An extremal problem for a graphic sequence to have a realization containing every 2-tree with prescribed size

De-Yan Zeng, Jian-Hua Yin

Discrete Mathematics and Theoretical Computer Science DMTCS vol. **17:3**, 2016, 315–326

An extremal problem for a graphic sequence to have a realization containing every 2-tree with prescribed size

De-Yan Zeng[1,2] Jian-Hua Yin[1*]

[1] *Dep. of Mathematics, College of Information Science and Technology, Hainan University, Haikou, P.R. China*
[2] *Institute of Technology, Sanya University, Sanya, P.R. China*

received 23rd Mar. 2015, revised 10th Apr. 2016, accepted 3rd Aug. 2016.

A graph G is a 2-*tree* if $G = K_3$, or G has a vertex v of degree 2, whose neighbors are adjacent, and $G - v$ is a 2-tree. Clearly, if G is a 2-tree on n vertices, then $|E(G)| = 2n - 3$. A non-increasing sequence $\pi = (d_1, \ldots, d_n)$ of nonnegative integers is a *graphic sequence* if it is realizable by a simple graph G on n vertices. Yin and Li (Acta Mathematica Sinica, English Series, 25(2009)795–802) proved that if $k \geq 2$, $n \geq \frac{9}{2}k^2 + \frac{19}{2}k$ and $\pi = (d_1, \ldots, d_n)$ is a graphic sequence with $\sum_{i=1}^{n} d_i > (k-2)n$, then π has a realization containing every tree on k vertices as a subgraph. Moreover, the lower bound $(k-2)n$ is the best possible. This is a variation of a conjecture due to Erdős and Sós. In this paper, we investigate an analogue extremal problem for 2-trees and prove that if $k \geq 3$, $n \geq 2k^2 - k$ and $\pi = (d_1, \ldots, d_n)$ is a graphic sequence with $\sum_{i=1}^{n} d_i > \frac{4kn}{3} - \frac{5n}{3}$, then π has a realization containing every 2-tree on k vertices as a subgraph. We also show that the lower bound $\frac{4kn}{3} - \frac{5n}{3}$ is almost the best possible.

Keywords: degree sequences; graphic sequences; realization; 2-trees.

1 Introduction

Let K_m be the complete graph on m vertices. A graph G is a 2-*tree* if $G = K_3$, or G has a vertex v of degree 2, whose neighbors are adjacent, and $G - v$ is a 2-tree. It is easy to see that if G is a 2-tree on n vertices, then $|E(G)| = 2n - 3$. An *ear* in a 2-tree is a vertex of degree 2 whose neighbors are adjacent.

The set of all non-increasing sequences $\pi = (d_1, \ldots, d_n)$ of nonnegative integers with $d_1 \leq n - 1$ is denoted by NS_n. A sequence $\pi \in NS_n$ is said to be *graphic* if it is the degree sequence of a simple graph G on n vertices, and such a graph G is called a *realization* of π. The set of all graphic sequences in NS_n is denoted by GS_n. For a nonnegative integer sequence $\pi = (d_1, \ldots, d_n)$, we denote $\sigma(\pi) = d_1 + \cdots + d_n$. Yin and Li [12] investigated a variation of a conjecture due to Erdős and Sós (see [1], Problem 12 in page 247), that is, an extremal problem for a sequence $\pi \in GS_n$ to have a realization containing every tree on k vertices as a subgraph, and obtained the following Theorem 1.1.

*Corresponding author. Email: yinjh@ustc.edu

Theorem 1.1 ([12]) *If $k \geq 2$, $n \geq \frac{9}{2}k^2 + \frac{19}{2}k$ and $\pi = (d_1, \ldots, d_n) \in GS_n$ with $\sigma(\pi) > (k-2)n$, then π has a realization H containing every tree on k vertices as a subgraph. Moreover, the lower bound $(k-2)n$ is the best possible.*

This kind of extremal problem was firstly introduced by Erdős et al. (see [5–6]). The purpose of this paper is to investigate an analogous extremal problem for a sequence $\pi \in GS_n$ to have a realization containing every 2-tree on k vertices as a subgraph. We establish the following Theorems 1.2 and 1.3.

Theorem 1.2 *If $k \geq 3$, $n \geq 2k^2 - k$ and $\pi = (d_1, \ldots, d_n) \in GS_n$ with $\sigma(\pi) > \frac{4kn}{3} - \frac{5n}{3}$, then π has a realization H containing every 2-tree on k vertices as a subgraph.*

The lower bound $\frac{4kn}{3} - \frac{5n}{3}$ in Theorem 1.2 is almost the best possible.

Theorem 1.3 *For $k \equiv i (mod\ 3)$, there exists a sequence $\pi \in GS_n$ with $\sigma(\pi) = 2\lfloor \frac{2k}{3} \rfloor n - 2n - \lfloor \frac{2k}{3} \rfloor^2 + \lfloor \frac{2k}{3} \rfloor + 1 - (-1)^i$ such that π has no realization containing every 2-tree on k vertices.*

2 Proof of Theorem 1.2

In order to prove Theorem 1.2, we need some known results. Let $\pi = (d_1, \ldots, d_n) \in NS_n$ and k be an integer with $1 \leq k \leq n$. Let

$$\pi_k'' = \begin{cases} (d_1 - 1, \ldots, d_{k-1} - 1, d_{k+1} - 1, \ldots, d_{d_k+1} - 1, d_{d_k+2}, \ldots, d_n), & \text{if } d_k \geq k, \\ (d_1 - 1, \ldots, d_{d_k} - 1, d_{d_k+1}, \ldots, d_{k-1}, d_{k+1}, \ldots, d_n), & \text{if } d_k < k. \end{cases}$$

Let $\pi_k' = (d_1', \ldots, d_{n-1}')$, where $d_1' \geq \cdots \geq d_{n-1}'$ is a rearrangement in non-increasing order of the $n-1$ terms of π_k''. We say that π_k' is the *residual sequence* obtained from π by laying off d_k. It is easy to see that if π_k' is graphic then so is π, since a realization G of π can be obtained from a realization G' of π_k' by adding a new vertex of degree d_k and joining it to the vertices whose degrees are reduced by one in going from π to π_k'. In fact, more is true:

Theorem 2.1 ([7]) *$\pi \in GS_n$ if and only if $\pi_k' \in GS_{n-1}$.*

Theorem 2.2 ([4]) *Let $\pi = (d_1, \ldots, d_n) \in NS_n$, where $\sigma(\pi)$ is even. Then $\pi \in GS_n$ if and only if $\sum\limits_{i=1}^{t} d_i \leq t(t-1) + \sum\limits_{i=t+1}^{n} \min\{t, d_i\}$ for any t with $1 \leq t \leq n-1$.*

Theorem 2.3 ([11]) *Let $\pi = (d_1, \ldots, d_n) \in NS_n$, where $d_1 = m$ and $\sigma(\pi)$ is even. If there exist an integer $n_1 \leq n$ and some integer $h \geq 1$ such that $d_{n_1} \geq h$ and $n_1 \geq \frac{1}{h}\lfloor \frac{(m+h+1)^2}{4} \rfloor$, then $\pi \in GS_n$.*

Theorem 2.4 ([6]) *If $\pi = (d_1, \ldots, d_n) \in NS_n$ has a realization G containing H as a subgraph, then there exists a realization G' of π containing H as a subgraph so that the vertices of H have the largest degrees of π.*

Theorem 2.5 ([10]) *Let $n \geq r$ and $\pi = (d_1, \ldots, d_n) \in GS_n$ with $d_r \geq r-1$. If $d_i \geq 2r - 2 - i$ for $i = 1, \ldots, r-2$, then π has a realization containing K_r.*

Theorem 2.6 ([9]) *If $r \geq 1$, $n \geq 2r - 1$ and $\pi = (d_1, \ldots, d_n) \in GS_n$ with $\sigma(\pi) \geq 2n(r-2) + 2$, then π has a realization containing K_r.*

Theorem 2.7 *Let $\pi = (d_1, \ldots, d_n) \in GS_n$.*

(1) [5] If $n \geq 6$ and $\sigma(\pi) \geq 2n$, then π has a realization containing K_3.

