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An extremal problem for a graphic sequence to have a realization containing every 2-tree with prescribed size

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A graph G is a 2-tree if $G = K_3$, or G has a vertex v of degree 2, whose neighbors are adjacent, and $G - v$ is a 2-tree. Clearly, if G is a 2-tree on n vertices, then $|E(G)| = 2n - 3$. A non-increasing sequence $\pi = (d_1, \dots, d_n)$ of nonnegative integers is a *graphic sequence* if it is realizable by a simple graph G on n vertices. Yin and Li (Acta Mathematica Sinica, English Series, 25(2009)795–802) proved that if $k \geq 2$, $n \geq \frac{9}{2}k^2 + \frac{19}{2}k$ and $\pi = (d_1, \dots, d_n)$ is a graphic sequence with $\sum_{i=1}^n d_i > (k - 2)n$, then π has a realization containing every tree on k vertices as a subgraph. Moreover, the lower bound $(k - 2)n$ is the best possible. This is a variation of a conjecture due to Erdős and Sós. In this paper, we investigate an analogue extremal problem for 2-trees and prove that if $k \geq 3$, $n \geq 2k^2 - k$ and $\pi = (d_1, \dots, d_n)$ is a graphic sequence with $\sum_{i=1}^n d_i > \frac{4kn}{3} - \frac{5n}{3}$, then π has a realization containing every 2-tree on k vertices as a subgraph. We also show that the lower bound $\frac{4kn}{3} - \frac{5n}{3}$ is almost the best possible.

Keywords: degree sequences; graphic sequences; realization; 2-trees.

1 Introduction

Let K_m be the complete graph on m vertices. A graph G is a 2-tree if $G = K_3$, or G has a vertex v of degree 2, whose neighbors are adjacent, and $G - v$ is a 2-tree. It is easy to see that if G is a 2-tree on n vertices, then $|E(G)| = 2n - 3$. An *ear* in a 2-tree is a vertex of degree 2 whose neighbors are adjacent.

The set of all non-increasing sequences $\pi = (d_1, \dots, d_n)$ of nonnegative integers with $d_1 \leq n - 1$ is denoted by NS_n . A sequence $\pi \in NS_n$ is said to be *graphic* if it is the degree sequence of a simple graph G on n vertices, and such a graph G is called a *realization* of π . The set of all graphic sequences in NS_n is denoted by GS_n . For a nonnegative integer sequence $\pi = (d_1, \dots, d_n)$, we denote $\sigma(\pi) = d_1 + \dots + d_n$. Yin and Li [12] investigated a variation of a conjecture due to Erdős and Sós (see [1], Problem 12 in page 247), that is, an extremal problem for a sequence $\pi \in GS_n$ to have a realization containing every tree on k vertices as a subgraph, and obtained the following Theorem 1.1.

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Theorem 1.1 ([12]) If $k \geq 2$, $n \geq \frac{9}{2}k^2 + \frac{19}{2}k$ and $\pi = (d_1, \dots, d_n) \in GS_n$ with $\sigma(\pi) > (k-2)n$, then π has a realization H containing every tree on k vertices as a subgraph. Moreover, the lower bound $(k-2)n$ is the best possible.

This kind of extremal problem was firstly introduced by Erdős et al. (see [5–6]). The purpose of this paper is to investigate an analogous extremal problem for a sequence $\pi \in GS_n$ to have a realization containing every 2-tree on k vertices as a subgraph. We establish the following Theorems 1.2 and 1.3.

Theorem 1.2 If $k \geq 3$, $n \geq 2k^2 - k$ and $\pi = (d_1, \dots, d_n) \in GS_n$ with $\sigma(\pi) > \frac{4kn}{3} - \frac{5n}{3}$, then π has a realization H containing every 2-tree on k vertices as a subgraph.

The lower bound $\frac{4kn}{3} - \frac{5n}{3}$ in Theorem 1.2 is almost the best possible.

Theorem 1.3 For $k \equiv i \pmod{3}$, there exists a sequence $\pi \in GS_n$ with $\sigma(\pi) = 2\lfloor \frac{2k}{3} \rfloor n - 2n - \lfloor \frac{2k}{3} \rfloor^2 + \lfloor \frac{2k}{3} \rfloor + 1 - (-1)^i$ such that π has no realization containing every 2-tree on k vertices.

