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# The double competition multigraph of a digraph

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In this article, we introduce the notion of the double competition multigraph of a digraph. We give characterizations of the double competition multigraphs of arbitrary digraphs, loopless digraphs, reflexive digraphs, and acyclic digraphs in terms of edge clique partitions of the multigraphs.

**Keywords:** competition graph, competition multigraph, competition-common enemy graph, double competition multigraph, edge clique partition

## 1 Introduction

The competition graph of a digraph is defined to be the intersection graph of the family of the out-neighborhoods of the vertices of the digraph (see [6] for intersection graphs). A *digraph*  $D$  is a pair  $(V(D), A(D))$  of a set  $V(D)$  of *vertices* and a set  $A(D)$  of ordered pairs of vertices, called *arcs*. An arc of the form  $(v, v)$  is called a *loop*. For a vertex  $x$  in a digraph  $D$ , we denote the *out-neighborhood* of  $x$  in  $D$  by  $N_D^+(x)$  and the *in-neighborhood* of  $x$  in  $D$  by  $N_D^-(x)$ , i.e.,  $N_D^+(x) := \{v \in V(D) \mid (x, v) \in A(D)\}$  and  $N_D^-(x) := \{v \in V(D) \mid (v, x) \in A(D)\}$ . A *graph*  $G$  is a pair  $(V(G), E(G))$  of a set  $V(G)$  of *vertices* and a set  $E(G)$  of unordered pairs of vertices, called *edges*. The *competition graph* of a digraph  $D$  is the graph which has the same vertex set as  $D$  and has an edge between two distinct vertices  $x$  and  $y$  if and only if  $N_D^+(x) \cap N_D^+(y) \neq \emptyset$ . R. D. Dutton and R. C. Brigham [3] and F. S. Roberts and J. E. Steif [8] gave characterizations of competition graphs by using edge clique covers of graphs. The notion of competition graphs was introduced by J. E. Cohen [2] in 1968 in connection with a problem in ecology, and several variants and generalizations of competition graphs have been studied.

In 1987, D. D. Scott [11] introduced the notion of double competition graphs as a variant of the notion of competition graphs. The *double competition graph* (or the *competition-common enemy graph* or the *CCE graph*) of a digraph  $D$  is the graph which has the same vertex set as  $D$  and has an edge between two distinct vertices  $x$  and  $y$  if and only if both  $N_D^+(x) \cap N_D^+(y) \neq \emptyset$  and  $N_D^-(x) \cap N_D^-(y) \neq \emptyset$  hold. See [4, 5, 10, 12] for recent results on double competition graphs.

A *multigraph*  $M$  is a pair  $(V(M), E(M))$  of a set  $V(M)$  of *vertices* and a multiset  $E(M)$  of unordered pairs of vertices, called *edges*. Note that, in our definition, multigraphs have no loops. We may consider

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a multigraph  $M$  as the pair  $(V(M), m_M)$  of the vertex set  $V(M)$  and the nonnegative integer-valued function  $m_M : \binom{V}{2} \rightarrow \mathbb{Z}_{\geq 0}$  on the set  $\binom{V}{2}$  of all unordered pairs of  $V$  where  $m_M(\{x, y\})$  is defined to be the number of multiple edges between the vertices  $x$  and  $y$  in  $M$ . The notion of competition multigraphs was introduced by C. A. Anderson, K. F. Jones, J. R. Lundgren, and T. A. McKee [1] in 1990 as a variant of the notion of competition graphs. The *competition multigraph* of a digraph  $D$  is the multigraph which has the same vertex set as  $D$  and has  $m_{xy}$  multiple edges between two distinct vertices  $x$  and  $y$ , where  $m_{xy}$  is the nonnegative integer defined by  $m_{xy} = |N_D^+(x) \cap N_D^+(y)|$ . See [9, 13] for recent results on competition multigraphs.