(2) [8] If $n \geq 7$ and $\sigma(\pi) \geq 3n-1$, then π has a realization containing $K_4 - e$, where $K_4 - e$ is the graph obtained from K_4 by removing one edge.

(3) [13] If $n \geq 9$ and $\sigma(\pi) \geq 5n-6$, then π has a realization containing $K_5 - e$, where $K_5 - e$ is the graph obtained from K_5 by removing one edge.

We note that a 2-tree can be constructed from an edge by repeatedly adding a new vertex and making it adjacent to the two ends of an edge in the graph formed so far. We refer to the initial edge in constructing such a 2-tree as a *base* of the 2-tree. Some properties of 2-trees can be summarized as follows.

Theorem 2.8 ([2, 3]) *Let G be a 2-tree with $n \geq 3$ vertices. Then*

(1) G has at least two ears,

(2) Every vertex of degree 2 in G is an ear,

(3) No two ears in G are adjacent unless $G = K_3$,

(4) Every edge of G can be a base.

We know that G is a 2-tree if either $G = K_3$, or G has an ear u such that $G' = G - u$ is a 2-tree. In other words, every 2-tree $G \neq K_3$ can be obtained from some 2-tree G' by adding a new vertex u adjacent to two vertices, v and w, where $vw \in E(G')$. We call this process *attaching* u to vw and denote $vw = e(u)$. For a 2-tree G, we denote $B(G)$ to be the set of all ears in G and $C(G) = \{e(u) | u \in B(G)\}$. For $xy \in C(G)$, we denote $B(xy) = \{u | u \in B(G) \text{ and } e(u) = xy\}$. Denote $T(k) = K_2 + \overline{K_{k-2}}$ (a star in 2-trees), where $\overline{K_{k-2}}$ is the complement of K_{k-2} and $+$ denotes 'join'. Clearly, $T(k)$ is a 2-tree with k vertices and $k-2$ ears, and every ear attaches to the edge of K_2. We also need the following lemmas.

Lemma 2.1 *Let G be a 2-tree on $k \geq 6$ vertices and $G \neq T(k)$. Then $|C(G)| \geq 2$.*

Proof: If $|C(G)| = 1$, let $C(G) = \{xy\}$, then u attaches to xy for each $u \in B(G)$. Let $G' = G \setminus B(G)$. Since $G \neq T(k)$, we have that $|V(G')| \geq 3$, G' is a 2-tree and each vertex of $V(G') \setminus \{x, y\}$ has degree at least 3 in G'. This implies that $G' \neq K_3$, and x and y are exactly two ears in G' by Theorem 2.8 (1). This is impossible by Theorem 2.8 (3). □

Lemma 2.2 *Let G be a 2-tree on $k \geq 6$ vertices. Let $xy \in C(G)$ so that xy is attached to as few ears as possible, and let s be the number of these ears. Denote $H = G \setminus (B(xy) \cup \{x, y\})$. Then H is a spanning subgraph of some 2-tree on $k - s - 2$ vertices.*

Proof: Clearly, Lemma 2.2 is trivial for $G = T(k)$. Assume $G \neq T(k)$. Let $G' = G \setminus B(xy)$, where $|B(xy)| = s$. Then G' is a 2-tree on $k - s$ vertices. If $s = 1$, then by $k \geq 6$, we have $k - s \geq 5$. If $s \geq 2$, then by $|C(G)| \geq 2$ (Lemma 2.1) and the minimality of s, we have $k - s \geq (s+1) + 2 \geq 5$. By Theorem 2.8 (4), G' can be constructed from xy by repeatedly adding a new vertex and making it adjacent to the two ends of an edge in the graph formed so far. In the process of constructing G' from xy, we let y' be the first vertex that is attached to xy. Since xy can not be attached to an ear in G', we have that $d_{G'}(y') \geq 3$. This implies that xy' or yy' must be attached to a new vertex. Let x' be the first vertex that is attached to xy' or yy'. Without loss of generality, we assume that x' is attached to xy'. Let

$\{x_1, \ldots, x_t\}$ be the subset of $V(G')$ so that x_i is attached to xx' for $i = 1, \ldots, t$ and $\{y_1, \ldots, y_{t'}\}$ be the subset of $V(G')$ so that y_j is attached to yy' for $j = 1, \ldots, t'$. Denote

$$G'' = G' - \{xx_1, \ldots, xx_t\} - \{yy_1, \ldots, yy_{t'}\} + \{y'x_1, \ldots, y'x_t\} + \{x'y_1, \ldots, x'y_{t'}\} - \{xy, xy'\}.$$

In G'', we first delete edges xx' and yy', and then identify the vertex x to the vertex x' and identify the vertex y to the vertex y', the resulting graph is denoted by G'''. Then G''' is a simple graph and is a 2-tree on $k - s - 2$ vertices. Moreover, $H = G \setminus (B(xy) \cup \{x, y\}) = G' \setminus \{x, y\}$ is a spanning subgraph of G'''. □

Lemma 2.3 *Let $k \geq 6$, $n \geq k$ and $\pi = (d_1, \ldots, d_n) \in GS_n$ with $\sigma(\pi) > \frac{4kn}{3} - \frac{5n}{3}$. Then $d_i \geq k - \lceil \frac{i}{2} \rceil$ for $i = 1, \ldots, \lceil \frac{2k}{3} \rceil$.*

Proof: If there is an even r with $2 \leq r \leq \lceil \frac{2k}{3} \rceil$ such that $d_r \leq k - \lceil \frac{r}{2} \rceil - 1 = k - \frac{r}{2} - 1$, then

$$\begin{array}{rcl} \sigma(\pi) & \leq & (r-1)(n-1) + (k - \frac{r}{2} - 1)(n - r + 1) \\ & = & \frac{r^2}{2} - r(k - \frac{n}{2} + \frac{1}{2}) + kn - 2n + k. \end{array}$$

Denote $f(r) = \frac{r^2}{2} - r(k - \frac{n}{2} + \frac{1}{2}) + kn - 2n + k$. Since $2 \leq r \leq \frac{2k+2}{3}$, we have that

$$\begin{array}{rcl} \sigma(\pi) & \leq & f(r) \leq \max\{f(2), f(\frac{2k+2}{3})\} \\ & = & \max\{\frac{4kn}{3} - \frac{5n}{3} - (\frac{(k-2)n}{3} + k - 1), \frac{4kn}{3} - \frac{5n}{3} - \frac{4(k^2-k)+1}{9}\} \\ & < & \frac{4kn}{3} - \frac{5n}{3}, \end{array}$$

a contradiction.

