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► **To cite this version:**

Yoshio Sano, Jeongmi Park. The double competition multigraph of a digraph. *Discrete Mathematics and Theoretical Computer Science, DMTCS*, 2015, Vol. 17 no.2 (2), pp.303-310. <hal-01349048>

HAL Id: hal-01349048

<https://hal.inria.fr/hal-01349048>

Submitted on 26 Jul 2016

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The double competition multigraph of a digraph

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In this article, we introduce the notion of the double competition multigraph of a digraph. We give characterizations of the double competition multigraphs of arbitrary digraphs, loopless digraphs, reflexive digraphs, and acyclic digraphs in terms of edge clique partitions of the multigraphs.

Keywords: competition graph, competition multigraph, competition-common enemy graph, double competition multigraph, edge clique partition

1 Introduction

The competition graph of a digraph is defined to be the intersection graph of the family of the out-neighborhoods of the vertices of the digraph (see [6] for intersection graphs). A *digraph* D is a pair $(V(D), A(D))$ of a set $V(D)$ of *vertices* and a set $A(D)$ of ordered pairs of vertices, called *arcs*. An arc of the form (v, v) is called a *loop*. For a vertex x in a digraph D , we denote the *out-neighborhood* of x in D by $N_D^+(x)$ and the *in-neighborhood* of x in D by $N_D^-(x)$, i.e., $N_D^+(x) := \{v \in V(D) \mid (x, v) \in A(D)\}$ and $N_D^-(x) := \{v \in V(D) \mid (v, x) \in A(D)\}$. A *graph* G is a pair $(V(G), E(G))$ of a set $V(G)$ of *vertices* and a set $E(G)$ of unordered pairs of vertices, called *edges*. The *competition graph* of a digraph D is the graph which has the same vertex set as D and has an edge between two distinct vertices x and y if and only if $N_D^+(x) \cap N_D^+(y) \neq \emptyset$. R. D. Dutton and R. C. Brigham [3] and F. S. Roberts and J. E. Steif [8] gave characterizations of competition graphs by using edge clique covers of graphs. The notion of competition graphs was introduced by J. E. Cohen [2] in 1968 in connection with a problem in ecology, and several variants and generalizations of competition graphs have been studied.

In 1987, D. D. Scott [11] introduced the notion of double competition graphs as a variant of the notion of competition graphs. The *double competition graph* (or the *competition-common enemy graph* or the *CCE graph*) of a digraph D is the graph which has the same vertex set as D and has an edge between two distinct vertices x and y if and only if both $N_D^+(x) \cap N_D^+(y) \neq \emptyset$ and $N_D^-(x) \cap N_D^-(y) \neq \emptyset$ hold. See [4, 5, 10, 12] for recent results on double competition graphs.

A *multigraph* M is a pair $(V(M), E(M))$ of a set $V(M)$ of *vertices* and a multiset $E(M)$ of unordered pairs of vertices, called *edges*. Note that, in our definition, multigraphs have no loops. We may consider

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a multigraph M as the pair $(V(M), m_M)$ of the vertex set $V(M)$ and the nonnegative integer-valued function $m_M : \binom{V}{2} \rightarrow \mathbb{Z}_{\geq 0}$ on the set $\binom{V}{2}$ of all unordered pairs of V where $m_M(\{x, y\})$ is defined to be the number of multiple edges between the vertices x and y in M . The notion of competition multigraphs was introduced by C. A. Anderson, K. F. Jones, J. R. Lundgren, and T. A. McKee [1] in 1990 as a variant of the notion of competition graphs. The *competition multigraph* of a digraph D is the multigraph which has the same vertex set as D and has m_{xy} multiple edges between two distinct vertices x and y , where m_{xy} is the nonnegative integer defined by $m_{xy} = |N_D^+(x) \cap N_D^+(y)|$. See [9, 13] for recent results on competition multigraphs.

In this article, we introduce the notion of the double competition multigraph of a digraph, and we give characterizations of the double competition multigraphs of arbitrary digraphs, loopless digraphs, reflexive digraphs, and acyclic digraphs in terms of edge clique partitions of the multigraphs.