In this article, we introduce the notion of the double competition multigraph of a digraph, and we give characterizations of the double competition multigraphs of arbitrary digraphs, loopless digraphs, reflexive digraphs, and acyclic digraphs in terms of edge clique partitions of the multigraphs.

## 2 Main Results

We define the double competition multigraph of a digraph as follows.

**Definition.** Let  $D$  be a digraph. The *double competition multigraph* of  $D$  is the multigraph which has the same vertex set as  $D$  and has  $m_{xy}$  multiple edges between two distinct vertices  $x$  and  $y$ , where  $m_{xy}$  is the nonnegative integer defined by

$$m_{xy} = |N_D^+(x) \cap N_D^+(y)| \cdot |N_D^-(x) \cap N_D^-(y)|,$$

i.e., the multigraph  $M$  defined by  $V(M) = V(D)$  and  $m_M(\{x, y\}) = m_{xy}$ .

Recall that a *clique* of a multigraph  $M$  is a set of vertices of  $M$  which are pairwise adjacent. We consider the empty set  $\emptyset$  as a clique of any multigraph for convenience. A multiset is also called a *family*. An *edge clique partition* of a multigraph  $M$  is a family  $\mathcal{F}$  of cliques of  $M$  such that any two distinct vertices  $x$  and  $y$  are contained in exactly  $m_M(\{x, y\})$  cliques in the family  $\mathcal{F}$ . For a positive integer  $n$ , let  $[n]$  denote the set  $\{1, 2, \dots, n\}$ .

**Theorem 1.** *Let  $M$  be a multigraph with  $n$  vertices. Then,  $M$  is the double competition multigraph of an arbitrary digraph if and only if there exist an ordering  $(v_1, \dots, v_n)$  of the vertices of  $M$  and a double indexed edge clique partition  $\{S_{ij} \mid i, j \in [n]\}$  of  $M$  such that the following condition holds:*

(I) *for any  $i, j \in [n]$ , if  $|A_i \cap B_j| \geq 2$ , then  $A_i \cap B_j = S_{ij}$ ,*

where  $A_i$  and  $B_j$  are the sets defined by

$$A_i = S_{i*} \cup T_i^+, \quad S_{i*} := \bigcup_{p \in [n]} S_{ip}, \quad T_i^+ := \{v_b \mid a, b \in [n], v_i \in S_{ab}\}, \quad (1)$$

$$B_j = S_{*j} \cup T_j^-, \quad S_{*j} := \bigcup_{q \in [n]} S_{qj}, \quad T_j^- := \{v_a \mid a, b \in [n], v_j \in S_{ab}\}. \quad (2)$$

**Proof:** First, we show the only-if part. Let  $M$  be the double competition multigraph of an arbitrary digraph  $D$ . Let  $(v_1, \dots, v_n)$  be an ordering of the vertices of  $D$ . For  $i, j \in [n]$ , we define

$$S_{ij} := \{v_k \in V(D) \mid (v_i, v_k), (v_k, v_j) \in A(D)\}. \quad (3)$$

Then  $S_{ij}$  is a clique of  $M$ . Let  $\mathcal{F}$  be the family of  $S_{ij}$ 's whose size is at least two, i.e.,

$$\mathcal{F} := \{S_{ij} \mid i, j \in [n], |S_{ij}| \geq 2\}. \quad (4)$$

By the definition of a double competition multigraph,  $\mathcal{F}$  is an edge clique partition of  $M$ .