2 Proof of Theorem 1.2

In order to prove Theorem 1.2, we need some known results. Let $\pi = (d_1, \dots, d_n) \in NS_n$ and k be an integer with $1 \leq k \leq n$. Let

$$\pi_k'' = \begin{cases} (d_1 - 1, \dots, d_{k-1} - 1, d_{k+1} - 1, \dots, d_{d_k+1} - 1, d_{d_k+2}, \dots, d_n), & \text{if } d_k \geq k, \\ (d_1 - 1, \dots, d_{d_k} - 1, d_{d_k+1}, \dots, d_{k-1}, d_{k+1}, \dots, d_n), & \text{if } d_k < k. \end{cases}$$

Let $\pi_k' = (d_1', \dots, d_{n-1}')$, where $d_1' \geq \dots \geq d_{n-1}'$ is a rearrangement in non-increasing order of the $n-1$ terms of π_k'' . We say that π_k' is the *residual sequence* obtained from π by laying off d_k . It is easy to see that if π_k' is graphic then so is π , since a realization G of π can be obtained from a realization G' of π_k' by adding a new vertex of degree d_k and joining it to the vertices whose degrees are reduced by one in going from π to π_k' . In fact, more is true:

Theorem 2.1 ([7]) $\pi \in GS_n$ if and only if $\pi_k' \in GS_{n-1}$.

Theorem 2.2 ([4]) Let $\pi = (d_1, \dots, d_n) \in NS_n$, where $\sigma(\pi)$ is even. Then $\pi \in GS_n$ if and only if $\sum_{i=1}^t d_i \leq t(t-1) + \sum_{i=t+1}^n \min\{t, d_i\}$ for any t with $1 \leq t \leq n-1$.

Theorem 2.3 ([11]) Let $\pi = (d_1, \dots, d_n) \in NS_n$, where $d_1 = m$ and $\sigma(\pi)$ is even. If there exist an integer $n_1 \leq n$ and some integer $h \geq 1$ such that $d_{n_1} \geq h$ and $n_1 \geq \frac{1}{h} \lfloor \frac{(m+h+1)^2}{4} \rfloor$, then $\pi \in GS_n$.

Theorem 2.4 ([6]) If $\pi = (d_1, \dots, d_n) \in NS_n$ has a realization G containing H as a subgraph, then there exists a realization G' of π containing H as a subgraph so that the vertices of H have the largest degrees of π .

Theorem 2.5 ([10]) Let $n \geq r$ and $\pi = (d_1, \dots, d_n) \in GS_n$ with $d_r \geq r-1$. If $d_i \geq 2r-2-i$ for $i = 1, \dots, r-2$, then π has a realization containing K_r .

Theorem 2.6 ([9]) If $r \geq 1$, $n \geq 2r-1$ and $\pi = (d_1, \dots, d_n) \in GS_n$ with $\sigma(\pi) \geq 2n(r-2) + 2$, then π has a realization containing K_r .

Theorem 2.7 Let $\pi = (d_1, \dots, d_n) \in GS_n$.

- (1) [5] If $n \geq 6$ and $\sigma(\pi) \geq 2n$, then π has a realization containing K_3 .
- (2) [8] If $n \geq 7$ and $\sigma(\pi) \geq 3n - 1$, then π has a realization containing $K_4 - e$, where $K_4 - e$ is the graph obtained from K_4 by removing one edge.
- (3) [13] If $n \geq 9$ and $\sigma(\pi) \geq 5n - 6$, then π has a realization containing $K_5 - e$, where $K_5 - e$ is the graph obtained from K_5 by removing one edge.

We note that a 2-tree can be constructed from an edge by repeatedly adding a new vertex and making it adjacent to the two ends of an edge in the graph formed so far. We refer to the initial edge in constructing such a 2-tree as a *base* of the 2-tree. Some properties of 2-trees can be summarized as follows.

Theorem 2.8 ([2, 3]) Let G be a 2-tree with $n \geq 3$ vertices. Then

- (1) G has at least two ears,
- (2) Every vertex of degree 2 in G is an ear,
- (3) No two ears in G are adjacent unless $G = K_3$,
- (4) Every edge of G can be a base.

We know that G is a 2-tree if either $G = K_3$, or G has an ear u such that $G' = G - u$ is a 2-tree. In other words, every 2-tree $G \neq K_3$ can be obtained from some 2-tree G' by adding a new vertex u adjacent to two vertices, v and w , where $vw \in E(G')$. We call this process *attaching* u to vw and denote $vw = e(u)$. For a 2-tree G , we denote $B(G)$ to be the set of all ears in G and $C(G) = \{e(u) | u \in B(G)\}$. For $xy \in C(G)$, we denote $B(xy) = \{u | u \in B(G) \text{ and } e(u) = xy\}$. Denote $T(k) = K_2 + \overline{K_{k-2}}$ (a star in 2-trees), where $\overline{K_{k-2}}$ is the complement of K_{k-2} and $+$ denotes 'join'. Clearly, $T(k)$ is a 2-tree with k vertices and $k - 2$ ears, and every ear attaches to the edge of K_2 . We also need the following lemmas.

Lemma 2.1 Let G be a 2-tree on $k \geq 6$ vertices and $G \neq T(k)$. Then $|C(G)| \geq 2$.

