Y} \\
 &\geq \frac{1 - 3C(c_1)K}{1 - C(c_1)K} \|F(x_n^\delta) - y^\delta\|^p \\
 &\quad - \|F(x_n^\delta) - y^\delta\|^{p-1} \left( \frac{2C(c_1)K}{1 - C(c_1)K} D_q^{x_0}(x^\dagger, x_n^\delta)^{\frac{c_2}{1-c_1}} + \frac{1 + C(c_1)K}{1 - C(c_1)K} \delta \right) \\
 &= \frac{1 - 3C(c_1)K}{1 - C(c_1)K} \|F(x_n^\delta) - y^\delta\|^p \\
 &\quad - \left( \frac{1 - 3C(c_1)K}{3C(\frac{p-1}{p})(1 - C(c_1)K)} \|F(x_n^\delta) - y^\delta\|^p \right)^{\frac{p-1}{p}} \left( \frac{(3C(\frac{p-1}{p}))^{p-1}}{(1 - C(c_1)K)} \right)^{\frac{1}{p}} \\
 &\quad \left( 2C(c_1)K D_q^{x_0}(x^\dagger, x_n^\delta)^{\frac{c_2}{1-c_1}} + (1 + C(c_1)K)\delta \right)
 \end{aligned}$$



$$\begin{aligned}
&\geq \frac{1-3C(c_1)K}{1-C(c_1)K} \|F(x_n^\delta) - y^\delta\|^p - C\left(\frac{p-1}{p}\right) \left\{ \frac{1-3C(c_1)K}{3C\left(\frac{p-1}{p}\right)(1-C(c_1)K)} \|F(x_n^\delta) - y^\delta\|^p \right. \\
&\quad \left. + \frac{(3C\left(\frac{p-1}{p}\right))^{p-1}}{(1-C(c_1)K)} 2^{p-1} \left( (2C(c_1)K)^p D_q^{x_0}(x^\dagger, x_n^\delta)^{\frac{c_2 p}{1-c_1}} + (1+C(c_1)K)^p \delta^p \right) \right\}. \tag{35}
\end{aligned}$$

Using the variational inequality (11), (29), and

$$(a+b)^\lambda \leq (a^\lambda + b^\lambda) \quad \text{for } a, b \geq 0, \lambda \in [0, 1], \tag{36}$$