We show that the condition (I) holds. Fix  $i$  and  $j$  in  $[n]$  and let  $A_i$  and  $B_j$  be sets as defined in (1) and (2). Since  $S_{ij} \subseteq A_i$  and  $S_{ij} \subseteq B_j$ , it holds that  $S_{ij} \subseteq A_i \cap B_j$ . Now we assume that  $|A_i \cap B_j| \geq 2$  and take any vertex  $v_k \in A_i \cap B_j$ . There are four cases for  $v_k$  arising from the definitions of  $A_i$  and  $B_j$  as follows: (i)  $v_k \in S_{i*} \cap S_{*j}$ ; (ii)  $v_k \in S_{i*} \cap T_j^-$ ; (iii)  $v_k \in T_i^+ \cap S_{*j}$ ; (iv)  $v_k \in T_i^+ \cap T_j^-$ . To show  $A_i \cap B_j \subseteq S_{ij}$ , we will check that  $v_k \in S_{ij}$  for each case.

**case (i):** Since  $v_k \in S_{i*}$ , there exists  $p \in [n]$  such that  $v_k \in S_{ip}$ . Since  $v_k \in S_{*j}$ , there exists  $q \in [n]$  such that  $v_k \in S_{qj}$ . By (3),  $v_k \in S_{ip}$  implies  $(v_i, v_k), (v_k, v_p) \in A(D)$ , and  $v_k \in S_{qj}$  implies  $(v_q, v_k), (v_k, v_j) \in A(D)$ . Therefore we have  $(v_i, v_k), (v_k, v_j) \in A(D)$ , which implies  $v_k \in S_{ij}$ .

**case (ii):** Since  $v_k \in S_{i*}$ , there exists  $p \in [n]$  such that  $v_k \in S_{ip}$ . Since  $v_k \in T_j^-$ , there exists  $b \in [n]$  such that  $v_j \in S_{kb}$ . By (3),  $v_k \in S_{ip}$  implies  $(v_i, v_k), (v_k, v_p) \in A(D)$ , and  $v_j \in S_{kb}$  implies  $(v_k, v_j), (v_j, v_b) \in A(D)$ . Therefore we have  $(v_i, v_k), (v_k, v_j) \in A(D)$ , which implies  $v_k \in S_{ij}$ .

**case (iii):** Since  $v_k \in T_i^+$ , there exists  $a \in [n]$  such that  $v_i \in S_{ak}$ . Since  $v_k \in S_{*j}$ , there exists  $q \in [n]$  such that  $v_k \in S_{qj}$ . By (3),  $v_i \in S_{ak}$  implies  $(v_a, v_i), (v_i, v_k) \in A(D)$ , and  $v_k \in S_{qj}$  implies  $(v_q, v_k), (v_k, v_j) \in A(D)$ . Therefore we have  $(v_i, v_k), (v_k, v_j) \in A(D)$ , which implies  $v_k \in S_{ij}$ .

**case (iv):** Since  $v_k \in T_i^+$ , there exists  $a \in [n]$  such that  $v_i \in S_{ak}$ . Since  $v_k \in T_j^-$ , there exists  $b \in [n]$  such that  $v_j \in S_{kb}$ . By (3),  $v_i \in S_{ak}$  implies  $(v_a, v_i), (v_i, v_k) \in A(D)$ , and  $v_j \in S_{kb}$  implies  $(v_k, v_j), (v_j, v_b) \in A(D)$ . Therefore we have  $(v_i, v_k), (v_k, v_j) \in A(D)$ , which implies  $v_k \in S_{ij}$ .

Thus we obtain  $A_i \cap B_j \subseteq S_{ij}$ , and so  $A_i \cap B_j = S_{ij}$ . Hence the condition (I) holds.

Next, we show the if part. Let  $M$  be a multigraph with  $n$  vertices, and suppose that there exist an ordering  $(v_1, \dots, v_n)$  of the vertices of  $M$  and a double indexed edge clique partition  $\mathcal{F} = \{S_{ij} \mid i, j \in [n]\}$  of  $M$  such that the condition (I) holds.