Proof: If $|C(G)| = 1$, let $C(G) = \{xy\}$, then u attaches to xy for each $u \in B(G)$. Let $G' = G \setminus B(G)$. Since $G \neq T(k)$, we have that $|V(G')| \geq 3$, G' is a 2-tree and each vertex of $V(G') \setminus \{x, y\}$ has degree at least 3 in G' . This implies that $G' \neq K_3$, and x and y are exactly two ears in G' by Theorem 2.8 (1). This is impossible by Theorem 2.8 (3). \square

Lemma 2.2 Let G be a 2-tree on $k \geq 6$ vertices. Let $xy \in C(G)$ so that xy is attached to as few ears as possible, and let s be the number of these ears. Denote $H = G \setminus (B(xy) \cup \{x, y\})$. Then H is a spanning subgraph of some 2-tree on $k - s - 2$ vertices.

Proof: Clearly, Lemma 2.2 is trivial for $G = T(k)$. Assume $G \neq T(k)$. Let $G' = G \setminus B(xy)$, where $|B(xy)| = s$. Then G' is a 2-tree on $k - s$ vertices. If $s = 1$, then by $k \geq 6$, we have $k - s \geq 5$. If $s \geq 2$, then by $|C(G)| \geq 2$ (Lemma 2.1) and the minimality of s , we have $k - s \geq (s + 1) + 2 \geq 5$. By Theorem 2.8 (4), G' can be constructed from xy by repeatedly adding a new vertex and making it adjacent to the two ends of an edge in the graph formed so far. In the process of constructing G' from xy , we let y' be the first vertex that is attached to xy . Since xy can not be attached to an ear in G' , we have that $d_{G'}(y') \geq 3$. This implies that xy' or yy' must be attached to a new vertex. Let x' be the first vertex that is attached to xy' or yy' . Without loss of generality, we assume that x' is attached to xy' . Let

$\{x_1, \dots, x_t\}$ be the subset of $V(G')$ so that x_i is attached to xx' for $i = 1, \dots, t$ and $\{y_1, \dots, y_{t'}\}$ be the subset of $V(G')$ so that y_j is attached to yy' for $j = 1, \dots, t'$. Denote

$$G'' = G' - \{xx_1, \dots, xx_t\} - \{yy_1, \dots, yy_{t'}\} + \{y'x_1, \dots, y'x_t\} + \{x'y_1, \dots, x'y_{t'}\} - \{xy, xy'\}.$$

In G'' , we first delete edges xx' and yy' , and then identify the vertex x to the vertex x' and identify the vertex y to the vertex y' , the resulting graph is denoted by G''' . Then G''' is a simple graph and is a 2-tree on $k - s - 2$ vertices. Moreover, $H = G \setminus (B(xy) \cup \{x, y\}) = G' \setminus \{x, y\}$ is a spanning subgraph of G''' . \square

Lemma 2.3 Let $k \geq 6, n \geq k$ and $\pi = (d_1, \dots, d_n) \in GS_n$ with $\sigma(\pi) > \frac{4kn}{3} - \frac{5n}{3}$. Then $d_i \geq k - \lceil \frac{i}{2} \rceil$ for $i = 1, \dots, \lceil \frac{2k}{3} \rceil$.

Proof: If there is an even r with $2 \leq r \leq \lceil \frac{2k}{3} \rceil$ such that $d_r \leq k - \lceil \frac{r}{2} \rceil - 1 = k - \frac{r}{2} - 1$, then

$$\begin{aligned} \sigma(\pi) &\leq (r-1)(n-1) + (k - \frac{r}{2} - 1)(n-r+1) \\ &= \frac{r^2}{2} - r(k - \frac{n}{2} + \frac{1}{2}) + kn - 2n + k. \end{aligned}$$

Denote $f(r) = \frac{r^2}{2} - r(k - \frac{n}{2} + \frac{1}{2}) + kn - 2n + k$. Since $2 \leq r \leq \frac{2k+2}{3}$, we have that

$$\begin{aligned} \sigma(\pi) &\leq f(r) \leq \max\{f(2), f(\frac{2k+2}{3})\} \\ &= \max\{\frac{4kn}{3} - \frac{5n}{3} - (\frac{(k-2)n}{3} + k - 1), \frac{4kn}{3} - \frac{5n}{3} - \frac{4(k^2-k)+1}{9}\} \\ &< \frac{4kn}{3} - \frac{5n}{3}, \end{aligned}$$

a contradiction.