If there is an odd r with $1 \leq r \leq \lceil \frac{2k}{3} \rceil$ such that $d_r \leq k - \lceil \frac{r}{2} \rceil - 1 = k - \frac{r+1}{2} - 1$, then

$$\begin{array}{rcl} \sigma(\pi) & \leq & (r-1)(n-1) + (k - \frac{r+1}{2} - 1)(n - r + 1) \\ & = & \frac{r^2}{2} - r(k - \frac{n}{2}) + kn + k - \frac{5n}{2} - \frac{1}{2}. \end{array}$$

Denote $g(r) = \frac{r^2}{2} - r(k - \frac{n}{2}) + kn + k - \frac{5n}{2} - \frac{1}{2}$. Since $1 \leq r \leq \frac{2k+2}{3}$, we have that

$$\begin{array}{rcl} \sigma(\pi) & \leq & g(r) \leq \max\{g(1), g(\frac{2k+2}{3})\} \\ & = & \max\{\frac{4kn}{3} - \frac{5n}{3} - \frac{kn}{3} - \frac{n}{3}, \frac{4kn}{3} - \frac{13n}{6} - \frac{4k^2}{9} + \frac{7k}{9} - \frac{5}{18}\} \\ & < & \frac{4kn}{3} - \frac{5n}{3}, \end{array}$$

a contradiction. □

Lemma 2.4 *Let $k \geq 6$, $n \geq k$ and $\pi = (d_1, \ldots, d_n) \in GS_n$ with $\sigma(\pi) > \frac{4kn}{3} - \frac{5n}{3}$. Then $d_i \geq 2(k + 1 - i)$ for $i = \lceil \frac{2k}{3} \rceil + 1, \ldots, k$.*

Proof: If there is an r with $\lceil \frac{2k}{3} \rceil + 1 \leq r \leq k$ such that $d_r \leq 2k - 2r + 1$, then by Theorem 2.2,

$$\begin{array}{rcl} \sigma(\pi) & = & \sum\limits_{i=1}^{n} d_i = \sum\limits_{i=1}^{r-1} d_i + \sum\limits_{i=r}^{n} d_i \leq ((r-2)(r-1) + \sum\limits_{i=r}^{n} \min\{r-1, d_i\}) + \sum\limits_{i=r}^{n} d_i \\ & = & (r-2)(r-1) + 2\sum\limits_{i=r}^{n} d_i \leq (r-2)(r-1) + 2(2k - 2r + 1)(n - r + 1) \\ & = & 5r^2 - (4k + 4n + 9)r + 4kn + 4k + 2n + 4. \end{array}$$

Denote $f(r) = 5r^2 - (4k + 4n + 9)r + 4kn + 4k + 2n + 4$. Since $\frac{2k+3}{3} \leq r \leq k$, we have that

$$\begin{array}{rcl} \sigma(\pi) & \leq & f(r) \leq \max\{f(\frac{2k+3}{3}), f(k)\} \\ & = & \max\{\frac{4kn}{3} - 2n - \frac{4k^2}{9} + \frac{2k}{3}, k^2 - 5k + 2n + 4\} \\ & < & \max\{\frac{4kn}{3} - \frac{5n}{3} - [(\frac{2k}{3})^2 - \frac{2k}{3}] - \frac{n}{3}, \frac{4kn}{3} - \frac{5n}{3} - [(k-3)(\frac{4n}{3} - k) + \frac{n}{3} + 2k - 4]\} \\ & < & \frac{4kn}{3} - \frac{5n}{3}, \end{array}$$

a contradiction. $\square$

We now define a new graph $G(k)$ as follows: Let $V(K_{\lceil \frac{2k}{3} \rceil}) = \{v_1, v_2, \ldots, v_{\lceil \frac{2k}{3} \rceil}\}$, and $G(k)$ be the graph obtained from $K_{\lceil \frac{2k}{3} \rceil}$ by adding new vertices $x_1, x_2, \ldots, x_{\lfloor \frac{k}{3} \rfloor}$ and joining x_i to $v_1, v_2, \ldots, v_{2i}$ for $1 \leq i \leq \lfloor \frac{k}{3} \rfloor$. It is easy to see that $|V(G(k))| = k$.

Lemma 2.5 *If G is a 2-tree on k vertices, then $G(k)$ contains G as a subgraph.*

Proof: We use induction on k. It is easy to check that Lemma 2.5 holds for $k = 3, 4, 5$. If $G = T(k)$, then it is easy to see that $G(k)$ contains G as a subgraph. Assume that $k \geq 6$ and $G \neq T(k)$. Let $xy \in C(G)$ so that xy is attached to as few ears as possible, and let s be the number of these ears. Denote $H = G \setminus (B(xy) \cup \{x, y\})$. By Lemma 2.2, H is a spanning subgraph of some 2-tree G' on $k - s - 2$ vertices. Denote $m = k - s - 2$. We consider the following cases.

Case 1. $k \equiv 0(mod\,3)$ and $m \equiv 0(mod\,3)$.

Let
$$M = G(k) - \{v_1, \ldots, v_{\frac{2k-2m}{3}}\} - \{x_1, \ldots, x_{\frac{k-m}{3}}\}.$$

Then $M = G(m)$. By the induction hypothesis, $G(m)$ contains G' as a subgraph. This implies that $G(m)$ contains H as a subgraph. Putting x and y on v_1 and v_2 respectively and taking $B(xy) = \{v_3, \ldots, v_{\frac{2k-2m}{3}}, x_1, \ldots, x_{\frac{k-m}{3}}\}$, we can see that $G(k)$ contains G as a subgraph.

Case 2. $k \equiv 0(mod\,3)$ and $m \equiv 1(mod\,3)$.

Let
$$M = G(k) - \{v_1, \ldots, v_{\frac{2k-2m-4}{3}}, v_{\frac{2k}{3}}\} - \{x_1, \ldots, x_{\frac{k-m-2}{3}}, x_{\frac{k}{3}}\}.$$

Then $M = G(m)$. By the induction hypothesis, $G(m)$ contains G' as a subgraph, and hence contains H as a subgraph. Putting x and y on v_1 and v_2 respectively and taking

$$B(xy) = \{v_3, \ldots, v_{\frac{2k-2m-4}{3}}, v_{\frac{2k}{3}}, x_1, \ldots, x_{\frac{k-m-2}{3}}, x_{\frac{k}{3}}\},$$

we can see that $G(k)$ contains G as a subgraph.

Case 3. $k \equiv 0(mod\,3)$ and $m \equiv 2(mod\,3)$.

Let
$$M = G(k) - \{v_1, \ldots, v_{\frac{2k-2m-2}{3}}\} - \{x_1, \ldots, x_{\frac{k-m-1}{3}}, x_{\frac{k}{3}}\}.$$

Then $M = G(m)$. By the induction hypothesis, $G(m)$ contains H as a subgraph. Clearly, $G(k)$ contains G as a subgraph.

Case 4. $k \equiv 1(mod\,3)$ and $m \equiv 0(mod\,3)$.

Let
$$M = G(k) - \{v_1, \ldots, v_{\frac{2k-2m-2}{3}}, v_{\frac{2k+1}{3}}\} - \{x_1, \ldots, x_{\frac{k-m-1}{3}}\}.$$

Then $M = G(m)$. By the induction hypothesis, $G(m)$ contains H as a subgraph. Clearly, $G(k)$ contains G as a subgraph.

Case 5. $k \equiv 1(mod\, 3)$ and $m \equiv 1(mod\, 3)$.

Let
$$M = G(k) - \{v_1, \ldots, v_{\frac{2k-2m}{3}}\} - \{x_1, \ldots, x_{\frac{k-m}{3}}\}.$$
Then $M = G(m)$. By the induction hypothesis, $G(m)$ contains H as a subgraph. Clearly, $G(k)$ contains G as a subgraph.

Case 6. $k \equiv 1(mod\, 3)$ and $m \equiv 2(mod\, 3)$.

Let
$$M = G(k) - \{v_1, \ldots, v_{\frac{2k-2m-4}{3}}, v_{\frac{2k+1}{3}}\} - \{x_1, \ldots, x_{\frac{k-m-2}{3}}, x_{\frac{k-1}{3}}\}.$$
Then $M = G(m)$. By the induction hypothesis, $G(m)$ contains H as a subgraph. Clearly, $G(k)$ contains G as a subgraph.

Case 7. $k \equiv 2(mod\, 3)$ and $m \equiv 0(mod\, 3)$.

