Doubly probabilistic representation for the stochastic porous media type equation.

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Summary: The purpose of the present paper consists in proposing and discussing a doubly probabilistic representation for a stochastic porous media equation in the whole space $\mathbb{R}^1$ perturbed by a multiplicative colored noise. For almost all random realizations ω, one associates a stochastic differential equation in law with random coefficients, driven by an independent Brownian motion.

Key words: stochastic partial differential equations; infinite volume; singular porous media type equation; doubly probabilistic representation; multiplicative noise; singular random Fokker-Planck type equation; filtering.

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1 Introduction

We consider a function $\psi : \mathbb{R} \to \mathbb{R}$ and real functions $e^0, \ldots, e^N$ on $\mathbb{R}$, for some strictly positive integer N. In the whole paper, the following assumption will be in force.

Assumption 1.1.
- $|\psi(u)| \leq \mathrm{const}|u|,\ u \geq 0$. *In particular,* $\psi(0) = 0$.
- $\psi : \mathbb{R} \to \mathbb{R}$ *is a continuous function such that its restriction to* $\mathbb{R}_+$ *is monotone increasing. Moreover we also suppose that* $\lim_{u\to 0} \frac{\psi(u)}{u}$ *exists.*
- *Let* $e^i \in C^2_{\mathrm{b}}(\mathbb{R}), 0 \leq i \leq N$.

Let $T > 0$ and $(\Omega, \mathcal{F}, P)$, be a fixed probability space. A generic element of Ω will be denoted by ω. $(\mathcal{F}_t, t \in [0,T])$ will stand for a filtration, fulfilling the usual conditions and we suppose $\mathcal{F} = \mathcal{F}_T$. Let $\mu(t,\xi), t \in [0,T], \xi \in \mathbb{R}$, be a random field of the type

$$\mu(t,\xi) = \sum_{i=1}^N e^i(\xi) W^i_t + e^0(\xi)t,\ t \in [0,T], \xi \in \mathbb{R},$$

where $W^i, 1 \leq i \leq N$, are independent continuous $(\mathcal{F}_t)$-Brownian motions on $(\Omega, \mathcal{F}, P)$, which are fixed from now on until the end of the paper.

For technical reasons we will sometimes set $W^0_t \equiv t$. We focus on a stochastic partial differential equation of the following type:

$$\left\{ \begin{array}{rcl} \partial_t X(t,\xi) & = & \frac{1}{2}\partial^2_{\xi\xi}(\psi(X(t,\xi)) + X(t,\xi)\partial_t\mu(t,\xi), \\ X(0,\mathrm{d}\xi) & = & x_0(\mathrm{d}\xi), \end{array} \right. \tag{1.1}$$

which holds in the sense of Definition 2.9, where x_0 is a a given probability measure on $\mathbb{R}$. The stochastic multiplication above is of Itô type. We look for a solution of (1.1) with time evolution in $L^1(\mathbb{R})$. Since ψ restricted to $\mathbb{R}_+$ is non-negative, Assumption 1.1 implies $\psi(u) = \Phi^2(u)u,\ u \geq 0$, $\Phi : \mathbb{R}_+ \to \mathbb{R}$ being a non-negative continuous function which is bounded on $\mathbb{R}_+$.

Remark 1.2.
1. *In the sequel we will consider, without further comments extensions of* ψ *(and* Φ*) to the real line which fulfill the first two items of Assumption 1.1 for* $u \in \mathbb{R}$ *instead of* $u \geq 0$.
2. *The restriction on* $u \mapsto \Phi(u)$ *introduced in Assumption 1.1 to be continuous is not always necessary, but here we assume this for simplicity.*

When $\psi(u) = |u|^{m-1}u$, $m > 1$, (1.1) and $\mu \equiv 0$, (1.1) is nothing else but the classical *porous media equation*. When ψ is a general increasing function (and

$\mu \equiv 0$), there are several contributions to the analytical study of (1.1), starting from [12] for existence, [15] for uniqueness in the case of bounded solutions and [13] for continuous dependence on the coefficients. Those are the classical references when the space variable varies on the real line. For equations in a bounded domain and Dirichlet boundary conditions, for simplicity, we only refer to monographs, e.g. [28, 26, 1, 2].

As far as the stochastic porous media is concerned, most of the work for existence and uniqueness concerned the case of bounded domain, see for instance [4, 5, 3]. In the infinite volume case, i.e. when the underlying domain is $\mathbb{R}^d$, well-posedness was fully analyzed in [22], when ψ is polynomially bounded (including the fast diffusion case) when the space dimension is $d \geq 3$. [8] established existence and uniqueness for any dimension $d \geq 1$ and the authors obtained estimates for finite time extinction. To the best of our knowledge, except for [22] and [8], this seems to be the only work concerning a stochastic porous type equation in infinite volume.

We provide a probabilistic representation of solutions to (1.1) extending the results of [14, 6] which treated the deterministic case $\mu \equiv 0$. In the deterministic case, it seems that the first author who considered a probabilistic representation (of the type studied in this paper) for the solutions of a non-linear deterministic PDE was McKean [19], particularly in relation with the so called propagation of chaos. In his case, however, the coefficients were smooth. From then on the literature steadily grew and nowadays there is a vast amount of contributions to the subject, see the reference list of [14, 6]. A probabilistic representation when $\psi(u) = |u|u^{m-1}, m > 1$, was provided for instance in [11], in the case of the classical porous media equation. When $m < 1$, i.e. in the case of the fast diffusion equation, [9] provides a probabilistic representation of the so called **Barenblatt solution**, i.e. the solution whose initial condition is concentrated at zero.

[14, 6] discussed the probabilistic representation when $\mu = 0$ in the so called non-degenerate and degenerate case respectively (see Definition 6.1), where ψ also may have jumps.

In the case $\mu = 0$, the equation (1.1) models a non-linear phenomenon macroscopically. Let us denote by $u : [0, T] \times \mathbb{R} \to \mathbb{R}$ the solution of that equation. The idea of the probabilistic representation is to find a process $(Y_t, t \in [0, T])$ whose law at time t has $u(t, \cdot)$ for its density. In this case the equation (1.1) is conservative, in the sense that the integral (mass) of the solution is conserved along the time.

The process Y turns out to be the weak solution of the non-linear stochastic differential equation

$$\begin{cases} Y_t & = & Y_0 + \int_0^t \Phi(u(s,Y_s))dB_s, \\ \mathrm{Law}(Y_t) & = & u(t,\cdot), \quad t \geq 0, \end{cases} \tag{1.2}$$

where B is a classical Brownian motion. The behavior of Y is the microscopic counterpart of the phenomenon described by (1.1), describing the evolution of a single particle, whose law behaves according to (1.1).

The idea of this paper is to consider the case when $\mu \neq 0$. This includes the case when μ is not vanishing but it is deterministic; it happens when only e^0 is non-zero, and $e^i \equiv 0, 1 \leq i \leq n$. In this case our technique gives a sort of forward Feynman-Kac formula for a non-linear PDE. One of the main interests of this paper is that it provides a (forward) probabilistic representation for *non conservative* (random) PDE.

We introduce a doubly stochastic representation on which one can represent the solution of (1.1) as the weighted-law with respect to the random field μ (or simply the μ-weighted law) of a solution to a non-linear SDE.

Intuitively, it describes the microscopic aspect of the SPDE (1.1) for almost all quenched ω. The terminology strongly refers to the case where the probability space $(\Omega, \mathcal{F}, P)$ on which the SPDE is defined, remains fixed.

We represent a solution X to (1.1) making use of another independent source of randomness described by another probability space based on some set Ω_1.

The analog of the process Y, obtained when μ is zero in [6, 14], is a doubly stochastic process, still denoted by Y defined on $(\Omega_1 \times \Omega, Q)$, for which, X constitutes the so-called family of *μ-marginal weighted laws of* Y, see Definition 2.4. Y is the solution of a *doubly stochastic non-linear diffusion* problem, see Definition 3.1. It will be a (doubly) stochastic process $(\omega_1, \omega) \mapsto Y(\omega_1, \omega)$ solution of

$$Y_t = Y_0 + \int_0^t \Phi(X(s, Y_s, \omega))dB_s, \tag{1.3}$$

and $B(\cdot, \omega)$ is a Brownian motion on Ω_1 for almost any fixed $\omega \in \Omega$. The solution of (1.3) is in the following sense: fixing a realization $\omega \in \Omega$, $Y(\cdot, \omega)$ is a weak solution to the first line of (1.2) with $u(t, \xi) = X(t, \xi, \omega)$. Moreover $X(t, \xi, \omega)$ is the μ-marginal weighted law of $Y_t(\cdot, \omega)$.

The paper includes the following main achievements.

1. If we replace in (1.3) $a(s, \xi, \omega) = \Phi(X(s, \xi, \omega))$ and a is bounded and non-degenerate, we show existence and uniqueness of the solution, strongly in ω, weakly in $\omega_1 \in \Omega_1$, see Proposition 4.1. We also show the existence of law densities, for P-almost all quenched ω, see Proposition 4.4.

2. Theorem 3.3 states that the μ-marginal weighted laws X of a solution Y of a *doubly stochastic non-linear diffusion* problem constitute a solution of the stochastic porous media equation (1.1).

3. Conversely, given a solution X of (1.1), under suitable conditions, there is a solution Y of the doubly stochastic non-linear diffusion. This is discussed in Theorem 6.3 and in Theorem 7.1, distinguishing respectively the cases when ψ is non-degenerate and degenerate, see Definition 6.1.

4. When ψ is non-degenerate, then the doubly stochastic non-linear diffusion problem also admits uniqueness, see Theorem 6.3.

5. Section 3.2 illustrates a filtering interpretation for a solution of SPDE (1.1). Indeed, the μ-marginal weighted laws X of a solution Y of a doubly stochastic non-linear diffusion problem (1.3) can be seen as *conditional densities* of $Y_t, t \in [0,T]$ with respect to some probability measure.

6. Uniqueness of the stochastic Fokker-Planck equation obtained replacing Φ^2 by a function $a(t,\omega,\xi)$ in (1.1), see Theorem 5.1.

7. Existence of a density to the solution of (1.3), see Proposition 4.4.

2 Preliminaries

2.1 Basic notations

First we introduce some basic recurrent notations. $\mathcal{M}(\mathbb{R})$ denotes the space of finite real measures.
We recall that $\mathcal{S}(\mathbb{R})$ is the space of the Schwartz fast decreasing test functions. $\mathcal{S}'(\mathbb{R})$ is its dual, i.e. the space of Schwartz tempered distributions. On $\mathcal{S}'(\mathbb{R})$, the map $(I-\Delta)^{\frac{s}{2}}, s \in \mathbb{R}$, is well-defined. For $s \in \mathbb{R}$, $H^s(\mathbb{R})$ denotes the classical Sobolev space consisting of all functions $f \in \mathcal{S}'(\mathbb{R})$ such that $(I-\Delta)^{\frac{s}{2}} f \in L^2(\mathbb{R})$. We introduce the norm

$$\|f\|_{H^s} := \|(I-\Delta)^{\frac{s}{2}} f\|_{L^2},$$

where $\|\cdot\|_{L^p}$ is the classical $L^p(\mathbb{R})$-norm for $1 \le p \le \infty$. In the sequel, we will often simply denote $H^{-1}(\mathbb{R})$, by H^{-1} and $L^2(\mathbb{R})$ by L^2. Furthermore, $W^{r,p}$ denote the classical Sobolev space of order $r \in \mathbb{N}$ in $L^p(\mathbb{R})$ for $1 \le p \le \infty$.

Definition 2.1. *Given a function e belonging to $L^1_{\mathrm{loc}}(\mathbb{R}) \cap \mathcal{S}'(\mathbb{R})$, we say that it is an H^{-1}-**multiplier**, if the map $\varphi \mapsto \varphi e$ is continuous from $\mathcal{S}(\mathbb{R})$ to H^{-1} with respect to the H^{-1}-topology on both spaces.*

In the following lines we give some other sufficient conditions on a function e to be an H^{-1}**-multiplier**.

Lemma 2.2. *Let $e : \mathbb{R} \to \mathbb{R}$. If $e \in W^{1,\infty}$ (for instance if $e \in W^{2,1}$), then e is a $H^{-1}(\mathbb{R})$-multiplier. In particular the functions $e^i, 0 \leq i \leq N$ of Definition 1.1 are $H^{-1}(\mathbb{R})$-multipliers.*

Proof. By duality arguments, we observe that it is enough to show the existence of a constant $\mathcal{C}(e)$ such that

$$\|eg\|_{H^1} \leqslant \mathcal{C}(e) \|g\|_{H^1}, \ \forall\, g \in \mathcal{S}(\mathbb{R}). \tag{2.1}$$

(2.1) follows by product derivation rules, with for instance $\mathcal{C}(e) = \sqrt{2}\left(\|e\|_\infty^2 + \|e'\|_\infty^2\right)^{\frac{1}{2}}$. ☐

With respect to the random field μ, we introduce a notation for the Itô type stochastic integral below.

Let $Z = (Z(s,\xi), s \in [0,T], \xi \in \mathbb{R})$ be a random field on $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ such that $\int_0^T \left(\int_\mathbb{R} |Z(s,\xi)| \mathrm{d}\xi\right)^2 \mathrm{d}s < \infty$ a.s. and it is an $L^1(\mathbb{R})$-valued $(\mathcal{F}_s)$-progressively measurable process. Then, the stochastic integral

$$\int_{[0,t]\times\mathbb{R}} Z(s,\xi)\mu(\mathrm{d}s,\xi)d\xi := \sum_{i=0}^N \int_0^t \left(\int_\mathbb{R} Z(s,\xi)e^i(\xi)\mathrm{d}\xi\right) \mathrm{d}W_s^i,$$

is well-defined.

More generally, if $s \mapsto Z(s,\cdot)$ is a measurable map $[0,T]\times\Omega \mapsto \mathcal{M}(\mathbb{R})$, where $\mathcal{M}(\mathbb{R})$ is the space of signed finite measures, such that $\int_0^T \|Z(s,\cdot)\|_{\mathrm{var}}^2 \mathrm{d}s < \infty$, then the stochastic integral

$$\int_{[0,t]\times\mathbb{R}} Z(s,\xi)\mu(\mathrm{d}s,\xi)d\xi := \sum_{i=0}^N \int_0^t \left(\int_\mathbb{R} Z(s,\mathrm{d}\xi)\right) e^i(\xi)\mathrm{d}W_s^i,$$

is well-defined.

We specify now better the filtration $(\mathcal{F}_t)_{t\in[0,T]}$ of the introduction. We will consider a fixed filtered probability space $(\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{t\in[0,T]})$, where $(\mathcal{F}_t)_{t\in[0,T]}$ is the canonical filtration of a standard Brownian motion $(W^1, \ldots, W^N)$ enlarged with the σ-field generated by x_0. We also suppose that $\mathcal{F}_0$ contains the P-null sets and $\mathcal{F} = \mathcal{F}_T$.

Let $(\Omega_1, \mathcal{H})$ be a measurable space. In the sequel, we will also consider another filtered probability space $(\Omega_0, \mathcal{G}, \mathbf{Q}, (\mathcal{G}_t)_{t\in[0,T]})$, where $\Omega_0 = \Omega_1 \times \Omega$, $\mathcal{G} = \mathcal{H} \otimes \mathcal{F}$.

Clearly any random element Z on $(\Omega, \mathcal{F})$ will be implicitly extended to $(\Omega_0, \mathcal{G})$ setting $Z(\omega_1, \omega) = Z(\omega)$. In particular $W^i, i = 1 \ldots N$ will be extended in that way.

Here we fix some conventions concerning measurability. Any topological space E is naturally equipped with its Borel σ-algebra $\mathcal{B}(E)$. For instance $\mathcal{B}(\mathbb{R})$ (resp. $\mathcal{B}([0,T])$ denotes the Borel σ-algebra of $\mathbb{R}$ (resp. $[0,T]$).

Given any probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$, the σ-field $\mathcal{F}$ will always be omitted. When we will say that a map $T : \Omega \times E \to \mathbb{R}$ is measurable, we will implicitly suppose that the corresponding σ-algebras are $\mathcal{F} \otimes \mathcal{B}(E)$ and $\mathcal{B}(\mathbb{R})$.

All the processes on any generic measurable space $(\Omega_2, \mathcal{F}_2)$ will be considered to be measurable with respect to both variables (t, ω). In particular any processes on $\Omega_1 \times \Omega$ is supposed to be measurable with respect to $([0,T] \times \Omega_1 \times \Omega, \mathcal{B}([0,T]) \otimes \mathcal{H} \otimes \mathcal{F})$.

A function $(A, \omega) \mapsto Q(A, \omega)$ from $\mathcal{H} \times \Omega \to \mathbb{R}_+$ is called **random kernel** (resp. **random probability kernel**) if for each $\omega \in \Omega$, $Q(\cdot, \omega)$ is a finite positive (resp. probability) measure and for each $A \in \mathcal{H}$, $\omega \mapsto Q(A, \omega)$ is $\mathcal{F}$-measurable. The finite measure $Q(\cdot, \omega)$ will also be denoted by Q^ω. To that random kernel we can associate a specific finite measure (resp. probability) denoted by $\mathbf{Q}$ on $(\Omega_0, \mathcal{G})$ setting $\mathbf{Q}(A \times F) = \int_F Q(A, \omega) P(\mathrm{d}\omega) = \int_F Q^\omega(A) P(\mathrm{d}\omega)$, for $A \in \mathcal{H}, F \in \mathcal{F}$. The probability Q from above will be supposed here and below to be associated with a random probability kernel.

Definition 2.3. *If there is a measurable space* $(\Omega_1, \mathcal{H})$ *and a random kernel* Q *as before, then the probability space* $(\Omega_0, \mathcal{G}, \mathbf{Q})$ *will be called* **suitable enlarged probability space** *(of* $(\Omega, \mathcal{F}, P)$*).*

As said above, any random variable on $(\Omega, \mathcal{F})$ will be considered as a random variable on $\Omega_0 = \Omega_1 \times \Omega$. Then, obviously, $W^1, \ldots, W^N$ are independent Brownian motions also $(\Omega_0, \mathcal{G}, Q)$.