2 Main Results

We define the double competition multigraph of a digraph as follows.

Definition. Let D be a digraph. The *double competition multigraph* of D is the multigraph which has the same vertex set as D and has m_{xy} multiple edges between two distinct vertices x and y , where m_{xy} is the nonnegative integer defined by

$$m_{xy} = |N_D^+(x) \cap N_D^+(y)| \cdot |N_D^-(x) \cap N_D^-(y)|,$$

i.e., the multigraph M defined by $V(M) = V(D)$ and $m_M(\{x, y\}) = m_{xy}$.

Recall that a *clique* of a multigraph M is a set of vertices of M which are pairwise adjacent. We consider the empty set \emptyset as a clique of any multigraph for convenience. A multiset is also called a *family*. An *edge clique partition* of a multigraph M is a family \mathcal{F} of cliques of M such that any two distinct vertices x and y are contained in exactly $m_M(\{x, y\})$ cliques in the family \mathcal{F} . For a positive integer n , let $[n]$ denote the set $\{1, 2, \dots, n\}$.

Theorem 1. *Let M be a multigraph with n vertices. Then, M is the double competition multigraph of an arbitrary digraph if and only if there exist an ordering (v_1, \dots, v_n) of the vertices of M and a double indexed edge clique partition $\{S_{ij} \mid i, j \in [n]\}$ of M such that the following condition holds:*

(I) *for any $i, j \in [n]$, if $|A_i \cap B_j| \geq 2$, then $A_i \cap B_j = S_{ij}$,*

where A_i and B_j are the sets defined by

$$A_i = S_{i*} \cup T_i^+, \quad S_{i*} := \bigcup_{p \in [n]} S_{ip}, \quad T_i^+ := \{v_b \mid a, b \in [n], v_i \in S_{ab}\}, \quad (1)$$

$$B_j = S_{*j} \cup T_j^-, \quad S_{*j} := \bigcup_{q \in [n]} S_{qj}, \quad T_j^- := \{v_a \mid a, b \in [n], v_j \in S_{ab}\}. \quad (2)$$

Proof: First, we show the only-if part. Let M be the double competition multigraph of an arbitrary digraph D . Let (v_1, \dots, v_n) be an ordering of the vertices of D . For $i, j \in [n]$, we define

$$S_{ij} := \{v_k \in V(D) \mid (v_i, v_k), (v_k, v_j) \in A(D)\}. \quad (3)$$

Then S_{ij} is a clique of M . Let \mathcal{F} be the family of S_{ij} 's whose size is at least two, i.e.,

$$\mathcal{F} := \{S_{ij} \mid i, j \in [n], |S_{ij}| \geq 2\}. \quad (4)$$

By the definition of a double competition multigraph, \mathcal{F} is an edge clique partition of M .

We show that the condition (I) holds. Fix i and j in $[n]$ and let A_i and B_j be sets as defined in (1) and (2). Since $S_{ij} \subseteq A_i$ and $S_{ij} \subseteq B_j$, it holds that $S_{ij} \subseteq A_i \cap B_j$. Now we assume that $|A_i \cap B_j| \geq 2$ and take any vertex $v_k \in A_i \cap B_j$. There are four cases for v_k arising from the definitions of A_i and B_j as follows: (i) $v_k \in S_{i*} \cap S_{*j}$; (ii) $v_k \in S_{i*} \cap T_j^-$; (iii) $v_k \in T_i^+ \cap S_{*j}$; (iv) $v_k \in T_i^+ \cap T_j^-$. To show $A_i \cap B_j \subseteq S_{ij}$, we will check that $v_k \in S_{ij}$ for each case.