We define a digraph  $D$  by  $V(D) := V(M)$  and

$$A(D) := \bigcup_{i, j \in [n]} \left( \bigcup_{v_k \in S_{ij}} \{(v_i, v_k), (v_k, v_j)\} \right). \quad (5)$$

Let  $M'$  denote the double competition multigraph of  $D$ . We show that  $M = M'$ . Since  $V(M) = V(M')$ , it is enough to show  $m_M = m_{M'}$ . Take any two distinct vertices  $v_k$  and  $v_l$  and let  $t := m_M(\{v_k, v_l\})$ . Since  $\mathcal{F}$  is an edge clique partition of  $M$ , the vertices  $v_k$  and  $v_l$  are contained in exactly  $t$  cliques  $S_{ij} \in \mathcal{F}$ . So, for some nonnegative integers  $r$  and  $s$  with  $rs = t$ , there are  $r$  common in-neighbors  $v_{i_1}, \dots, v_{i_r}$  and  $s$  common out-neighbors  $v_{j_1}, \dots, v_{j_s}$  of the vertices  $v_k$  and  $v_l$  in  $D$ . Therefore it follows that  $m_{M'}(\{v_k, v_l\}) = |N_D^-(v_k) \cap N_D^-(v_l)| \cdot |N_D^+(v_k) \cap N_D^+(v_l)| \geq rs = t$ . Thus  $m_M(\{v_k, v_l\}) \leq m_{M'}(\{v_k, v_l\})$ . Again, take any two distinct vertices  $v_k$  and  $v_l$  and let  $t' := m_{M'}(\{v_k, v_l\})$ . Then, for some nonnegative integers  $r'$  and  $s'$  with  $r's' = t'$ , there are  $r'$  common in-neighbors  $v_{i_1}, \dots, v_{i_{r'}}$  and  $s'$  common out-neighbors  $v_{j_1}, \dots, v_{j_{s'}}$  of the vertices  $v_k$  and  $v_l$  in  $D$ . For each  $i \in \{i_1, \dots, i_{r'}\}$ ,

since  $(v_i, v_k), (v_i, v_l) \in A(D)$ , it follows that  $\{v_k, v_l\} \subseteq A_i$ . Similarly, for each  $j \in \{j_1, \dots, j_{s'}\}$ , since  $(v_k, v_j), (v_l, v_j) \in A(D)$ , it follows that  $\{v_k, v_l\} \subseteq B_j$ . Therefore,  $\{v_k, v_l\} \subseteq A_i \cap B_j = S_{ij}$  for any  $i \in \{i_1, \dots, i_{r'}\}$  and any  $j \in \{j_1, \dots, j_{s'}\}$ . By the condition (I), we have  $A_i \cap B_j = S_{ij}$ . Therefore  $\{v_k, v_l\} \subseteq S_{ij}$  for any  $i \in \{i_1, \dots, i_{r'}\}$  and any  $j \in \{j_1, \dots, j_{s'}\}$  and this implies that  $m_M(\{v_k, v_l\}) = |\{S_{i,j} \in \mathcal{F} \mid \{v_k, v_l\} \subseteq S_{i,j}\}| \geq r's' = t'$ . Thus  $m_M(\{v_k, v_l\}) \geq m_{M'}(\{v_k, v_l\})$ . Hence it holds that  $m_M(\{v_k, v_l\}) = m_{M'}(\{v_k, v_l\})$  for any two distinct vertices  $v_k$  and  $v_l$ , that is,  $m_M = m_{M'}$ , i.e.,  $M = M'$ . So  $M$  is the double competition multigraph of  $D$ .  $\square$

A digraph  $D$  is said to be *loopless* if  $D$  has no loops, i.e.,  $(v, v) \notin A(D)$  holds for any  $v \in V(D)$ .