Given a local martingale M on any filtered probability space, the process $Z := \mathcal{E}(M)$ denotes its Doléans exponential, which is a local martingale. In particular it is the unique solution of $dZ_t = Z_{t-} dM_t, \quad Z_0 = 1$. When M is continuous we have $Z_t = e^{M_t - \frac{1}{2}\langle M \rangle_t}$.

2.2 The concept of marginal weighted laws

Let us consider a suitably enlarged probability space as in Definition 2.3.

Definition 2.4. *Let* $Y : \Omega_1 \times \Omega \times [0,T] \to \mathbb{R}$ *be a measurable process, progressively measurable on* $(\Omega_0, \mathcal{G}, \mathbf{Q}, (\mathcal{G}_t))$*, where* $(\mathcal{G}_t)$ *is some filtration on* $(\Omega_0, \mathcal{G}, \mathbf{Q})$ *such that* $W^1, \ldots, W^N$ *are* $(\mathcal{G}_t)$*-Brownian motions on* $(\Omega_0, \mathcal{G}, \mathbf{Q})$*. We will make use of the* *stochastic integral notation*

$$\int_0^t \mu(ds, Y_s) = \sum_{i=0}^{N} \int_0^t e^i(Y_s)dW_s^i, t \in [0,T]. \tag{2.2}$$

As we shall see below in Proposition 2.6, for every $t \in [0,T]$

$$E^{\mathbf{Q}}\left(\mathcal{E}_t\left(\int_0^{\cdot} \mu(\mathrm{d}s, Y_s)\right)\right) < \infty. \tag{2.3}$$

To Y, we will associate its **family of μ-marginal weighted laws***, (or simply* **family of μ-weighted laws***) i.e. the family of random kernels ($t \in [0,T]$),*

$$\Gamma_t = \left(\Gamma_t^Y(A,\omega),\ A \in \mathcal{B}(\mathbb{R}),\ \omega \in \Omega\right)$$

defined by

$$\varphi \mapsto E^{Q^\omega}\left(\varphi(Y_t(\cdot,\omega))\mathcal{E}_t(\int_0^{\cdot} \mu(\mathrm{d}s, Y_s)(\cdot,\omega))\right) = \int_{\mathbb{R}} \varphi(r)\Gamma_t^Y(\mathrm{d}r,\omega), \tag{2.4}$$

where φ is a generic bounded real Borel function. We will also say that for fixed $t \in [0,T]$, Γ_t is **the μ-marginal weighted law** *of Y_t.*

Remark 2.5. *i) If Ω is a singleton $\{\omega_0\}$, $e^i = 0$, $1 \leqslant i \leqslant N$, the μ-marginal weighted laws coincide with the weighted laws*

$$\varphi \mapsto E^{\mathbf{Q}}\left(\varphi(Y_t)\exp\left(\int_0^t \mathrm{e}^0(Y_s)\mathrm{d}s\right)\right),$$

with $\mathbf{Q} = Q^{\omega_0}$. In particular if $\mu \equiv 0$ then the μ-marginal weighted laws are the classical laws.

ii) By (2.3)*, for any $t \in [0,T]$, for P almost all $\omega \in \Omega$,*

$$E^{Q^\omega}\left(\mathcal{E}_t(\int_0^{\cdot} \mu(\mathrm{d}s, Y_s)(\cdot\ ,\omega))\right) < \infty.$$

iii) The function $(t,\omega) \mapsto \Gamma_t(A,\omega)$ is measurable, for any $A \in \mathcal{B}(\mathbb{R})$, because Y is a measurable process.

iv) In the case $\mathrm{e}^0 = 0$, the situation is the following. For each fixed $\omega \in \Omega$, (2.4) *is a (random) non-negative measure which is not a probability. However the expectation of its total mass is indeed* 1*.*

Proposition 2.6. *Consider the situation of Definition 2.4. Then we have the following.*

i) The process $M_t := \mathcal{E}_t\left(\sum_{i=1}^N \int_0^{\cdot} \mathrm{e}^i(Y_s)\mathrm{d}W_s^i\right)$ is a martingale. We emphasize that the sum starts indeed at $i = 1$.

ii) The quantity (2.3) *is bounded by* $\exp\left(T\left\|e^0\right\|_\infty\right)$.

iii) $E^{\mathbf{Q}}(M_t^2) \le \exp(3T\sum_{i=1}^N \|e^i\|_\infty^2), t \in [0,T]$. *Consequently* M *is a uniformly integrable martingale.*

iv) For P*-a.e.* $\omega \in \Omega$, $\sup_{0\le t\le T}\|\Gamma_t(\cdot,\omega)\|_{\mathrm{var}} < \infty$, *where we remind that* $\|\cdot\|_{\mathrm{var}}$ *stands for the total variation.*

Remark 2.7. *Proposition 2.6 ii) yields in particular that* Y *always admits* μ*-marginal weighted laws.*

Proof. i) The result follows since the Novikov condition $E^{\mathbf{Q}}\left(\exp\left(\frac{1}{2}\sum_{i=1}^N\int_s^t e^i(Y_s)^2\mathrm{d}s\right)\right) < \infty$ is verified, because the functions $e^i,\ i = 1\dots N$, are bounded.

ii) This follows because $E^{\mathbf{Q}}(M_t) = 1, \forall t \in [0,T]$.

iii) M_t^2 is equal to $N_t \exp\left(3\sum_{i=1}^N\int_0^t (e^i)^2(Y_s)\mathrm{d}s\right)$, where N is a positive martingale with $N_0 = 1$.

iv) For $t \in [0,T]$,

$$\sup_{t\leqslant T}\|\Gamma_t(\cdot\ ,\omega)\|_{\mathrm{var}} = \sup_{t\leqslant T} E^{Q^\omega}\left(M_t\exp\left(\int_0^t e^0(Y_s)\mathrm{d}s\right)\right)$$
$$\leqslant \exp\left(T\left\|e^0\right\|_\infty\right)\sup_{t\le T}E^{Q^\omega}\left(M_t\right).$$

Taking the expectation with respect to P it implies

$$\begin{aligned} E^P\left(\sup_{t\leqslant T}\left\|\Gamma_t^Y(\cdot\ ,\omega)\right\|_{\mathrm{var}}\right) &\le \exp\left(T\left\|e^0\right\|_\infty\right)E^P\left(\sup_{t\leqslant T}E^{Q^\omega}\left(M_t\right)\right) \\ &\le \exp\left(T\left\|e^0\right\|_\infty\right)E^P\left(E^{Q^\omega}\left(\sup_{t\leqslant T}M_t\right)\right). \end{aligned}$$

By the Burkholder-Davis-Gundy (BDG) inequality this is bounded by

$$\begin{aligned} 3\exp\left(T\left\|e^0\right\|_\infty\right)E^{\mathbf{Q}}\left(\langle M\rangle_T^{\frac{1}{2}}\right) &\le 3\exp\left(T\left\|e^0\right\|_\infty\right)E^{\mathbf{Q}}\left(\left[\int_0^T \mathrm{d}s\sum_{i=1}^N M_s^2 e^i(Y_s)^2\right]^{\frac{1}{2}}\right) \\ &\le \mathrm{C}(e,N,T)\left\{E^{\mathbf{Q}}\left(\int_0^T \mathrm{d}s M_s^2\right)\right\}^{\frac{1}{2}}, \end{aligned}$$

where the last inequality is due to Jensen's inequality; $\mathrm{C}(e,N,T)$ is a constant depending on N, T and $e^i,\ i = 0\dots N$,. By Fubini's Theorem and item iii), we have

$$E^{\mathbf{Q}}\left(\int_0^T \mathrm{d}s M_s^2\right) \le T\exp(3T\sum_{i=1}^N\|e^i\|_\infty).$$

□

The lemma below gives a characterization of the μ-weighted laws of a process Y living on an enlarged probability space.

Lemma 2.8. *Let Y (resp. $\tilde{Y}$) be a process on a suitable enlarged probability space $(\Omega_0, \mathcal{G}, \mathbf{Q})$ (resp. $(\tilde{\Omega}_0, \tilde{\mathcal{G}}, \tilde{\mathbf{Q}})$). Set $W = (W^1, \dots, W^N)$. Suppose that the law of (Y, W) under $\mathbf{Q}$ and the law of $(\tilde{Y}, W)$ under $\tilde{\mathbf{Q}}$ are the same. Then, the μ-marginal weighted laws of Y under $\mathbf{Q}$ coincide a.s. with the μ-marginal weighted laws of $\tilde{Y}$ under $\tilde{\mathbf{Q}}$.*

Proof. Let $0 \le t \le T$. Using the assumption, we deduce that for any bounded continuous function $f : \mathbb{R} \to \mathbb{R}$, and every $F \in \mathcal{F}_t$, we have

$$E^{\mathbf{Q}}\left(1_F f(Y_t)\mathcal{E}_t\left(\sum_{i=0}^{N}\int_0^{\cdot} e^i(Y_s)dW_s^i\right)\right) = E^{\tilde{\mathbf{Q}}}\left(1_F f(\tilde{Y}_t)\mathcal{E}_t\left(\sum_{i=0}^{N}\int_0^{\cdot} e^i(\tilde{Y}_s)dW_s^i\right)\right). \tag{2.5}$$

To show this, using classical regularization properties of Itô integral, see e.g. Theorem 2 in [25], and uniform integrability arguments, we first observe that

$$\mathcal{E}_t\left(\sum_{i=0}^{N}\int_0^{\cdot} e^i(Y_s)dW_s^i\right)$$

is the limit in $L^2(\Omega_0, \mathbf{Q})$ of

$$\mathcal{E}_t\left(\sum_{i=0}^{N}\int_0^{\cdot} e^i(Y_s)\frac{W_{s+\varepsilon}^i - W_s^i}{\varepsilon}\mathrm{d}s\right).$$

A similar approximation property arises replacing Y with $\tilde{Y}$ and $\mathbf{Q}$ with $\tilde{\mathbf{Q}}$. Then (2.5) easily follows.

To conclude, it will be enough to show the existence of a countable family $(f_j)_{j\in\mathbb{N}}$ of bounded continuous real functions for which, for P almost all $\omega \in \Omega$, for any $j \in \mathbb{N}$, we have $R_j = \tilde{R}_j$ where

$$\begin{aligned} R_j(\omega) &= E^{Q^\omega}\left(f_j(Y_t(\cdot,\omega))\mathcal{E}_t\left(\sum_{i=0}^{N}\int_0^{\cdot} e^i(Y_s(\cdot,\omega))dW_s^i\right)\right) \\ \tilde{R}_j(\omega) &= E^{\tilde{Q}^\omega}\left(f_j(\tilde{Y}_t(\cdot,\omega))\mathcal{E}_t\left(\sum_{i=0}^{N}\int_0^{\cdot} e^i(\tilde{Y}_s(\cdot,\omega))dW_s^i\right)\right). \end{aligned}$$

This will follow, since applying (2.5), for any $F \in \mathcal{F}_t$, we have $E^P(1_F R_j) = E^P(1_F \tilde{R}_j)$. ☐

2.3 SPDE, weak-strong existence of SDEs

In this section we introduce the basic concepts related to the stochastic porous media equation and the related non-linear diffusion.

Definition 2.9. *A random field $X = (X(t,\xi,\omega), t \in [0,T], \xi \in \mathbb{R}, \omega \in \Omega)$ is said to be a solution to* (1.1) *if P a.s. we have the following.*

1. *$X \in C([0,T]; \mathcal{S}'(\mathbb{R})) \cap L^2([0,T]; L^1_{\mathrm{loc}}(\mathbb{R}))$.*
2. *X is an $\mathcal{S}'(\mathbb{R})$ -valued $(\mathcal{F}_t)$-progressively measurable process.*
3. *For any test function $\varphi \in \mathcal{S}(\mathbb{R})$ with compact support, $t \in]0,T]$ a.s. we have*

$$\begin{aligned}\int_{\mathbb{R}} X(t,\xi)\varphi(\xi)\mathrm{d}\xi &= \int_{\mathbb{R}} x_0(\mathrm{d}\xi)\varphi(\xi) + \frac{1}{2}\int_0^t \mathrm{d}s \int_{\mathbb{R}} \psi(X(s,\xi,\cdot))\varphi''(\xi)\mathrm{d}\xi \\ &+ \int_{[0,t]\times\mathbb{R}} X(s,\xi)\varphi(\xi)\mu(\mathrm{d}s,\xi)\mathrm{d}\xi.\end{aligned}$$

At Definition 3.1, we will present the concept of *double stochastic non-linear diffusion* which is a McKean type equation with a supplementary source of randomness. Before this, as a first step, we will introduce a particular the case of simple *double stochastic differential equation* (DSDE). Let $\gamma : [0,T] \times \mathbb{R} \times \Omega \to \mathbb{R}$ be an $(\mathcal{F}_t)$-progressively measurable random fields and x_0 be a probability on $\mathcal{B}(\mathbb{R})$.

Definition 2.10. *a) We say that (DSDE)(γ, x_0) admits* **weak-strong existence** *if there is a suitable extended probability space $(\Omega_0, \mathcal{G}, \mathbf{Q})$, i.e. a measurable space $(\Omega_1, \mathcal{H})$, a probability kernel $(Q(\cdot\,,\omega), \omega \in \Omega)$ on $\mathcal{H} \times \Omega$, two $\mathbf{Q}$-a.s. continuous processes Y, B on $(\Omega_0, \mathcal{G})$ where $\Omega_0 = \Omega_1 \times \Omega$, $\mathcal{G} = \mathcal{H} \otimes \mathcal{F}$ such that the following holds.*

1) For almost all ω, $Y(\cdot,\omega)$ is a (weak) solution to

$$\begin{cases} Y_t(\cdot\,,\omega) = Y_0 + \int_0^t \gamma(s, Y_s(\cdot\,,\omega),\omega)\mathrm{d}B_s(\cdot\,,\omega), \\ Law(Y_0) = x_0, \end{cases} \tag{2.7}$$

with respect to Q^ω, where $B(\cdot\,,\omega)$ is a Q^ω-Brownian motion for almost all ω.

2) We denote $(\mathcal{Y}_t)$ the canonical filtration associated with $(Y_s, 0 \le s \le t)$ and $\mathcal{G}_t = \mathcal{Y}_t \vee (\{\emptyset, \Omega_1\} \otimes \mathcal{F}_t)$. $W^1, \dots, W^N$ is a $(\mathcal{G}_t)$-martingale under $\mathbf{Q}$.

3) For every $0 \le s \le T$, for every bounded continuous $\mathcal{A} : C([0,s]) \to \mathbb{R}$, the r.v. $\omega \mapsto E^{Q^\omega}(\mathcal{A}(Y_r(\cdot,\omega), r \in [0,s]))$ is $\mathcal{F}_s$-measurable.

b) We say that (DSDE)(γ, x_0) admits **weak-strong uniqueness** *if the following holds. Consider a measurable space $(\Omega_1, \mathcal{H})$ (resp. $(\widetilde{\Omega}_1, \widetilde{\mathcal{H}})$), a probability kernel $(Q(\cdot\,,\omega), \omega \in \Omega)$ (resp. $(\widetilde{Q}(\cdot\,,\omega), \omega \in \Omega)$), with processes (Y, B) (resp. $(\widetilde{Y}, \widetilde{B})$) such that (2.7) holds (resp. (2.7) holds with $(\Omega_0, \mathcal{G}, \mathbf{Q})$ replaced with*

($\widetilde{\Omega}_0, \widetilde{\mathcal{G}}_0, \widetilde{\mathbf{Q}}$, $\widetilde{\mathbf{Q}}$ *being associated with* $(\widetilde{Q}(\cdot\ ,\omega))$*). Moreover we suppose that item 2) is verified for* Y *and* $\tilde{Y}$.
Then $(Y, W^1, \dots, W^N)$ *and* $(\widetilde{Y}, W^1, \dots, W^N)$ *have the same law.*

c) *A process* Y *fulfilling items 1) and 2) under (a) will be called* **weak-strong solution of** *(DSDE)*(γ, x_0).

Remark 2.11. *Let* Y *be a weak-strong solution of (DSDE)*(γ, x_0) *with corresponding* B.

a) *Since for almost all* $\omega \in \Omega$, $B(\cdot, \omega)$ *is a Brownian motion under* Q^ω, *it is clear that* B *is a Brownian motion under* Q, *which is independent of* $\mathcal{F}_T$, *i.e. independent of* $W^1, \dots, W^N$.
Indeed let $\mathcal{A} : C([0,T]) \to \mathbb{R}$ *be a continuous bounded functional, and denote by* $\mathcal{W}$ *the Wiener measure on* $C([0,T])^N$. *Let* F *be a bounded* $\mathcal{F}_T$*-measurable r.v. Since for each* ω, $B(\cdot, \omega)$ *is a Wiener process with respect to* Q^ω, *we get*

$$\begin{aligned} E^{\mathbf{Q}}(F\mathcal{A}(B)) &= \int_\Omega F E^{Q^\omega}(\mathcal{A}(B(\cdot,\omega)))\mathrm{d}P(\omega) = \int_\Omega F(\omega)\mathrm{d}P(\omega) \int_{\Omega_1} \mathcal{A}(\omega_1) d\mathcal{W}(\omega_1) \\ &= \int_{\Omega_0} F(\omega)\mathrm{d}\mathbf{Q}(\omega_0) \int_{\Omega_0} \mathcal{A}(\omega_1)\mathrm{d}\mathbf{Q}(\omega_0). \end{aligned}$$

This shows that $(W^1, \dots, W^N)$ *and* B *are independent. Taking* $F = 1_\Omega$ *in previous expression, the equality between the left-hand side and the third term, shows that* B *is a Brownian motion under* Q.

b) *Since for any* $1 \le i, j \le N$,

$$[W^i, W^j]_t = \delta_{ij} t, \ \ [W^i, B] = 0, \ \ [B, B]_t = t,$$

Lévy's characterization theorem, implies that $(W^1, \dots, W^N, B)$ *is a* $\mathbf{Q}$*-Brownian motion.*

c) *An equivalent formulation to 1) in item a) of Definition 2.10 is the following. For* P *a.e.,* $\omega \in \Omega$, $Y(\cdot\ ,\omega)$ *solves the* Q^ω*-martingale problem with respect to the (random) PDE operator*

$$L_t^\omega f(\xi) = \frac{1}{2}\gamma^2(t, \xi, \omega) f''(\xi),$$

and initial distribution x_0. *Indeed, we remark that the construction can be performed on the canonical space* $\Omega_1 = C([0,T]; \mathbb{R})$.