case (i): Since $v_k \in S_{i*}$, there exists $p \in [n]$ such that $v_k \in S_{ip}$. Since $v_k \in S_{*j}$, there exists $q \in [n]$ such that $v_k \in S_{qj}$. By (3), $v_k \in S_{ip}$ implies $(v_i, v_k), (v_k, v_p) \in A(D)$, and $v_k \in S_{qj}$ implies $(v_q, v_k), (v_k, v_j) \in A(D)$. Therefore we have $(v_i, v_k), (v_k, v_j) \in A(D)$, which implies $v_k \in S_{ij}$.

case (ii): Since $v_k \in S_{i*}$, there exists $p \in [n]$ such that $v_k \in S_{ip}$. Since $v_k \in T_j^-$, there exists $b \in [n]$ such that $v_j \in S_{kb}$. By (3), $v_k \in S_{ip}$ implies $(v_i, v_k), (v_k, v_p) \in A(D)$, and $v_j \in S_{kb}$ implies $(v_k, v_j), (v_j, v_b) \in A(D)$. Therefore we have $(v_i, v_k), (v_k, v_j) \in A(D)$, which implies $v_k \in S_{ij}$.

case (iii): Since $v_k \in T_i^+$, there exists $a \in [n]$ such that $v_i \in S_{ak}$. Since $v_k \in S_{*j}$, there exists $q \in [n]$ such that $v_k \in S_{qj}$. By (3), $v_i \in S_{ak}$ implies $(v_a, v_i), (v_i, v_k) \in A(D)$, and $v_k \in S_{qj}$ implies $(v_q, v_k), (v_k, v_j) \in A(D)$. Therefore we have $(v_i, v_k), (v_k, v_j) \in A(D)$, which implies $v_k \in S_{ij}$.

case (iv): Since $v_k \in T_i^+$, there exists $a \in [n]$ such that $v_i \in S_{ak}$. Since $v_k \in T_j^-$, there exists $b \in [n]$ such that $v_j \in S_{kb}$. By (3), $v_i \in S_{ak}$ implies $(v_a, v_i), (v_i, v_k) \in A(D)$, and $v_j \in S_{kb}$ implies $(v_k, v_j), (v_j, v_b) \in A(D)$. Therefore we have $(v_i, v_k), (v_k, v_j) \in A(D)$, which implies $v_k \in S_{ij}$.

Thus we obtain $A_i \cap B_j \subseteq S_{ij}$, and so $A_i \cap B_j = S_{ij}$. Hence the condition (I) holds.

Next, we show the if part. Let M be a multigraph with n vertices, and suppose that there exist an ordering (v_1, \dots, v_n) of the vertices of M and a double indexed edge clique partition $\mathcal{F} = \{S_{ij} \mid i, j \in [n]\}$ of M such that the condition (I) holds.

We define a digraph D by $V(D) := V(M)$ and

$$A(D) := \bigcup_{i, j \in [n]} \left(\bigcup_{v_k \in S_{ij}} \{(v_i, v_k), (v_k, v_j)\} \right). \quad (5)$$

Let M' denote the double competition multigraph of D . We show that $M = M'$. Since $V(M) = V(M')$, it is enough to show $m_M = m_{M'}$. Take any two distinct vertices v_k and v_l and let $t := m_M(\{v_k, v_l\})$. Since \mathcal{F} is an edge clique partition of M , the vertices v_k and v_l are contained in exactly t cliques $S_{ij} \in \mathcal{F}$. So, for some nonnegative integers r and s with $rs = t$, there are r common in-neighbors v_{i_1}, \dots, v_{i_r} and s common out-neighbors v_{j_1}, \dots, v_{j_s} of the vertices v_k and v_l in D . Therefore it follows that $m_{M'}(\{v_k, v_l\}) = |N_D^-(v_k) \cap N_D^-(v_l)| \cdot |N_D^+(v_k) \cap N_D^+(v_l)| \geq rs = t$. Thus $m_M(\{v_k, v_l\}) \leq m_{M'}(\{v_k, v_l\})$. Again, take any two distinct vertices v_k and v_l and let $t' := m_{M'}(\{v_k, v_l\})$. Then, for some nonnegative integers r' and s' with $r's' = t'$, there are r' common in-neighbors $v_{i_1}, \dots, v_{i_{r'}}$ and s' common out-neighbors $v_{j_1}, \dots, v_{j_{s'}}$ of the vertices v_k and v_l in D . For each $i \in \{i_1, \dots, i_{r'}\}$,