**Theorem 2.** *Let  $M$  be a multigraph with  $n$  vertices. Then,  $M$  is the double competition multigraph of a loopless digraph if and only if there exist an ordering  $(v_1, \dots, v_n)$  of the vertices of  $M$  and a double indexed edge clique partition  $\{S_{ij} \mid i, j \in [n]\}$  of  $M$  such that the following conditions hold:*

- (I) *for any  $i, j \in [n]$ , if  $|A_i \cap B_j| \geq 2$ , then  $A_i \cap B_j = S_{ij}$ ;*
- (II) *for any  $i, j \in [n]$ ,  $v_i \notin S_{ij}$  and  $v_j \notin S_{ij}$ ,*

where  $A_i$  and  $B_j$  are the sets defined as (1) and (2).

**Proof:** First, we show the only-if part. Let  $M$  be the double competition multigraph of a loopless digraph  $D$ . Let  $(v_1, \dots, v_n)$  be an ordering of the vertices of  $D$ . Let  $S_{ij}$  ( $i, j \in [n]$ ) be the sets defined as (3), and let  $\mathcal{F}$  be the family defined as (4). Then  $S_{ij}$  is a clique of  $M$ , and  $\mathcal{F}$  is an edge clique partition of  $M$ . Moreover, we can show, as in the proof of Theorem 1, that the condition (I) holds. Now we show that the condition (II) holds. Take any vertex  $v_k \in S_{ij}$ . Then  $(v_i, v_k), (v_k, v_j) \in A(D)$ . Since  $D$  is loopless, we have  $v_i \neq v_k$  and  $v_i \neq v_k$ . Therefore it follows that  $v_i \notin S_{ij}$  and  $v_j \notin S_{ij}$ . Thus the condition (II) holds.

Next, we show the if part. Let  $M$  be a multigraph with  $n$  vertices, and suppose that there exists an ordering  $(v_1, \dots, v_n)$  of the vertices of  $M$  and a double indexed edge clique partition  $\{S_{ij} \mid i, j \in [n]\}$  of  $M$  such that the conditions (I) and (II) hold. We define a digraph  $D$  by  $V(D) := V(M)$  and  $A(D)$  given in (5). By the condition (II), it follows from the definition of  $D$  that  $(v_i, v_i) \notin A(D)$  for any  $i \in [n]$ . Therefore  $D$  is a loopless digraph. Moreover we can show, as in the proof of Theorem 1, that  $M$  is the double competition multigraph of  $D$ .  $\square$

A digraph  $D$  is said to be *reflexive* if all the vertices of  $D$  have loops, i.e.,  $(v, v) \in A(D)$  holds for any  $v \in V(D)$ .

**Theorem 3.** *Let  $M$  be a multigraph with  $n$  vertices. Then,  $M$  is the double competition multigraph of a reflexive digraph if and only if there exist an ordering  $(v_1, \dots, v_n)$  of the vertices of  $M$  and a double indexed edge clique partition  $\{S_{ij} \mid i, j \in [n]\}$  of  $M$  such that the following conditions hold:*

- (I) *for any  $i, j \in [n]$ , if  $|A_i \cap B_j| \geq 2$ , then  $A_i \cap B_j = S_{ij}$ ;*
- (III) *for any  $i \in [n]$ ,  $v_i \in S_{i*} \cup S_{*i}$ ,*

where  $A_i$ ,  $B_j$ ,  $S_{i*}$ , and  $S_{*i}$  are the sets defined as (1) and (2).

**Proof:** First, we show the only-if part. Let  $M$  be the double competition multigraph of a reflexive digraph  $D$ . Let  $(v_1, \dots, v_n)$  be an ordering of the vertices of  $D$ . Let  $S_{ij}$  ( $i, j \in [n]$ ) be the sets defined as (3), and let  $\mathcal{F}$  be the family defined as (4). Then  $S_{ij}$  is a clique of  $M$ , and  $\mathcal{F}$  is an edge clique partition of  $M$ .