Proposition 2.12. *Let* Y *be a process as in Definition 2.10 a). We have the following.*

1. *Y is a $(\mathcal{G}_t)$-martingale on the product space $(\Omega_0, \mathcal{G}, \mathbf{Q})$.*

2. $[Y, W^i] = 0, \ \forall 1 \le i \le N$.

Proof. Let $0 \le s < t \le T$, $F_s \in \mathcal{F}_s$ and $G : C([0,s]) \to \mathbb{R}$ be continuous and bounded. We will prove below that, for $1 \le i \le N+1$, setting $W_t^{N+1} = 1$, for all $t \ge 0$,

$$E^{\mathbf{Q}}(Y_t W_t^i G(Y_r, r \le s) 1_{F_s}) = E^{\mathbf{Q}}(Y_s W_s^i 1_{F_s} G(Y_r, r \le s)). \tag{2.8}$$

Then (2.8) with $i = N+1$ shows item 1. Considering (2.8) with $1 \le i \le N$, shows that YW^i is a $(\mathcal{G}_t)$-martingale, which shows item 2. Therefore, it remains to show (2.8).

The left-hand side of that equality gives

$$\begin{aligned} \int_\Omega \mathrm{d}P(\omega) \qquad & W_t^i(\omega) 1_{F_s}(\omega) E^{Q^\omega}\left(Y_t(\cdot,\omega) G(Y_r(\cdot,\omega), r \le s)\right) \\ = \quad & \int_\Omega \mathrm{d}P(\omega) 1_{F_s}(\omega) W_t^i(\omega) E^{Q^\omega}\left(Y_s(\cdot,\omega) G(Y_r(\cdot,\omega), r \le s)\right), \end{aligned}$$

because $Y(\cdot,\omega)$ is a Q^ω-martingale for P-almost all ω. To obtain the right-hand side of (2.8) it is enough to remember that W^i are $(\mathcal{G}_t)$-martingales and that item a) 3) in Definition 2.10 holds. This concludes the proof of Proposition 2.12. □

Remark 2.13. *Lemma 2.8 shows that, whenever weak-strong uniqueness holds, then the μ-weighted marginal laws of any weak solution Y are uniquely determined.*

3 The concept of doubly probabilistic representation

3.1 The doubly stochastic non-linear diffusion.

We come back to the notations and conventions of the introduction and of Section 2. Let x_0 be a probability on $\mathbb{R}$. The doubly probabilistic representation is based on the following idea. Let $Y : \Omega_0 \times [0,T] \to \mathbb{R}$ be a measurable process where $\Omega_0 = \Omega_1 \times \Omega$ is the usual enlarged probability space as introduced in Definition 2.3. Let $\mathbf{Q}$ be a probability inherited from a random kernel Q^ω as before Definition 2.3. Let $(\mathcal{G}_t)$, where $(\mathcal{G}_t)$ is some filtration on $(\Omega_0, \mathcal{G})$ such that $W^1, \ldots, W^N$ are $(\mathcal{G}_t)$-Brownian motions on $(\Omega_0, \mathcal{G}, \mathbf{Q})$.

Suppose that

$$\left\{\begin{array}{rcl} Y_t & = & Y_0 + \int_0^t \Phi(X(s, Y_s)) dB_s, \\ \mu - \text{Weighted Law}(Y_t) & = & X(t, \xi)\mathrm{d}\xi, \quad t \in]0, T], \\ \mu - \text{Weighted Law}(Y_0) & = & x_0(\mathrm{d}\xi), \end{array}\right. \tag{3.1}$$

where B is a Q-standard Brownian motion with respect to $(\mathcal{G}_t)$. Then X solves the SPDE (1.1). This will be the object of Theorem 3.3. Vice versa, if X is a solution of (1.1) then there is a process Y solving (3.1), see Theorem 7.1.

Definition 3.1. *1) We say that the doubly stochastic non-linear diffusion (DSNLD) driven by Φ (on the space $(\Omega, \mathcal{F}, P)$ with initial condition x_0, related to the random field μ (shortly (DSNLD)(Φ, μ, x_0)) admits* **weak existence** *if there is a measurable random field $X : [0, T] \times \mathbb{R} \times \Omega \to \mathbb{R}$ with the following properties.*

a) The problem (DSDE)(γ, x_0) with $\gamma(t, \xi, \omega) = \Phi(X(t, \xi, \omega))$ admits weak-strong existence.

b) $X = X(t, \xi, \cdot)\mathrm{d}\xi, t \in]0, T]$, is the family of μ-marginal weighted laws of Y, where Y is the solution of (2.7) *in Definition 2.10. In other words X constitutes the densities of those μ-marginal weighted laws.*

2) A couple (Y, X), such that Y is a (weak-strong) solution to the (DSDE)(γ, x_0), is called **weak solution** *to the (DSNLD)(Φ, μ, x_0). Y is also called doubly stochastic representation of the random field X.*

3) Suppose that, given two measurable random fields $X^i : [0, T] \times \mathbb{R} \times \Omega \to \mathbb{R}, i = 1, 2$ on $(\Omega, \mathcal{F}, P, (\mathcal{F}_t))$, and Y^i, on extended probability space $(\Omega_0^i, \mathbf{Q}^i), i = 1, 2$, such that (Y^i, X^i) is a weak-strong solution of (DSDE)$(\Phi(X^i), x_0), i = 1, 2$, where we always have that $(Y^1, W^1, \dots, W^N)$ and $(Y^2, W^1, \dots, W^N)$ have the same law. Then we say that the (DSNLD)(Φ, μ, x_0) admits **weak uniqueness**.

Remark 3.2. *If (DSNLD)(Φ, μ, x_0) admits* **weak uniqueness** *then the μ-marginal weighted laws of Y are uniquely determined, P-a.s., see Lemma 2.8.*

Theorem 3.3. *Let (Y, X) be a solution of (DSNLD)(Φ, μ, x_0). Then X is a solution to the SPDE* (1.1).

Remark 3.4. *1. Let $t \in [0, T]$. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be Borel and bounded. Then*

$$\int_{\mathbb{R}} \varphi(\xi) X(t, \xi, \omega)\mathrm{d}\xi = E^{Q^\omega}\left(\varphi(Y_t(\omega))\mathcal{E}_t\left(\int_0^\cdot \mu(ds, Y_s(\omega))\right)\right).$$

So

$$\int_{\mathbb{R}} X(t, \xi, \omega) d\xi = E^{Q^\omega}\left(\mathcal{E}_t\left(\int_0^\cdot \mu(ds, Y_s(\omega))\right)\right).$$

Even though for a.e. $\omega \in \Omega$, the previous expression is not necessarily a probability measure, of course,

$$\nu_\omega : \varphi \mapsto \frac{\int_{\mathbb{R}} \varphi(\xi) X(t,\xi,\omega) d\xi}{\int_{\mathbb{R}} X(t,\xi,\omega) d\xi}$$

is one. It can be expressed as

$$\nu_\omega(A) = \frac{E^{Q^\omega}(1_A(Y_t)\mathcal{E}_t(M(\cdot,\omega)))}{E^{Q^\omega}\mathcal{E}_t(M(\cdot,\omega))},$$

where $M_t(\cdot,\omega) = \int_0^t \mu(ds, Y_s(\cdot,\omega)), t \in [0,T]$, is defined in (2.2).

2. *Consider the particular case $e_0 = 0, e_1 = c$, c being some constant. In this case, the μ-marginal laws are given by*

$$A \mapsto E^{Q^\omega}(1_A(Y_t)c\mathcal{E}_t(W)) = c\mathcal{E}_t(W)E^{Q^\omega}(1_A(Y_t)) = c\mathcal{E}_t(W)\nu_\omega(t,A)$$

and $\nu_\omega(t,\cdot)$ is the law of $Y_t(\cdot,\omega)$ under Q^ω.

Proof. Let B denote the Brownian motion associated to Y as a solution to (DSDE)(γ, x_0), mentioned in item a)1) of Definition 3.1. For $t \in [0,T]$, we set

$$Z_t = \mathcal{E}_t\left(\int_0^\cdot \mu(\mathrm{d}s, Y_s)\right), \quad M_t = Z_t \exp\left(-\int_0^t e^0(Y_s)\mathrm{d}s\right),\ t \in [0,T].$$

1. Proof of Definition 2.9 1. By Proposition 2.6, $(M_t, t \in [0,T])$ is a uniformly integrable martingale. Consequently $t \mapsto Z_t$ is continuous in $L^1(\Omega_0, \mathbf{Q})$. On the other hand the process Y is continuous. This implies that P a.e. $\omega \in \Omega$, $X \in C([0,T]; \mathcal{M}(\mathbb{R}))$, where $\mathcal{M}(\mathbb{R})$ is equipped with the weak topology. This implies that $X \in C([0,T]; \mathcal{S}'(\mathbb{R}))$. Furthermore, for P a.e. $\omega \in \Omega$, and $t \in]0,T]$, $X(t,\cdot,\omega) \in L^1(\mathbb{R})$ and $\int_{\mathbb{R}} X(t,\xi,\omega)\mathrm{d}\xi = \|\Gamma(t,\cdot,\omega)\|_{\mathrm{var}}$. By Proposition 2.6 iv), it follows that P-a.s. $X(\cdot,\cdot,\omega) \in L^\infty([0,T]; L^1(\mathbb{R})) \subset L^2([0,T]; L^1_{\mathrm{loc}}(\mathbb{R}))$.

2. Definition 2.9 2. follows from Remark 3.4 1) and Definition 2.10 a) 3).

3. Proof of Definition 2.9 3. Let $\varphi \in \mathcal{S}(\mathbb{R})$ with compact support. Taking into account Proposition 2.12, we apply Itô's formula to get

$$\varphi(Y_t)Z_t = \varphi(Y_0) + \int_0^t \varphi'(Y_s)Z_s \mathrm{d}Y_s + \int_0^t \varphi(Y_s)Z_s\left(\mu(\mathrm{d}s, Y_s) - \frac{1}{2}\sum_{i=1}^N (e^i(Y_s))^2 \mathrm{d}s\right)$$
$$+ \frac{1}{2}\int_0^t \varphi''(Y_s)\Phi^2(X(s,Y_s))Z_s \mathrm{d}s + \frac{1}{2}\int_0^t \varphi(Y_s)Z_s\left(\sum_{i=1}^N (e^i(Y_s))^2\right)\mathrm{d}s.$$

Indeed we remark that $\int_0^t \varphi'(Y_s)\mathrm{d}[Z,Y]_s = 0$, because $[Z,Y]_t = \sum_{i=1}^N \int_0^t e^i(Y_s)Z_s\mathrm{d}[W^i,Y]_s = 0$; in fact $[W^i,Y]=0$ by Proposition 2.12. So

$$\begin{aligned}\varphi(Y_t)Z_t &= \varphi(Y_0) + \int_0^t \varphi'(Y_s)Z_s\Phi(X(s,Y_s))\mathrm{d}B_s \\ &+ \int_0^t \varphi(Y_s)Z_s\mu(\mathrm{d}s,Y_s) + \frac{1}{2}\int_0^t \varphi''(Y_s)\Phi^2(X(s,Y_s))Z_s\mathrm{d}s.\end{aligned}$$

Taking the expectation with respect to Q^ω we get $\mathrm{d}P$-a.s.,

$$\begin{aligned}\int_{\mathbb{R}} \mathrm{d}\xi\varphi(\xi)X(t,\xi) &= \int_{\mathbb{R}} \varphi(\xi)x_0(\mathrm{d}\xi) + \sum_{i=0}^N \int_0^t \mathrm{d}W_s^i\left(\int_{\mathbb{R}} \mathrm{d}\xi\varphi(\xi)e^i(\xi)X(s,\xi)\right) \\ &+ \frac{1}{2}\int_0^t \mathrm{d}s\int_{\mathbb{R}} \mathrm{d}\xi\varphi''(\xi)\Phi^2(X(s,\xi))X(s,\xi),\end{aligned}$$

which implies the result. Indeed, in the previous equality, we have used Lemma 3.5 below.

□

Lemma 3.5. *Let $1 \le i \le N$. For P a.e. $\omega \in \Omega$, we have*

$$E^{Q^\omega}\left(\int_0^t \varphi(Y_s)Z_se^i(Y_s)\mathrm{d}W_s^i\right)(\cdot,\omega) = \int_0^t \mathrm{d}W_s^i(\omega)\int_{\mathbb{R}} \varphi(\xi)e^i(\xi)X(s,\xi,\omega)\mathrm{d}\xi.$$

Proof. Since the Brownian motions W^i are not random for Q^ω, it is possible to justify the permutation of the stochastic integral with respect to W^i and E^{Q^ω} by a Fubini argument approximating the stochastic integrals via Lebesgue integral, see e.g. Theorem 2 of [25]. A complete proof is given in [7]. □

3.2 Filtering interpretation

Item 1. of Remark 3.4 has an interpretation in the framework of filtering theory, see e.g. [20] for a comprehensive introduction on that subject.
Suppose $e^0 = 0$. Let $\hat{\mathbf{Q}}$ be a probability on some probability space $(\Omega_0, \mathcal{G}_T)$, and consider the non-linear diffusion problem (1.2) as a basic dynamical phenomenon. We suppose now that there are N observations $Y^1,\ldots,Y^N$ related to the process Y generating a filtration $(\mathcal{F}_t)$. We suppose in particular that $\mathrm{d}Y_t^i = \mathrm{d}W_t^i + e^i(Y_t)\mathrm{d}t, 1 \le i \le N$, and $W^1,\ldots,W^N$ be $(\mathcal{F}_t)$-Brownian motions. Consider the following dynamical system of non-linear diffusion type:

$$\left\{\begin{array}{ccc} Y_t &=& Y_0 + \int_0^t \Phi(X(s,Y_s))dB_s \\ \mathrm{d}Y_t^i &=& \mathrm{d}W_t^i + e^i(Y_t)\mathrm{d}t, 1 \le i \le N, \\ X(t,\cdot) &:& \text{conditional law of } Y_t \text{ under } \mathcal{F}_t. \end{array}\right. \tag{3.1}$$

The third equality of (3.1) means, under $\hat{\mathbf{Q}}$, that we have,

$$\int_{\mathbb{R}} \varphi(\xi) X(t,\xi) \mathrm{d}\xi = E(\varphi(Y_t)|\mathcal{F}_t). \tag{3.2}$$

We remark that, under the new probability Q defined by $\mathrm{d}\mathbf{Q} = \mathrm{d}\hat{\mathbf{Q}}\mathcal{E}(\int_0^T \mu(\mathrm{d}s, Y_s))$, $Y^1, \ldots, Y^N$ are standard $(\mathcal{F}_t)$-independent Brownian motions. Then (3.2) becomes

$$\int_{\mathbb{R}} \varphi(\xi) X(t,\xi) \mathrm{d}\xi = E^{\hat{\mathbf{Q}}}(\varphi(Y_t)|\mathcal{F}_t) = \frac{E^{\mathbf{Q}}(\varphi(Y_t)\mathcal{E}_t(\int_0^{\cdot} \mu(ds, Y_s)|\mathcal{F}_t))}{E^{\mathbf{Q}}(\mathcal{E}_t(\int_0^{\cdot} \mu(ds, Y_s)|\mathcal{F}_t))}.$$

Consequently, by Theorem 3.3, X will be the solution of the SPDE (1.1), with x_0 being the law of Y_0; so (1.1) constitutes the Zakai type equation associated with our filtering problem.

4 The densities of the μ-marginal weighted laws

This section constitutes an important step towards the doubly probabilistic representation of a solution to (1.1), when ψ is non-degenerate. Let x_0 be a fixed probability on $\mathbb{R}$. We recall that a process Y (on a suitable enlarged probability space $(\Omega_0, \mathcal{G}, \mathbf{Q})$), which is a weak solution to the (DSNLD)(Φ, μ, x_0), is in particular a weak-strong solution of a (DSDE)(γ, x_0) where $\gamma : [0,T] \times \mathbb{R} \times \Omega \to \mathbb{R}$ is some suitable progressively measurable random field on $(\Omega, \mathcal{F}, P)$. The aim of this section is twofold.

A) To show that whenever γ is a.s. bounded and non-degenerate, (DSDE)(γ, x_0) admit weak-strong existence and uniqueness.

B) The marginal μ-laws of the solution to (DSDE)(γ, x_0) admit a density for P ω a.s.

A) We start discussing well-posedness.

Proposition 4.1. *We suppose the existence of random variables A_1, A_2 such that*

$$0 < A_1(\omega) \leqslant \gamma(t,\xi,\omega) \leqslant A_2(\omega), \quad \forall (t,\xi) \in [0,T] \times \mathbb{R}, \quad \mathrm{d}P\text{-}a.s.$$

Then (DSDE)(γ, x_0) admits weak-strong existence and uniqueness.

Proof. Uniqueness. This is the easy part. Let Y and $\widetilde{Y}$ be two solutions. Then for ω outside a P-null set $N_0, Y(\cdot\ ,\omega)$ and $\widetilde{Y}(\cdot\ ,\omega)$ are solutions to the same one-dimensional classical SDE with measurable bounded and non-degenerate (i.e. greater than a strictly positive constant) coefficients. Then, by Exercise 7.3.3 of [27] the law of $Y(\cdot\ ,\omega)$ equals the law of $\widetilde{Y}(\cdot\ ,\omega)$. Then obviously the law of Y equals the law of $\widetilde{Y}$.