we get

$$\begin{aligned}
&|\alpha_n \langle J_q^X(x^\dagger - x_0), x^\dagger - x_n^\delta \rangle_{X^* \times X}| \\
&\leq \beta \alpha_n D_q^{x_0}(x^\dagger, x_n^\delta)^{\frac{1-\nu}{2}} \|F'(x^\dagger)(x_n^\delta - x^\dagger)\|^\nu \\
&\leq \beta \alpha_n D_q^{x_0}(x^\dagger, x_n^\delta)^{\frac{1-\nu}{2}} \frac{1}{(1-C(c_1)K)^\nu} \left( \|F(x_n^\delta) - y^\delta\| + \delta + C(c_1)K D_q^{x_0}(x^\dagger, x_n^\delta)^{\frac{c_2}{1-c_1}} \right)^\nu \\
&\leq \beta \alpha_n D_q^{x_0}(x^\dagger, x_n^\delta)^{\frac{1-\nu}{2}} \epsilon^{-\nu} \left( \epsilon \frac{1}{(1-C(c_1)K)^\nu} (\|F(x_n^\delta) - y^\delta\| + \delta) \right)^\nu \\
&\quad + \beta \alpha_n \left( \frac{C(c_1)K}{(1-C(c_1)K)} \right)^\nu D_q^{x_0}(x^\dagger, x_n^\delta)^{\frac{1-\nu}{2} + \frac{\nu c_2}{1-c_1}} \\
&\leq C\left(\frac{\nu}{p}\right) \left\{ \left( \beta \alpha_n D_q^{x_0}(x^\dagger, x_n^\delta)^{\frac{1-\nu}{2}} \epsilon^{-\nu} \right)^{\frac{p}{p-\nu}} + \left( \epsilon \frac{1}{(1-C(c_1)K)^\nu} (\|F(x_n^\delta) - y^\delta\| + \delta) \right)^p \right\} \\
&\quad + \beta \alpha_n \left( \frac{C(c_1)K}{(1-C(c_1)K)} \right)^\nu D_q^{x_0}(x^\dagger, x_n^\delta)^{\frac{1-\nu}{2} + \frac{\nu c_2}{1-c_1}} \\
&= C\left(\frac{\nu}{p}\right) \left\{ \left( \beta \epsilon^{-\nu} \right)^{\frac{p}{p-\nu}} \left( 3C\left(\frac{\nu}{p}\right) C\left(\frac{p(1-\nu)}{2(p-\nu)}\right) \right)^{\frac{p(1-\nu)}{2(p-\nu)}} \alpha_n^{\frac{p(1+\nu)}{2(p-\nu)}} \left( \frac{\alpha_n D_q^{x_0}(x^\dagger, x_n^\delta)}{3C\left(\frac{\nu}{p}\right) C\left(\frac{p(1-\nu)}{2(p-\nu)}\right)} \right)^{\frac{p(1-\nu)}{2(p-\nu)}} \right\} \\
&\quad + \left( \epsilon \frac{1}{(1-C(c_1)K)^\nu} (\|F(x_n^\delta) - y^\delta\| + \delta) \right)^p \\
&\quad + \beta \alpha_n \left( \frac{C(c_1)K}{(1-C(c_1)K)} \right)^\nu D_q^{x_0}(x^\dagger, x_n^\delta)^{\frac{1-\nu-c_1+\nu c_1+2\nu c_2}{2(1-c_1)}} \\
&\leq C\left(\frac{\nu}{p}\right) \left\{ C\left(\frac{p(1-\nu)}{2(p-\nu)}\right) \left[ \left( \beta \epsilon^{-\nu} \left( 3C\left(\frac{\nu}{p}\right) C\left(\frac{p(1-\nu)}{2(p-\nu)}\right) \right)^{\frac{1-\nu}{2}} \right)^{\frac{2p}{p(\nu+1)-2\nu}} \alpha_n^{\frac{p(1+\nu)}{p(\nu+1)-2\nu}} \right. \right. \\
&\quad \left. \left. + \left( \frac{\alpha_n D_q^{x_0}(x^\dagger, x_n^\delta)}{3C\left(\frac{\nu}{p}\right) C\left(\frac{p(1-\nu)}{2(p-\nu)}\right)} \right) \right] \right\}
\end{aligned}$$

$$\begin{aligned}
 & + \left( \epsilon \frac{1}{(1 - C(c_1)K)^\nu} (\|F(x_n^\delta) - y^\delta\| + \delta) \right)^p \Big\} \\
 & + \frac{1}{3} \alpha_n D_q^{x_0}(x^\dagger, x_n^\delta)
 \end{aligned} \tag{37}$$

where we have used (28) two times and  $\epsilon > 0$  will be chosen as a sufficiently small number below. Moreover, by (15), the exponent  $\frac{1-\nu-c_1+\nu c_1+2\nu c_2}{2(1-c_1)} = 1 + \frac{1+\nu}{2(1-c_1)}(c_1 + \frac{2\nu}{\nu+1}c_2 - 1)$  is larger or equal to one and  $\beta$  is sufficiently small so that  $\beta \left( \frac{C(c_1)K}{(1-C(c_1)K)} \right)^\nu \rho^{\frac{1-\nu-c_1+\nu c_1+2\nu c_2}{2(1-c_1)}-1} < \frac{1}{3}$ .

Finally, we have that

$$\begin{aligned}
 \langle J_q^X(x^\dagger - x_0) - J_q^X(x_n^\delta - x_0), x^\dagger - x_n^\delta \rangle_{X^* \times X} &= D_q^{x_0}(x^\dagger, x_n^\delta) + D_q^{x_0}(x_n^\delta, x^\dagger) \\
 &\geq D_q^{x_0}(x^\dagger, x_n^\delta)
 \end{aligned} \tag{38}$$

