

On an Algorithm for Dynamic Reconstruction in Systems with Delay in Control

Marina Blizorukova

► **To cite this version:**

Marina Blizorukova. On an Algorithm for Dynamic Reconstruction in Systems with Delay in Control. 25th System Modeling and Optimization (CSMO), Sep 2011, Berlin, Germany. pp.225-234, 10.1007/978-3-642-36062-6_23 . hal-01347542

HAL Id: hal-01347542

<https://hal.inria.fr/hal-01347542>

Submitted on 21 Jul 2016

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



On an Algorithm for Dynamic Reconstruction in Systems with Delay in Control

Marina Blizorukova*

Ural Federal University and
Institute of Mathematics and Mechanics,
Ural Branch of the Russian Academy of Sciences,
Ekaterinburg, 620990 Russia
msb@imm.uran.ru

Abstract. We discuss a problem of the dynamic reconstruction of unknown input controls in nonlinear vector equations. A regularizing algorithm is proposed for reconstructing these controls simultaneously with the processes. The algorithm is stable with respect to informational noises and computational errors.

Keywords: dynamic reconstruction, method of auxiliary models

1 Introduction. Problem statement.

Consider a controlled system described by the following equation

$$\dot{x}(t) = f_1(t, u_t(s), x_t(s)) + f_2(t, x_t(s))u(t) \quad (1)$$

with the initial state

$$u_{t_0}(s) = u_0(s) \in C([- \tau_m^u, 0]; R^{n_1}), \quad x_{t_0}(s) = x_0(s) \in C([- \tau_n^x, 0]; R^{n_2}). \quad (2)$$

Here t is time from a fixed interval $T = [t_0, \vartheta]$ ($t_0 < \vartheta < +\infty$); $x(t) = (x_1(t), \dots, x_{n_2}(t))$ is the phase state of the system; $u(t) = (u_1(t), \dots, u_{n_1}(t))$ is a control; the symbols $x_t(s)$ and $u_t(s)$ mean the functions $x_t(s) = x(t+s)$ for $s \in [-\tau_n^x, 0]$ and $u_t(s) = u(t+s)$ for $s \in [-\tau_m^u, 0]$, respectively. We assume that initial state (2) is Lipschitz. For simplicity, we assume also that the initial state $x_0(s)$, $u_0(s)$ is fixed and known. The control $u = u(t) = (u_1(t), \dots, u_{n_1}(t))$ is called an admissible control if its components $u_i(t)$, $i \in [1 : n_1]$, are Lebesgue measurable functions on the interval T and values $u(t)$ belong to a given compact set P from Euclidean space R^{n_1} for almost all $t \in T$. The set of all admissible controls is denoted by $P(\cdot)$. Therefore, $P(\cdot) = \{u(\cdot) \in L_2(T; R^{n_1}) : u(t) \in P \text{ for a. a. } t \in T\}$. By the trajectory (or the solution) $x(\cdot)$ of equation (1) with

* This work was supported by the Russian Foundation for Basic Research (12-01-00175-a), by the Ural-Siberian Integration Project (12-C-1-1017), and by the Program for support of leading scientific schools of Russia (6512.2012.1).

initial state (2) corresponding to some admissible control $u(\cdot)$, we call absolutely continuous on T function $x = x(t)$ satisfying (1) for a.a. $t \in T$.

Condition 1. The elements of matrix function

$$f_{2ij}(t, x_t(s)) = f_{2ij}(t, x(t), x(t - \tau_1^x), \dots, x(t - \tau_n^x)), \quad i \in [1 : n_2], \quad j \in [1 : n_1],$$

and vector-valued function

$$\begin{aligned} f_{1i}(t, u_t(s), x_t(s)) &= \\ &= f_{1i}(t, u(t - \tau_1^u), \dots, u(t - \tau_m^u), x(t), x(t - \tau_1^x), \dots, x(t - \tau_n^x)), \quad i \in [1 : n_2] \end{aligned}$$

satisfy the Lipschitz conditions

$$|f_{2ij}(t_1, x_0^{(1)}, x_1^{(1)}, \dots, x_n^{(1)}) - f_{2ij}(t_2, x_0^{(1)}, x_1^{(2)}, \dots, x_n^{(2)})| \leq \quad (3)$$

$$\leq C_1(|t_2 - t_1| + \sum_{j=0}^n |x_j^{(1)} - x_j^{(2)}|),$$

$$|f_{1i}(t_1, u_1^{(1)}, \dots, u_m^{(1)}, x_0^{(1)}, x_1^{(1)}, \dots, x_n^{(1)}) - f_{1i}(t_2, u_1^{(2)}, \dots, u_m^{(2)}, x_0^{(2)}, x_1^{(2)}, \dots, x_n^{(2)})|$$

$$\leq d_1(|t_2 - t_1| + \sum_{i=1}^m |u_i^{(1)} - u_i^{(2)}| + \sum_{j=0}^n |x_j^{(1)} - x_j^{(2)}|). \quad (4)$$

In this case, under this condition for any pair, i.e., for initial state (2) and the control $u(\cdot) \in P(\cdot)$, there exists a unique solution of equation (1).