Existence. This point is more delicate. In fact one needs to solve the random SDE for P almost all ω but in such a way that the solution produces bimeasurable processes Y and B.

First we regularize the coefficient γ. Let ϕ be a mollifier with compact support; we set $\phi_n(x) = n\phi(nx),\ x \in \mathbb{R}\ ,\ n \in \mathbb{N}$. We consider the random fields $\gamma_n : [0,T] \times \mathbb{R} \times \Omega \to \mathbb{R}$ by $\gamma_n(t,x,\omega) := \int_{\mathbb{R}} \gamma(t, x-y, \omega)\phi_n(y)\mathrm{d}y$.
Let $(\widetilde{\Omega}_1, \widetilde{\mathcal{H}}_1, \widetilde{P})$ be a probability space where we can construct a random variable Y_0 distributed according to x_0 and an independent Brownian motion B.

In this way on $(\widetilde{\Omega}_1 \times \Omega, \widetilde{\mathcal{H}}_1 \otimes \mathcal{F}, \widetilde{P} \otimes P)$ we dispose of a random variable Y_0 and a Brownian motion independent of $\{\emptyset, \widetilde{\Omega}_1\} \otimes \mathcal{F}$. By usual fixed point techniques, it is possible to exhibit a (strong) solution of (DSDE)(γ_n, x_0) on the over mentioned product probability space. We can show that there is a unique solution $Y = Y^n$ of $Y_t = Y_0 + \int_0^t \gamma_n(s, Y_s, \cdot)\mathrm{d}B_s$. In fact, the maps $\Gamma_n : Z \mapsto \int_0^\cdot \gamma_n(s, Z_s, \omega)\mathrm{d}B_s + Y_0$, where $\Gamma_n : L^2(\widetilde{\Omega}_1 \times \Omega;\ \widetilde{P} \otimes P) \to L^2(\widetilde{\Omega}_1 \times \Omega, \widetilde{P} \otimes P)$ are Lipschitz; by usual Picard fixed point arguments one can show the existence of a unique solution $Z = Z^n$ in $L^2(\widetilde{\Omega}_1 \times \Omega;\ \widetilde{P} \otimes P)$. We observe that, by usual regularization arguments for Itô integral as in Lemma 3.5, for ω-a.s., $Y(\cdot, \omega)$ solves for P a.e. $\omega \in \Omega$, equation

$$Y_t(\cdot\ ,\omega) = Y_0 + \int_0^t \gamma_n(s, Y_s(\cdot\ ,\omega), \omega)\mathrm{d}B_s, \tag{4.1}$$

on $(\widetilde{\Omega}_1, \widetilde{\mathcal{H}}_1, \widetilde{P})$. We consider now the measurable space $\Omega_0 = \Omega_1 \times \Omega$, where $\Omega_1 = C([0,T], \mathbb{R})$, equipped with product σ-field $\mathcal{G} = \mathcal{B}(\Omega_1) \otimes \mathcal{F}$. On that measurable space, we introduce the probability measures $\mathbf{Q}_n$ where $\mathbf{Q}_n(\mathrm{d}\omega_1, \omega) = Q_n(\mathrm{d}\omega_1, \omega)P(\mathrm{d}\omega)$ and $Q_n(\cdot, \omega)$ being the law of $Y^n(\cdot\ ,\omega)$ for almost all fixed ω.
We set $Y_t(\omega_1, \omega) = \omega_1(t)$, where $\omega_1 \in C([0,T]; \mathbb{R})$. We denote by $(\mathcal{Y}_t,\ t \in [0,T])$ (resp. $(\mathcal{Y}_t^1)$) the canonical filtration associated with Y on Ω_0 (resp. Ω_1). The next step will be the following.

Lemma 4.2. *For almost all ω dP a.s. $Q_n(\omega,\ \cdot)$ converges weakly to $Q(\omega, \cdot)$, where under $Q(\cdot, \omega)$, $Y(\cdot\ ,\omega)$ solves the SDE*

$$Y_t(\cdot\ ,\omega) = Y_0 + \int_0^t \gamma(s, Y_s(\cdot\ ,\omega), \omega)\mathrm{d}B_s(\cdot, \omega),$$

where $B(\cdot, \omega)$ is an $(\mathcal{Y}_t^1)$-Brownian motion on Ω_1.

Proof. It follows directly from Proposition A.1 of the Appendix. □

This shows the validity of 1) if Definition 2.10 a).

Remark 4.3. *1) Since $Q_n(\cdot,\omega)$ converges weakly to $Q(\cdot,\omega)$, ω dP a.s., then the limit (up to an obvious modification) is a measurable random kernel.*

2) This also implies that $Y_n(\cdot,\omega)$ converges stably to $Q(\cdot,\omega)$. For details about the stable convergence the reader can consult [17, section VIII 5. c] and the recent monograph [16].

The considerations above allow to complete the proof of Proposition 4.1. By Lemma 4.2, $Q^\omega = Q(\cdot,\omega)$ is a random kernel, being a limit of random kernels. Let us consider the associated probability measure on the suitable enlarged probability space $(\Omega_0, \mathcal{G}, Q)$. We observe that Y on $(\Omega_0, \mathcal{G})$ is obviously measurable, because it is the canonical process $Y(\omega_1,\omega) = \omega_1$. Setting

$$B_t(\cdot,\omega) = \int_0^t \frac{\mathrm{d}Y_s(\cdot,\omega)}{\gamma(s, Y_s(\cdot,\omega),\omega)},$$

we get $[B]_t(\cdot,\omega) = t$ under $Q(\cdot\ ,\omega)$, so, by Lévy characterization theorem, it is a Brownian motion. Moreover B is bimeasurable.
Let $G = \mathcal{A}(Y_r(\cdot,\omega), r \in [0,s])$, where $\mathcal{A}$ is a bounded functional $C([0,s]) \to \mathbb{R}$. We first observe that the r.v. $\omega \mapsto E^{Q^\omega}(G)$ is $\mathcal{F}_s$-measurable. This happens because Y is, under Q^ω, a martingale with quadratic variation $\left(\int_0^t \gamma^2(s, Y_s(\cdot,\omega),\omega)\mathrm{d}s, 0 \le t \le T\right)$, i.e. with (random) coefficient which is $(\mathcal{F}_t)$-progressively measurable. This shows item 3) of Definition 2.10 a).

The last point to check is that $W^1,\dots,W^N$ are $(\mathcal{G}_t)$-martingales, where $\mathcal{G}_t = \mathcal{Y}_t \vee (\{\emptyset,\Omega_1\} \otimes \mathcal{F}_t),\ 0 \le t \le T$, i.e. item 2) of Definition 2.10.
Indeed, we justify this immediately. Consider $0 \le s \le t \le T$. Taking into account monotone class arguments, given $F \in \mathcal{F}_s$, $G \in \mathcal{Y}_s^1$, $1 \le i \le N$, it is enough to prove that

$$E^{\mathbf{Q}}(FGW_t^i) = E^{\mathbf{Q}}(FGW_s^i). \tag{4.2}$$

Using the fact that W^i is an $(\mathcal{F}_t)$-martingale and that $E^{Q^\omega}(G)$ is $\mathcal{F}_s$-measurable by item a) 3) of Definition 2.10 (established above), the left-hand side of (4.2) gives

$$E^P(FW_t^i E^{Q^\omega}(G)) = E^P(FW_s^i E^{Q^\omega}(G)),$$

which constitutes the right-hand side of (4.2). This concludes the proof of the proposition. ☐

We go on now with step B) of the beginning of Section 4.

Proposition 4.4. *We suppose the existence of r.v. A_1, A_2 such that*

$$0 < A_1(\omega) \leqslant \gamma(t,\xi,\omega) \leqslant A_2(\omega), \forall (t,\xi) \in [0,T]\times\mathbb{R}, \qquad a.s.$$

Let Y be a weak-strong solution to (DSDE)(γ, x_0) and we denote by $(\nu_t(dy,\cdot), t \in [0,T])$, the μ-marginal weighted laws of process Y.

1. *There is a measurable function $q : [0,T]\times\mathbb{R}\times\Omega \to \mathbb{R}_+$ such that* $\mathrm{d}t\mathrm{d}P$ *a.e., $\nu_t(\mathrm{d}y,\cdot) = q_t(y,\cdot)\mathrm{d}y$. In other words the μ-marginal weighted laws admit densities.*
2. $\int_{[0,T]\times\mathbb{R}} q_t^2(y,\ \cdot)\mathrm{d}t\mathrm{d}y < \infty \quad \mathrm{d}P$*-a.s..*
3. *q is an $L^2(\mathbb{R})$-valued progressively measurable process.*

Proof. By 3) of Definition 2.10, the μ-marginal laws constitute an $\mathcal{S}'(\mathbb{R})$-valued progressively measurable process. Consequently 3. holds if 1. and 2. hold.

Let

$$B_t(\cdot,\omega) := \int_0^t \frac{\mathrm{d}Y_s(\cdot,\omega)}{\gamma(s, Y_s(\cdot,\omega),\omega)}.$$

We denote again $Q^\omega := Q(\cdot\ ,\omega)$ according to Definition 2.10, $\omega \in \Omega$.

Let $\omega \in \Omega$ be fixed. Let $\varphi : [0,T]\times\mathbb{R} \to \mathbb{R}$ be a continuous function with compact support. We need to evaluate

$$E^{Q^\omega}\left(\int_0^T \varphi(s,Y_s)Z_s\mathrm{d}s\right), \tag{4.3}$$

where $Z_s = M_s \exp\left(\int_0^s e^0(Y_r)\mathrm{d}r\right)$ where $M_s = \mathcal{E}_s\left(\sum_{i=1}^N \int_0^\cdot e^i(Y_r)\mathrm{d}W_r^i\right)$.

M_s is smaller or equal than $\exp\left(\sum_{j=1}^N \int_0^s e^j\left(Y_r\right)\mathrm{d}W_r^j\right)$ which equals

$$\exp\left(\sum_{j=1}^N \left\{W_s^j e^j(Y_s) - \int_0^s W_r^j (e^j)'(Y_r)\mathrm{d}Y_r\right\} - \frac{1}{2}\int_0^s \left\{\sum_{j=1}^N W_r^j (e^j)''(Y_r)\gamma^2(r,Y_r,\cdot)\right\}\mathrm{d}r\right), \tag{4.4}$$

taking into account the fact that $[Y, W^j] = 0$ for any $1 \le j \le n$, by Proposition 2.12.

Denoting $\|g\|_\infty := \sup_{t\in[0,T]}|g(t)|$, for a function $g : [0,T] \to \mathbb{R}$, (4.4) is smaller or equal than

$$\exp\left(\sum_{j=1}^N \|W^j\|_\infty(\left\|e^j\right\|_\infty + \frac{T}{2}\left\|(e^j)''\right\|_\infty A_2^2)\right)\exp\left(-\int_0^s\left[\sum_{j=1}^N W_r^j (e^j)'(Y_r)\gamma(r,Y_r,\cdot)\right]\mathrm{d}B_r\right).$$

So (4.3) is bounded by

$$\varrho(\omega)E^{Q^\omega}\left(\int_0^T |\varphi|(s,Y_s(\cdot,\omega)R_s(\cdot,\omega)\mathrm{d}s\right), \tag{4.5}$$

where

$$\begin{aligned}\varrho(\omega) &= \exp\left(T\|e_0\|_\infty + \sum_{i=1}^N \|W^i\|_\infty \|e^i\|_\infty\right.\\ &+ \left. T\frac{A_2^2(\omega)}{2}\sum_{i=1}^N (\|W^i\|_\infty^2 \|(e^i)'\|_\infty^2 + \|W^i\|_\infty \|(e^i)''\|_\infty)\right)\end{aligned}$$

and R is the Q^ω-exponential martingale

$$R_t(\,\cdot\,,\omega) = \exp\big(-\int_0^t \delta(r,\,\cdot\,,\omega)\mathrm{d}B_r - \frac{1}{2}\int_0^t \delta^2(r,\,\cdot\,,\omega)\mathrm{d}r\big).$$

where $\delta(r,\,\cdot\,,\omega) = \sum_{j=1}^N W_r^j (e^j)'\,(Y_r(\,\cdot\,,\omega))\,\gamma\,(r, Y_r(\,\cdot\,,\omega),\omega)$. So there is a random (depending on $\omega \in \Omega$) constant

$$\varrho_1(\omega) := \mathrm{const}\left(T, W^j, \|e^j\|_\infty, \|(e^j)'\|_\infty, \|(e^j)''\|_\infty,\ 1 \leqslant j \leqslant N,\ A_2(\omega)\right), \quad (4.6)$$

so that (4.5) is smaller than

$$\varrho_1(\omega)E^{Q^\omega}\left(\int_0^T |\varphi(s, Y_s(\,\cdot\,,\omega))|\mathrm{d}sR_T(\,\cdot\,,\omega)\right), \quad (4.7)$$

where we remind that $R(\cdot,\omega)$ is a Q^ω-martingale. By Girsanov theorem, $\widetilde{B}_t(\cdot,\omega) = B_t(\cdot,\omega) + \int_0^t \delta(r,\,\cdot\,,\omega)\mathrm{d}r$ is a $\widetilde{Q}^\omega$-Brownian motion with $\mathrm{d}\widetilde{Q}^\omega = R_T(\,\cdot\,,\omega)\mathrm{d}Q^\omega$. At this point, the expectation in (4.7) gives

$$E^{\widetilde{Q}^\omega}\left(\int_0^T |\varphi|(s, Y_s(\,\cdot\,,\omega))\mathrm{d}s\right), \quad (4.8)$$

where

$$Y_t(\,\cdot\,,\omega) = Y_0 + \int_0^t \gamma(s, Y_s(\,\cdot\,,\omega),\omega)\mathrm{d}\widetilde{B}_s - \int_0^t \gamma(s, Y_s(\,\cdot\,,\omega),\omega)\delta(s,\,\cdot\,,\omega)\mathrm{d}s.$$

For fixed $\omega \in \Omega$, δ is bounded by a random constant $\varrho_2(\omega)$ of the type (4.6). Moreover we keep in mind assumption (4.1) on γ. By Exercise 7.3.3 of [27], (4.8) is bounded by $\varrho_3(\omega)\,\|\varphi\|_{L^2([0,T]\times\mathbb{R})}$, where $\varrho_3(\omega)$ again depends on the same quantities as in (4.6) and Φ. So for ω dP-a.s., the map $\varphi \mapsto E^{Q^\omega}\left(\int_0^T \varphi(s, Y_s(\,\cdot\,,\omega))Z_s(\,\cdot\,,\omega)\mathrm{d}s\right)$ prolongs to $L^2([0,T]\times\mathbb{R})$. Using Riesz' theorem it is not difficult to show the existence of an $L^2([0,T]\times\mathbb{R})$ function $(s,y) \mapsto q_s(y,\omega)$ which constitutes indeed the density of the family of the μ-marginal weighted laws. □

5 On the uniqueness of a Fokker-Planck type SPDE

The next result is an extension of Theorem 3.8 of [14] to the stochastic case. It has an independent interest since it is a Fokker-Planck SPDE with possibly degenerate measurable coefficients.

Theorem 5.1. *Let z_0 be a distribution in $\mathcal{S}'(\mathbb{R})$. Let z^1, z^2 be two measurable random fields belonging ω a.s. to $C([0,T], \mathcal{S}'(\mathbb{R}))$ such that $z^1, z^2 :]0,T] \times \Omega \to \mathcal{M}(\mathbb{R})$. Let $a : [0,T] \times \mathbb{R} \times \Omega \to \mathbb{R}_+$ be a bounded measurable random field such that, for any $t \in [0,T]$, $a(t,\cdot)$ is $\mathcal{B}([0,t]) \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{F}_t$-measurable. We suppose moreover the following.*

i) $z^1 - z^2 \in L^2([0,T] \times \mathbb{R})$ *a.s.*

ii) $t \mapsto (z^1 - z^2)(t,\cdot)$ *is an* $(\mathcal{F}_t)$*-progressively measurable* $\mathcal{S}'(\mathbb{R})$*-valued process.*

iii) $\int_0^T \|z^i(s,\cdot\|^2_{\mathrm{var}} \mathrm{d}s < \infty$ *a.s.*

iv) z^1, z^2 *are solutions to*

$$\begin{cases} \partial_t z(t,\xi) = \partial^2_{\xi\xi}((az)(t,\xi)) + z(t,\xi)\mu(\mathrm{d}t,\xi), \\ z(0,\ \cdot\ ) = z_0. \end{cases} \tag{5.1}$$

Then $z^1 \equiv z^2$.

Remark 5.2. *By solution of equation* (5.1) *we intend, as expected, the following: for every* $\varphi \in \mathcal{S}(\mathbb{R})$, $\forall t \in [0,T]$,

$$\int_{\mathbb{R}} \varphi(\xi) z(t,\mathrm{d}\xi) = \langle z_0, \varphi\rangle + \int_0^t \mathrm{d}s \int_{\mathbb{R}} a(s,\xi)\varphi''(\xi) z(s,\mathrm{d}\xi) + \sum_{j=0}^N \int_0^t \mathrm{d}W^j_s \int_{\mathbb{R}} \varphi(\xi) e^j(\xi) z(s,d\xi).$$

Proof of Theorem 5.1. The proof makes use of the similar arguments as in Theorem 3.8 of [14] or Theorem 3.1 in [10], in a randomized form. The full proof is given in [24] Theorem 4.2, see also [7].

□

6 The non-degenerate case

We are now able to discuss the doubly probabilistic representation of a solution to (1.1) when ψ is non-degenerate provided that its solution fulfills some properties.

Definition 6.1.
- *We will say that equation (1.1) (or ψ) is* **non-degenerate** *if on each compact, there is a constant $c_0 > 0$ such that $\Phi \geq c_0$.*
- *We will say that equation (1.1) or ψ is* **degenerate** *if* $\lim_{u\to 0_+} \Phi(u) = 0$.

One of the typical examples of degenerate ψ is the case of ψ being **strictly increasing after some zero**. This notion was introduced in [6] and it means the following. There is $0 \leq u_c$ such that $\psi_{[0,u_c]} \equiv 0$ and ψ is strictly increasing on $]u_c, +\infty[$.

