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Discrete Mathematics and Theoretical Computer Science DMTCS vol. **17:3**, 2016, 307–314

On degree-sequence characterization and the extremal number of edges for various Hamiltonian properties under fault tolerance[*]

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received 4th Mar. 2015, revised 16th June 2016, 30th June 2016, 18th July 2016, accepted 25th July 2016.

Assume that n, δ, k are integers with $0 \leq k < \delta < n$. Given a graph $G = (V, E)$ with $|V| = n$. The symbol $G - F$, $F \subseteq V$, denotes the graph with $V(G - F) = V - F$, and $E(G - F)$ obtained by E after deleting the edges with at least one endvertex in F. G is called *k-vertex fault traceable*, *k-vertex fault Hamiltonian*, or *k-vertex fault Hamiltonian-connected* if $G - F$ remains traceable, Hamiltonian, and Hamiltonian-connected for all F with $0 \leq |F| \leq k$, respectively. The notations $h_1(n, \delta, k)$, $h_2(n, \delta, k)$, and $h_3(n, \delta, k)$ denote the minimum number of edges required to guarantee an n-vertex graph with minimum degree $\delta(G) \geq \delta$ to be k-vertex fault traceable, k-vertex fault Hamiltonian, and k-vertex fault Hamiltonian-connected, respectively. In this paper, we establish a theorem which uses the degree sequence of a given graph to characterize the k-vertex fault traceability/hamiltonicity/Hamiltonian-connectivity, respectively. Then we use this theorem to obtain the formulas for $h_i(n, \delta, k)$ for $1 \leq i \leq 3$, which improves and extends the known results for $k = 0$.

Keywords: graph size, Hamiltonian, fault-tolerant Hamiltonian, Hamiltonian-connected, degree sequence.

1 Introduction

In this paper, all graphs are undirected, simple, and without loops. For graph definitions and notations, we refer to Hsu and Lin (2009). We denote any graph by $G = (V, E)$, where V is the vertex set and $E \subseteq \{(u, v) \mid (u, v)$ is an unordered pair of $V\}$ the edge set of G. The *order* of a graph G, denoted by $|G|$, is the number of vertices of G. An edge of G is denoted by (u, v), where $u, v \in V$ and $(u, v) \in E$. Two vertices u and v are *adjacent* in G if there is an edge (u, v) in G. The *degree* of a vertex u in G, denoted by $deg_G(u)$, is the number of vertices adjacent to u. The notation $\delta(G)$ represents the minimum degree of vertices of the graph G. A *walk* of *length* k is denoted by $\langle v_0, v_1, \ldots, v_k \rangle$, where v_i's are vertices such that $(v_{i-1}, v_i) \in E$ for all i. A walk from u to v starts from the first vertex u and ends at the last vertex v; u and v are called the *endvertices*. A *path* is a walk with no repeated vertex. A graph G is *traceable* if G contains a Hamiltonian path. A *cycle* is a closed walk in which the first vertex and

[*]This research was partially supported by the Ministry of Science and Technology of the Republic of China under contract: MOST 104-2115-M-033-002- and MOST 105-2115-M-033-003.
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the last vertex are the only vertex repetition. A *Hamiltonian cycle* of G is a cycle that traverses every vertex of G exactly once. A *Hamiltonian graph* is a graph with a Hamiltonian cycle. A graph G is *connected* if there is a path between any two distinct vertices in G and is *Hamiltonian-connected* if there is a Hamiltonian path between any two distinct vertices in G. The symbol $G - F$, $F \subseteq V$, denotes the graph with $V(G - F) = V - F$, and $E(G - F)$ obtained by E after deleting the edges with at least one endvertex in F. A graph G is *k-vertex fault traceable* (resp. *k-vertex fault Hamiltonian*, *k-vertex fault Hamiltonian-connected*) if $G - F$ remains traceable (resp. Hamiltonian, Hamiltonian-connected) for any set $F \subseteq V$ with $|F| \leq k$. However, it is obvious that for a graph G to be k-vertex fault traceable (resp. k-vertex fault Hamiltonian, k-vertex fault Hamiltonian-connected), it must be $k \leq \delta(G) - 1$ (resp. $k \leq \delta(G) - 2$, $k \leq \delta(G) - 3$). See Kao et al. (2006).