Let $u(\cdot)$ be an admissible control realizing during the given time interval T ; $x(\cdot)$ be the real motion generated by this control. We assume that the phase states $x(\tau_i)$ of the system are inaccurately measured at frequent enough time moments $\tau_i \in T$ in the process. Measurement results $\xi^h(\tau_i) \in R^{n_2}$ satisfy the inequalities

$$|\xi^h(\tau_i) - x(\tau_i)| \leq h. \quad (5)$$

Here, the quantity $h \in (0, 1)$ specifies the measurement error.

In the present paper, we construct an algorithm that reconstructs the control $u(\cdot)$ on the basis of the current information $\xi^h(\cdot)$ in real time. Since the exact reconstruction is impossible due to the error of measurements $\xi^h(\cdot)$ we require that the algorithm should generate some approximation. Namely, it is required to construct an algorithm allowing us, on the basis of the inaccurate measurements $\xi^h(\cdot)$, and in real time, to form the admissible control $v^h(\cdot)$ such that the mean-square deviation of $v^h(\cdot)$ from $u(\cdot)$; i.e.,

$$|v^h(\cdot) - u(\cdot)|_{L_2(T)}^2 = \int_{t_0}^{\vartheta} |v^h(t) - u(t)|^2 dt, \quad (6)$$

is arbitrarily small for the sufficiently small measurement error h . Since the measurements are inaccurate it is in general impossible to identify $u(t)$ precisely, therefore the problem is to approximate the input by some function $v^h(t)$.

Here and below, the symbol $|\cdot|$ stands for both the Euclidean norm and the corresponding matrix norm and for the modulo of a number. In what follows, we set $\tau_m^u = \tau_n^x = \tau$ for simplicity, and by $\xi^h(\cdot)$ we denote the function $\xi^h(t)$, $t \in [t_0 - \tau, \vartheta]$ such that $\xi^h(t) = x_0(t - t_0)$ for $t \in [t_0 - \tau, t_0)$, $\xi^h(t) = \xi^h(\tau_i)$ for $t \in [\tau_i, \tau_{i+1})$, $i \in [0 : d - 1]$, where $\tau_i = \tau_{h,i}$, $d = d_h$, $\xi^h(\tau_i)$ satisfies (5).

The suggested solution outline is the following ([1–6]). An auxiliary control system (model M) described by equation of the form

$$\dot{w}(t) = F(t, \xi_t^h(s), v_t^h(s)), \quad w_{t_0}(s) = w_0(s), \quad t \in T \quad (7)$$

is associated with the real dynamical system (1). Here the vector $w \in R^{n_2}$ characterizes state of the model, the form of function F is corrected below, vector v^h is control action. After that, the problem of reconstruction of input $u(\cdot)$ is replaced by the problem of positional control of the model. This process is realized on the time interval T in such a way that control $v^h(\cdot)$ “approximates” appropriately $u(\cdot)$. First, one takes a uniform net $\Delta = \{\tau_i\}_{i=0}^m$, $\tau_{i+1} = \tau_i + \delta$, $\delta > 0$, $i \in [0 : m]$, $\tau_0 = 0$, $\tau_m = T$ with the step δ . Then, on the interval $t \in [\tau_i, \tau_{i+1})$ the model is acted upon the controls

$$v_i^h = V_h(\tau_i, w_{\tau_i}(s), \xi_{\tau_i}^h(s)) \quad (8)$$

calculated at the moment τ_i by use of some rule, which hereinafter we shall identify with mapping V_h . Thus, the controls in the model are realized by the method of feedback control. Its value on the interval $[\tau_i, \tau_{i+1}]$ depends on the measurement results $\xi^h(\cdot)$ corresponding to the phase state $x(\cdot)$ of the system (1) and state w of the model (7). The described process forms the piece-wise function

$$v^h(t) = v_i^h, \quad t \in [\tau_i, \tau_{i+1})$$

in real time synchro with the motion of real system (1). Thus, to solve the problem above, we should specify a model and a control law for this model.