Let
$$M = G(k) - \{v_1, \ldots, v_{\frac{2k-2m-4}{3}}, v_{\frac{2k-1}{3}}, v_{\frac{2k+2}{3}}\} - \{x_1, \ldots, x_{\frac{k-m-2}{3}}\}.$$
Then $M = G(m)$. By the induction hypothesis, $G(m)$ contains H as a subgraph. Clearly, $G(k)$ contains G as a subgraph.

Case 8. $k \equiv 2(mod\, 3)$ and $m \equiv 1(mod\, 3)$.

Let
$$M = G(k) - \{v_1, \ldots, v_{\frac{2k-2m-2}{3}}, v_{\frac{2k+2}{3}}\} - \{x_1, \ldots, x_{\frac{k-m-1}{3}}\}.$$
Then $M = G(m)$. By the induction hypothesis, $G(m)$ contains H as a subgraph. Clearly, $G(k)$ contains G as a subgraph.

Case 9. $k \equiv 2(mod\, 3)$ and $m \equiv 2(mod\, 3)$.

Let
$$M = G(k) - \{v_1, \ldots, v_{\frac{2k-2m}{3}}\} - \{x_1, \ldots, x_{\frac{k-m}{3}}\}.$$
Then $M = G(m)$. By the induction hypothesis, $G(m)$ contains H as a subgraph. Clearly, $G(k)$ contains G as a subgraph. □

We now define sequence $\pi_0, \pi_1, \ldots, \pi_k$ as follows. Let $\pi_0 = \pi$. We define the sequence
$$\pi_1 = (d_2^{(1)}, \ldots, d_k^{(1)}, d_{k+1}^{(1)}, \ldots, d_n^{(1)})$$
from π_0 by deleting d_1, decreasing the first d_1 remaining nonzero terms each by one unity, and then reordering the last $n-k$ terms to be non-increasing. Note that the definition of the residual sequence obtained from π by laying off d_k is to reorder all the remaining terms to be non-increasing.

For $2 \leq i \leq k$, we define the sequence
$$\pi_i = (d_{i+1}^{(i)}, \ldots, d_k^{(i)}, d_{k+1}^{(i)}, \ldots, d_n^{(i)})$$
from
$$\pi_{i-1} = (d_i^{(i-1)}, \ldots, d_k^{(i-1)}, d_{k+1}^{(i-1)}, \ldots, d_n^{(i-1)})$$
by deleting $d_i^{(i-1)}$, decreasing the first $d_i^{(i-1)}$ remaining nonzero terms each by one unity, and then reordering the last $n-k$ terms to be non-increasing.

Lemma 2.6 *Let $k \geq 6$, $n \geq k$ and $\pi = (d_1, \ldots, d_{\lceil \frac{2k}{3} \rceil}, d_{\lceil \frac{2k}{3} \rceil + 1}, \ldots, d_k, d_{k+1}, \ldots, d_n) \in GS_n$ satisfy $d_i \geq k - \lceil \frac{i}{2} \rceil$ for $i = 1, \ldots, \lceil \frac{2k}{3} \rceil$. If π_k is graphic, then π has a realization containing $G(k)$ as a subgraph.*

Proof: Suppose that π_k is realized by graph G_k with vertex set $V(G_k) = \{v_{k+1}, \ldots, v_n\}$ such that $d_{G_k}(v_i) = d_i^{(k)}$ for $k+1 \leq i \leq n$. For $i = k, \ldots, 1$ in turn, form G_{i-1} from G_i by adding a new vertex v_i that is adjacent to the vertices of G_i whose degrees are reduced by one in going from π_{i-1} to π_i. Then, for each i, G_i has degrees given by π_i. In particular, G_0 has degrees given by π. Since π satisfies $d_i \geq k - \lceil \frac{i}{2} \rceil$ for $i = 1, \ldots, \lceil \frac{2k}{3} \rceil$, by the definition of π_i for $i = 1, \ldots, k$ in turn, we can see that $G_0[\{v_1, \ldots, v_k\}]$ contains $G(k)$ as a subgraph. $\square$

Lemma 2.7 *Let $k \geq 6$, $n \geq k$ and $\pi = (d_1, \ldots, d_{\lceil \frac{2k}{3} \rceil}, d_{\lceil \frac{2k}{3} \rceil + 1}, \ldots, d_k, d_{k+1}, \ldots, d_n) \in GS_n$. Let $\pi_1' = (d_1', \ldots, d_{n-1}')$ be the residual sequence obtained from π by laying off d_1 and $\rho = (\rho_1, \ldots, \rho_{n-2})$ be the residual sequence obtained from π_1' by laying off the term $d_2 - 1$. If π satisfies one of (a)–(c), where*

(a) $d_1 = d_2 = n - 1$,
(b) $d_1 = n - 1$, $d_2 \leq n - 2$ and $d_k > d_{d_2+2}$,
(c) $d_1 \leq n - 2$, $d_k > d_{d_2+2}$ and $d_k - d_{d_1+2} \geq 2$,

then $\rho_1 = d_3 - 2, \rho_2 = d_4 - 2, \ldots, \rho_{k-2} = d_k - 2$.

Proof: If π satisfies (a), then $\rho = (d_3 - 2, \ldots, d_n - 2)$, and so $\rho_1 = d_3 - 2, \rho_2 = d_4 - 2, \ldots, \rho_{k-2} = d_k - 2$.

If π satisfies (b), then $\pi_1' = (d_2 - 1, d_3 - 1, \ldots, d_n - 1)$. By $d_k - 2 \geq d_{d_2+2} - 1$, we further have that $\rho_1 = d_3 - 2, \rho_2 = d_4 - 2, \ldots, \rho_{k-2} = d_k - 2$.

Assume that π satisfies (c). If $d_{d_2+2} > d_{d_1+2}$, then $d_{d_2+2} - 1 \geq d_{d_1+2}$, and hence $d_1' = d_2 - 1, \ldots, d_{d_2+1}' = d_{d_2+2} - 1$. By $d_k > d_{d_2+2}$, we have $d_k - 2 \geq d_{d_2+2} - 1$, implying that $\rho_1 = d_3 - 2, \rho_2 = d_4 - 2, \ldots, \rho_{k-2} = d_k - 2$. If $d_{d_2+2} = \cdots = d_{d_1+2}$, then $d_{d_2+2} - 1 < d_{d_1+2}$. By $d_k - d_{d_1+2} \geq 2$, we have $d_1' = d_2 - 1, \ldots, d_{k-1}' = d_k - 1$ and $d_{d_2+1}' \leq d_{d_1+2}$, implying that $\rho_1 = d_3 - 2, \rho_2 = d_4 - 2, \ldots, \rho_{k-2} = d_k - 2$. $\square$

Lemma 2.8 *Let $k \geq 6$, $n \geq k$ and $\pi = (d_1, \ldots, d_{\lceil \frac{2k}{3} \rceil}, d_{\lceil \frac{2k}{3} \rceil + 1}, \ldots, d_k, d_{k+1}, \ldots, d_n) \in GS_n$. For each $\pi_i = (d_{i+1}^{(i)}, \ldots, d_k^{(i)}, d_{k+1}^{(i)}, \ldots, d_n^{(i)})$, let $t_i = \max\{j | d_{k+1}^{(i)} - d_{k+j}^{(i)} \leq 1\}$.*
(1) If π satisfies (d) or (e), where

(d) $d_1 \leq n - 2$, $d_k > d_{d_2+2}$ and $d_k - d_{d_1+2} \leq 1$,
(e) $d_1 \leq n - 2$, $d_k = d_{d_2+2}$ and $d_{d_2+2} = d_{d_1+2}$,

then $d_{k+r}^{(k)} = d_{k+r}$ for $r > t_k$.
(2) If π satisfies (f) or (g), where

(f) $d_1 = n - 1$, $d_2 \leq n - 2$ and $d_k = d_{d_2+2}$,
(g) $d_1 \leq n - 2$, $d_k = d_{d_2+2}$ and $d_{d_2+2} > d_{d_1+2}$,

then $d_{k+r}^{(k)} = d_{k+r}^{(1)}$ for $r > t_k$.