Moreover, we can show, as in the proof of Theorem 1, that the condition (I) holds. Now we show that the condition (III) holds. Since  $D$  is reflexive, we have  $(v_i, v_i) \in A(D)$  for any  $i \in [n]$ . Then it follows that there exists  $p \in [n]$  such that  $v_i \in S_{ip}$  or  $v_i \in S_{pi}$ . Therefore  $v_i \in S_{i*} \cup S_{*i}$ . Thus the condition (III) holds.

Next, we show the if part. Let  $M$  be a multigraph with  $n$  vertices, and suppose that there exist an ordering  $(v_1, \dots, v_n)$  of the vertices of  $M$  and a double indexed edge clique partition  $\mathcal{F} = \{S_{ij} \mid i, j \in [n]\}$  of  $M$  such that the conditions (I) and (III) hold. We define a digraph  $D$  by  $V(D) := V(M)$  and  $A(D)$  given in (5). Fix any  $i \in [n]$ . By the condition (III), there exists  $p \in [n]$  such that  $v_i \in S_{ip}$  or  $v_i \in S_{pi}$ . Then it follows from the definition of  $D$  that  $(v_i, v_i) \in A(D)$ . Therefore  $D$  is a reflexive digraph. Moreover we can show, as in the proof of Theorem 1, that  $M$  is the double competition multigraph of  $D$ .  $\square$

A digraph  $D$  is said to be *acyclic* if  $D$  has no directed cycles. An ordering  $(v_1, \dots, v_n)$  of the vertices of a digraph  $D$ , where  $n$  is the number of vertices of  $D$ , is called an *acyclic ordering* of  $D$  if  $(v_i, v_j) \in A(D)$  implies  $i < j$ . It is well known that a digraph  $D$  is acyclic if and only if  $D$  has an acyclic ordering.

**Theorem 4.** *Let  $M$  be a multigraph with  $n$  vertices. Then,  $M$  is the double competition multigraph of an acyclic digraph if and only if there exist an ordering  $(v_1, \dots, v_n)$  of the vertices of  $M$  and a double indexed edge clique partition  $\{S_{ij} \mid i, j \in [n]\}$  of  $M$  such that the following conditions hold:*

(I) *for any  $i, j \in [n]$ , if  $|A_i \cap B_j| \geq 2$ , then  $A_i \cap B_j = S_{ij}$ ;*

(IV) *for any  $i, j, k \in [n]$ ,  $v_k \in S_{ij}$  implies  $i < k < j$ ,*

where  $A_i$  and  $B_j$  are the sets defined as (1) and (2).

**Proof:** First, we show the only-if part. Let  $M$  be the double competition multigraph of an acyclic digraph  $D$ . Let  $(v_1, \dots, v_n)$  be an acyclic ordering of the vertices of  $D$ . Let  $S_{ij}$  ( $i, j \in [n]$ ) be the sets defined as (3), and let  $\mathcal{F}$  be the family defined as (4). Then  $S_{ij}$  is a clique of  $M$ , and  $\mathcal{F}$  is an edge clique partition of  $M$ . Moreover, we can show, as in the proof of Theorem 1, that the condition (I) holds. Now we show that the condition (IV) holds. Suppose that  $v_k \in S_{ij}$ . Then  $(v_i, v_k), (v_k, v_j) \in A(D)$ . Since  $(v_1, \dots, v_n)$  is an acyclic ordering of  $D$ ,  $(v_i, v_k) \in A(D)$  implies  $i < k$  and  $(v_k, v_j) \in A(D)$  implies  $k < j$ . Therefore  $i < k < j$ . Thus the condition (IV) holds.