2 Algorithm for solving the problem

As a model, we take the following system of linear ordinary differential equation

$$\dot{w}(t) = f_1(\tau_i, v_{\tau_i}^h(s), \xi_{\tau_i}^h(s)) + f_2(\tau_i, \xi_{\tau_i}^h(s))v_i^h + 2(\xi^h(\tau_i) - w(\tau_i)), \quad (9)$$

$$w \in R^{n_2}, \quad t \in [\tau_i, \tau_{i+1}), \quad \tau_i = \tau_{h,i}, \quad v_{t_0}^h(s) = u_0(s),$$

with the initial state $w(t_0) = \xi^h(t_0)$. The solution of this equation $w(\cdot) = w(\cdot; t_0, w_{t_0}(s), v^h(\cdot))$ is understood in the sense of Caratheodory. So, the right-hand side of equation of the model (7) has the form

$$\begin{aligned} F(t, \xi_t^h(s), v_t^h(s)) &= f_1(\tau_i, v_{\tau_i}^h(s), \xi_{\tau_i}^h(s)) + f_2(\tau_i, \xi_{\tau_i}^h(s))v_i^h + \\ &+ 2(\xi^h(\tau_i) - w(\tau_i)), \quad t \in [\tau_i, \tau_{i+1}). \end{aligned}$$

Introduce the following notation: $\Delta^{(j)} = [t_j, t_{j+1}]$, $t_j = t_0 + \tau_1^x j$; the symbol l stands for the integer part of the number τ/τ_1^x ; $j_* = \max\{j : t_j < \vartheta\}$,

$$g_j(h) = h^{(1/3)^j}, \quad j \in [1 : j_*].$$

Fix a partition of the interval T with a step $\delta = \delta(h)$ depending on the measurement error h , i.e.,

$$\Delta_h = \{\tau_{h,i}\}_{i=0}^{d_h}, \quad \tau_i = \tau_{h,i}, \quad \tau_{h,0} = t_0, \quad \tau_{h,d_h} = \vartheta, \quad (10)$$

(for simplicity, we assume that $\tau_i - \tau_{i-1} = \delta = \delta(h)$). Without loss of generality, we can suppose that the partition Δ_h is chosen in such a way that $t_j \in \Delta_h$. Define the law of forming the control v_i^h in the model (for $\tau_i \in [t_j, t_{j+1}) \cap T$) by the relations

$$\begin{aligned} V_h(\tau_i, w_{\tau_i}(s), \xi_{\tau_i}^h(s)) &= V_j(\tau_i, w_{\tau_i}(s), \xi_{\tau_i}^h(s)) \\ &= \arg \min\{2(l_i, f_2(\tau_i, \xi_{\tau_i}^h(s))v) + \alpha_j |v|^2 : v \in P\}. \end{aligned} \quad (11)$$

Here α_j is a parameter, $j \in [0 : j_*]$, $l_i = w(\tau_i) - \xi^h(\tau_i)$.

Condition 2. Let $n_2 \geq n_1$, and let there exists a number $c_* > 0$ such that the matrix $f_2(t, x_t(s))$ has a minor of order n_1 with the property: the $n_1 \times n_1$ -dimensional matrix $\bar{f}_2(t) = \bar{f}_2(t, x_t(s))$ corresponding to this minor satisfies the inequality

$$|\bar{f}_2(t)u| \geq c_* |u|$$

for each $t \in T$ and all $u \in R^{n_1}$.

We choose the parameter α_j which plays the role of the regularizer, as follows:

$$\alpha_0 = Ch^{2/3}, \quad \alpha_j = Cg_j^{2/3}(h), \quad j \geq 1, \quad C = \text{const} > 0. \quad (12)$$

Let us describe the algorithm for solving the problem above.

Before the initial moment the value h and the partition $\Delta = \Delta_h$ with diameter $\delta = \delta(h)$ are fixed. The work of the algorithm starting at time $t = 0$ is decomposed into $m_h - 1$ steps. At the i -th step carried out during the time interval $\delta_i = [\tau_i, \tau_{i+1})$, $\tau_i = \tau_{h,i}$, the following actions take place. First, at time moment τ_i vector v_i^h is calculated by formula (11). Then the control $v^h(t) = v_i^h$ is fed onto the input of the model (9). After that, we transform the state $w_{\tau_i}(s)$ of the model into $w_{\tau_{i+1}}(s)$. The procedure stops at time ϑ .

The following theorem is true.