Proof: (1) If π satisfies (d) or (e), then $k + t_0 \geq d_1 + 2$. Since $d_{k+1}^{(i-1)} - d_{k+t_{i-1}}^{(i-1)} \leq 1$ implies that $d_{k+1}^{(i)} - d_{k+t_{i-1}}^{(i)} \leq 1$ for $1 \leq i \leq k$, we have that $t_k \geq t_{k-1} \geq \cdots \geq t_0 \geq d_1 + 2 - k$. By $\min\{d_{k+1}^{(i-1)} - 1, \ldots, d_{d_i+1}^{(i-1)} - 1, d_{d_i+2}^{(i-1)}, \ldots, d_{k+t_{i-1}}^{(i-1)}\} \geq d_{k+1}^{(i-1)} - 2 \geq d_{k+t_{i-1}+1}^{(i-1)} \geq \cdots \geq d_n^{(i-1)}$, we have that $d^{(i)}_{k+t_{i-1}+m} = d^{(i-1)}_{k+t_{i-1}+m}$ for $m \geq 1$. Thus, $d^{(i)}_{k+r} = d^{(i-1)}_{k+r}$ for $r > t_i$. This implies that $d^{(k)}_{k+r} = d_{k+r}$ for $r > t_k$.

(2) If π satisfies (f) or (g), then $t_k \geq t_{k-1} \geq \cdots \geq t_1 \geq t_0 \geq d_2 + 2 - k$. Since $\min\{d^{(i-1)}_{k+1} - 1, \ldots, d^{(i-1)}_{d_i+1} - 1, d^{(i-1)}_{d_i+2}, \ldots, d^{(i-1)}_{k+t_{i-1}}\} \geq d^{(i-1)}_{k+1} - 2 \geq d^{(i-1)}_{k+t_{i-1}+1} \geq \cdots \geq d^{(i-1)}_n$ for $i \geq 2$, we have that $d^{(i)}_{k+t_{i-1}+m} = d^{(i-1)}_{k+t_{i-1}+m}$ for $i \geq 2$ and $m \geq 1$. Thus, $d^{(i)}_{k+r} = d^{(i-1)}_{k+r}$ for $i \geq 2$ and $r > t_i$. This implies that $d^{(k)}_{k+r} = d^{(1)}_{k+r}$ for $r > t_k$. □

If $\pi = (d_1, \ldots, d_n) \in GS_n$ has a realization containing every 2-tree on k vertices as a subgraph, then π is *potentially $A'(k)$-graphic*. If π has a realization in which the subgraph induced by the k vertices of largest degrees contains every 2-tree on k vertices as a subgraph, then π is *potentially $A''(k)$-graphic*. It is easy to see that if π is potentially $A''(k)$-graphic, then π is potentially $A'(k)$-graphic.

Lemma 2.9 *Let $k \geq 3$, $n \geq 6k$ and $\pi = (d_1, \ldots, d_n) \in GS_n$ with $d_n \geq \frac{2k}{3} - 2$ and $\sigma(\pi) > \frac{4kn}{3} - \frac{5n}{3}$. Then π is potentially $A''(k)$-graphic.*

Proof: We use induction on k. If $k = 3$, then by $\sigma(\pi) > \frac{4kn}{3} - \frac{5n}{3} \geq 2n$ and Theorem 2.7 (1), π has a realization containing K_3. By Theorem 2.4, π is potentially $A''(3)$-graphic. If $k = 4$, then by $\sigma(\pi) > \frac{4kn}{3} - \frac{5n}{3} \geq 3n - 1$ and Theorem 2.7 (2), π has a realization containing $K_4 - e$. By Theorem 2.4, π is potentially $A''(4)$-graphic. If $k = 5$, then by $\sigma(\pi) > \frac{4kn}{3} - \frac{5n}{3} \geq 5n - 6$ and Theorem 2.7 (3), π has a realization containing $K_5 - e$. Since $K_5 - e$ contains every 2-tree on 5 vertices, by Theorem 2.4, π is potentially $A''(5)$-graphic. Assume $k \geq 6$. We only need to prove that $\pi = (d_1, \ldots, d_n)$ has a realization in which the subgraph induced by the vertices with degrees $d_1, \ldots, d_k$ contains every 2-tree on k vertices. Let $\pi'_1 = (d'_1, \ldots, d'_{n-1})$ be the residual sequence obtained from π by laying off d_1 and $\rho = (\rho_1, \ldots, \rho_{n-2})$ be the residual sequence obtained from π'_1 by laying off the term $d_2 - 1$. Then $n - 2 \geq 6(k-3)$, $\rho_{n-2} \geq (\frac{2k}{3} - 2) - 2 = \frac{2(k-3)}{3} - 2$ and $\sigma(\rho) = \sigma(\pi) - 2d_1 - 2d_2 + 2 > \frac{4kn}{3} - \frac{5n}{3} - 4(n-1) + 2 > \frac{4(k-3)(n-2)}{3} - \frac{5(n-2)}{3}$. By the induction hypothesis, ρ has a realization G_1 in which the subgraph induced by the vertices with degrees $\rho_1, \ldots, \rho_{k-3}$ contains every 2-tree on $k-3$ vertices. Denote F to be the subgraph induced by the vertices with degrees $\rho_1, \ldots, \rho_{k-3}$ in G_1, and let F' be the graph obtained from F by adding three new vertices x, y, u such that x, y are adjacent to each vertex of F and $xy, xu, yu \in E(F')$.

Claim *F' contains every 2-tree on k vertices.*

Proof of Claim. Let G be any one 2-tree on k vertices. Take $xy \in C(G)$ and $u \in B(xy)$, and denote $H = G \setminus \{x, y, u\}$. By Lemma 2.2, it is easy to get that H is a spanning subgraph of some 2-tree on $k-3$ vertices. Since F contains every 2-tree on $k-3$ vertices, we have that F contains H as a subgraph. By the definition of F', we can see that F' contains G as a subgraph. By the arbitrary of G, F' contains every 2-tree on k vertices. This proves Claim.

If π satisfies one of (a)–(c), by Lemma 2.7, then $\rho_1 = d_3 - 2, \rho_2 = d_4 - 2, \ldots, \rho_{k-2} = d_k - 2$. Now by the definitions of ρ and π'_1, it is easy to get that π has a realization G' in which the subgraph induced by the vertices with degrees $d_1, \ldots, d_k$ contains F' as a subgraph. Thus by Claim, π has a realization in which the subgraph induced by the vertices with degrees $d_1, \ldots, d_k$ contains every 2-tree on k vertices.