Inserting estimates (32)-(38) with  $\epsilon = 2^{p-1} \mu_n^{1/p} \left( \frac{1-3C(c_1)K}{3(1-C(c_1)K)} \right)^{1/p} \frac{(1-C(c_1)K)^\nu}{C(\frac{\nu}{p})}$  into (31) and using boundedness away from zero of  $\mu_n$  and the abbreviations

$$\begin{aligned}
 d_n &= D_q^{x_0}(x^\dagger, x_n^\delta)^{1/2} \\
 C_0 &= 6^{p-1} C \left( \frac{p-1}{p} \right)^p \frac{(2C(c_1)K)^p}{(1-C(c_1)K)} \\
 C_1 &= 2^{q^*+q-2} \frac{C_{q^*}}{c_q} \\
 C_2 &= C \left( \frac{\nu}{p} \right) C \left( \frac{p(1-\nu)}{2(p-\nu)} \right) \left( \beta \epsilon^{-\nu} (3C \left( \frac{\nu}{p} \right) C \left( \frac{p(1-\nu)}{2(p-\nu)} \right)^{\frac{1-\nu}{2}}) \right)^{\frac{2p}{p(\nu+1)-2\nu}} \\
 C_3 &= 2^{q^*+q-2} C_{q^*} \|x^\dagger - x_0\|^q \\
 C_4 &= 2^{p-1} C \left( \frac{\nu}{p} \right) \bar{\epsilon} \frac{1}{(1-C(c_1)K)^\nu} + 6^{p-1} C \left( \frac{p-1}{p} \right)^p \frac{(1+C(c_1)K)^p}{1-C(c_1)K} \\
 \underline{\epsilon} &= 2^{p-1} \underline{\mu}^{1/p} \left( \frac{1-3C(c_1)K}{3(1-C(c_1)K)} \right)^{1/p} \frac{(1-C(c_1)K)^\nu}{C(\frac{\nu}{p})} \\
 \bar{\epsilon} &= 2^{p-1} \bar{\mu}^{1/p} \left( \frac{1-3C(c_1)K}{3(1-C(c_1)K)} \right)^{1/p} \frac{(1-C(c_1)K)^\nu}{C(\frac{\nu}{p})}
 \end{aligned}$$

we obtain

$$\begin{aligned}
 d_{n+1}^2 &\leq C_0 d_n^{\frac{2c_2 p}{1-c_1}} + (1 - \frac{1}{3} \alpha_n + C_1 \alpha_n^{q^*}) d_n^2 + C_2 \alpha_n^{\frac{p(1+\nu)}{p(\nu+1)-2\nu}} + C_3 \alpha_n^{q^*} + C_4 \delta^p \\
 &- \left( \mu_n \frac{1-3C(c_1)K}{3(1-C(c_1)K)} \|F(x_n^\delta) - y^\delta\|^p - 2^{q^*+q-2} C_{q^*} \mu_n^{q^*} \|A_n^* J_p^Y(F(x_n^\delta) - y^\delta)\|^{q^*} \right).
 \end{aligned}$$

Here the last term is nonpositive due to the choice (18) of  $\mu_n$ , so that we arrive at

$$d_{n+1}^2 \leq C_0 d_n^{\frac{2c_2 p}{1-c_1}} + \left(1 - \frac{1}{3}\alpha_n + C_1 \alpha_n^{q^*}\right) d_n^2 + \underbrace{(C_2 + C_3 + C_4 \tau^{-p})}_{=: C_5} \alpha_n^{\frac{p(1+\nu)}{p(\nu+1)-2\nu}} \quad (39)$$

where we have used (24) and the stopping rule (22). Denoting

$$\gamma_n := \frac{d_n^2}{\alpha_n^{\frac{2\nu}{p(\nu+1)-2\nu}}}$$

we get the following recursion

$$\gamma_{n+1} \leq C_0 \left(\frac{\alpha_n}{\alpha_{n+1}}\right)^\theta \alpha_n^{\theta\omega} \gamma_n^\omega + \left(\frac{\alpha_n}{\alpha_{n+1}}\right)^\theta \left(1 - \frac{1}{3}\alpha_n + C_1 \alpha_n^{q^*}\right) \gamma_n + C_5 \left(\frac{\alpha_n}{\alpha_{n+1}}\right)^\theta \alpha_n \quad (40)$$

with

$$\theta = \frac{2\nu}{p(\nu+1)-2\nu} \quad \omega = \frac{c_2 p}{1-c_1},$$

where

$$\omega \geq 1$$

by (14) and

$$\theta\omega = \frac{p}{p - \frac{2\nu}{\nu+1}} \frac{c_2 \frac{2\nu}{\nu+1}}{1-c_1} \geq 1$$