Theorem 1. *Let $\delta = \delta(h) \leq h$. Then the inequalities*

$$\nu^{(j)} \equiv |v^h(\cdot) - u(\cdot)|_{L_2(\Delta^{(j-1)}; R^{n_1})}^2 \leq c_j g_j(h), \quad j \in [1 : j_*],$$

are valid. Here, $v^h(t) = u(t)$ for $t \in [t_0 - \tau, t_0]$, $v^h(t) = u_0(-\tau)$ for $t \in [t_0 - \tau - \tau_1^u, t_0 - \tau)$.

The proof of the theorem is based on auxiliary statements, which are used in forthcoming considerations. Introduce two systems

$$\begin{aligned} \dot{p}(t) &= f_1(t) + f_2(t)u_1(t), \quad t \in T, \\ \dot{q}(t) &= F_1(t) + F_2(t)u_2(t), \end{aligned}$$

where $p(t), q(t) \in R^n$, $f_1(\cdot), F_1(\cdot) \in L_2(T; R^n)$, $f_2(\cdot) \in L_2(T; R^{n \times r})$, $F_2(\cdot) \in L_2(T; R^{n \times r})$, $u_1(\cdot), u_2(\cdot) \in L_2(T; R^r)$, $|u_l(\cdot)|_{L_\infty(T; R^r)} \leq K$, $l = 1, 2$.

Introduce the notation: $\Delta_*^{(j)} = [t_j^*, t_{j+1}^*] \cap T$, $t_j^* = t_0 + \tau_* j$, $j \in [0 : j_0]$, $\Delta^{(-1)} = [t_0 - \tau_*, t_0]$, $\tau_* = \text{const} \in (0, \vartheta - t_0)$, $j_0 = \max\{j : t_j^* \leq \vartheta\}$. Let $r \leq n$ and let there exists a number $c > 0$ such that the matrix $f_2(t)$ has a minor of order r such that the $r \times r$ -matrix $\bar{f}_2(t)$ corresponding to this minor satisfies the following inequality: $|\bar{f}_2(t)u| \geq c|u|$ for each $t \in T$ and all $u \in R^r$.

It is easy to verify the following lemmas.

Lemma 1. *Let the function $t \rightarrow (\bar{f}_2(t))^{-1}u_1(t)$ be a function of bounded variation on T and let the conditions*

$$\begin{aligned} |f_1(\cdot) - F_1(\cdot)|_{L_2(\Delta_*^{(j)}; R^n)}^2 &\leq a_1^{(j)}, \quad |f_2(\cdot) - F_2(\cdot)|_{L_2(\Delta_*^{(j)}; R^{n \times r})}^2 \leq a_2^{(j)}, \\ |p(t) - q(t)|^2 + \tilde{\alpha}_j \int_{t_j^*}^t \{|u_2(\nu)|^2 - |u_1(\nu)|^2\} d\nu &\leq a_3^{(j)} \quad t \in [t_j^*, t_{j+1}^*], \\ |p(t_j^*) - q(t_j^*)|^2 &\leq a_4^{(j)}, \quad \tilde{\alpha}_j = \text{const} \in (0, +\infty) \end{aligned}$$

be true. Then the inequality

$$|u_1(\cdot) - u_2(\cdot)|_{L_2(\Delta_*^{(j)}; R^r)}^2 \leq K_j \left\{ \sum_{l=1}^4 (a_l^{(j)})^{1/2} + \tilde{\alpha}_j^{1/2} \right\} + a_3^{(j)} / \tilde{\alpha}_j$$

is valid.

Lemma 2. *The bunches of solutions of systems (1) and (9) are bounded in the space $W^{1, \infty}(T; R^{n_2}) = \{x(\cdot) \in L_2(T; R^{n_2}); \dot{x}(\cdot) \in L_2(T; R^{n_2})\}$.*

We use the relation

$$\varepsilon_j(t) = |x(t) - w(t)|^2 + \alpha_j \int_{t_j}^t \{|v^h(\nu)|^2 - |u(\nu)|^2\} d\nu, \quad j \in [0 : j_*], \quad t \in T.$$

Lemma 3. *The following inequalities*

$$\varepsilon_j(t) \leq b_j, \quad t \in \Delta^{(j)} \cap T, \quad j \in [0 : j_*],$$

are valid, where

$$b_j = |x(t_j) - w(t_j)|^2 + c_j^{(1)}(h + \delta) + c_j^{(2)} \sum_{k=j-l}^j \nu^{(k)},$$

$c_j^{(1)}, c_j^{(2)}$ are some constants, which can be explicitly written.