We now assume that π satisfies one of (d)–(g). If $d_k \ge 2k-3$, then by Theorem 2.5, π has a realization containing K_k, and hence π is potentially $A''(k)$-graphic by Theorem 2.4. Assume that $d_k \le 2k-4$. By $\sigma(\pi) > \frac{4kn}{3} - \frac{5n}{3}$ and Lemmas 2.3 and 2.4, we have that $d_i \ge k - \lceil \frac{i}{2} \rceil$ for $i = 1, \ldots, \lceil \frac{2k}{3} \rceil$ and $d_{\lceil \frac{2k}{3} \rceil + 1} \ge 2\lfloor \frac{k}{3} \rfloor$. It is enough to prove that π_k is graphic by Theorem 2.4 and Lemmas 2.5 and 2.6. If π satisfies (d) or (e), by Lemma 2.8 (1), then

$$\pi_k = (d_{k+1}^{(k)}, \ldots, d_{k+t_k}^{(k)}, d_{k+t_k+1}, \ldots, d_n).$$

If π satisfies (f) or (g), by Lemma 2.8 (2), then

$$\pi_k = (d_{k+1}^{(k)}, \ldots, d_{k+t_k}^{(k)}, d_{k+t_k+1}^{(1)}, \ldots, d_n^{(1)}).$$

If $t_k < n-k$, then $k + t_k < n$. By $d_{k+1}^{(k)} \le d_{k+1} \le d_k \le 2k-4$ and $d_n \ge d_n^{(1)} \ge d_n - 1 \ge \frac{2k}{3} - 3 \ge 1$, we have that $d_{k+1}^{(k)} \le 2k-4$ and $d_n \ge d_n^{(1)} \ge \lceil \frac{2k}{3} - 3 \rceil \ge 1$. Since $\frac{(2k-3+x)^2}{4x}$ is a monotone decreasing function of x on the interval $(0, 2k-3]$, by $\lceil \frac{2k}{3} - 3 \rceil \ge \frac{2k}{3} - 3$, we have that

$$\begin{aligned}
\frac{1}{\lceil \frac{2k}{3} - 3 \rceil} \left\lfloor \frac{(2k-4+\lceil \frac{2k}{3} - 3 \rceil + 1)^2}{4} \right\rfloor &\le \frac{(2k-3+\lceil \frac{2k}{3} - 3 \rceil)^2}{4\lceil \frac{2k}{3} - 3 \rceil} \\
&\le \frac{(2k-3+\frac{2k}{3}-3)^2}{4(\frac{2k}{3}-3)} \\
&= \frac{\frac{16k^2}{3} - 24k + 27}{2k-9} \\
&= \frac{\frac{8}{3}k(2k-9)+27}{2k-9} \\
&\le \frac{8k}{3} + 9 \le n-k.
\end{aligned}$$

By Theorem 2.3, π_k is graphic. If $t_k = n-k$, then $d_{k+1}^{(k)} - d_n^{(k)} \le 1$. Denote $d_n^{(k)} = m$. If $m = 0$, then by $d_{k+1}^{(k)} \le 1$ and $\sigma(\pi_k)$ being even, π_k is clearly graphic. If $m \ge 1$, then $d_{k+1}^{(k)} \le m+1$, and hence

$$\frac{1}{m} \left\lfloor \frac{(m+1+m+1)^2}{4} \right\rfloor = \frac{(m+1)^2}{m} \le m+3 \le 2k-4+3 \le n-k.$$

By Theorem 2.3, π_k is also graphic. □

Lemma 2.10 *Let $k \ge 6$, $n = 6k$ and $\pi = (d_1, \ldots, d_n) \in GS_n$ with $\sigma(\pi) > \frac{4kn}{3} - \frac{5n}{3} + 4k^2 - 14k$. Then π is potentially $A''(k)$-graphic.*

Proof: By $\sigma(\pi) \ge \frac{4kn}{3} - \frac{5n}{3} + 4k^2 - 14k + 2 = 2n(k-2) + 2$ and Theorem 2.6, π has a realization containing K_k. By Theorem 2.4, π is potentially $A''(k)$-graphic. □

Lemma 2.11 *Let $k \ge 6$ and $n = 6k + t$, where $0 \le t \le 2k^2 - 7k$. If $\pi = (d_1, \ldots, d_n) \in GS_n$ with $\sigma(\pi) > \frac{4kn}{3} - \frac{5n}{3} + 4k^2 - 14k - 2t$, then π is potentially $A'(k)$-graphic.*

Proof: We use induction on t. It is known from Lemma 2.10 that Lemma 2.11 holds for $t = 0$. Suppose now that $1 \le t \le 2k^2 - 7k$. Then $\sigma(\pi) > \frac{4kn}{3} - \frac{5n}{3}$. If $d_n \ge \frac{2k}{3} - 2$, then π is potentially $A''(k)$-graphic by Lemma 2.9. If $d_n < \frac{2k}{3} - 2$, then the residual sequence $\pi_n' = (d_1', \ldots, d_{n-1}')$ obtained by laying off d_n from π satisfies $\sigma(\pi'_n) = \sigma(\pi) - 2d_n > \frac{4kn}{3} - \frac{5n}{3} + 4k^2 - 14k - 2t - 2(\frac{2k}{3} - 2) > \frac{4k(n-1)}{3} - \frac{5(n-1)}{3} + 4k^2 - 14k - 2(t-1)$. By the induction hypothesis, π'_n is potentially $A'(k)$-graphic, and hence so is π. □

We now prove Theorem 1.2.

Proof of Theorem 1.2: Let $k \geq 3$, $n \geq 2k^2 - k$ and $\pi = (d_1, \ldots, d_n) \in GS_n$ with $\sigma(\pi) > \frac{4kn}{3} - \frac{5n}{3}$. We only need to prove that π is potentially $A'(k)$-graphic. If $k = 3$, then by $\sigma(\pi) > \frac{4kn}{3} - \frac{5n}{3} \geq 2n$ and Theorem 2.7 (1), π has a realization containing K_3, and hence π is potentially $A'(3)$-graphic. If $k = 4$, then by $\sigma(\pi) > \frac{4kn}{3} - \frac{5n}{3} \geq 3n - 1$ and Theorem 2.7 (2), π has a realization containing $K_4 - e$, and hence π is potentially $A'(4)$-graphic. If $k = 5$, then by $\sigma(\pi) > \frac{4kn}{3} - \frac{5n}{3} \geq 5n - 6$ and Theorem 2.7 (3), π has a realization containing $K_5 - e$. Since $K_5 - e$ contains every 2-tree on 5 vertices, π is potentially $A'(5)$-graphic. Assume that $k \geq 6$. We now use induction on n. If $n = 2k^2 - k$, then by Lemma 2.11 ($t = 2k^2 - 7k$), π is potentially $A'(k)$-graphic. Assume that $n \geq 2k^2 - k + 1$. If $d_n \geq \frac{2k}{3} - 2$, then by Lemma 2.9, π is potentially $A'(k)$-graphic. If $d_n < \frac{2k}{3} - 2$, then the residual sequence $\pi'_n = (d'_1, \ldots, d'_{n-1})$ obtained from π by laying off d_n satisfies $\sigma(\pi'_n) = \sigma(\pi) - 2d_n > \frac{4kn}{3} - \frac{5n}{3} - 2(\frac{2k}{3} - 2) > \frac{4k(n-1)}{3} - \frac{5(n-1)}{3}$. By the induction hypothesis, π'_n is potentially $A'(k)$-graphic, and hence so is π. □

3 Proof of Theorem 1.3

In order to prove Theorem 1.3, we recursively define a new graph $F(k)$ on $k \geq 3$ vertices as follows. Let $F(3) = K_3$, and let $V(F(k-1)) = \{x_1, \ldots, x_{k-1}\}$ for $k \geq 4$. Define $F(k)$ be the graph obtained from $F(k-1)$ by adding a new vertex x_k and joining x_k to x_{k-2}, x_{k-1}. Clearly, $F(k)$ is a 2-tree on k vertices. Let $\alpha(G)$ denote the independence number of G. We need the following Lemma 3.1.

Lemma 3.1 *Let $k \geq 3$ and $e \in E(F(k))$. Then*
(1) $\alpha(F(k)) \leq \lceil \frac{k}{3} \rceil$;
(2) If $k \equiv 1 (mod\, 3)$, then $\alpha(F(k) - e) \leq \lceil \frac{k}{3} \rceil$.