Remark 6.2. *1. ψ is non-degenerate if and only if $= \lim_{u \to 0+} \Phi(u) > 0$.*

2. *Of course, if ψ is strictly increasing after some zero, with $u_c > 0$ then ψ is degenerate.*

3. *If ψ is degenerate, then $\psi^\kappa(u) = (\Phi^2(u) + \kappa)u$, for every $\kappa > 0$, is non-degenerate.*

As announced the theorem below also holds when ψ is multi-valued.

Theorem 6.3. *We suppose the following assumptions.*

1. *x_0 is a real probability measure.*
2. *ψ is non-degenerate.*
3. *There is only one random field $X : [0,T] \times \mathbb{R} \times \Omega \to \mathbb{R}$ solution of (1.1) (see Definition 2.9) such that*

$$\int_{[0,T]\times\mathbb{R}} X^2(s,\xi)\mathrm{d}s\mathrm{d}\xi < \infty \quad a.s. \tag{6.1}$$

Then there is a unique weak solution to the (DSNLD)(Φ, μ, x_0).

Remark 6.4. *1. An easy adaptation of Theorem 3.4 of [8] (taking into account e^0), when ψ is Lipschitz and $e^0, \ldots, e^N$ belong to H^1 allows to show that there is a solution to (1.1) such that $E\left(\int_{[0,T]\times\mathbb{R}} X^2(s,\xi)\mathrm{d}s\mathrm{d}\xi\right) < \infty$. This holds even if x_0 belongs to $H^{-1}(\mathbb{R})$. According to Theorem B.1, that solution is unique. In particular item 3. in Theorem 6.3 statement holds.*

2. *Theorem 6.3 constitutes the converse of Theorem 3.3 when ψ is non-degenerate.*

3. *The theorem also holds if ψ is multi-valued. For implementing this, we need to adapt the techniques of [14].*

4. *As side-effect of the proof of the weak-strong existence Proposition 4.1, the space $(\Omega_0, \mathcal{G}, \mathbf{Q})$ can be chosen as $\Omega_0 = \Omega_1 \times \Omega$, $\Omega_1 = C([0,T]; \mathbb{R}) \times \mathbb{R}$, $\mathcal{G} = \mathcal{B}(\Omega_1) \times \mathcal{F}$, $\mathbf{Q}(H \times F) = \int_{\Omega_1 \times \Omega} \mathrm{d}P(\omega) 1_F(\omega) Q(\mathrm{d}\omega_1, \omega)$.*

Proof. 1) We set $\gamma(t,\xi,\omega) = \Phi\left(X(t,\xi,\omega)\right)$. According to Proposition 4.1 there is a weak-strong solution Y to (DSDE)(γ, x_0). By Proposition 4.4 ω a.s. the μ-marginal weighted laws of Y admit densities $(q_t(\xi,\omega), t \in]0,T],\ \xi \in \mathbb{R},\ \omega \in \Omega)$ such that dP-a.s. $\int\limits_{[0,T]\times\mathbb{R}} \mathrm{d}s\mathrm{d}\xi q_s^2(\xi,\ \cdot\ ) < \infty$ a.s.

2) Setting

$$\nu_t(\xi,\omega) = \left(\begin{array}{ccc} q_t(\xi,\omega)\mathrm{d}\xi & : & t \in]0,T], \\ x_0 & : & t = 0, \end{array} \right.$$

ν is a solution to (5.1) with $\nu_0 = x_0,\ a(t,\xi,\omega) = \Phi^2(X(t,\xi,\omega))$. This can be shown applying Itô's formula similarly as in the proof of Theorem 3.3.

3) On the other hand X is obviously also a solution of (5.1), which in particular verifies (6.1). Consequently $z^1 = \nu,\ z^2 = X$ verify items i), ii), iii) of Theorem 5.1. So Theorem 5.1 implies that $\nu \equiv X$; this shows that Y provides a solution to (DSNLD)(Φ, μ, x_0).

4) Concerning uniqueness, let Y^1, Y^2 be two solutions to the (DSNLD) related to (Φ, μ, x). The corresponding random fields X^1, X^2 constitute the μ-marginal laws of Y^1, Y^2 respectively.

Now $Y^i,\ i = 1,2$, is a weak-strong solution of (DSDE)(γ_i, x) with $\gamma_i(t,\xi,\omega) = \Phi(X_i(t,\xi,\omega))$, so by Proposition 4.4 $X_i,\ i = 1,2$ fulfills (6.1). By Theorem 3.3, X_1 and X_2 are solutions to (1.1). By assumption 3. of the statement, $X_1 = X_2$. The conclusion follows by Proposition 4.1, which guarantees the uniqueness of the weak-strong solution of (DSDE)(γ, x_0) with $\gamma = \gamma_1 = \gamma_2$. □

Remark 6.5. *One side-effect of Theorem 6.3 is the following. Suppose ψ to be non-degenerate. Let $X : [0,T] \times \mathbb{R} \times \Omega \to \mathbb{R}$ be a solution such that $\mathrm{d}P$-a.s. $\int\limits_{[0,T]\times\mathbb{R}} X^2(s,\xi)\mathrm{d}s\mathrm{d}\xi < \infty$ a.s. We have the following for ω $\mathrm{d}P$-a.s.*

i) $X(t,\ \cdot\ ,\omega) \geqslant 0$ *a.e.* $\forall\, t \in [0,T]$.

ii) $E\left(\int\limits_{\mathbb{R}} X(t,\xi)\mathrm{d}\xi\right) = 1,\ \forall\, t \in [0,T]$ *if* $e_0 = 0$.

Remark 6.6. *If* (1.1) *has a solution, not necessarily unique, then (DSNLD) with respect to* (Φ, μ, x_0) *still admits existence.*

7 The degenerate case

The idea consists in proceeding similarly to [6], which treated the case $\mu = 0$ and the case when x_0 is absolutely continuous with bounded density. ψ will be assumed to be strictly increasing after some zero $u_c \geq 0$, see Definition 6.1. We recall that if ψ is degenerate, then necessarily $\Phi(0) := \lim_{x\to 0} \Phi(x) = 0$.

The theorem below concerns existence, we do not know any uniqueness result in the degenerate case.

Theorem 7.1. *We suppose the following.*

1. *The functions $e^i, 1 \leq i \leq N$ belong to $H^1(\mathbb{R})$.*
2. *We suppose that $\psi : \mathbb{R} \to \mathbb{R}$ is non-decreasing, Lipschitz and strictly increasing after some zero.*
3. *x_0 belongs to $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$.*

Then there is a weak solution to the (DSNLD)(Φ, μ, x_0).

Remark 7.2. *If $u_c > 0$ then ψ is necessarily degenerate and also Φ restricted to $[0, u_c]$ vanishes.*

Proof (of Theorem 7.1).

1) We proceed by approximation rendering Φ non-degenerate. Let $\kappa > 0$. We define $\Phi_\kappa : \mathbb{R} \to \mathbb{R}_+$ by $\Phi_\kappa(u) = \sqrt{\Phi^2(u) + \kappa}$, $\psi_\kappa(u) = \Phi_\kappa^2(u) \cdot u$. Let X^κ be the solution so (1.1) with ψ_κ instead of ψ. According to Theorem 6.3 and Remark 6.4 4., setting

$$\widetilde{\Omega}_1 = C([0,T], \mathbb{R}) \times \mathbb{R}, \quad Y(\omega_1, \omega) = \omega_1, \tag{7.1}$$

$\mathcal{H}$ the Borel σ-algebra of $\widetilde{\Omega}_1$, there are families of probability kernels Q^κ on $\mathcal{H} \times \Omega$, and measurable processes B^κ on $\widetilde{\Omega}_0 = \widetilde{\Omega}_1 \times \Omega$ such that

i) $B^\kappa(\,\cdot\,, \omega)$ is a $Q^\kappa(\,\cdot\,, \omega)$-Brownian motion;

ii) Y is a (weak) solution, on $(\tilde{\Omega}_1, Q^\kappa(\cdot, \omega))$, of
$Y_t = Y_0 + \int_0^t \Phi_\kappa(X^\kappa(s, Y_s, \omega)) \mathrm{d}B^\kappa_s(\cdot, \omega),\ t \in [0, T]$;

iii) Y_0 is distributed according to $x_0 = X^\kappa(0, \,\cdot\,)$.

iv) The μ-marginal weighted laws of Y under $\mathbf{Q}^\kappa$ are $(X^\kappa(t, \,\cdot\,))$.

In agreement with Definition 3.1 and Definition 2.10, we need to show the existence of a suitable measurable space $(\Omega_1, \mathcal{H})$, a probability kernel Q on $\mathcal{H} \times \Omega$, a process B on $\Omega_0 := \Omega_1 \times \Omega$ such that the following holds.

i) $B(\,\cdot\,, \omega)$ is a $Q(\,\cdot\,, \omega)$-Brownian motion.

ii) Y is a (weak) solution on $(\Omega_1, Q(\cdot,\omega))$ of $Y_t = Y_0 + \int_0^t \Phi(X(s,Y_s,\omega))\mathrm{d}B_s(\cdot,\omega)$, $t \in [0,T]$, i.e. item 1) of Definition 2.10. Moreover items 2), 3) of the same Definition are fulfilled.

iii) Y_0 is distributed according to x_0.

iv) For every $t \in]0,T]$, $\varphi \in C_b(\mathbb{R})$, if we denote $Q^\omega = Q(\,\cdot\,,\omega)$, we have

$$\int_{\mathbb{R}} X(t,\xi)\varphi(\xi)\mathrm{d}\xi = E^{Q^\omega}\left(\varphi(Y_t)\mathcal{E}_t\left(\int_0^{\cdot} \mu(\mathrm{d}s, Y_s)X(s,Y_s)\right)\right).$$

2) We show now that X^κ approaches X in some sense when $\kappa \to 0$, where X is the solution to (1.1). This is given in the Lemma 7.3 below.

Lemma 7.3. *Under the assumptions of Theorem 7.1, according to Remark B.2, let X (resp. X^κ) be a solution of* (1.1) *verifying* (2.1) *with $\psi(u) = u\Phi^2(u)$ (resp. $\psi_\kappa(u) = u(\Phi^2(u)+\kappa)$), for $u > 0$. We have the following.*

a) $\lim_{\kappa\to 0}\sup_{t\in[0,T]} E\left(\|X^\kappa(t,\,\cdot\,) - X(t,\,\cdot\,)\|^2_{H^{-1}}\right) = 0;$

b) $\lim_{\kappa\to 0} E\left(\int_0^T \mathrm{d}t\,\|\psi\left(X^\kappa(t,\,\cdot\,)\right) - \psi\left(X(t,\,\cdot\,)\right)\|^2_{L^2}\right) = 0;$

c) $\lim_{\kappa\to 0} \kappa E\left(\int_{[0,T]\times\mathbb{R}} \mathrm{d}t\mathrm{d}\xi\left(X^\kappa(t,\xi) - X(t,\xi)\right)^2\right) = 0.$

Remark 7.4. *1) a) implies of course* $\lim_{\kappa\to 0} E\left(\int_0^T \mathrm{d}t\,\|X^\kappa(t,\,\cdot\,) - X(t,\,\cdot\,)\|^2_{H^{-1}}\right) = 0.$

2) In particular Lemma 7.3 b) implies that for each sequence $(\kappa_n) \to 0$ there is a subsequence, still denoted by the same notation, that $\int_{[0,T]\times\mathbb{R}} \left(\psi(X^{\kappa_n}(t,\xi)) - \psi(X(t,\xi))\right)^2 \mathrm{d}t\mathrm{d}\xi \underset{n\to\infty}{\longrightarrow} 0$ a.s.

3) For every $t \in [0,T]$ $X(t,\,\cdot\,) \geqslant 0 \quad d\xi \otimes dP$ a.e. Indeed, for this it will be enough to show that a.s.

$$\int_{\mathbb{R}} \mathrm{d}\xi\varphi(\xi)X(t,\xi) \geq 0 \textit{ for every } \varphi \in \mathcal{S}(\mathbb{R}), \tag{7.2}$$

for every $t \in [0,T]$. Since $X \in C\left([0,T];\, \mathcal{S}'(\mathbb{R})\right)$ it will be enough to show (7.2) *for almost all $t \in [0,T]$. This holds true since item 1) in this Remark 7.4, implies the existence of a sequence (κ_n) such that* $\int_0^T \mathrm{d}t\,\|X^{\kappa_n}(t,\cdot) - X(t,\,\cdot\,)\|^2_{H^{-1}} \underset{n\to\infty}{\longrightarrow} 0,\quad$ a.s.

4) Since ψ is strictly increasing after u_c, then, for P almost all ω, for almost all $(t,\xi)\in[0,T]\times\mathbb{R}$, there is a sequence (κ_n) such that $(X^{\kappa_n}(t,\xi)-X(t,\xi))\,1_{\{X(t,\xi)>u_c\}}\underset{n\to\infty}{\longrightarrow}0$. This follows from item 2) of Remark 7.4.

Since $\Phi^2(u)=0$ for $0\leqslant u\leqslant u_c$ and X is a.e. non-negative, this implies that $dtd\xi dP$ a.e. we have

$$\Phi^2\left(X(t,\xi)\right)\left(X^{\kappa_n}(t,\xi)-X(t,\xi)\right)\underset{n\to\infty}{\longrightarrow}0. \tag{7.3}$$

Proof (of Lemma 7.3). By Remark B.2 3. we can write $\mathrm{d}P$-a.s. the following $H^{-1}(\mathbb{R})$-valued equality.

$$\left(X^\kappa-X\right)(t,\,\cdot\,)=\int_0^t \mathrm{d}s\left(\psi_\kappa\left(X^\kappa(s,\,\cdot\,)\right)-\psi\left(X(s,\,\cdot\,)\right)\right)''+\sum_{i=0}^N\int_0^t\left(X^\kappa(s,\,\cdot\,)-X(s,\,\cdot\,)\right)e^i\mathrm{d}W_s^i.$$

So

$$\begin{aligned}(I-\Delta)^{-1}\left(X^\kappa-X\right)(t,\,\cdot\,)=&-\int_0^t \mathrm{d}s\left(\psi_\kappa\left(X^\kappa(s,\,\cdot\,)\right)-\psi\left(X(s,\,\cdot\,)\right)\right)\\&+\int_0^t \mathrm{d}s(I-\Delta)^{-1}\left(\psi_\kappa\left(X^\kappa(s,\,\cdot\,)\right)-\psi\left(X(s,\,\cdot\,)\right)\right)\\&+\sum_{i=0}^N\int_0^t(I-\Delta)^{-1}\left(e^i\left(X^\kappa(s,\,\cdot\,)-X(s,\,\cdot\,)\right)\right)\mathrm{d}W_s^i.\end{aligned}$$

After regularization and application of Itô calculus with values in H^{-1}, we will be able to estimate $g^\kappa(t)=\|\left(X^\kappa-X\right)(t,\,\cdot\,)\|_{H^{-1}}^2$. Taking advantage of the form of $\psi_\kappa-\psi$, we obtain

$$\begin{aligned}g^\kappa(t)=&\sum_{i=1}^N\int_0^t\left\|e^i\left(X^\kappa-X\right)(s,\,\cdot\,)\right\|_{H^{-1}}^2\mathrm{d}s \qquad (7.4)\\&-2\int_0^t\langle\left(X^\kappa-X\right)(s,\,\cdot\,),\psi_\kappa\left(X^\kappa(s,\,\cdot\,)\right)-\psi\left(X(s,\,\cdot\,)\right)\rangle_{L^2}\\&+2\int_0^t\mathrm{d}s\left\langle\left(X^\kappa-X\right)(s,\,\cdot\,),(I-\Delta)^{-1}\left(\psi_\kappa\left(X^\kappa(s,\,\cdot\,)\right)-\psi\left(X(s,\,\cdot\,)\right)\right)\right\rangle_{L^2}\\&+2\int_0^t\mathrm{d}s\left\langle\left(X^\kappa-X\right)(s,\,\cdot\,),(I-\Delta)^{-1}e^0\left(X^\kappa-X\right)(s,\,\cdot\,)\right\rangle_{L^2}+M_t^\kappa,\end{aligned}$$

where M^κ is the local martingale

$$M_t^\kappa = 2\sum_{i=1}^{N}\int_0^t \left\langle (I-\Delta)^{-1}(X^\kappa - X)(s,\,\cdot\,),(X^\kappa - X)(s,\,\cdot\,)e^i\right\rangle_{L^2} \mathrm{d}W_s^i.$$

Indeed, M^κ is a well-defined local martingale because, taking into account (B.1) and Remark B.2, using classical arguments, we can prove that

$$\sum_{i=1}^{N}\int_0^t |\left\langle (X^\kappa - X|)(s,\,\cdot\,),(I-\Delta)^{-1}(X^\kappa - X)(s,\,\cdot\,)e^i\right\rangle_{L^2}|^2 \mathrm{d}s < \infty \text{ a.s.}$$

(7.4) gives

$$\begin{aligned}
g^\kappa(t) &+ 2\int_0^t \langle (X^\kappa - X)(s,\,\cdot\,),\ \psi(X^\kappa(s,\,\cdot\,)) - \psi(X(s,\,\cdot\,))\rangle_{L^2}\,\mathrm{d}s\\
&+ 2\kappa\int_0^t \langle (X^\kappa - X)(s,\,\cdot\,),\ (X^\kappa - X)(s,\,\cdot\,)\rangle_{L^2}\,\mathrm{d}s\\
\leqslant &- 2\kappa\int_0^t \mathrm{d}s\,\langle (X^\kappa - X)(s,\,\cdot\,),\ X(s,\,\cdot\,)\rangle_{L^2}\,\mathrm{d}s + \sum_{i=1}^{N}\int_0^t \left\| e^i(X^\kappa - X)(s,\,\cdot\,)\right\|_{H^{-1}}^2 \mathrm{d}s\\
&+ 2\int_0^t \mathrm{d}s\left\langle (I-\Delta)^{-1}(X^\kappa - X)(s,\,\cdot\,),\ \psi(X^\kappa(s,\,\cdot\,)) - \psi(X(s,\,\cdot\,))\right\rangle_{L^2}\\
&+ 2\kappa\int_0^t \mathrm{d}s\left\langle (I-\Delta)^{-1}(X^\kappa - X)(s,\,\cdot\,),\ (X^\kappa - X)(s,\,\cdot\,)\right\rangle_{L^2}\\
&+ 2\kappa\int_0^t \mathrm{d}s\left\langle (I-\Delta)^{-1}(X^\kappa - X)(s,\,\cdot\,),\ X(s,\,\cdot\,)\right\rangle_{L^2}\\
&+ 2\int_0^t \mathrm{d}s\left\langle (I-\Delta)^{-1}(X^\kappa - X)(s,\,\cdot\,),\ \left(e^0(X^\kappa - X)(s,\,\cdot\,)\right)\right\rangle_{L^2} + M_t^\kappa.
\end{aligned}$$