Proof. Fix $\tau_i \in \Delta^{(j)}$. Then for $t \in \Delta^{(j)} \cap \delta_i = [\tau_i, \tau_{i+1}]$, we obtain

$$\varepsilon_j(t) \leq \varepsilon_j(\tau_i) + \sum_{j=1}^4 \Lambda_{ji}(t), \quad (13)$$

where

$$\Lambda_{1i}(t) = 2(s_i, \int_{\tau_i}^t \{f_1(\nu, u_\nu(s), x_\nu(s)) - f_1(\tau_i, v_\nu^h(s), \xi_{\tau_i}^h(s))\} d\nu), \quad s_i = x(\tau_i) - w(\tau_i),$$

$$\begin{aligned} \Lambda_{2i}(t) = & 2(s_i, \int_{\tau_i}^t \{f_2(\nu, x_\nu(s))u(\nu) - \\ & - f_2(\tau_i, \xi_{\tau_i}^h(s))v_i^h\} d\nu) + \alpha_j \int_{\tau_i}^t \{|v^h(\nu)|^2 - |u(\nu)|^2\} d\tau, \end{aligned}$$

$$\Lambda_{3i}(t) = -2(t - \tau_i)(s_i, \xi^h(\tau_i) - w(\tau_i)), \quad \Lambda_{4i}(t) = (t - \tau_i) \int_{\tau_i}^t |\dot{w}(\tau) - \dot{x}(\tau)|^2 d\tau.$$

By virtue of lemma 2, we have

$$\Lambda_{4i}(t) \leq K_*^{(j)}(t - \tau_i)^2, \quad t \in \delta_i. \quad (14)$$

Note that $v^h(\tau_i + s) = v^h(t + s)$ for $s \geq t_0 - \tau_i$, $t \in [\tau_i, \tau_{i+1}]$ and in addition

$$|\xi^h(\tau_i + s) - x(t + s)| \leq K_*(h + t - \tau_i) \quad \text{for } \tau_i + s \geq t_0 - \tau. \quad (15)$$

Taking into account lemma 2, as well as the Lipschitz property of the functions $u_0(s)$ and $x_0(s)$, inequalities (4) and the relation

$$|\xi^h(\tau_i + s) - x(t + s)| \leq K_*(h + t - \tau_i) \quad \text{for } \tau_i + s \geq t_0 - \tau, \quad (16)$$

we obtain for $t \in \delta_i$ the estimate

$$\begin{aligned} & \int_{\tau_i}^t |f_1(\nu, u_\nu(s), x_\nu(s)) - f_1(\tau_i, v_\nu^h(s), \xi_{\tau_i}^h(s))| d\nu \leq \\ & \leq K_1^{(j)}(t - \tau_i)(h + t - \tau_i) + K_2^{(j)}(t - \tau_i)^{1/2} \sum_{k=1}^m \left(\int_{\tau_i - \tau_k^u}^{\tau_i - \tau_k^u} |u(\nu) - v^h(\nu)|^2 d\nu \right)^{1/2}. \end{aligned}$$

Here, $\tau_0^x = 0$. In this case, the inequality

$$\Lambda_{1i}(t) \leq 2(t - \tau_i)|x(\tau_i) - w(\tau_i)|^2 + K_3^{(j)} \{(t - \tau_i)(h + t - \tau_i)^2 +$$

$$+ \sum_{k=1}^m \int_{\tau_i - \tau_k^u}^{t - \tau_k^u} |u(\nu) - v^h(\nu)|^2 d\nu \} \quad (17)$$

holds for $t \in \delta_i$. In view of (5), we have

$$\Lambda_{3i}(t) \leq -2(t - \tau_i)|x(\tau_i) - w(\tau_i)|^2 + K_4^{(j)}h(t - \tau_i), \quad t \in \delta_i. \quad (18)$$

Moreover, from (5), (3), and (16), we derive

$$|f_2(\nu, x_\nu(s))u(\nu) - f_2(\tau_i, \xi_{\tau_i}^h(s))u(\nu)| \leq K_0(h + \nu - \tau_i)$$

for $\nu \in [\tau_i, \tau_{i+1}]$. In this case,

$$\begin{aligned} \Lambda_{2i}(t) &\leq K_5^{(j)}(t - \tau_i)(h + t - \tau_i) + \\ &+ \int_{\tau_i}^t \{2(l_i, f_2(\tau_i, \xi_{\tau_i}^h(s))\{v_i^h - u(\nu)\} + \alpha_j\{|v_i^h|^2 - |u(\nu)|^2\}\} d\nu. \end{aligned}$$