Proof: (1) We use induction on k. It is easy to check that Lemma 3.1(1) holds for $k = 3, 4, 5$. Assume that $k \geq 6$. Let $V(F(k)) = \{x_1, \ldots, x_k\}$. By the construction of $F(k)$, we have that the subgraph induced by $\{x_{k-2}, x_{k-1}, x_k\}$ in $F(k)$ is K_3. Let X be a maximum independent set of $F(k)$. Then $|\{x_{k-2}, x_{k-1}, x_k\} \cap X| \leq 1$. If $|\{x_{k-2}, x_{k-1}, x_k\} \cap X| = 0$, then X is an independent set of $F(k) - \{x_{k-2}, x_{k-1}, x_k\} = F(k-3)$. By the induction hypothesis, we have that $\alpha(F(k)) = |X| \leq \alpha(F(k-3)) \leq \lceil \frac{k-3}{3} \rceil \leq \lceil \frac{k}{3} \rceil$. If $|\{x_{k-2}, x_{k-1}, x_k\} \cap X| = 1$, let $\{x_{k-2}, x_{k-1}, x_k\} \cap X = \{x\}$, then $X \setminus \{x\}$ is an independent set of $F(k) - \{x_{k-2}, x_{k-1}, x_k\} = F(k-3)$. By the induction hypothesis, we have that $\alpha(F(k)) - 1 = |X \setminus \{x\}| \leq \alpha(F(k-3)) \leq \lceil \frac{k-3}{3} \rceil$, i.e., $\alpha(F(k)) \leq \lceil \frac{k-3}{3} \rceil + 1 = \lceil \frac{k}{3} \rceil$.

(2) Clearly, Lemma 3.1(2) holds for $k = 4$. Assume that $k \geq 7$. By the construction of $F(k)$, we have that $e = x_i x_{i+1}$ for $1 \leq i \leq k-1$ or $e = x_j x_{j+2}$ for $1 \leq j \leq k-2$. Let X be a maximum independent set of $F(k) - e$.

Firstly, we assume that $e = x_i x_{i+1}$ for $1 \leq i \leq k-1$. If $|\{x_i, x_{i+1}\} \cap X| \leq 1$, then X is an independent set of $F(k)$, and hence $\alpha(F(k) - e) = |X| \leq \alpha(F(k)) \leq \lceil \frac{k}{3} \rceil$. Assume that $|\{x_i, x_{i+1}\} \cap X| = 2$, i.e., $\{x_i, x_{i+1}\} \subseteq X$. If $i = 1$ (or $i = k-1$), then $X \setminus \{x_1, x_2\}$ (or $X \setminus \{x_{k-1}, x_k\}$) is an independent set of

$F(k)-\{x_1,x_2,x_3,x_4\}=F(k-4)$ (or $F(k)-\{x_k,x_{k-1},x_{k-2},x_{k-3}\}=F(k-4)$). This implies that $\alpha(F(k)-e)-2=|X|-2\leq\alpha(F(k-4))\leq\lceil\frac{k-4}{3}\rceil$, i.e., $\alpha(F(k)-e)\leq\lceil\frac{k-4}{3}\rceil+2=\lceil\frac{k+2}{3}\rceil=\lceil\frac{k}{3}\rceil$ (as $k\equiv 1(mod\,3)$). If $i=2$ (or $i=k-2$), then $X\setminus\{x_2,x_3\}$ (or $X\setminus\{x_{k-2},x_{k-1}\}$) is an independent set of $F(k)-\{x_1,x_2,x_3,x_4,x_5\}=F(k-5)$ (or $F(k)-\{x_k,x_{k-1},x_{k-2},x_{k-3},x_{k-4}\}=F(k-5)$). This implies that $\alpha(F(k)-e)-2=|X|-2\leq\alpha(F(k-5))\leq\lceil\frac{k-5}{3}\rceil$, i.e., $\alpha(F(k)-e)\leq\lceil\frac{k-5}{3}\rceil+2=\lceil\frac{k+1}{3}\rceil=\lceil\frac{k}{3}\rceil$ (as $k\equiv 1(mod\,3)$). If $3\leq i\leq k-3$, then $X\setminus\{x_i,x_{i+1}\}$ is an independent set of $F(k)-\{x_{i-2},x_{i-1},x_i,x_{i+1},x_{i+2},x_{i+3}\}$. For convenience, we denote $F(i)=K_i$ for $i=1,2$. Clearly, $\alpha(F(i))\leq\lceil\frac{i}{3}\rceil$ for $i=1,2$. Since $F(k)-\{x_{i-2},x_{i-1},x_i,x_{i+1},x_{i+2},x_{i+3}\}$ is the disjoint union of $F(i-3)$ and $F(k-i-3)$, we have that $\alpha(F(k)-e)-2=|X|-2\leq\alpha(F(k)-\{x_{i-2},x_{i-1},x_i,x_{i+1},x_{i+2},x_{i+3}\})=\alpha(F(i-3))+\alpha(F(k-i-3))\leq\lceil\frac{i-3}{3}\rceil+\lceil\frac{k-i-3}{3}\rceil=\lceil\frac{i}{3}\rceil+\lceil\frac{k-i}{3}\rceil-2$. Hence $\alpha(F(k)-e)\leq\lceil\frac{i}{3}\rceil+\lceil\frac{k-i}{3}\rceil=\frac{k+2}{3}=\lceil\frac{k}{3}\rceil$ (as $k\equiv 1(mod\,3)$).

We now assume that $e=x_jx_{j+2}$ for $1\leq j\leq k-2$. If $|\{x_j,x_{j+2}\}\cap X|\leq 1$, then X is an independent set of $F(k)$, and hence $\alpha(F(k)-e)=|X|\leq\alpha(F(k))\leq\lceil\frac{k}{3}\rceil$. Assume that $|\{x_j,x_{j+2}\}\cap X|=2$, i.e., $\{x_j,x_{j+2}\}\subseteq X$. If $j=1$ (or $j=k-2$), then $X\setminus\{x_1,x_3\}$ (or $X\setminus\{x_{k-2},x_k\}$) is an independent set of $F(k)-\{x_1,x_2,x_3,x_4,x_5\}=F(k-5)$ (or $F(k)-\{x_k,x_{k-1},x_{k-2},x_{k-3},x_{k-4}\}=F(k-5)$). This implies that $\alpha(F(k)-e)-2=|X|-2\leq\alpha(F(k-5))\leq\lceil\frac{k-5}{3}\rceil$, i.e., $\alpha(F(k)-e)\leq\lceil\frac{k-5}{3}\rceil+2=\lceil\frac{k+1}{3}\rceil=\lceil\frac{k}{3}\rceil$ (as $k\equiv 1(mod\,3)$). If $j=2$ (or $j=k-3$), then $X\setminus\{x_2,x_4\}$ (or $X\setminus\{x_{k-3},x_{k-1}\}$) is an independent set of $F(k)-\{x_1,x_2,x_3,x_4,x_5,x_6\}=F(k-6)$ (or $F(k)-\{x_k,x_{k-1},x_{k-2},x_{k-3},x_{k-4},x_{k-5}\}=F(k-6)$). This implies that $\alpha(F(k)-e)-2=|X|-2\leq\alpha(F(k-6))\leq\lceil\frac{k-6}{3}\rceil$, i.e., $\alpha(F(k)-e)\leq\lceil\frac{k-6}{3}\rceil+2=\lceil\frac{k}{3}\rceil$. If $3\leq j\leq k-4$, then $X-\{x_j,x_{j+2}\}$ is an independent set of $F(k)-\{x_{j-2},x_{j-1},x_j,x_{j+1},x_{j+2},x_{j+3},x_{j+4}\}$. Since $F(k)-\{x_{j-2},x_{j-1},x_j,x_{j+1},x_{j+2},x_{j+3},x_{j+4}\}$ is the disjoint union of $F(j-3)$ and $F(k-j-4)$, we have that $\alpha(F(k)-e)-2=|X|-2\leq\alpha(F(k)-\{x_{j-2},x_{j-1},x_j,x_{j+1},x_{j+2},x_{j+3},x_{j+4}\})=\alpha(F(j-3))+\alpha(F(k-j-4))\leq\lceil\frac{j-3}{3}\rceil+\lceil\frac{k-j-4}{3}\rceil=\lceil\frac{j}{3}\rceil+\lceil\frac{k-j-1}{3}\rceil-2$. Hence $\alpha(F(k)-e)\leq\lceil\frac{j}{3}\rceil+\lceil\frac{k-j-1}{3}\rceil\leq\lceil\frac{k}{3}\rceil$ (as $k\equiv 1(mod\,3)$). □