We use Cauchy-Schwarz and the inequality $2\sqrt{\kappa}b\sqrt{\kappa}c \leqslant \kappa b^2 + \kappa c^2$, with first $b = \|X^\kappa(s,\,\cdot\,) - X(s,\,\cdot\,)\|_{L^2}$, $c = \|X(s,\,\cdot\,)\|_{L^2}$ and then $b = \|X^\kappa(s,\cdot) - X(s,\cdot)\|_{H^{-2}}$, $c = \|X(s,\,\cdot\,)\|_{L^2}$. We also take into account the property of H^{-1}-multiplier for e^i, $0 \leqslant i \leqslant N$. Consequently there is a constant

$\mathcal{C}(e)$ depending on $(e^i,\ 0 \leqslant i \leqslant N)$ such that

$$g^\kappa(t) + 2\int_0^t \langle (X^\kappa - X)(s,\ \cdot),\ \psi(X^\kappa(s,\ \cdot)) - \psi(X(s,\ \cdot))\rangle_{L^2}\,\mathrm{d}s \tag{7.5}$$

$$+ 2\kappa \int_0^t \|X^\kappa(s,\ \cdot) - X(s,\ \cdot)\|_{L^2}^2\,\mathrm{d}s$$

$$\leqslant \kappa \int_0^t \|(X^\kappa - X)(s,\ \cdot)\|_{L^2}^2\,\mathrm{d}s + \kappa\int_0^t \mathrm{d}s\,\|X(s,\ \cdot)\|_{L^2}^2$$

$$+ \mathrm{C}(e)\int_0^t \mathrm{d}s\,\|X^\kappa(s,\ \cdot) - X(s,\ \cdot)\|_{H^{-1}}^2$$

$$+ 2\int_0^t \|(X^\kappa - X)(s,\ \cdot)\|_{H^{-2}}\,\|\psi(X^\kappa(s,\ \cdot)) - \psi(X(s,\ \cdot))\|_{L^2}$$

$$+ 2\kappa\int_0^t \mathrm{d}s g^\kappa(s) + \kappa\int_0^t \mathrm{d}s\,\|(X^\kappa - X)(s,\ \cdot)\|_{H^{-2}}^2 + \kappa\int_0^t \mathrm{d}s\,\|X(s,\ \cdot)\|_{L^2}^2 + M_t^\kappa.$$

Since ψ is Lipschitz, it follows $(\psi(r) - \psi(r_1))(r - r_1) \geqslant \alpha(\psi(r) - \psi(r_1))^2$, for any $r, r_1 \geq 0$, for some $\alpha > 0$. Consequently, the inequality $2bc \leqslant b^2\alpha + \frac{c^2}{\alpha}$, with $b, c \in \mathbb{R}$ and the fact that $\|\cdot\|_{H^{-2}} \leqslant \|\cdot\|_{H^{-1}}$ give

$$2\int_0^t \mathrm{d}s\,\|(X^\kappa - X)(s,\ \cdot)\|_{H^{-2}}\,\|\psi(X^\kappa(s,\ \cdot)) - \psi(X(s,\ \cdot))\|_{L^2}$$

$$\leqslant \int_0^t \mathrm{d}s \alpha g^\kappa(s,\ \cdot) + \int_0^t \mathrm{d}s\,\langle \psi(X^\kappa(s,\ \cdot)) - \psi(X(s,\ \cdot)),\ X^\kappa(s,\ \cdot) - X(s,\ \cdot)\rangle_{L^2}.$$

So (7.5) yields

$$g^\kappa(t) + \int_0^t \langle X^\kappa(s,\ \cdot) - X(s,\ \cdot),\ \psi(X^\kappa(s,\ \cdot)) - \psi(X(s,\ \cdot))\rangle_{L^2}\,\mathrm{d}s \tag{7.6}$$

$$+ \kappa\int_0^t \mathrm{d}s\,\|X^\kappa(s,\ \cdot) - X(s,\ \cdot)\|_{L^2}^2\,\mathrm{d}s$$

$$\leqslant 2\kappa\int_0^t \mathrm{d}s\,\|X(s,\ \cdot)\|_{L^2}^2 + M_t^\kappa + (\mathrm{C}(e) + \alpha + 3\kappa)\int_0^t g^\kappa(s)\mathrm{d}s.$$

Taking the expectation we get

$$E(g^{\kappa}(t)) \le (\mathrm{C}(e)+\alpha+3\kappa)\int_0^t E(g^{\kappa}(s))\mathrm{d}s + 2\kappa \int_0^t E(\|X(s,\cdot)\|_{L^2}^2)\mathrm{d}s,$$

for every $t \in [0,T]$. By Gronwall lemma we get

$$E\left(g_{\kappa}(t)\right) \leqslant 2\kappa E\left\{\int_0^T \mathrm{d}s\, \|X(s,\,\cdot\,)\|_{L^2}^2\right\} e^{(\mathrm{C}(e)+\alpha+3\kappa)T}, \quad \forall\, t\in[0,T]. \tag{7.7}$$

Taking the supremum and letting $\kappa \to 0$, item a) of Lemma 7.3 is now established. We go on with item b). Since ψ is Lipschitz, (7.6) implies that, for $t\in[0,T]$,

$$\begin{aligned}
&\int_0^t \mathrm{d}s\, \|\psi\left(X^{\kappa}(s,\,\cdot\,)\right)-\psi\left(X(s,\,\cdot\,)\right)\|_{L^2}^2 \\
\leqslant &\frac{1}{\alpha}\mathrm{d}s \left\langle \psi\left(X^{\kappa}(s,\,\cdot\,)\right)-\psi\left(X(s,\,\cdot\,)\right),\, X^{(\kappa)}(s,\,\cdot\,)-X(s,\,\cdot\,)\right\rangle_{L^2} \\
\leqslant &\frac{\kappa}{2\alpha}\int_0^t \mathrm{d}s\,\|X(s,\,\cdot\,)\|_{L^2}^2 + \mathrm{C}(e,\alpha)\int_0^t g_{\kappa}(s)\mathrm{d}s + M_t^{\kappa},
\end{aligned}$$

where $\mathrm{C}(e,\alpha)$ is a constant depending on $e^i, 0\leqslant i\leqslant N$ and α. Taking the expectation for $t=T$, we get

$$E\left(\int_0^T \mathrm{d}s\,\|\psi\left(X^{\kappa}(s,\,\cdot\,)\right)-\psi\left(X(s,\,\cdot\,)\right)\|_{L^2}^2\right) \leqslant \frac{\kappa}{2\alpha}E\left(\int_0^T \mathrm{d}s\,\|X(s,\,\cdot\,)\|_{L^2}^2\right) + \mathrm{C}(e,\alpha)\int_0^T E(g_{\kappa}(s))\mathrm{d}s.$$

Taking $\kappa\to 0$, (2.1) and (7.7) provide the conclusion of item b) of Lemma 7.3.

c) Coming back to (7.6), and $t=T$, we have

$$\kappa\int_0^T \mathrm{d}s\,\|X^{\kappa}(s,\,\cdot\,)-X(s,\,\cdot\,)\|_{L^2}^2 \leqslant 2\kappa\int_0^T \mathrm{d}s\,\|X(s,\,\cdot\,)\|_{L^2}^2 + M_T^{\kappa} + (\mathrm{C}(e)+\alpha+3\kappa)\int_0^T \mathrm{d}s g^{\kappa}(s).$$

Taking the expectation we have

$$\kappa E\left(\int_0^T \mathrm{d}s\,\|X^{\kappa}(s,\,\cdot\,)-X(s,\,\cdot\,)\|_{L^2}^2\right) \leqslant 2\kappa E\left(\int_0^T \mathrm{d}s\,\|X(s,\,\cdot\,)\|_{L^2}^2\right) + (\mathrm{C}(e)+\alpha+3\kappa)\,E\left(\int_0^T g^{\kappa}(s)\mathrm{d}s\right).$$

Using item a) and the fact that $E\left(\int_{[0,T]\times\mathbb{R}} X^2(s,\xi)\mathrm{d}s\mathrm{d}\xi\right) < \infty$, the result follows. Lemma 7.3 is finally completely established.

□

We need now another intermediate lemma concerning the paths of a solution to (1.1).

Lemma 7.5. *For almost all $\omega \in \Omega$, almost all $t \in [0,T]$,*

1) $\xi \mapsto \psi(X(t,\xi,\omega)) \in H^1(\mathbb{R})$,

2) $\xi \mapsto \Phi(X(t,\xi,\omega))$ *is continuous.*

Proof. Item 1) is established in [8], see Definition 3.2 and Theorem 3.4. 1) implies that $\xi \mapsto \psi(X(t,\xi,\omega))$ is continuous. See also Remark B.2 1. By the same arguments as in Proposition 4.22 in [6], we can deduce item 2). □

3) We go on with the proof of Theorem 7.1. We keep in mind i), ii), iii), iv) at the beginning of item 1) of the proof. Since Φ is bounded, for P-almost all ω, using Burkholder-Davis-Gundy inequality one obtains

$$E^{Q^\kappa(\,\cdot\,,\omega)}\,(Y_t - Y_s)^4 \leqslant \mathrm{const}(t-s)^2, \tag{7.8}$$

where const does not depend on ω. On the other hand, for all $Q^\kappa(\,\cdot\,,\omega)$, Y_0 is distributed according to x_0.

At this point, we need a version of Kolmogorov-Centsov theorem for the stable convergence. Let $\tilde\Omega_0 = \tilde\Omega_1 \times \Omega$ as at the beginning of the proof of Theorem 7.1. We recall that $\tilde\Omega_1 = C([0,T]) \times \mathbb{R}$, $Y(\omega_1,\omega) = \omega_1$, $\mathcal{H}$ is the Borel σ-field on $\tilde\Omega_1$.

Lemma 7.6. *Let be a sequence $Q^\kappa(\cdot,\omega)$ of random kernel on $\mathcal{H}\times\Omega$. Let us denote by $\mathbf{Q}^\kappa$ the sequence of marginal laws of the probabilities on $(\tilde\Omega_0, \mathcal{H}\otimes\mathcal{F})$ given by $Q^\kappa(\cdot,\omega)P(\mathrm{d}\omega)$. Suppose the following.*

- *The sequences of marginal laws of the probabilities $\mathbf{Q}^\kappa$ at zero are tight.*
- *There are $\alpha,\beta > 0$ such that*

$$E^{Q^\kappa(\cdot,\omega)}|Y_t - Y_s|^\alpha \le C(\omega)(t-s)^{1+\beta}, \quad 0 \le s \le T,$$

for some positive P-integrable random constant C.

Then there is a random kernel Q^∞ on $\mathcal{H}\times\Omega$ and a subsequence (κ_n) such that for every bounded continuous functional $G : \tilde\Omega_1 \to \mathbb{R}$, for every bounded $\mathcal{F}$-measurable r.v. $F : \Omega \to \mathbb{R}$, we have

$$\begin{array}{ll} \int_\Omega F(\omega)\mathrm{d}P(\omega) & \int_{\tilde\Omega_1} G(Y(\omega_1))Q^{\kappa_n}(\mathrm{d}\omega_1,\omega) \to_{n\to\infty} \\ & \int_\Omega F(\omega)\mathrm{d}P(\omega)\int_{\tilde\Omega_1} G(Y(\omega_1))Q^\infty(\mathrm{d}\omega_1,\omega). \end{array} \tag{7.9}$$

Proof. Taking the expectation with respect to P we obtain

$$E^{\mathbf{Q}^\kappa}(Y_t - Y_s)^\alpha \le C_0(t-s)^{1+\beta}, \quad 0 \le s \le T,$$

where C_0 is the expectation of C. First, by usual arguments as Chebyshev inequality, one can show the following:

$$\begin{aligned}\lim_{\lambda\to\infty}\sup_\kappa \mathbf{Q}^\kappa\{(\omega_1,\omega)||(W^1,\dots,W^N)(\omega)(0)| > \lambda; |\omega_1(0)| > \lambda\} &= 0,\\ \lim_{\delta\to 0}\sup_\kappa \mathbf{Q}^\kappa\{(\omega_1,\omega)|m((W^1,\dots,W^N,\omega_1);\delta) > \varepsilon\} &= 0, \forall\varepsilon > 0,\end{aligned}$$

where m denotes the modulus of continuity. By Theorem 4.10 of [18], the sequences of probabilities $\mathbf{Q}^\kappa, \kappa > 0$, on $\tilde\Omega_1 \times \Omega$ are tight. Let $\mathbf{Q}^{\kappa_n}$ be a sequence converging weakly to a probability $\mathbf{Q}^\infty$ on $\mathcal{H}\otimes\mathcal{F}$. Since $\mathcal{F}$ is separable and $C([0,T])^N$, which is space value of process W, is a Polish space equipped with its Borel σ-algebra, according to [23], it is possible to desintegrate $\mathbf{Q}^\infty$, i.e. there is random kernel $Q^\infty(\cdot,\omega)$ such that for every bounded continuous functional $G : \tilde\Omega_1 \to \mathbb{R}$, for every bounded continuous $\tilde F : C([0,T])^N \to \mathbb{R}$ such that (7.9) holds for every $F = \tilde F(W)$, where $W = (W^1,\dots,W^N)$. Since continuous bounded functionals $\tilde F$ are dense in $L^2(C([0,T])^N$ equipped with Wiener measure, (7.9) holds also for any F bounded $\mathcal{F}$-measurable r.v. with $\mathbf{Q}^\infty(\mathrm{d}\omega_1,\mathrm{d}\omega) = Q^\infty(\mathrm{d}\omega_1,\omega)P(\mathrm{d}\omega)$. □

By (7.8), we apply Lemma 7.6 with $\alpha = 2, \beta = 1$ and we consider the corresponding $Q^{\kappa_n}(\cdot,\omega)$ and the limit random kernel $Q(\cdot,\omega) := Q^\infty(\cdot,\omega)$. We define also the probability $\mathbf{Q} := \mathbf{Q}^\infty$ on $\tilde\Omega_0 = \tilde\Omega_1 \times \Omega$ according to the conventions introduced before Definition 2.3. In the sequel we denote again by $\mathrm{d}\mathbf{Q}^\kappa(\omega_1,\omega) := \mathrm{d}P(\omega)Q^\kappa(\mathrm{d}\omega_1,\omega)$ and also $Q^{\omega,\kappa} := Q^\kappa(\cdot,\omega)$, $Q^\omega := Q(\cdot,\omega)$.

From Lemma 7.6 derives the following.

Corollary 7.7. *For any bounded random element $F : \tilde\Omega_1 \times \Omega \to \mathbb{R}$ such that for almost all $\omega \in \Omega$, $F(\cdot,\omega) \in C(\tilde\Omega_1)$. Then*
$\int_\Omega \mathrm{d}P(\omega)\int_{\tilde\Omega_1}(\mathrm{d}Q^{\omega,\kappa_n}(\omega_1) - \mathrm{d}Q^\omega(\omega_1))\,F(Y,\omega)$ converges to zero.

Proof. See Appendix A. □

We need here a technical lemma.

Lemma 7.8. *Let $t \in [0,T]$, $p \in \mathbb{R}$.*

1. *There is $C(p) > 0$ such that*

$$E^{\mathbf{Q}^\kappa}\left(\mathcal{E}_t\left(\int_0^\cdot \mu(\mathrm{d}s, Y_s)\right)^p\right) \le C(p),\ \forall\kappa > 0.$$

2. *For almost all $\omega \in \Omega$, and every $p \in \mathbb{R}$ there is a random constant $C(p,\omega)$ such that the random variables*

$$E^{Q^{\omega,\kappa}}\left(\mathcal{E}_t\left(\int_0^{\cdot}\mu(\mathrm{d}s, Y_s)\right)^p\right) \leq C(p,\omega),\ \forall \kappa > 0. \tag{7.10}$$

Proof. Without restriction of generality we can of course suppose $e^0 = 0$.

1. We can write

$$\begin{aligned}\mathcal{E}_t\left(\int_0^{\cdot}\mu(\mathrm{d}s, Y_s)\right)^p &= \mathcal{E}_t\left(p\int_0^{\cdot}\mu(\mathrm{d}s, Y_s)\right)\exp\left(\frac{p^2-p}{2}\sum_{i=1}^N\left(\int_0^t e^i(Y_s)^2\mathrm{d}s\right)\right)\\ &\leq \mathcal{E}_t\left(p\int_0^{\cdot}\mu(\mathrm{d}s, Y_s)\right)\exp\left(T\frac{p^2-p}{2}\sum_{i=1}^N\|e^i\|_\infty^2\right).\end{aligned}$$

   Since $p\int_0^t \mu(\mathrm{d}s, Y_s)$ is a $(\mathcal{G}_t)$-$\mathbf{Q}^\kappa$-martingale, the result follows.