The rule for forming the control v_i^h (11) and the last inequality imply

$$\Lambda_{2i}(t) \leq K_5^{(j)}(t - \tau_i)(h + t - \tau_i). \quad (19)$$

Finally, taking into account (13)–(19), we conclude that for $t \in \Delta^{(j)} \cap \delta_i$

$$\varepsilon_j(t) \leq \varepsilon_j(\tau_i) + K_6^{(j)}\delta(h + \delta) + K_3^{(j)} \sum_{k=1}^m \int_{\tau_i - \tau_k^u}^{t - \tau_k^u} |u(\nu) - v^h(\nu)|^2 d\nu,$$

i.e., for $t \in \Delta^{(j)} = [t_j, t_{j+1}]$,

$$\varepsilon_j(t) \leq \varepsilon_j(t_j) + K_7^{(j)}(h + \delta) + K_8^{(j)} \int_{t_j - \tau}^{t_{j+1} - \tau_1^u} |u(\nu) - v^h(\nu)|^2 d\nu.$$

Note that $\tau = l\tau_1^u + \gamma$, $\gamma \geq 0$. Therefore, $t_{j+1} - \tau_1^u = t_j$, $t_{j-l-1} \leq t_j - \tau \leq t_{j-l}$. In this case, for $t \in \Delta^{(j)}$ we have

$$\varepsilon_j(t) \leq \varepsilon_j(t_j) + K_7^{(j)}(h + \delta) + K_9^{(j)} \sum_{k=j-l}^j \nu^{(k)}.$$

Here, constants $K_k^{(j)}$, $k \in [0 : 9]$ are written explicitly. Thus, one can assume that $c_j^{(1)} = K_7^{(j)}$ and $c_j^{(2)} = K_9^{(j)}$. The lemma is proved.

Lemma 4. *Let $\delta \leq h$ and values α_j be given by (12). Then the inequalities*

$$\nu^{(j)} \leq c_j g_j(h), \quad (20)$$

$$b_j \leq c_j^{(0)} g_j(h) \quad (21)$$

are valid.

Proof. For simplicity, set $t_{j_*+1} = \vartheta$. By virtue of lemma 3, we have for $t \in \Delta^{(j)}$

$$|x(t) - w(t)| \leq \left(\varepsilon_j(t) + \alpha_j \int_{t_j}^t \{ |v^h(\nu)|^2 + |u(\nu)|^2 \} d\nu \right)^{1/2} \leq \left(b_j + \alpha_j \rho_A \right)^{1/2}, \quad (22)$$

where $\rho_A = 2\tau_* d^2(P)$ and $d(P) = \sup\{|u| : u \in P\}$. Taking into account the inclusion $t_j \in \Delta_h$, we conclude that for any $j \in [0 : j_*]$, one can specify the number $i = i_j(h)$ such that $t_j = \tau_{i_j(h)}$. Introduce the notation $\varrho_j \equiv |f_1(\cdot) - F_1(\cdot)|_{L_2(\Delta^{(j)}; R^{n_2})}^2$. In this case, by virtue of lemma 2, as well as of (4) and (16), we obtain

$$\varrho_j \leq d_j^{(1)} \sum_{i=i_j(h)}^{i=i_{j+1}(h)-1} \int_{\tau_i}^{\tau_{i+1}} \{ \delta^2 + h^2 + \gamma^h(\nu) + \gamma_i^h(\nu) + |\xi^h(\tau_i) - w(\tau_i)|^2 \} d\nu,$$

where

$$\gamma^h(\nu) = \sum_{k=1}^m |u(\nu - \tau_k^u) - v^h(\nu - \tau_k^u)|^2, \quad \gamma_i^h(\nu) = \sum_{k=0}^n |x(\nu - \tau_k^x) - \xi^h(\tau_i - \tau_k^x)|^2.$$

Note that

$$\int_{t_j}^{t_{j+1}} \gamma^h(\nu) d\nu \leq d_j^{(2)} \int_{t_{j-l-1}}^{t_j} |u(\nu) - v^h(\nu)|^2 d\nu = d_j^{(2)} \sum_{k=j-l}^j \nu^{(k)}, \quad (23)$$

$$\int_{t_j}^{t_{j+1}} \gamma_i^h(\nu) d\nu \leq d_j^{(3)} (h^2 + \delta^2). \quad (24)$$