due to assumption (15). Hence as sufficient conditions for uniform boundedness of  $\{\gamma_n\}_{n \leq n_*}$  by  $\bar{\gamma}$  and for  $x_{n+1}^\delta \in \mathcal{B}_\rho^D(x^\dagger)$  we get

$$\bar{\gamma} \leq \rho^2 \quad (41)$$

$$C_0 \alpha_n^{\theta\omega-1} \bar{\gamma}^\omega - \left\{ \left(\frac{\alpha_{n+1}}{\alpha_n}\right)^\theta + \frac{1}{3}\alpha_n - 1 - C_1 \alpha_n^{q^*} \right\} \alpha_n^{-1} \bar{\gamma} + C_5 \leq 0, \quad (42)$$

where by  $q^* > 1$ , (15) the factors  $C_0 \alpha_n^{\theta\omega-1}$ ,  $C_1 \alpha_n^{q^*-1}$  and  $C_5$  can be made small for small  $\alpha_{\max}$ ,  $\beta$ ,  $\|x^\dagger - x_0\|$  and large  $\tau$ . We use this fact to achieve

$$C_0 \alpha_n^{\theta\omega-1} \rho^{\omega-1} + C_1 \alpha_n^{q^*-1} \leq \tilde{c} < c$$

with  $\tilde{c}$  independent of  $n$ , which together with (23) yields sufficiency of

$$\frac{C_5}{c - \tilde{c}} \leq \bar{\gamma} \leq \rho^2$$

for (41), (42), which for any (even small) prescribed  $\rho$  is indeed enabled by possibly decreasing  $\beta$ ,  $\|x^\dagger - x_0\|$ ,  $\tau^{-1}$ , and therewith  $C_5$ .

In case  $c_1 = 1$ , estimates (29), (30) simplify to

$$\|A(x_n^\delta - x^\dagger)\| \leq \frac{1}{1 - \rho^{2c_2} K} \|F(x_n^\delta) - F(x^\dagger)\| \quad (43)$$

and

$$\|F(x_n^\delta) - F(x^\dagger) - A_n(x_n^\delta - x^\dagger)\| \leq \frac{2\rho^{2c_2} K}{1 - \rho^{2c_2} K} \|F(x_n^\delta) - F(x^\dagger)\|. \quad (44)$$

Therewith, the terms containing  $D_q^{x_0}(x^\dagger, x_n^\delta)^{\frac{c_2}{1-c_1}}$  are removed and  $C(c_1)$  is replaced by  $\rho^{2c_2}$  in (32)-(38), so that we end up with a recursion of the form (40) (with  $C_0$  replace by zero) as before. Hence the remainder of the proof of uniform boundedness of  $\gamma_n$  can be done in the same way as in case  $c_1 < 1$ .

In case  $\delta = 0$ , i.e.,  $n_* = \infty$ , uniform boundedness of  $\{\gamma_n\}_{n \in \mathbb{N}}$  implies (26). For  $\delta > 0$  we get (25) by using (22) in

$$D_q^{x_0}(x^\dagger, x_{n_*}) = \gamma_{n_*} \alpha_{n_*}^{\frac{2\nu}{p(\nu+1)-2\nu}} \leq \bar{\gamma} \alpha_{n_*}^{\frac{2\nu}{p(\nu+1)-2\nu}} \leq \bar{\gamma}(\tau\delta)^{\frac{2\nu}{\nu+1}}$$

*Remark 3.* In view of estimate (39), an optimal choice of  $\alpha_n$  would be one that minimizes the right hand side. At least in the special case that the same power of  $\alpha_n$  appears in the last two terms, i.e.,  $\frac{p(1+\nu)}{p(\nu+1)-2\nu} = q^*$ , elementary calculus yields

$$(\alpha_n^{opt})^{\frac{2\nu}{p(\nu+1)-2\nu}} = \frac{D_q^{x_0}(x^\dagger, x_n^\delta)}{3q^*(C_1 D_q^{x_0}(x^\dagger, x_n^\delta) + C_5)},$$

which shows that the obtained relation  $D_q^{x_0}(x^\dagger, x_n^\delta) \sim \alpha_n^{\frac{2\nu}{p(\nu+1)-2\nu}}$  is indeed reasonable and probably even optimal.

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