2. Let $\omega \in \Omega$ excepted on a P-null set. The integrand of the expectation in (7.10) equals $\exp\left(J_1(n) + J_2(n)\right)$, where

$$J_1(n) := p\sum_{i=1}^N\left(W_t^i e^i(Y_t) - \frac{1}{2}\int_0^t e^i(Y_s)^2\mathrm{d}s - \frac{1}{2}\int_0^t W_s^i (e^i)''(Y_s)\Phi^2\left(X(s,Y_s,\omega)\right)\mathrm{d}s\right)$$

   and $J_2(n) = -p\sum_{i=1}^N\int_0^t W_s^i(e^i)'(Y_s)\mathrm{d}Y_s$. For each ω, $\exp\left((J_1(n)\right)$ is bounded, so it remains to prove the existence of a random constant $C(p,\omega)$ such that for every $0 \leqslant i \leqslant N$

$$E^{Q^{\omega,\kappa}}\left(\exp\left(-p\int_0^t W_s^i(e^i)'(Y_s)\mathrm{d}Y_s\right)\right) \leq C(p,\omega). \tag{7.11}$$

   Since $-p\int_0^t W_s^i(e^i)'(Y_s)\mathrm{d}Y_s$ is a $Q^{\omega,\kappa}$-martingale,

$$\mathcal{E}_t^\kappa := \exp\left(-p\int_0^t W_s^i(e^i)'(Y_s)\mathrm{d}Y_s - \frac{p^2}{2}\int_0^t (W^i)_s^2(e^i)'^2(Y_s)\Phi_\kappa^2\left(X^\kappa(s,Y_s,\omega)\right)\mathrm{d}s\right)$$

   is an (exponential) martingale, with respect to $Q^{\omega,\kappa}$. Consequently the left-hand side of (7.11) is bounded by

$$E^{Q^{\omega,\kappa}}\left(\mathcal{E}_t^\kappa \exp\left(\frac{p^2}{2}\int\limits_0^t (W^i)_s^2((e^i)')^2(Y_s)\Phi_\kappa^2(X^\kappa(s,Y_s,\omega))\mathrm{d}s\right)\right)$$

$$\leqslant C(p,\cdot) := \exp\left(\frac{p^2}{2}\left\|(e^i)'\right\|_\infty^2\left(\|\Phi\|_\infty^2+1\right)\int\limits_0^T (W_s^i)^2\mathrm{d}s\right).$$

This concludes the proof of Lemma 7.8.

□

Lemma 7.9. *We fix $\omega \in \Omega$ excepted on some P-null set. Let $\varphi : [0,T]\times\mathbb{R}\to\mathbb{R}$ continuous with compact support. The random variables*

$$E^{Q^{\omega,\kappa}}\left(\int_0^T |\Phi_\kappa\left(X^\kappa(r,Y_r,\omega)\right)-\Phi\left(X(r,Y_r,\omega)\right)|\,\varphi(r,Y_r)\mathrm{d}r\right)$$

converge to zero a.s. and in $L^p(\Omega,P)$ for every $p\geq 1$, when $\kappa\to 0$.

Proof. Let $\omega\in\Omega$. Since φ has compact support, by Cauchy-Schwarz with respect to the measure $\varphi(r,Y(r))\mathrm{d}r$ on $[0,T]$, it is enough to prove that

$$E^{Q^{\omega,\kappa}}\left(\int_0^T \left(\Phi_\kappa\left(X^\kappa(r,Y_r,\omega)\right)-\Phi\left(X(r,Y_r,\omega)\right)\right)^2\varphi(r,Y_r)\mathrm{d}r\right) \tag{7.12}$$

converges to zero. Since Φ is bounded it is enough to prove the convergence to zero for almost all $\omega\in\Omega$. In order not to overcharge the notation, in this proof we will omit the argument of ω of Y. By Fubini's theorem the left-hand side of (7.12) equals

$$\int_0^T \mathrm{d}r E^{Q^{\omega,\kappa}}\left((\Phi_\kappa(X^\kappa(r,Y_r,\omega))-\Phi(X(r,Y_r,\omega)))^2\varphi(r,Y_r)\right).$$

Using also Lebesgue dominated convergence theorem, given a sequence (κ_n), when $n\to\infty$, it is enough to find a subsequence (κ_{n_ℓ}) such that for all $r\in[0,T]$ outside a possible Lebesgue null set

$$E^{Q^{\omega,\kappa_{n_\ell}}}\left\{\left(\Phi_{\kappa_{n_\ell}}\left(X^{\kappa_{n_\ell}}(r,Y_r,\omega)\right)-\Phi\left(X(r,Y_r,\omega)\right)\right)^2\varphi(r,Y_r)\right\}\to_{\ell\to\infty}0.$$

We set $Z_r(\omega_1,\omega)=\mathcal{E}_r\left(\int\limits_0^\cdot \mu(\omega)(\mathrm{d}s,Y_s(\omega_1))\right)$. We will substitute from now on (n_ℓ) with n.

Taking into account Lemma 7.8 and Cauchy-Schwarz with respect to the finite measure $Z_r(\omega_1,\omega)Q^{\omega,\kappa_n}(\mathrm{d}\omega_1)$, it is enough to prove that for r a.e.

$$E^{Q^{\omega,\kappa_n}}\left\{\left(\Phi_{\kappa_n}(X^{\kappa_n}(r,Y_r,\omega))-\Phi(X(r,Y_r,\omega))\right)^2\varphi(r,Y_r)Z_r(\cdot,\omega)\right\}$$

converges to zero when n goes to infinity.

Since X^κ constitutes the family of μ-marginal weighted laws of Y under $Q^{\omega,\kappa}$, previous expression gives

$$\begin{aligned}&\int_{\mathbb{R}}|\varphi|(r,y)\left(\Phi_{\kappa_n}\left(X^{\kappa_n}(r,y,\omega)\right)-\Phi\left(X(r,y,\omega)\right)\right)^2X^{\kappa_n}(r,y,\omega)\mathrm{d}y\\ \leqslant&I_{11}(\kappa_n,r)+I_{12}(\kappa_n,r)+I_{13}(\kappa_n,r)+I_{14}(\kappa_n,r),\end{aligned}\tag{7.13}$$

where we have developed the square in the first line of (7.13) using the definition of ψ and Φ_κ. Indeed we get

$$\begin{aligned}I_{11}(\kappa,r)&=\int_{\mathbb{R}}\mathrm{d}y|\varphi|(r,y)||\psi\left(X^\kappa(r,y,\omega)\right)-\psi\left(X(r,y,\omega)\right)|,\\ I_{12}(\kappa,r)&=\int_{\mathbb{R}}\mathrm{d}y|\varphi|(r,y)|\Phi^2\left(X^\kappa(r,y,\omega)\right)|(X-X^\kappa)(r,y,\omega)|,\\ I_{13}(\kappa,r)&=\int_{\mathbb{R}}\mathrm{d}y|\varphi(r,y)|\kappa|X^{\kappa_n}-X|(r,y,\omega),\\ I_{14}(\kappa,r)&=\int_0^T\mathrm{d}r\int_{\mathbb{R}}\mathrm{d}y\kappa|X(r,y,\omega)||\varphi(r,y)|.\end{aligned}$$

We denote $I_{1j}(\kappa):=\int_0^T I_{1j}(\kappa,r)\mathrm{d}r,\ j=1,2,3,4$. It is of course enough to prove that, up to a subsequence $I_{1j}(\kappa_n)\to 0,\ j=1,2,3,4$, where $n\to\infty$. By Cauchy-Schwarz, $I_{11}^2(\kappa)$ is bounded by

$$\|\varphi\|^2_{L^2([0,T]\times\mathbb{R})}\int_0^T\mathrm{d}r\int_{\mathbb{R}}\left(\psi\left(X^\kappa(r,y,\omega)\right)-\psi\left(X(r,y,\omega)\right)\right)^2\mathrm{d}y.$$

This converges to zero according to Remark 7.4 2), after extracting a subsequence κ_n) (not depending on ω). The square of the expectation of $I_{12}(\kappa)$ is bounded by

$$\|\varphi\|^2_{L^2([0,T]\times\mathbb{R})}\int_{[0,T]\times\mathbb{R}}\mathrm{d}r\mathrm{d}y\Phi^4\left(X(r,y,\omega)\right)|X^\kappa-X|^2(r,y,\omega).$$

The expectation of previous expression is indeed uniformly bounded in κ because of (7.6) and (7.7). So the family of r.v.

$\Phi^2\left(X^{\kappa_n}(r,y,\omega)\right)|(X-X^\kappa)(r,y,\omega)|$ is uniformly integrable with respect to the finite measure $\mathrm{d}P(\omega)|\varphi|(t,y)\mathrm{d}t\mathrm{d}y$. Consequently $I_{12}(\kappa)$ goes to zero because of (7.3) in Remark 7.4 4).

$I_{13}^2(\kappa)$ is bounded by $\kappa\,\|\varphi\|^2_{L^2([0,T]\times\mathbb{R})}\,\kappa\int\limits_{[0,T]\times\mathbb{R}}\mathrm{d}r\mathrm{d}y|X^\kappa-X|^2(r,y,\omega)$. After extracting a subsequence κ_n, previous expression converges to zero because of Lemma 7.3 c). Finally $I_{14}(\kappa)\underset{n\to\infty}{\longrightarrow}0$ by Cauchy-Schwarz and the fact that $\int\limits_{[0,T]\times\mathbb{R}}\mathrm{d}r\mathrm{d}yX^2(r,y,\omega)<\infty$ $\mathrm{d}P$-a.s. This establishes the proof of Lemma 7.9. ☐

Let (κ_n) be the sequence introduced by the statement of Lemma 7.6. Previous Corollary 7.7 and Lemma 7.9 have the following consequences . Let $Q(\mathrm{d}\omega_1,\omega)$ be the random kernel introduced in Lemma 7.5 and the related probability $\mathbf{Q}(\mathrm{d}\omega_1,\mathrm{d}\omega)=\mathrm{d}P(\omega)Q(\mathrm{d}\omega_1,\omega)$.

Corollary 7.10. *Let $R:\Omega\to\mathbb{R}$ be a bounded measurable r.v. Let $\varphi:[0,T]\times\mathbb{R}\to\mathbb{R}$ be a function with compact support. The sequence*

$$\int_\Omega R(\omega)\mathrm{d}P(\omega)\int_{\tilde\Omega_1}\mathrm{d}Q^{\omega,\kappa_n}(\omega_1)\int_0^T\varphi(r,Y_r)\Phi^2_{\kappa_n}(X^{\kappa_n}(r,Y_r,\omega))\mathrm{d}r \tag{7.14}$$

converges, when $n\to\infty$, to

$$\int_\Omega R(\omega)\mathrm{d}P(\omega)\int_{\tilde\Omega_1}\mathrm{d}Q(\omega_1,\omega)\int_0^T\varphi(r,Y_r)\Phi^2(X(r,Y_r,\omega))\mathrm{d}r. \tag{7.15}$$

Proof. We split the difference between (7.14) and (7.15) which gives $I_1(n)+I_2(n)$ where

$$I_1(n)=\int_\Omega R(\omega)\mathrm{d}P(\omega)\int_{\tilde\Omega_1}\mathrm{d}Q^{\omega,\kappa_n}(\omega_1)\left(\int_0^T\varphi(r,Y_r)(\Phi^2_{\kappa_n}(X^{\kappa_n}(r,Y_r,\omega))\mathrm{d}r-\Phi^2(X(r,Y_r,\omega))\mathrm{d}r\right),$$

and

$$I_2(n)=\int_\Omega R(\omega)\mathrm{d}P(\omega)\int_{\tilde\Omega_1}(Q^{\omega,\kappa_n}(\mathrm{d}\omega_1)-Q(\mathrm{d}\omega_1,\omega))\left(\int_0^T\varphi(r,Y_r)\Phi^2(X(r,Y_r,\omega))\mathrm{d}r\right).$$

We have

$$|I_1(n)|\le 2\|\Phi\|_\infty\|R\|_\infty\int_\Omega\mathrm{d}P(\omega)\int_{\tilde\Omega_1}\mathrm{d}Q^{\omega,\kappa_n}(\omega_1)\left(\int_0^T|\varphi(r,Y_r)||\Phi_{\kappa_n}(X^{\kappa_n}(r,Y_r,\omega))\mathrm{d}r-\Phi(X(r,Y_r,\omega))|\mathrm{d}r\right).$$

$I_1(n)$ converges to zero by Lemma 7.9. Concerning $I_2(n)$, by Fubini's theorem, we first observe that

$$I_2(n)=\int_0^T\mathrm{d}r\int_\Omega\mathrm{d}P(\omega)\left(\int_{\tilde\Omega_1}(Q^{\omega,\kappa_n}(\mathrm{d}\omega_1)-Q(\mathrm{d}\omega_1,\omega))\varphi(r,Y_r)\Phi^2(X(r,Y_r,\omega))R(\omega)\right).$$

We apply now Corollary 7.7, setting for fixed r, $F(\omega_1,\omega) = R(\omega)\varphi(r,\omega_1(r))\Phi^2(X(r,\omega_1(r),\omega))$ and the result follows.

□

5) We go on with the proof of Theorem 7.1.

We want now to prove that $Y(\cdot,\omega)$ is a (weak) solution of

$$Y_t = Y_0 + \int_0^t \Phi\left(X(s,Y_s,\cdot)\right) \mathrm{d}\beta^\omega_s,$$

for some Brownian motion β^ω. This is related to item 1) of Definition 2.10 with $\gamma(t,\xi,\omega) = \Phi(X(t,\xi,\omega))$. According to Remark 2.11 c), for this it is enough to show that for $\mathrm{d}P$-a.s.ω $Y(\ \cdot\ ,\omega)$ is a solution of the following (local) martingale problem. For every $f \in C^{1,2}([0,T]\times\mathbb{R})$ with compact support, the process

$$Z^f_t := f(t,Y_t) - f(0,Y_0) - \frac{1}{2}\int_0^t \partial^2_{xx} f(r,Y_r)\Phi^2\left(X(r,Y,\omega)\right)\mathrm{d}r - \int_0^t \partial_r f(r,Y_r)dr,$$

is a (local) martingale under Q^ω.

This will be a consequence of the lemma below.

Lemma 7.11. *Let F be a bounded $\mathcal{F}_s$-measurable, let $\mathcal{A} : C([0,s]) \to \mathbb{R}$ bounded continuous functional. Let $G = \mathcal{A}(Y_r, r\le s)$. Then, for $0 \le s \le t \le T$ we have*

$$E(FE^{Q^\omega}(GZ^f_t)) = E(FE^{Q^\omega}(GZ^f_s)). \tag{7.16}$$

Proof. We set

$$Z^{\kappa,f}_t = f(t,Y_t) - f(0,Y_0) - \frac{1}{2}\int_0^t \partial^2_{xx} f(r,Y_r)\Phi^2_\kappa\left(X^\kappa(r,Y,\omega)\right)\mathrm{d}r - \int_0^t \partial_r f(r,Y_r)\mathrm{d}r.$$

Let (κ_n) be the sequence introduced by Lemma 7.6. The difference of the right and left-hand side of (7.16) is the sum $(I_1+I_2+I_3)(\kappa_n)$ where

$$\begin{aligned}
I_1(\kappa) &= E\left(F(E^{Q^\omega}(GZ^f_t) - E^{Q^{\omega,\kappa}}(GZ^{\kappa,f}_t))\right)\\
I_2(\kappa) &= E\left(FE^{Q^{\omega,\kappa}}(G(Z^{\kappa,f}_t - Z^{\kappa,f}_s))\right)\\
I_3(\kappa) &= E\left(F(E^{Q^{\omega,\kappa}}(GZ^{\kappa,f}_s) - E^{Q^\omega}(GZ^f_s))\right).
\end{aligned}$$

$I_1(\kappa_n) + I_3(\kappa_n)$ converges to zero by Lemma 7.6, Corollary 7.10 and Lemma 7.9. $I_2(\kappa_n) = 0$ since $Z^{\kappa,f}$ is a $Q^{\kappa,\omega}$-martingale. □

6) After previous intermediary step 5) we need to show that Y defined in (7.1) is a weak-strong solution of DSDE(γ, x_0) with $\gamma(s,\xi,\omega) = \Phi(X(s,\xi,\omega))$ and X is a solution of (1.1). We recall that the kernel $Q(\cdot,\omega)$ has been introduced through Lemma 7.6 on $(\tilde{\Omega}_1 \times \Omega, \mathcal{H} \otimes \mathcal{F})$. So, according to step 5), under $Q^\omega := Q(\cdot,\omega)$, Y is a martingale with $[Y]_t = \int\limits_0^t \Phi^2\left(X(s,Y_s,\omega)\right) \mathrm{d}s$. To conclude the proof of item 1) in Definition 2.10, it remains to construct the suitable required process B. For this, we need to enlarge the probability space $\tilde{\Omega}_1$ as follows. We set $\Omega_1 = \tilde{\Omega}_1 \times C([0,T];\mathbb{R})$; the second component allows to define a Brownian motion. By an abuse of notation, we set again $Y_t(\omega_1,\omega) = \omega_1^0(t)$, this time with $\omega_1 = (\omega_1^0, \omega_1^1)$. In spite of adding the component ω_1^1, in step 5) we have already shown $Q^\omega := Q(\,\cdot\,,\omega)$, is by construction the law of $Y(\cdot,\omega)$. We need to construct a process B on $\Omega \times \Omega_1$, such that for almost all ω, $B(\cdot,\omega)$ is a Q^ω-Brownian motion and (2.7) holds for $\gamma(t,\cdot,\omega) = \Phi(X(t,\cdot,\omega))$.

On Ω_1 we set $\beta_t(\omega_1) = \omega_1^1(t)$. We equip $C([0,T];\mathbb{R})$ in Ω_1 with the Wiener measure $\mathcal{W}$ so that β is a standard Brownian motion on Ω_1. β can also be considered to be a Brownian motion on $\Omega_0 = \Omega_1 \times \Omega$ which is Q^ω-independent of Y for P-almost all $\omega \in \Omega$. Of course β is also independent of Y on the probability space $(\Omega_1 \times \Omega, \mathcal{B}(\Omega_1) \times \mathcal{F}, \mathrm{d}\mathbf{Q}(\omega_1,\omega) := Q^\omega(\mathrm{d}\omega_1)\mathrm{d}P(\omega))$. β is also independent of $(\mathcal{F}_t)$.