Proof of Theorem 1.3: Let $k\geq 3$ with $k\equiv i(\mathrm{mod}\,3)$. Denote $H=K_{\lfloor\frac{2k}{3}\rfloor-1}+\overline{K_{n-\lfloor\frac{2k}{3}\rfloor+1}}$. If H contains $F(k)$ on the vertices $u_1,\ldots,u_k$, then $k-(\lfloor\frac{2k}{3}\rfloor-1)\leq\alpha(H[\{u_1,\ldots,u_k\}])\leq\alpha(F(k))\leq\lceil\frac{k}{3}\rceil$ (Lemma 3.1(1)). This is impossible as $k-(\lfloor\frac{2k}{3}\rfloor-1)=\lceil\frac{k}{3}\rceil+1$. Hence H contains no $F(k)$.

For $i=0$ or 2, we let $\pi=((n-1)^{\lfloor\frac{2k}{3}\rfloor-1},(\lfloor\frac{2k}{3}\rfloor-1)^{n-\lfloor\frac{2k}{3}\rfloor+1})$. Then $\pi\in GS_n$, $\sigma(\pi)=2\lfloor\frac{2k}{3}\rfloor n-2n-\lfloor\frac{2k}{3}\rfloor^2+\lfloor\frac{2k}{3}\rfloor$ and H is the unique realization of π. Since H contains no $F(k)$, we have that π has no realization containing $F(k)$. This implies that π has no realization containing every 2-tree on k vertices.

For $i=1$, we let $\pi=((n-1)^{\lfloor\frac{2k}{3}\rfloor-1},(\lfloor\frac{2k}{3}\rfloor)^2,(\lfloor\frac{2k}{3}\rfloor-1)^{n-\lfloor\frac{2k}{3}\rfloor-1})$. Then $\pi\in GS_n$, $\sigma(\pi)=2\lfloor\frac{2k}{3}\rfloor n-2n-\lfloor\frac{2k}{3}\rfloor^2+\lfloor\frac{2k}{3}\rfloor+2$ and $H+e$ (a simple graph is obtained from H by adding an edge e) is the unique realization of π. Assume that $H+e$ contains $F(k)$. Since H contains no $F(k)$, we have that H contains $F(k)-e$. If H contains $F(k)-e$ on the vertices $u_1,\ldots,u_k$, then $k-(\lfloor\frac{2k}{3}\rfloor-1)\leq\alpha(H[\{u_1,\ldots,u_k\}])\leq\alpha(F(k)-e)\leq\lceil\frac{k}{3}\rceil$ (Lemma 3.1(2)), a contradiction. Hence π has no realization containing $F(k)$. This proves Theorem 1.3. □

Since

$$\lim_{n\to+\infty}\frac{\frac{4kn}{3}-\frac{5n}{3}}{2\lfloor\frac{2k}{3}\rfloor n-2n-\lfloor\frac{2k}{3}\rfloor^2+\lfloor\frac{2k}{3}\rfloor+1-(-1)^i}=\frac{\frac{4k}{3}-\frac{5}{3}}{2\lfloor\frac{2k}{3}\rfloor-2}\approx 1,$$

we have that $\frac{4kn}{3}-\frac{5n}{3}$ is almost the best possible lower bound in Theorem 1.2.

For $k \equiv i(\mathrm{mod}\ 3)$, we feel that $2\lfloor\frac{2k}{3}\rfloor n - 2n - \lfloor\frac{2k}{3}\rfloor^2 + \lfloor\frac{2k}{3}\rfloor + 1 - (-1)^i$ is the best possible lower bound for sufficiently large n, thus we propose the following conjecture.

Conjecture *If $k \geq 3$ with $k \equiv i(mod\ 3)$, n is sufficiently large, and $\pi \in GS_n$ with $\sigma(\pi) > 2\lfloor\frac{2k}{3}\rfloor n - 2n - \lfloor\frac{2k}{3}\rfloor^2 + \lfloor\frac{2k}{3}\rfloor + 1 - (-1)^i$, then π has a realization H containing every 2-tree on k vertices. Moreover, the lower bound $2\lfloor\frac{2k}{3}\rfloor n - 2n - \lfloor\frac{2k}{3}\rfloor^2 + \lfloor\frac{2k}{3}\rfloor + 1 - (-1)^i$ is the best possible.*

Acknowledgements

This work is supported by National Natural Science Foundation of China (No. 11561017) and Natural Science Foundation of Hainan Province (No. 2016CXTD004). We also thank the referees for their helpful suggestions and comments.

References

[1] J.A. Bondy and U.S.R. Murty. Graph Theory with Applications. The Macmillan Press, London, 1976.

[2] P. Bose, V. Dujmovic, D. Krizanc, S. Langerman, P. Morin, D.R. Wood and S. Wuhrer. A characterization of the degree sequences of 2-trees. J. Graph Theory, 58: 191–209, 2008.

[3] L.Z. Cai. On spanning 2-trees in a graph. Discrete Appl. Math., 74: 203-216, 1997.

[4] P. Erdős and T. Gallai. Graphs with prescribed degrees of vertices (Hungarian). Mat. Lapok, 11: 264–274, 1960.

[5] P. Erdős, M.S. Jacobson and J. Lehel. Graphs realizing the same degree sequences and their respective clique numbers, in: Y. Alavi et al., (Eds.). Graph Theory, Combinatorics and Applications, John Wiley & Sons, New York, Vol.1: 439–449, 1991.

[6] R.J. Gould, M.S. Jacobson and J. Lehel. Potentially G-graphical degree sequences, in: Y. Alavi et al., (Eds.). Combinatorics, Graph Theory, and Algorithms, New Issues Press, Kalamazoo Michigan, Vol.1: 451–460, 1999.

[7] D.J. Kleitman and D.L. Wang. Algorithm for constructing graphs and digraphs with given valences and factors. Discrete Math., 6: 79–88, 1973.

[8] C.H. Lai. A note on potentially $K_4 - e$ graphical sequences. Australas. J. Combin., 24: 123–127, 2001.

[9] J.S. Li and Z.X. Song. On the potentially P_k-graphic sequence. Discrete Math., 195: 255–262, 1999.

[10] J.H. Yin and J.S. Li. Two sufficient conditions for a graphic sequence to have a realization with prescribed clique size. Discrete Math., 301: 218–227, 2005.

[11] J.H. Yin and J.S. Li. An extremal problem on potentially $K_{r,s}$-graphic sequences. Discrete Math., 260: 295–305, 2003.

[12] J.H. Yin and J.S. Li. A variation of a conjecture due to Erdős and Sós. Acta Math. Sin. Engl. Ser., 25: 795–802, 2009.

[13] J.H. Yin, J.S. Li and R. Mao. An extremal problem on the potentially $K_{r+1}-e$-graphic sequences. Ars Combin., 74: 151–159, 2005.