We use $\binom{a}{b}$ to denote the binomial coefficient indexed by a and b, where a and b are positive integers and $a \geq b$. Let G_1 and G_2 be two graphs. We say that G_1 and G_2 are *disjoint* if G_1 and G_2 have no vertex in common. The *union* of two disjoint graphs G_1 and G_2, denoted by $G_1 + G_2$, is a graph with $V(G_1 + G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 + G_2) = E(G_1) \cup E(G_2)$. The *join* of two disjoint subgraphs G_1 and G_2, denoted by $G_1 \vee G_2$, is the graph obtained from $G_1 + G_2$ by connecting each vertex of G_1 to each vertex of G_2 with an edge. We use K_n for a complete graph with n vertices and $\overline{K}_n$ for the union of n isolated vertices.

If G is a graph with $|G| = n$ and degrees $d_1 \leq d_2 \leq \ldots \leq d_n$, then the sequence $(d_1, d_2, \ldots, d_n)$ is called the *degree sequence* of G. A sequence of real numbers $(p_1, p_2, \ldots, p_n)$ is said to be *majorized* by another sequence $(q_1, q_2, \ldots, q_n)$ if $p_i \leq q_i$ for $1 \leq i \leq n$. A graph G is *degree-majorized* by a graph H if $|G| = |H|$ and the nondecreasing degree sequence of G is majorized by that of H. For example, the 5-cycle is degree-majorized by the complete bipartite graph $K_{2,3}$ since $(2, 2, 2, 2, 2)$ is majorized by $(2, 2, 2, 3, 3)$.

Ever since Dirac's theorem for hamiltonicity was established, theorems for various Hamiltonian properties have been derived based on the degree conditions of a given graph. Some of the well-known results are presented below.

Theorem 1 *(Ore (1960, 1963)) Let $G = (V, E)$ be a graph with $|G| = n \geq 3$. If $deg_G(u) + deg_G(v) \geq n + k^*$ for any pair of non-adjacent vertices $\{u, v\}$ in V, then G is traceable if $k^* = -1$, Hamiltonian if $k^* = 0$, and Hamiltonian-connected if $k^* = 1$.*

Theorem 2 *(Chvátal (1972)) Let G be a graph with $|G| = n \geq 3$. Assume that $(d_1, d_2, \ldots, d_n)$ is the degree sequence of G. Then*
(i) G is traceable if for every $i \leq n/2$, $d_i < i \Rightarrow d_{n+1-i} \geq n - i$ holds.
(ii) G is Hamiltonian if for every $i < n/2$, $d_i \leq i \Rightarrow d_{n-i} \geq n - i$ holds.

Theorem 3 *(Lick (1970)) Let G be a graph with $|G| = n \geq 3$. Assume that $(d_1, d_2, \ldots, d_n)$ is the degree sequence of G. Then G is Hamiltonian-connected if for every $2 \leq i \leq n/2$, $d_{i-1} \leq i \Rightarrow d_{n-i} \geq n - i + 1$ holds.*

Obviously, compared with degree sums, the concept of degree sequences characterizes a graph in a more refined way. Consider $G_i = (2K_2 + K_1) \vee K_{i+1}$, where $1 \leq i \leq 2$, for example. It is easy to see that $|G_i| = 6 + i$. Applying Theorem 2(i) on G_1 and (ii) on G_2, we know that G_1 is traceable and G_2 is Hamiltonian. However, for any pair of non-adjacent vertices $\{u, v\}$ in G_i, G_1 fails to satisfy the condition $deg_{G_1}(u) + deg_{G_1}(v) \geq |G_1| - 1$, and G_2 fails to satisfy the condition $deg_{G_2}(u) + deg_{G_2}(v) \geq |G_2|$. Let G_3 be obtained as $(\overline{K}_2 + K_4) \vee K_3$ with an additional edge between a vertex of $\overline{K}_2$ and K_4. Then G_3 is a

Hamiltonian-connected graph, which satisfies Theorem 3 but fails to satisfy Theorem 1. These examples show that the degree sequence of a graph helps to determine the associated Hamiltonian properties when its degree sum condition (as in Theorem 1) gives no conclusion.