We set now

$$B_t(\cdot,\omega) = \int_0^t \mathrm{d}Y_s(\cdot,\omega) 1_{\{\gamma(s,\xi,\omega)\neq 0\}} \frac{1}{\gamma(s,\xi,\omega)} + \int_0^t 1_{\{\gamma(s,\xi,\omega)=0\}} \mathrm{d}\beta_s.$$

Now for Q^ω-a.s. the quadratic variation of the Q^ω-martingale $B(\cdot,\omega)$ is t, so that, by Lévy characterization theorem, $B(\cdot,\omega)$ is a Brownian motion under Q^ω.

It remains to show items 2) and 3) of the definition of weak-strong solution. Let $(\mathcal{Y}_t)$ be the canonical filtration of the process $Y(\cdot,\omega)$. Item 3) follows because of item 1) and because $\gamma(t,\cdot,\omega) = \Phi(X(t,\cdot,\omega))$ is progressively measurable. Concerning item 2) we see that under $\mathbf{Q}$ defined by P and the kernel $Q(\cdot,\omega)$, $W^1,\ldots,W^N$ are $\mathbf{Q}$-martingales with $(\mathcal{G}_t)$ as defined in Definition 2.10. Indeed let F be a bounded $\mathcal{F}_s$-measurable random variable and G be a bounded $\mathcal{Y}_s$-measurable r.v. Let $1 \le i \le N$. By item 3) $E^{Q^\omega}(G)$ is $\mathcal{F}_s$-measurable, so

$$E^{\mathbf{Q}}((W_t^i - W_s^i)FG) = E((W_t^i - W_s^i)FE^{Q^\omega}(G)) = 0,$$

since W^i is an $\mathcal{F}_s$-martingale.

7) The final step consists in proving that X is the family of μ-marginal weighted laws of Y. We need to show that for almost all ω, for every $t \in [0,T]$, $\varphi \in \mathcal{S}(\mathbb{R})$, that

$$\int_{\mathbb{R}} \mathrm{d}\xi \varphi(\xi) X(t,\xi,\omega) = E^{Q^\omega}\left(\varphi(Y_t)\mathcal{E}_t\left(\int_0^{\cdot} \mu(\mathrm{d}s, Y_s)\right)\right).$$

Since both sides of previous equality are $\mathcal{F}_t$-measurable, given a bounded $\mathcal{F}_t$-measurable random variable R it will be enough to show that

$$\int_\Omega \mathrm{d}P(\omega)R(\omega)\int_{\mathbb{R}} \mathrm{d}\xi \varphi(\xi) X(t,\xi,\omega) = \int_\Omega \mathrm{d}P(\omega)R(\omega)E^{Q^\omega}\left(\varphi(Y_t)\mathcal{E}_t\left(\int_0^{\cdot} \mu(\mathrm{d}s, Y_s)\right)\right). \tag{7.17}$$

Let $\omega \in \Omega$ outside some P-null set.

By step 1) of the proof of this Theorem 7.1, we know that X^κ fulfills, for almost all ω,

$$\int_{\mathbb{R}} \mathrm{d}\xi X^\kappa(t,\xi)\varphi(\xi) = E^{Q^\kappa(\,\cdot\,,\omega)}\left(\varphi(Y_t)\mathcal{E}_t\left(\int_0^{\cdot} \mu(\mathrm{d}s, Y_s)\right)\right),$$

for all $\varphi \in \mathcal{S}(\mathbb{R})$. Consequently if (κ_n) is the sequence obtained via Lemma 7.5, we have

$$\int_\Omega \mathrm{d}P(\omega)R(\omega)\int_{\mathbb{R}} \mathrm{d}\xi X^{\kappa_n}(t,\xi)\varphi(\xi) = \int_\Omega \mathrm{d}P(\omega)R(\omega)E^{Q^{\omega,\kappa_n}}\left(\varphi(Y_t)\mathcal{E}_t\left(\int_0^{\cdot} \mu(\mathrm{d}s, Y)\right)\right), \tag{7.18}$$

for every $\varphi \in \mathcal{S}(\mathbb{R})$.

Since $t \mapsto X(t,\,\cdot\,)$ is continuous from $[0,T]$ to $\mathcal{S}'(\mathbb{R})$ and the right-hand side of (7.17) is continuous on $[0,T]$ for fixed $\varphi \in \mathcal{S}(\mathbb{R})$, it is enough to show (7.17) for almost all $t \in [0,T]$.

Now for almost all t, the left-hand side of (7.17) is approached by the left-hand side of (7.18). Let us fix $t \in [0,T]$. It remains to show that the right-hand side of (7.17) is the limit of the right-hand side of (7.18). We fix $\omega \in \Omega$ outside a null set. We set $\mathcal{E}_t := \mathcal{E}_t\left(\int_0^{\cdot} \mu(\mathrm{d}s, Y_s)\right)$, $t \in [0,T]$. By Theorem 2 of [25] and uniform integrability arguments, similarly as after (2.5), we have

$$\mathcal{E}_t = \exp(\psi_\omega(Y)),$$

where $\psi_\omega : \tilde{\Omega}_1 \to \mathbb{R}$ is a continuous modification of

$$\omega \mapsto \left(\eta \mapsto \int_0^t e^i(\eta_s)\mathrm{d}W_s^i - \frac{1}{2}\int_0^t e^i(\eta_s)^2\mathrm{d}s)\right).$$

Indeed, previous random field, indexed by $\eta \in \tilde{\Omega}_1$, admits a continuous modification; to prove this we make use of Kolmogorov-Centsov theorem and Doob's inequality, which says that for any $0 \le i \le N$, there is a constant $\mathrm{const} = \mathrm{const}((e^i)')$ with

$$E\left(|\int_0^t (e^i(\eta_s^1) - e^i(\eta_s^2)) \mathrm{d}W_s^i|^4 \right) \le \mathrm{const} \sup_{s \in [0,T]} |\eta^1 - \eta^2|^2(s), \ \eta^1, \eta^2 \in \tilde{\Omega}_1.$$

At this point we fix $M > 0$. We decompose the difference of the right-hand sides of (7.18) and (7.17) as

$$J_1(n, M) + J_2(n, M) + J_3(n, M), \tag{7.19}$$

where

$$J_1(n, M) = \int_\Omega \mathrm{d}P(\omega) R(\omega) E^{Q^{\omega, \kappa_n}} \left(\varphi(Y_t) \mathcal{E}_t - \varphi(Y_t)(\mathcal{E}_t \wedge M) \right),$$

$$J_2(n, M) = \int_\Omega \mathrm{d}P(\omega) R(\omega) (E^{Q^{\omega, \kappa_n}} - E^{Q^\omega}) \left(\varphi(Y_t)(\mathcal{E}_t \wedge M) \right),$$

$$J_3(n, M) = \int_\Omega \mathrm{d}P(\omega) R(\omega) E^{Q^\omega} \left(\varphi(Y_t)(\mathcal{E}_t \wedge M) - \varphi(Y_t) \mathcal{E}_t \right).$$

Setting $\mathbf{Q}^{\kappa_n}(\mathrm{d}\omega, \mathrm{d}\omega_1) = \mathrm{d}P(\omega) Q^{\omega, \kappa_n}(\mathrm{d}\omega_1)$, by Cauchy-Schwarz and Chebyshev inequalities, for every $p > 1$, we have

$$|J_1(n, M)| = \left| \int_{\Omega_1 \times \Omega} \mathrm{d}Q^{\kappa_n} \varphi(Y_t) \mathcal{E}_t 1_{\{\mathcal{E}_t > M\}} \right| \le \|\varphi\|_\infty \frac{E^{Q^{\kappa_n}}(\mathcal{E}_t^p)}{M^{p-1}}.$$

By Lemma 7.8, we get $\sup_n |J_1(n, M)| \to 0$ if $M \to \infty$. By a similar reasoning, replacing $Q^{\kappa_n}(\mathrm{d}\omega, \mathrm{d}\omega_1)$ with $Q(\mathrm{d}\omega, \mathrm{d}\omega_1) = \mathrm{d}P(\omega) Q^\omega(\mathrm{d}\omega_1)$, we can prove that $\sup_n |J_3(n, M)| \to 0$. Let $\varepsilon > 0$. Let M such that $\sup_n |J_1(n, M) + J_3(n, M)| \le \varepsilon$. On the other hand we have

$$J_2(n, M) = \int_\Omega \mathrm{d}P(\omega) R(\omega) \left(E^{Q^{\omega, \kappa_n}} - E^{Q^\omega} \right) (\varphi(Y_t)(\psi_\omega(Y) \wedge M)).$$

Since for almost all ω, $F(\eta, \omega) := R(\omega) \varphi(\eta(t)) \psi_\omega(\eta)$ is bounded and continuous, Corollary 7.7 implies that $J_2(n, M)$ goes to zero when $n \to \infty$.

Taking the limsup in (7.19) we get

$$\limsup_{n \to \infty} |J_1(n, M) + J_2(n, M) + J_3(n, M)| \le \varepsilon.$$

Since ε is arbitrarily small, we get $\lim_{n \to \infty} |J_1(n, M) + J_2(n, M) + J_3(n, M)| = 0$ and the result follows.

$\square$

A Technicalities

Proposition A.1. *Let Y_0 be distributed according to x_0. Let $a : [0,T] \times \mathbb{R} \to \mathbb{R}$ be a Borel function such there are $0 < c < C$ with $c \leqslant a(s,\xi) \leqslant C, \quad \forall\, (s,\xi) \in [0,T] \times \mathbb{R}$. We fix $0 \le r \le t \le T$. We set $a_n(t,x) = \int_{\mathbb{R}} \rho_n(x-y) a(t,y) \mathrm{d}y$ where (ρ_n) is the usual sequence of mollifiers converging to the Dirac delta. The unique solutions S^n to $S_t^n = Y_0 + \int_r^t a_n(s, S_s^n) \mathrm{d}B_s$, B being a classical Wiener process, converges in law to the (weak unique solution) of $S_t = Y_0 + \int_r^t a(s, S_s) \mathrm{d}B_s$.*

Proof. The proof follows by standard arguments, see Stroock-Varadhan ([27], Problem 7.3.3), tightness and Kolmogorov-Centsov type arguments. For a detailed proof, the reader may consult [7]. □

Proof (of Corollary 7.7).

By (7.8), the family $(Q^{\kappa_\ell}, \ell \in \mathbb{N}, \omega \in \Omega)$ is tight. So, for every positive integer n there exists a compact subset K_n of $\tilde{\Omega}_1$ such that

$$Q^{\kappa_\ell}(K_n^c, \omega) < \frac{1}{n}, \forall \ell \in \mathbb{N}, \omega \in \Omega. \tag{1.1}$$

Since each $C(K_n) := C(K_n; \mathbb{R})$ is separable with respect to the sup-norm $\| \cdot \|_\infty$ then $C(K_n), \| \cdot \|_\infty$ is a separable Banach space. So we apply Appendix 1, Lemma A.1.4 in [21], to the map $\Omega \ni \omega \mapsto F(\cdot|_{K_n}, \omega) \in C(K_n)$, where $F(\cdot|_{K_n}, \omega)$ denotes the map $K_n \ni \eta \mapsto F(\eta, \omega)$. Therefore we can find a sequence $\tilde{F}_{n,k} : \Omega \to C(K_n)$, $\omega \mapsto \tilde{F}_{n,k}(\cdot, \omega) \in K_n$ such that for $\|F\|_\infty := \sup_{\eta \in \tilde{\Omega}_1, \omega \in \Omega} |F(\eta, \omega)|$, we have

$$\|\tilde{F}_{n,k}\|_\infty \le 1 + \|F\|_\infty, \quad \tilde{F}_{n,k}(\Omega) \subset \{\tilde{g}_{n,k}^{(1)}, \ldots, \tilde{g}_{n,k}^{(N_{n,k})}\} \subset C(K_n),$$

where $\tilde{g}_{n,k}^{(i)} \neq \tilde{g}_{n,k}^{(j)}$ if $i \neq j$, and for all $\omega \in \Omega$

$$\sup_{\eta \in K_n} |F(\eta, \omega) - F_{n,k}(\eta, \omega)| \to 0, \tag{1.2}$$

as $k \to \infty$. Clearly, for all $\omega \in \Omega$, $\tilde{F}_{n,k}(\cdot, \omega) = \sum_{j=1}^{N_{n,k}} \tilde{g}_{n,k}^{(j)} 1_{\{\tilde{g}_{n,k}^{(j)}\}} \circ \tilde{F}_{n,k}(\cdot, \omega)$. By Tietze's extension theorem there exist extensions $g_{n,k}^{(1)}, \ldots, g_{n,k}^{(N_{n,k})} \in C(\tilde{\Omega}_1)$ of $\tilde{g}_{n,k}^{(1)}, \ldots, \tilde{g}_{n,k}^{(N_{n,k})}$ such that for all $1 \le j \le N_{n,k}$, $\sup_{\eta \in \tilde{\Omega}_1} |g_{n,k}^{(j)}(\eta)| \le \sup_{\eta \in \tilde{K}_n} |\tilde{g}_{n,k}^{(j)}(\eta)|$. Now we define $F_{n,k} : \Omega \to C(\tilde{\Omega}_1)$, $\omega \mapsto F_{n,k}(\cdot, \omega)$ by

$$F_{n,k}(\cdot, \omega) = \sum_{j=1}^{N_{n,k}} g_{n,k}^{(j)} 1_{\{\tilde{g}_{n,k}^{(j)}\}} \circ \tilde{F}_{n,k}(\cdot, \omega).$$

Clearly, still

$$\|F_{n,k}\|_\infty \le 1 + \|F\|_\infty. \tag{1.3}$$

Note that for all $\eta \in \tilde{\Omega}_1$

$$\tilde{F}_{n,k}(\eta,\omega) = \sum_{j=1}^{N_{n,k}} g_{n,k}^{(j)}(\eta) 1(\eta)_{\{\tilde{g}_{n,k}^{(j)}\}} \circ \tilde{F}_{n,k}(\eta,\omega),$$

hence of the form that Lemma 7.6 applies. Therefore using the standard notation $\mu(f) := \int f \mathrm{d}\mu$, for a measure μ and a function f, we can argue as follows. Fix $n \in \mathbb{N}$. Then for all $\ell, k \in \mathbb{N}$

$$\begin{aligned}
& \left| \int Q^{\kappa_\ell}(F(\cdot,\omega),\omega) P(\mathrm{d}\omega) - \int Q(F(\cdot,\omega),\omega) P(\mathrm{d}\omega) \right| \\
\leq \; & \left| \int Q^{\kappa_\ell}(F(\cdot,\omega) 1_{K_n},\omega) P(\mathrm{d}\omega) - \int Q(F(\cdot,\omega) 1_{K_n},\omega) P(\mathrm{d}\omega) \right| + \frac{2}{n} \|F\|_\infty \\
\leq \; & \int \underbrace{Q^{\kappa_\ell}(|F(\cdot,\omega) - F_{n,k}(\cdot,\omega)| \, 1_{K_n},\omega)}_{\leq \sup_{\eta \in K_n} |F(\eta,\omega) - F_{n,k}(\eta,\omega)|} P(\mathrm{d}\omega) \\
+ \; & \left| \int Q^{\kappa_\ell}(F_{n,k}(\cdot,\omega),\omega) P(\mathrm{d}\omega) - \int Q(F_{n,k}(\cdot,\omega),\omega) P(\mathrm{d}\omega) \right| \\
+ \; & \frac{2}{n}(1 + \|F\|_\infty) + \int Q(|F(\cdot,\omega) - F_{n,k}(\cdot,\omega)|,\omega) 1_{K_n} P(\mathrm{d}\omega) + \frac{2}{n} \|F\|_\infty .
\end{aligned}$$

The first inequality is a consequence of (1.1), the second one of (1.1) and (1.3). Now, letting first $\ell \to \infty$ (using Lemma 7.6), then $k \to \infty$ (using (1.2)) and finally $n \to \infty$, the assertion follows. □

B Uniqueness for the porous media equation with noise

We state here a general uniqueness lemma which only holds under even weaker hypotheses than Assumption 1.1 i.e. $\psi : \mathbb{R} \to \mathbb{R}$ is Lipschitz and that the functions belong to $W^{1,\infty}$.

Theorem B.1. *Let $x_0 \in \mathcal{S}'(\mathbb{R}^d)$ and suppose $\psi : \mathbb{R} \to \mathbb{R}$ to be Lipschitz. Then equation* (1.1) *admits at most one solution among the random fields $X :]0,T] \times \mathbb{R} \times \Omega \to \mathbb{R}$ such that*

$$\int_{[0,T] \times \mathbb{R}} X^2(s,\xi) \mathrm{d}s \mathrm{d}\xi < \infty \quad a.s. \tag{B.1}$$

Remark B.2. *1. Suppose moreover that $e^i, 0 \leq i \leq N$,* belong to H^1. *If $x_0 \in L^2$ or ψ is non-degenerate then Theorem 3.4 of [8] provides an existence theorem for* (1.1). *It states the existence of a random field X such that*

$$E \left(\int_{[0,T] \times \mathbb{R}} X^2(s,\xi) \mathrm{d}s \mathrm{d}\xi \right) < \infty,$$

such that $t \mapsto X(t,\cdot)$ *belongs to* $C([0,T];H^{-1}(\mathbb{R}))$ *and* $t \mapsto \int_0^t \psi(X(s,\cdot))\mathrm{d}s \in C([0,T];H^1(\mathbb{R}))$ *a.s.*

2. *So, under the assumption of item 1., the solution* X *is unique among those fulfilling* (B.1).

3. X *of point ii) fulfills the equation, for almost all* ω*, in* H^{-1}

$$X(t,\ \cdot\ ) = x_0 + \int_0^t \Delta(\psi(X(s,\ \cdot\ )))\mathrm{d}s + \int_0^t \mu(\mathrm{d}s,\cdot)X(s,\cdot), \quad t \in [0,T]. \tag{2.1}$$

The proof of Theorem B.1 is a consequence of the result stated in Theorem B.1 of [24], see also [7].

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