since $(v_i, v_k), (v_i, v_l) \in A(D)$, it follows that $\{v_k, v_l\} \subseteq A_i$. Similarly, for each $j \in \{j_1, \dots, j_{s'}\}$, since $(v_k, v_j), (v_l, v_j) \in A(D)$, it follows that $\{v_k, v_l\} \subseteq B_j$. Therefore, $\{v_k, v_l\} \subseteq A_i \cap B_j = S_{ij}$ for any $i \in \{i_1, \dots, i_{r'}\}$ and any $j \in \{j_1, \dots, j_{s'}\}$. By the condition (I), we have $A_i \cap B_j = S_{ij}$. Therefore $\{v_k, v_l\} \subseteq S_{ij}$ for any $i \in \{i_1, \dots, i_{r'}\}$ and any $j \in \{j_1, \dots, j_{s'}\}$ and this implies that $m_M(\{v_k, v_l\}) = |\{S_{i,j} \in \mathcal{F} \mid \{v_k, v_l\} \subseteq S_{i,j}\}| \geq r's' = t'$. Thus $m_M(\{v_k, v_l\}) \geq m_{M'}(\{v_k, v_l\})$. Hence it holds that $m_M(\{v_k, v_l\}) = m_{M'}(\{v_k, v_l\})$ for any two distinct vertices v_k and v_l , that is, $m_M = m_{M'}$, i.e., $M = M'$. So M is the double competition multigraph of D . \square

A digraph D is said to be *loopless* if D has no loops, i.e., $(v, v) \notin A(D)$ holds for any $v \in V(D)$.

Theorem 2. *Let M be a multigraph with n vertices. Then, M is the double competition multigraph of a loopless digraph if and only if there exist an ordering (v_1, \dots, v_n) of the vertices of M and a double indexed edge clique partition $\{S_{ij} \mid i, j \in [n]\}$ of M such that the following conditions hold:*

- (I) *for any $i, j \in [n]$, if $|A_i \cap B_j| \geq 2$, then $A_i \cap B_j = S_{ij}$;*
- (II) *for any $i, j \in [n]$, $v_i \notin S_{ij}$ and $v_j \notin S_{ij}$,*

where A_i and B_j are the sets defined as (1) and (2).

Proof: First, we show the only-if part. Let M be the double competition multigraph of a loopless digraph D . Let (v_1, \dots, v_n) be an ordering of the vertices of D . Let S_{ij} ($i, j \in [n]$) be the sets defined as (3), and let \mathcal{F} be the family defined as (4). Then S_{ij} is a clique of M , and \mathcal{F} is an edge clique partition of M . Moreover, we can show, as in the proof of Theorem 1, that the condition (I) holds. Now we show that the condition (II) holds. Take any vertex $v_k \in S_{ij}$. Then $(v_i, v_k), (v_k, v_j) \in A(D)$. Since D is loopless, we have $v_i \neq v_k$ and $v_i \neq v_k$. Therefore it follows that $v_i \notin S_{ij}$ and $v_j \notin S_{ij}$. Thus the condition (II) holds.

Next, we show the if part. Let M be a multigraph with n vertices, and suppose that there exists an ordering (v_1, \dots, v_n) of the vertices of M and a double indexed edge clique partition $\{S_{ij} \mid i, j \in [n]\}$ of M such that the conditions (I) and (II) hold. We define a digraph D by $V(D) := V(M)$ and $A(D)$ given in (5). By the condition (II), it follows from the definition of D that $(v_i, v_i) \notin A(D)$ for any $i \in [n]$. Therefore D is a loopless digraph. Moreover we can show, as in the proof of Theorem 1, that M is the double competition multigraph of D . \square

A digraph D is said to be *reflexive* if all the vertices of D have loops, i.e., $(v, v) \in A(D)$ holds for any $v \in V(D)$.