In the past decade, some results regarding the minimum number of edges that guarantees various properties have been published. In Brandt (1997), Bollobás and Thomason (1999), and Erdős et al. (1996), for example, the minimum number of edges is given as a function of the total number of vertices of any graph. In Ho et al. (2010, 2011), Ho and his coauthors studied the minimum number of edges required to guarantee an n-vertex graph G with minimum degree $\delta(G) \geq \delta$ to be Hamiltonian or Hamiltonian-connected, and expressed it as a function of $|G| = n$ and the minimum degree $\delta(G) \geq \delta$. Such results have many applications in interconnection networks under conditional faults, and provide better lower bounds for the number of edges by taking δ into account. See Ho et al. (2010, 2011) and their references. Our present results extend the formulas of Ho et al.

Chvátal (1972) characterized the degree sequence behavior for a graph to remain Hamiltonian after the removal of up to k faulty vertices. To our knowledge, other than this study, no result about the vertex fault version of Theorem 2 and 3 has been published. Inspired by the above-mentioned works, we intend to establish two main theorems as follows. Note that Theorem 4(ii) was proved by Chvátal (1972).

Theorem 4 *Let G be a graph with $|G| = n \geq 3$. Assume that $(d_1, d_2, \ldots, d_n)$ is the degree sequence of G.*

(i) Let k be an integer with $0 \leq k \leq n-2$. G is k-vertex fault traceable if the degree sequence $(d_1, d_2, \ldots, d_n)$ satisfies

$$d_j < j + k' \leq \frac{n+k'}{2} \Rightarrow d_{n-j-k'+1} \geq n - j \quad for\ all\ integers\ k'\ with\ 0 \leq k' \leq k. \tag{1}$$

(ii) Let k be an integer with $0 \leq k \leq n-3$. G is k-vertex fault Hamiltonian if the degree sequence $(d_1, d_2, \ldots, d_n)$ satisfies

$$d_j \leq j + k' < \frac{n+k'}{2} \Rightarrow d_{n-j-k'} \geq n - j \quad for\ all\ integers\ k'\ with\ 0 \leq k' \leq k. \tag{2}$$

(iii) Let k be an integer with $0 \leq k \leq n-4$. G is k-vertex fault Hamiltonian-connected if the degree sequence $(d_1, d_2, \ldots, d_n)$ satisfies

$$d_j \leq j + k' + 1 \leq \frac{n+k'}{2} \Rightarrow d_{n-j-k'-1} \geq n - j \quad for\ all\ integers\ k'\ with\ 0 \leq k' \leq k. \tag{3}$$

In the sequel, the notation a **mod** b denotes the remainder of the division of a by b.

Theorem 5 *Assume that n, δ, and k are integers with $0 \leq k < \delta < n$. For $1 \leq i \leq 3$, let $\alpha_i = \frac{n-k-9+5i+3\times[(n-k+i+1)\ \mathbf{mod}\ 2]}{6}$ and $\beta_i = \frac{n+k-3+i}{2}$. Denote by $h_1(n,\delta,k)$, $h_2(n,\delta,k)$, and $h_3(n,\delta,k)$ the minimum number of edges required to guarantee the n-vertex graph G with minimum degree $\delta(G) \geq \delta$ to be k-vertex fault traceable, k-vertex fault Hamiltonian, and k-vertex fault Hamiltonian-connected, respectively. Then*

$$h_i(n,\delta,k) = \begin{cases} \binom{n+k-\delta-2+i}{2} + \delta^2 - (k-2+i)\delta + 1 & \text{if } k+i \leq \delta \leq \alpha_i + k; \\ \binom{n+k-\lfloor\beta_i\rfloor-2+i}{2} + \lfloor\beta_i\rfloor^2 - (k-2+i)\lfloor\beta_i\rfloor + 1 & \text{if } \alpha_i + k < \delta \leq \beta_i; \\ \lceil \frac{n\delta}{2} \rceil & \text{if } \delta > \beta_i. \end{cases}$$

It can be observed that Theorem 2 and Theorem 3 become special cases for Theorem 4 with $k = 0$; the formula in Ho et al. (2010, 2011) becomes a special case of Theorem 5 by taking $k = 0$. In other words, Theorem 5 further extends Ho's formulas for fault tolerant Hamiltonian graphs.