Next, we show the if part. Let  $M$  be a multigraph with  $n$  vertices, and suppose that there exist an ordering  $(v_1, \dots, v_n)$  of the vertices of  $M$  and a double indexed edge clique partition  $\{S_{ij} \mid i, j \in [n]\}$  of  $M$  such that the conditions (I) and (IV) hold. We define a digraph  $D$  by  $V(D) := V(M)$  and  $A(D)$  given in (5). By the condition (IV), it follows from the definition of  $D$  that  $(v_1, \dots, v_n)$  is an acyclic ordering of  $D$ . Therefore  $D$  is an acyclic digraph. Moreover we can show, as in the proof of Theorem 1, that  $M$  is the double competition multigraph of  $D$ .  $\square$

**Remark 5.** The condition (I) in Theorems 1, 2, 3, and 4 may be replaced by the following condition:

(I)' *for any  $i, j \in [n]$ ,  $A_i \cap B_j = S_{ij}$ .*

**Proof:** If the condition (I)' is satisfied, then so is the condition (I). Suppose that the condition (I) is satisfied. If  $|A_i \cap B_j| \geq 2$ , then it follows from the condition (I) that  $A_i \cap B_j = S_{ij}$ . If  $|A_i \cap B_j| = 0$ , then  $A_i \cap B_j = \emptyset$ . Since  $S_{ij} \subseteq A_i \cap B_j$ , we have  $S_{ij} = \emptyset$ . Therefore,  $A_i \cap B_j = S_{ij}$ . If  $|A_i \cap B_j| = 1$ ,

then  $A_i \cap B_j = \{v_k\}$  for some  $k \in [n]$ . Since  $S_{ij} \subseteq A_i \cap B_j$ , we have  $S_{ij} = \emptyset$  or  $S_{ij} = \{v_k\}$ . If  $S_{ij} = \{v_k\}$ , then  $A_i \cap B_j = S_{ij}$ . If  $S_{ij} = \emptyset$ , then we replace  $S_{ij} = \emptyset$  by  $S_{ij} = \{v_k\}$ . Then  $\mathcal{F}$  is still an edge clique partition of  $M$ , and  $A_i \cap B_j = S_{ij}$ . Thus the condition (I)' holds. Hence the remark holds.  $\square$

At the end of this paper, we mention two corollaries related to the edge clique partition number of a multigraph. Recall that the *edge clique partition number* of a multigraph  $M$  is the minimum size of an edge clique partition of  $M$  and is denoted by  $\theta^*(M)$ . As a corollary of Theorem 1, we obtain a necessary condition for multigraphs being the double competition multigraph of a digraph.

**Corollary 6.** *If a multigraph  $M$  with  $n$  vertices is the double competition multigraph of a digraph, then  $\theta^*(M) \leq n^2$ .*

For the double competition multigraphs of acyclic digraphs, we can improve the above upper bound for the edge clique partition numbers of multigraphs.

**Corollary 7.** *If a multigraph  $M$  with  $n$  vertices is the double competition multigraph of an acyclic digraph, then  $\theta^*(M) \leq \frac{1}{2}(n-2)(n-3)$ .*

**Proof:** Suppose that a multigraph  $M$  with  $n$  vertices is the double competition multigraph of an acyclic digraph. Then, by Theorem 4, there exist an ordering  $(v_1, \dots, v_n)$  of the vertices of  $M$  and a double indexed edge clique partition  $\{S_{ij} \mid i, j \in [n]\}$  of  $M$  satisfying the conditions (I) and (IV). It follows from the condition (IV) that, if  $j \leq i+1$ , then  $S_{ij} = \emptyset$ . If  $j = i+2$ , then  $S_{ij} = \emptyset$  or  $S_{ij} = \{v_{i+1}\}$ , which does not cover an edge of  $M$ . Therefore, the family  $\{S_{ij} \mid i, j \in [n], i+3 \leq j\}$  is an edge clique partition of  $M$ . Thus the corollary holds.  $\square$

**Remark 8.** In [7], the authors defined the *double multicompetition number*  $dk^*(M)$  of a multigraph  $M$  to be the minimum nonnegative integer  $k$  such that  $M$  together with  $k$  new isolated vertices is the double competition multigraph of some acyclic digraph. In this context, Theorem 4 gives a characterization of multigraphs whose double multicompetition number is equal to 0.

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