Theorem 3. *Let M be a multigraph with n vertices. Then, M is the double competition multigraph of a reflexive digraph if and only if there exist an ordering (v_1, \dots, v_n) of the vertices of M and a double indexed edge clique partition $\{S_{ij} \mid i, j \in [n]\}$ of M such that the following conditions hold:*

- (I) *for any $i, j \in [n]$, if $|A_i \cap B_j| \geq 2$, then $A_i \cap B_j = S_{ij}$;*
- (III) *for any $i \in [n]$, $v_i \in S_{i*} \cup S_{*i}$,*

where A_i , B_j , S_{i*} , and S_{*i} are the sets defined as (1) and (2).

Proof: First, we show the only-if part. Let M be the double competition multigraph of a reflexive digraph D . Let (v_1, \dots, v_n) be an ordering of the vertices of D . Let S_{ij} ($i, j \in [n]$) be the sets defined as (3), and let \mathcal{F} be the family defined as (4). Then S_{ij} is a clique of M , and \mathcal{F} is an edge clique partition of M .

Moreover, we can show, as in the proof of Theorem 1, that the condition (I) holds. Now we show that the condition (III) holds. Since D is reflexive, we have $(v_i, v_i) \in A(D)$ for any $i \in [n]$. Then it follows that there exists $p \in [n]$ such that $v_i \in S_{ip}$ or $v_i \in S_{pi}$. Therefore $v_i \in S_{i*} \cup S_{*i}$. Thus the condition (III) holds.

Next, we show the if part. Let M be a multigraph with n vertices, and suppose that there exist an ordering (v_1, \dots, v_n) of the vertices of M and a double indexed edge clique partition $\mathcal{F} = \{S_{ij} \mid i, j \in [n]\}$ of M such that the conditions (I) and (III) hold. We define a digraph D by $V(D) := V(M)$ and $A(D)$ given in (5). Fix any $i \in [n]$. By the condition (III), there exists $p \in [n]$ such that $v_i \in S_{ip}$ or $v_i \in S_{pi}$. Then it follows from the definition of D that $(v_i, v_i) \in A(D)$. Therefore D is a reflexive digraph. Moreover we can show, as in the proof of Theorem 1, that M is the double competition multigraph of D . \square

A digraph D is said to be *acyclic* if D has no directed cycles. An ordering (v_1, \dots, v_n) of the vertices of a digraph D , where n is the number of vertices of D , is called an *acyclic ordering* of D if $(v_i, v_j) \in A(D)$ implies $i < j$. It is well known that a digraph D is acyclic if and only if D has an acyclic ordering.

Theorem 4. *Let M be a multigraph with n vertices. Then, M is the double competition multigraph of an acyclic digraph if and only if there exist an ordering (v_1, \dots, v_n) of the vertices of M and a double indexed edge clique partition $\{S_{ij} \mid i, j \in [n]\}$ of M such that the following conditions hold:*

(I) *for any $i, j \in [n]$, if $|A_i \cap B_j| \geq 2$, then $A_i \cap B_j = S_{ij}$;*

(IV) *for any $i, j, k \in [n]$, $v_k \in S_{ij}$ implies $i < k < j$,*

where A_i and B_j are the sets defined as (1) and (2).

Proof: First, we show the only-if part. Let M be the double competition multigraph of an acyclic digraph D . Let (v_1, \dots, v_n) be an acyclic ordering of the vertices of D . Let S_{ij} ($i, j \in [n]$) be the sets defined as (3), and let \mathcal{F} be the family defined as (4). Then S_{ij} is a clique of M , and \mathcal{F} is an edge clique partition of M . Moreover, we can show, as in the proof of Theorem 1, that the condition (I) holds. Now we show that the condition (IV) holds. Suppose that $v_k \in S_{ij}$. Then $(v_i, v_k), (v_k, v_j) \in A(D)$. Since (v_1, \dots, v_n) is an acyclic ordering of D , $(v_i, v_k) \in A(D)$ implies $i < k$ and $(v_k, v_j) \in A(D)$ implies $k < j$. Therefore $i < k < j$. Thus the condition (IV) holds.