2 Proof of the main theorems

Let n be the total number of vertices in a graph and k be an integer with $k \geq 0$. We first define three graph families as follows.
(1) $T^k_{m,n} = (\overline{K}_{m+1} + K_{n-2m-k-1}) \vee K_{m+k}$, where $n \geq 4$, $0 \leq k \leq n-2$, and $0 \leq m \leq \frac{n-k-2}{2}$.
(2) $C^k_{m,n} = (\overline{K}_m + K_{n-2m-k}) \vee K_{m+k}$, where $n \geq 5$, $0 \leq k \leq n-3$, and $1 \leq m \leq \frac{n-k-1}{2}$.
(3) $H^k_{m,n} = (\overline{K}_{m-1} + K_{n-2m-k+1}) \vee K_{m+k}$, where $n \geq 6$, $0 \leq k \leq n-4$, and $2 \leq m \leq \frac{n-k}{2}$.
See Figure 1 for $H^k_{m,n}$ in two different layouts.

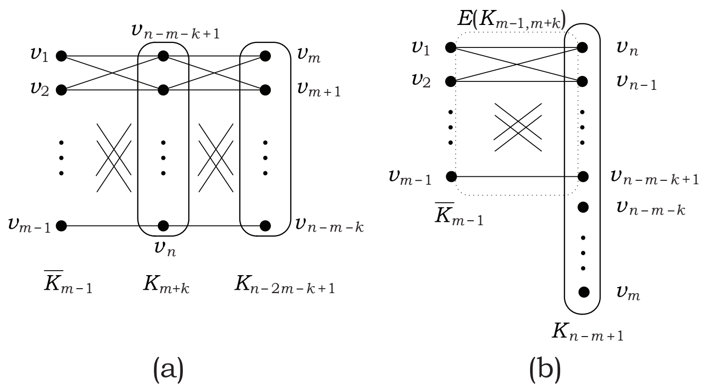

Fig. 1: (a)$H^k_{m,n}$; (b)$H^k_{m,n}$ with a different layout.

We will use these three graph families to establish the sharpness of the bounds in Theorem 5. Let the faulty vertex set $F \subseteq V(K_{m+k})$ with $|F| = k$. For example, $F = \{v_{n-k-m+i} \mid 1 \leq i \leq k\}$ as labeled in Figure 1 for $H^k_{m,n}$. Then it is easy to check that $T^k_{m,n} - F$ has no Hamiltonian path, $C^k_{m,n} - F$ has no Hamiltonian cycle, and $H^k_{m,n} - F$ has no Hamiltonian path between some pair of distinct vertices $\{u, v\}$ in $K_{m+k} - F$. Thus we have the following lemma.

Lemma 1 *(i) Let $n \geq 4$, $0 \leq k \leq n-2$, and $0 \leq m \leq \frac{n-k-2}{2}$. The graph $T^k_{m,n}$ is not k-vertex fault traceable.*
(ii) Let $n \geq 5$, $0 \leq k \leq n-3$, and $1 \leq m \leq \frac{n-k-1}{2}$. The graph $C^k_{m,n}$ is not k-vertex fault Hamiltonian.
(iii) Let $n \geq 6$, $0 \leq k \leq n-4$, and $2 \leq m \leq \frac{n-k}{2}$. The graph $H^k_{m,n}$ is not a k-vertex fault Hamiltonian-connected graph.

2.1 Proof of Theorem 4

To prove Theorem 4, we recall the following theorem by Chvátal (1972) first, which contains (ii) of Theorem 4. Note that any graph in the family $C^k_{m,n}$, which was introduced by Chvátal (1972), has the greatest degree sequence among all graphs with the same number of vertices and being not k-vertex fault Hamiltonian.

Theorem 6 *(Chvátal (1972)) Let k be an integer with* $0 \leq k \leq n-3$. *If the degree sequence* $(d_1, d_2, \ldots, d_n)$ *of a graph G satisfies*

$$d_j \leq j + k' < \frac{n+k'}{2} \Rightarrow d_{n-j-k'} \geq n - j \quad for\ all\ integers\ k'\ with\ 0 \leq k' \leq k, \tag{4}$$

then G is k-vertex fault Hamiltonian. On the other hand, if the degree sequence of G fails to satisfy (4), then it is majorized by the degree sequence of the graph $C^k_{m,n}$*, which is not k-vertex fault Hamiltonian.*

Proof of Theorem 4: (ii) is directly derived from Theorem 6. We will show (iii) and (i) in order using the result of (ii).