In addition,

$$\nu^{(k)} = 0 \quad k \in [-l : 0]. \quad (25)$$

Therefore, combining inequalities (22)–(24), we obtain the estimates

$$\varrho_j \leq d_j^{(5)} \{ h^2 + \delta^2 + \sum_{k=j-l}^j \nu^{(k)} + b_j + \alpha_j \}, \quad j \in [0 : j_*]. \quad (26)$$

One can easily see that the following estimates also hold:

$$|f_2(\cdot) - F_2(\cdot)|_{L_2(\Delta^{(j)}; R^{n_2 \times n_1})}^2 \leq d_j^{(5)} (h^2 + \delta^2), \quad j \in [0 : j_*]. \quad (27)$$

Here $d_j^{(1)} - d_j^{(5)}$ are some constants, which can be explicitly written. By lemma 3, (22), and (25), for $\delta \leq h$, we have the inequalities

$$\varepsilon_0(t) \leq b_0 \leq c_0^* h, \quad t \in \Delta^{(0)}, \quad (28)$$

$$|x(t_1) - w(t_1)|^2 \leq \rho_A \alpha_0 + c_0^* h \leq c_* h^{2/3}. \quad (29)$$

Taking into account (25)–(28), for $h \in (0, 1)$, we obtain

$$\varrho_0 \leq d_0^{(1)} \{h^2 + \delta^2 + b_0 + h^{2/3}\} \leq d_0^* h^{2/3}, \quad |f_2(\cdot) - F_2(\cdot)|_{L_2(\Delta^{(0)}; R^{n_2 \times n_1})}^2 \leq c_j^{(*)} h^2.$$

By virtue of condition 1, one can use lemma 1. Set $p = x$, $q = w$, $u_1 = u$, $u_2 = v^h$, $f_1(t) = f_1(t, u_t(s), x_t(s))$, $f_2(t) = f_2(t, x_t(s))$, $F_1(t) = f_1(\tau_i, v_{\tau_i}^h(s), \xi_{\tau_i}^h(s)) + 2(\xi^h(\tau_i) - w(\tau_i))$, $F_2(t) = f_2(\tau_i, \xi_{\tau_i}^h(s))$ $t \in [\tau_i, \tau_{i+1})$. Then, assuming $a_1^{(0)} = d_0^* h^{2/3}$, $a_2^{(0)} = c_j^{(*)} h^2$, $a_3^{(0)} = c_0^* h$, $a_4^{(0)} = c_* h^{2/3}$, $\tilde{\alpha}_0 = \alpha_0 = ch^{2/3}$, we have

$$\nu^{(1)} = |u(\cdot) - v^h(\cdot)|_{L_2(\Delta^{(0)}; R^{n_1})}^2 \leq \tilde{c}_1 h^{1/3} = c_1 g_1(h). \quad (30)$$

It means that inequality (20) holds for $j = 1$. Further, by using (29) and (30), we deduce that

$$b_1 = |x(t_1) - w(t_1)|^2 + c_1^{(1)}(h + \delta) + c_1^{(2)} \sum_{k=1-l}^1 \nu^{(k)} \leq \tilde{c}_1^{(0)} h^{1/3} = c_1^{(0)} g_1(h).$$

Inequality (21) for $j = 1$ is also verified. It follows from (22) that

$$|x(t_j) - w(t_j)|^2 \leq b_{j-1} + \rho_A \alpha_{j-1}, \quad j \in [1 : j_* - 1]. \quad (31)$$

Consequently, in view of relations (31), as well as of the rule for definition b_j , we have the inequality

$$b_j \leq b_{j-1} + d_j \left(h + \alpha_{j-1} + \sum_{k=j-l}^j \nu^{(k)} \right), \quad d_j = \text{const} \in (0, +\infty). \quad (32)$$

Setting $a_1^{(j)} = d_j^{(4)} \{h^2 + \delta^2 + \sum_{k=j-l}^j \nu^{(k)} + a_3^{(j)} + \alpha_j\}$, $a_3^{(j)} = b_j$, $a_2^{(j)} = d_j^{(5)} (h^2 + \delta^2)$, $a_4^{(j)} = b_{j-1} + \rho_A \alpha_{j-1}$, $j \in [1 : j_*]$ for $j \geq 1$ in lemma 1 and taking into account inequalities (32), we obtain

$$\nu^{(j+1)} \leq c^{(j)} \{h^{1/2} + \left(\sum_{k=j-l}^j \nu^{(k)} \right)^{1/2} + b_{j-1}^{1/2} + \alpha_{j-1}^{1/2} + \alpha_j^{1/2}\} + b_j \alpha_j^{-1}, \quad j \in [1 : j_*].$$

Here, we used lemma 3 and inequalities (27), (28), and (31)) for choosing values $a_i^{(j)}$. Now, to proof inequalities (20) and (21), one can use the proof by induction. The lemma is proved.