Next, we show the if part. Let M be a multigraph with n vertices, and suppose that there exist an ordering (v_1, \dots, v_n) of the vertices of M and a double indexed edge clique partition $\{S_{ij} \mid i, j \in [n]\}$ of M such that the conditions (I) and (IV) hold. We define a digraph D by $V(D) := V(M)$ and $A(D)$ given in (5). By the condition (IV), it follows from the definition of D that (v_1, \dots, v_n) is an acyclic ordering of D . Therefore D is an acyclic digraph. Moreover we can show, as in the proof of Theorem 1, that M is the double competition multigraph of D . \square

Remark 5. The condition (I) in Theorems 1, 2, 3, and 4 may be replaced by the following condition:

(I)' for any $i, j \in [n]$, $A_i \cap B_j = S_{ij}$.

Proof: If the condition (I)' is satisfied, then so is the condition (I). Suppose that the condition (I) is satisfied. If $|A_i \cap B_j| \geq 2$, then it follows from the condition (I) that $A_i \cap B_j = S_{ij}$. If $|A_i \cap B_j| = 0$, then $A_i \cap B_j = \emptyset$. Since $S_{ij} \subseteq A_i \cap B_j$, we have $S_{ij} = \emptyset$. Therefore, $A_i \cap B_j = S_{ij}$. If $|A_i \cap B_j| = 1$,

then $A_i \cap B_j = \{v_k\}$ for some $k \in [n]$. Since $S_{ij} \subseteq A_i \cap B_j$, we have $S_{ij} = \emptyset$ or $S_{ij} = \{v_k\}$. If $S_{ij} = \{v_k\}$, then $A_i \cap B_j = S_{ij}$. If $S_{ij} = \emptyset$, then we replace $S_{ij} = \emptyset$ by $S_{ij} = \{v_k\}$. Then \mathcal{F} is still an edge clique partition of M , and $A_i \cap B_j = S_{ij}$. Thus the condition (I)' holds. Hence the remark holds. \square

At the end of this paper, we mention two corollaries related to the edge clique partition number of a multigraph. Recall that the *edge clique partition number* of a multigraph M is the minimum size of an edge clique partition of M and is denoted by $\theta^*(M)$. As a corollary of Theorem 1, we obtain a necessary condition for multigraphs being the double competition multigraph of a digraph.

Corollary 6. *If a multigraph M with n vertices is the double competition multigraph of a digraph, then $\theta^*(M) \leq n^2$.*

For the double competition multigraphs of acyclic digraphs, we can improve the above upper bound for the edge clique partition numbers of multigraphs.

Corollary 7. *If a multigraph M with n vertices is the double competition multigraph of an acyclic digraph, then $\theta^*(M) \leq \frac{1}{2}(n-2)(n-3)$.*

Proof: Suppose that a multigraph M with n vertices is the double competition multigraph of an acyclic digraph. Then, by Theorem 4, there exist an ordering (v_1, \dots, v_n) of the vertices of M and a double indexed edge clique partition $\{S_{ij} \mid i, j \in [n]\}$ of M satisfying the conditions (I) and (IV). It follows from the condition (IV) that, if $j \leq i+1$, then $S_{ij} = \emptyset$. If $j = i+2$, then $S_{ij} = \emptyset$ or $S_{ij} = \{v_{i+1}\}$, which does not cover an edge of M . Therefore, the family $\{S_{ij} \mid i, j \in [n], i+3 \leq j\}$ is an edge clique partition of M . Thus the corollary holds. \square

Remark 8. In [7], the authors defined the *double multicompetition number* $dk^*(M)$ of a multigraph M to be the minimum nonnegative integer k such that M together with k new isolated vertices is the double competition multigraph of some acyclic digraph. In this context, Theorem 4 gives a characterization of multigraphs whose double multicompetition number is equal to 0.

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