To show (iii), we assume that the sequence $(d_1, d_2, \ldots, d_n)$ satisfies (3). Let k' be an arbitrary integer with $0 \leq k' \leq k$. For any faulty vertex set $F = \{y_1, y_2, \ldots, y_{k'}\}$ in G, we want to show that for any pair of vertices $\{u, v\}$ in $G - F$, there exists a Hamiltonian path between u and v in $G - F$. Let $G = (V, E)$. Define $G^* = (V^*, E^*)$, where $V^* = V \cup \{x\}$ and $E^* = E \cup \{(x, u), (x, v)\} \cup \{(x, y_i) | 1 \leq i \leq k'\}$. Let $(d^*_1, d^*_2, \ldots, d^*_{n+1})$ be the degree sequence of G^*. Note that $d^*_1 = deg_{G^*}(x) = k' + 2$ since $d^*_2 \geq d_1 \geq \delta(G) \geq k' + 3$, and for $1 \leq i \leq n$, $d_i \leq d^*_{i+1} \leq d_i + 1$. From (3), we have

$$d_{j-1} \leq (j-1) + k' + 1 \leq \frac{n+k'}{2} \Rightarrow d_{n-(j-1)-k'-1} \geq n - (j-1). \tag{5}$$

Note that n and k' are integers. Thus (5) is equivalent to

$$d_{j-1} \leq j + k' < \frac{(n+1)+k'}{2} \Rightarrow d_{(n+1)-j-k'-1} \geq (n+1) - j. \tag{6}$$

Note that $d^*_j = d_{j-1} + 1$. If $d^*_j > j + k'$, we do not need to check the above condition. If $d^*_j \leq j + k'$, since $d^*_{(n+1)-j-k'} \geq d_{(n+1)-j-k'-1}$, from (6), we have

$$d^*_j \leq j + k' < \frac{(n+1)+k'}{2} \Rightarrow d^*_{(n+1)-j-k'} \geq (n+1) - j. \tag{7}$$

Therefore, according to (7), the degree sequence $(d^*_1, d^*_2, \ldots, d^*_{n+1})$, with $n+1$ in place of n, satisfies (2). By (ii), $G^* - F$ is Hamiltonian. Since $G^* - F$ has a Hamiltonian cycle if and only if $G - F$ has a Hamiltonian path between u and v, $G - F$ is Hamiltonian-connected.

To show (i), we assume that the sequence $(d_1, d_2, \ldots, d_n)$ satisfies (1). Let k' be an arbitrary integer with $0 \leq k' \leq k$. For any faulty vertex set $F = \{v_1, v_2, \ldots, v_{k'}\}$ in G, we want to show there exists a Hamiltonian path in $G - F$. Let $\widehat{G}$ be a graph by adding to G a new vertex x and new edges joining x to all the vertices of G. Let $(\widehat{d}_1, \widehat{d}_2, \ldots, \widehat{d}_{n+1})$ be the degree sequence of $\widehat{G}$. Note that $\widehat{d}_i = d_i + 1$ for $1 \leq i \leq n$ and $\widehat{d}_{n+1} = deg_{\widehat{G}}(x) = n$. As in (iii), it is easy to show that the degree sequence $(\widehat{d}_1, \widehat{d}_2, \ldots, \widehat{d}_{n+1})$ satisfies (2)(with $n+1$ in place of n). According to (ii), $\widehat{G} - F$ is Hamiltonian. Since that $\widehat{G} - F$ has a Hamiltonian cycle if and only if $G - F$ has a Hamiltonian path, $G - F$ is traceable. □

Corollary 1 *Let G be a graph with* $|G| = n \geq 4$ *and* $k \geq 0$ *be an integer.*
(i) If G is not k-vertex fault traceable, then G is degree-majorized by $T^k_{m,n}$*, where* $0 \leq k \leq n-2$ *and* $0 \leq m \leq \frac{n-k-2}{2}$.