3 Example

The algorithm was tested by a model example. The following system

$$\begin{aligned} \dot{x}_1(t) &= 2x_1(t-1) + u(t) \\ \dot{x}_2(t) &= x_2(t-1) + x_1(t) + u(t-1), \quad t \in T = [0, 2], \end{aligned} \quad (33)$$

with initial conditions $x_0(s) = y_0(s) = 1$, $u(s) = 0$ for $s \in [-1, 0]$ and control $u(t) = t$ was considered. The solution $x(t) = \{x_1(t), x_2(t)\}$ of system (33) was calculated analytically. During the experiment, we assumed that $\xi^h(\tau_i) = x_1(\tau_i) + h$. As a model, we took the system (9), which has the form

$$\begin{aligned} \dot{w}^{(0)}(t) &= 2\xi_1^h(\tau_i - 1) + v_i^h + 2(\xi_1^h(\tau_i) - w^{(0)}(\tau_i)) \quad \text{for } t \in [\tau_i, \tau_{i+1}) \\ \dot{w}^{(1)}(t) &= \xi_2^h(\tau_i - 1) + \xi_1^h(\tau_i) + v^h(\tau_i - 1) + 2(\xi_2^h(\tau_i) - w^{(1)}(\tau_i)), \end{aligned} \quad (34)$$

with the initial condition $w^{(0)}(s) = w^{(1)}(s) = 1$, for $s \in [-1, 0]$. Here $v^h(\tau_i) = v_i^h$ for $t \in [\tau_i, \tau_{i+1})$, $i \geq 0$, $v^h(s) = 0$ for $s \in [-1, 0)$. The controls v_i^h in model (34) were calculated by the following formula (see (11))

$$v_i^h = \arg \min \{2l_i v + \alpha_j |v|^2 : |v| \leq K\},$$

where $l_i = w^{(0)}(\tau_i) - \xi_1^h(\tau_i)$.

In figures 1 and 2 the results of calculations are presented for the case when $\delta = 10^{-4}$, $\alpha_0 = Ch^{2/3}$, $\alpha_1 = Ch^{2/9}$, $C = 0.2$, $K = 10$. Fig. 1 corresponds to the case when $h = 0.001$, fig. 2 — $h = 0.02$. In these figures the solid (dashed) lines represent the model control $v^h(\cdot)$ (the real control $u(\cdot)$). The equations were solved by the Euler method with step δ .

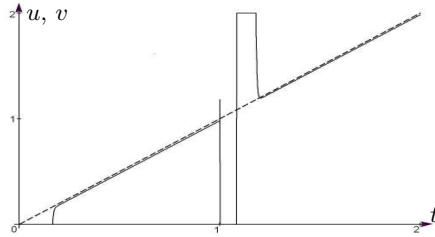


Fig. 1.

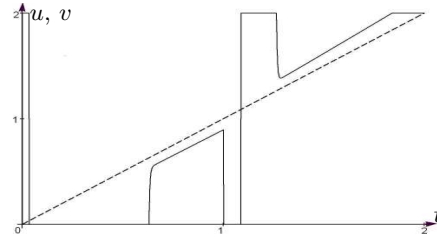


Fig. 2.

References

1. Osipov, Yu.S., Kryazhimskii, A.V.: Inverse Problems for Ordinary Differential Equations: Dynamical Solutions. Gordon and Breach, London (1995)
2. Maksimov, V.I.: Dynamical Inverse Problems of Distributed Systems. VSP, Utrecht–Boston (2002)
3. Maksimov, V., Pandolfi, L.: Dynamical reconstruction of inputs for construction semigroup systems: the boundary input case. J. Optim. Theor. Appl. 103, 401–420 (1999)
4. Maksimov V., Troltzsch F.: Dynamical state and control reconstruction for a phase field model. Dynamics of continuous, discrete and impulsive systems. A: Mathematical analysis. 13, 3-4, 419-444 (2006)
5. Osipov, Yu.S., Kryazhimskii, A.V., Maksimov, V.I.: Methods of Dinamical Reconstruction of Inputs of Controlled Systems. Ekaterinburg (2011) (in Russian)
6. Maksimov, V.: Lyapunov function method in input reconstruction problems of systems with aftereffect. J. Math. Sci. 140, 6, 832–849 (2007)