(ii) If $n \geq 5$ and G is not k-vertex fault Hamiltonian, then G is degree-majorized by $C^k_{m,n}$, where $0 \leq k \leq n-3$ and $1 \leq m \leq \frac{n-k-1}{2}$.
(iii) If $n \geq 6$ and G is not k-vertex fault Hamiltonian-connected, then G is degree-majorized by $H^k_{m,n}$, where $0 \leq k \leq n-4$ and $2 \leq m \leq \frac{n-k}{2}$.

Proof: We show (iii) only. The proofs for (i) and (ii) can be derived similarly.

According to Theorem 4(iii), if $G-F$ is not k-vertex fault Hamiltonian-connected, then for some k' with $0 \leq k' \leq k$, there exists m, $2 \leq m \leq \frac{n-k'}{2}$, such that

$$d_{m-1} \leq m+k' \text{ and } d_{n-m-k'} \leq n-m. \tag{8}$$

The greatest degree sequence satisfying (8) is of the following form:

$$(\underbrace{m+k', \cdots, m+k'}_{m-1}, \underbrace{n-m, \cdots, n-m}_{n-2m-k'+1}, \underbrace{n-1, \cdots, n-1}_{m+k'}).$$

For any graph being not k-vertex fault Hamiltonian-connected, its degree sequence must be degree-majorized by

$$(\underbrace{m+k, \cdots, m+k}_{m-1}, \underbrace{n-m, \cdots, n-m}_{n-2m-k+1}, \underbrace{n-1, \cdots, n-1}_{m+k}).$$

It is easy to see that $H^k_{m,n}$'s degree sequence is the same as above. Consequently, adding an extra edge to $H^k_{m,n}$ results in a graph where no index m in (8) exists, which means the sufficient condition of Theorem 4(iii) is satisfied. Therefore, $H^k_{m,n}+e$ is k-vertex fault Hamiltonian-connected for any extra edge e. □

2.2 Proof of Theorem 5

Theorem 5 consists of three results. We shall derive $h_3(n,\delta,k)$ in this section. The values $h_1(n,\delta,k)$ and $h_2(n,\delta,k)$ can be obtained by the similar derivations.

Theorem 7 *(Ore (1963)) Let $n \geq 3$. Any simple graph G, where $|G| = n$, with $\delta(G) \geq \frac{n+1}{2}$ is Hamiltonian-connected.*

Corollary 2 *Let n and k be integers with $n \geq 4$ and $0 \leq k \leq n-4$. If G is a graph with $|G| = n$ and $\delta(G) > \frac{n+k}{2}$, then G is k-vertex fault Hamiltonian-connected.*

Lemma 2 *Let n, m, k be integers with $n \geq 6$ and $0 \leq k \leq n-4$. Let G be a graph with $|G| = n$ and δ be an integer with $k+3 \leq \delta \leq \delta(G)$. If G is not k-vertex fault Hamiltonian-connected, then $\delta(G) \leq \frac{n+k}{2}$ and $|E(G)| \leq max\{|E(H^k_{\delta-k,n})|, |E(H^k_{\lfloor\frac{n-k}{2}\rfloor,n})|\}$.*

Proof: If G is not a k-vertex fault Hamiltonian-connected graph with $|G| = n \geq 6$, by Corollary 2, $\delta(G) \leq \frac{n+k}{2}$. By Corollary 1, G is degree-majorized by the graph $H^k_{m,n}$ for some positive integer m. Since $\delta(H^k_{m,n}) = m+k$, $\delta \leq \delta(G) \leq m+k$. Thus, $|E(G)| \leq max\{|E(H^k_{m,n})| \mid \delta-k \leq m \leq \frac{n-k}{2}\}$. A calculation of $|E(H^k_{m,n})|$ shows that

$$|E(H^k_{m,n})| = \frac{3}{2}m^2 + (-n+k-\frac{3}{2})m + (\frac{1}{2}n^2 + \frac{1}{2}n - k) \tag{9}$$

is a quadratic function of m and its maximum value occurs at the boundary $m = \delta - k$ or $m = \lfloor \frac{n-k}{2} \rfloor$. Therefore, $|E(G)| \leq max\{|E(H^k_{\delta-k,n})|, |E(H^k_{\lfloor \frac{n-k}{2} \rfloor,n})|\}$. □

Lemma 3 *Let n, k, t be integers with $n \geq 6$, $0 \leq k \leq n-4$, and $1 \leq t \leq \frac{n-k}{2}$. Then $|E(H^k_{t,n})| \geq |E(H^k_{\lfloor \frac{n-k}{2} \rfloor,n})|$ if and only if $1 \leq t \leq \frac{n-k+6+3\times[(n-k) \mathbf{\ mod\ } 2]}{6}$ or $t = \lfloor \frac{n-k}{2} \rfloor$.*

Proof: See (9) for $|E(H^k_{t,n})|$. We finish this proof by the following two cases.
Case 1: $n-k$ is even. In this case, $\lfloor \frac{n-k}{2} \rfloor = \frac{n-k}{2}$ and $|E(H^k_{\frac{n-k}{2},n})| = \frac{1}{2}[3(\frac{n-k}{2})^2 + (-2n+2k-3)(\frac{n-k}{2}) + (n^2+n-2k)]$. We claim that $|E(H^k_{t,n})| \geq |E(H^k_{\frac{n-k}{2},n})|$ if and only if $1 \leq t \leq \lfloor \frac{n-k}{6} \rfloor$ or $t = \frac{n-k}{2}$. Assume that $|E(H^k_{t,n})| \geq |E(H^k_{\frac{n-k}{2},n})|$. Then we obtain $3t^2 + (-2n+2k-3)t + \frac{1}{4}[n^2 - (2k-6)n + (k^2-6k)] \geq 0$, which is equivalent to $[t - \frac{1}{2}(n-k)][3t - \frac{1}{2}(n-k+6)] \geq 0$. That is, $t \leq \frac{n-k+6}{6}$ or $t \geq \frac{n-k}{2}$. Note that n and k are integers with $n-k$ even, $n \geq 6$, and $1 \leq t \leq \frac{n-k}{2}$. Therefore, we have $|E(H^k_{t,n})| \geq |E(H^k_{\frac{n-k}{2},n})|$ if and only if $1 \leq t \leq \frac{n-k+6}{6}$ or $t = \frac{n-k}{2}$.
Case 2: $n-k$ is odd. In this case, $\lfloor \frac{n-k}{2} \rfloor = \frac{n-k-1}{2}$ and $|E(H^k_{\frac{n-k-1}{2},n})| = \frac{1}{2}[3(\frac{n-k-1}{2})^2 + (-2n+2k-3)(\frac{n-k-1}{2}) + (n^2+n-2k)]$. Reasoning in the same way, we obtain that $|E(H^k_{t,n})| \geq |E(H^k_{\frac{n-k-1}{2},n})|$ if and only if $1 \leq t \leq \frac{n-k+9}{6}$ or $t = \frac{n-k-1}{2}$. □

Proof of the derivation of $h_3(n,\delta,k)$ of Theorem 5: We define $S_3(n,\delta,k)$ as the set of all integers M with $M \geq \frac{n\delta}{2}$ such that every graph H with $|H| = n$ and $\delta(H) \geq \delta$ satisfying $|E(H)| \geq M$ is k-vertex fault Hamiltonian-connected. Obviously, $h_3(n,\delta,k) = min\ S_3(n,\delta,k)$. It can be observed that if δ is large enough so that every n-vertex graph with minimum degree greater than or equal to δ is k-fault Hamiltonian-connected, then $h_3(n,\delta,k)$ is $\lceil \frac{n\delta}{2} \rceil$. For example, if $k = 0$ and $\delta \geq \frac{n+1}{2}$, then by Theorem 7, $h_3(n,\delta,0) = \lceil \frac{n\delta}{2} \rceil$. Note that $0 \leq k < \delta < n$ and $k \leq \delta - 3$. If $n = 4$, then $\delta = 3$ and $k = 0$. If $n = 5$, there are three combinations for δ and k, which are $\delta = 3$ and $k = 0$, $\delta = 4$ and $k = 0$, and $\delta = 4$ and $k = 1$. The theorem holds for $n = 4, 5$ since $h_3(4,3,0) = 6$, $h_3(5,3,0) = 8$, $h_3(5,4,0) = 10$, and $h_3(5,4,1) = 10$. We prove the theorem by deriving $h_3(n,\delta,k)$ for $n \geq 6$ in the following three cases.
Case 1: $3 \leq \delta - k \leq \frac{n-k+6+3\times[(n-k) \mathbf{\ mod\ } 2]}{6}$. According to Lemma 3, $|E(H^k_{\delta-k,n})| \geq |E(H^k_{\lfloor \frac{n-k}{2} \rfloor,n})|$. Since $H^k_{\delta-k,n} + e$ is a graph with $\delta(H^k_{\delta-k,n} + e) \geq \delta$ and $|E(H^k_{\delta-k,n})| + 1$ edges for any extra edge e, by Lemma 2, $H^k_{\delta-k,n} + e$ is k-vertex fault Hamiltonian-connected. Thus $|E(H^k_{\delta-k,n})| + 1 \in S_3(n,\delta,k)$. Hence, $h_3(n,\delta,k) \leq |E(H^k_{\delta-k,n})| + 1$. In addition, note that $3 \leq \delta - k \leq \delta(G) - k \leq \frac{n-k}{2}$. Thus, by Lemma 1, $H^k_{\delta-k,n}$ is not k-vertex fault Hamiltonian-connected. Hence, $h_3(n,\delta,k) > |E(H^k_{\delta-k,n})|$. Therefore, $h_3(n,\delta,k) = |E(H^k_{\delta-k,n})| + 1$. Since $|E(H^k_{\delta-k,n})| = \binom{n+k-\delta+1}{2} + \delta^2 - (k+1)\delta$, $h_3(n,\delta,k) = \binom{n+k-\delta+1}{2} + \delta^2 - (k+1)\delta + 1$.
Case 2: $\frac{n-k+6+3\times[(n-k) \mathbf{\ mod\ } 2]}{6} < \delta - k \leq \frac{n-k}{2}$. According to Lemma 3, $|E(H^k_{\delta-k,n})| < |E(H^k_{\lfloor \frac{n-k}{2} \rfloor,n})|$. Since $H^k_{\lfloor \frac{n-k}{2} \rfloor,n} + e$ is a graph with $\delta(H^k_{\lfloor \frac{n-k}{2} \rfloor,n} + e) \geq \lfloor \frac{n+k}{2} \rfloor \geq \delta$ and $|E(H^k_{\lfloor \frac{n-k}{2} \rfloor,n})| + 1$ edges for any extra edge e, by Lemma 2, $H^k_{\lfloor \frac{n-k}{2} \rfloor,n} + e$ is k-vertex fault Hamiltonian-connected. Thus $|E(H^k_{\lfloor \frac{n-k}{2} \rfloor,n})| + 1 \in S_3(n,\delta,k)$. Hence, $h_3(n,\delta,k) \leq |E(H^k_{\lfloor \frac{n-k}{2} \rfloor,n})| + 1$. In addition, since, by Lemma 1, $H^k_{\lfloor \frac{n-k}{2} \rfloor,n}$

is not k-vertex fault Hamiltonian-connected, $h_3(n,\delta,k) > |E(H^k_{\lfloor\frac{n-k}{2}\rfloor,n})|$. Therefore, $h_3(n,\delta,k) = |E(H^k_{\lfloor\frac{n-k}{2}\rfloor,n})| + 1$. Since $|E(H^k_{\lfloor\frac{n-k}{2}\rfloor,n})| = |E(H^k_{\lfloor\frac{n+k}{2}\rfloor-k,n})| = \binom{n+k-\lfloor\frac{n+k}{2}\rfloor+1}{2} + \lfloor\frac{n+k}{2}\rfloor^2 - (k+1)\lfloor\frac{n+k}{2}\rfloor$, $h_3(n,\delta,k) = \binom{n+k-\lfloor\frac{n+k}{2}\rfloor}{2} + \lfloor\frac{n+k}{2}\rfloor^2 - (k+1)\lfloor\frac{n+k}{2}\rfloor + 1$.

Case 3: $\delta - k > \frac{n-k}{2}$. Thus $\delta(G) \geq \delta > \frac{n+k}{2}$. By Corollary 2, G is k-vertex fault Hamiltonian-connected. Obviously, $|E(G)| \geq \lceil\frac{n\delta}{2}\rceil$. Therefore, $h_3(n,\delta,k) = \lceil\frac{n\delta}{2}\rceil$. □

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