If there is an odd r with $1 \leq r \leq \lceil \frac{2k}{3} \rceil$ such that $d_r \leq k - \lceil \frac{r}{2} \rceil - 1 = k - \frac{r+1}{2} - 1$, then

$$\begin{aligned} \sigma(\pi) &\leq (r-1)(n-1) + (k - \frac{r+1}{2} - 1)(n-r+1) \\ &= \frac{r^2}{2} - r(k - \frac{n}{2}) + kn + k - \frac{5n}{2} - \frac{1}{2}. \end{aligned}$$

Denote $g(r) = \frac{r^2}{2} - r(k - \frac{n}{2}) + kn + k - \frac{5n}{2} - \frac{1}{2}$. Since $1 \leq r \leq \frac{2k+2}{3}$, we have that

$$\begin{aligned} \sigma(\pi) &\leq g(r) \leq \max\{g(1), g(\frac{2k+2}{3})\} \\ &= \max\{\frac{4kn}{3} - \frac{5n}{3} - \frac{kn}{3} - \frac{n}{3}, \frac{4kn}{3} - \frac{13n}{6} - \frac{4k^2}{9} + \frac{7k}{9} - \frac{5}{18}\} \\ &< \frac{4kn}{3} - \frac{5n}{3}, \end{aligned}$$

a contradiction. \square

Lemma 2.4 Let $k \geq 6, n \geq k$ and $\pi = (d_1, \dots, d_n) \in GS_n$ with $\sigma(\pi) > \frac{4kn}{3} - \frac{5n}{3}$. Then $d_i \geq 2(k+1-i)$ for $i = \lceil \frac{2k}{3} \rceil + 1, \dots, k$.

Proof: If there is an r with $\lceil \frac{2k}{3} \rceil + 1 \leq r \leq k$ such that $d_r \leq 2k - 2r + 1$, then by Theorem 2.2,

$$\begin{aligned} \sigma(\pi) &= \sum_{i=1}^n d_i = \sum_{i=1}^{r-1} d_i + \sum_{i=r}^n d_i \leq ((r-2)(r-1) + \sum_{i=r}^n \min\{r-1, d_i\}) + \sum_{i=r}^n d_i \\ &= (r-2)(r-1) + 2 \sum_{i=r}^n d_i \leq (r-2)(r-1) + 2(2k-2r+1)(n-r+1) \\ &= 5r^2 - (4k+4n+9)r + 4kn + 4k + 2n + 4. \end{aligned}$$

Denote $f(r) = 5r^2 - (4k + 4n + 9)r + 4kn + 4k + 2n + 4$. Since $\frac{2k+3}{3} \leq r \leq k$, we have that

$$\begin{aligned} \sigma(\pi) &\leq f(r) \leq \max\{f(\frac{2k+3}{3}), f(k)\} \\ &= \max\{\frac{4kn}{3} - 2n - \frac{4k^2}{9} + \frac{2k}{3}, k^2 - 5k + 2n + 4\} \\ &< \max\{\frac{4kn}{3} - \frac{5n}{3} - [(\frac{2k}{3})^2 - \frac{2k}{3}] - \frac{n}{3}, \frac{4kn}{3} - \frac{5n}{3} - [(k-3)(\frac{4n}{3} - k) + \frac{n}{3} + 2k - 4]\} \\ &< \frac{4kn}{3} - \frac{5n}{3}, \end{aligned}$$

a contradiction. □

We now define a new graph $G(k)$ as follows: Let $V(K_{\lceil \frac{2k}{3} \rceil}) = \{v_1, v_2, \dots, v_{\lceil \frac{2k}{3} \rceil}\}$, and $G(k)$ be the graph obtained from $K_{\lceil \frac{2k}{3} \rceil}$ by adding new vertices $x_1, x_2, \dots, x_{\lfloor \frac{k}{3} \rfloor}$ and joining x_i to v_1, v_2, \dots, v_{2i} for $1 \leq i \leq \lfloor \frac{k}{3} \rfloor$. It is easy to see that $|V(G(k))| = k$.

Lemma 2.5 *If G is a 2-tree on k vertices, then $G(k)$ contains G as a subgraph.*

Proof: We use induction on k . It is easy to check that Lemma 2.5 holds for $k = 3, 4, 5$. If $G = T(k)$, then it is easy to see that $G(k)$ contains G as a subgraph. Assume that $k \geq 6$ and $G \neq T(k)$. Let $xy \in C(G)$ so that xy is attached to as few ears as possible, and let s be the number of these ears. Denote $H = G \setminus (B(xy) \cup \{x, y\})$. By Lemma 2.2, H is a spanning subgraph of some 2-tree G' on $k - s - 2$ vertices. Denote $m = k - s - 2$. We consider the following cases.

Case 1. $k \equiv 0 \pmod{3}$ and $m \equiv 0 \pmod{3}$.

Let

$$M = G(k) - \{v_1, \dots, v_{\frac{2k-2m}{3}}\} - \{x_1, \dots, x_{\frac{k-m}{3}}\}.$$

Then $M = G(m)$. By the induction hypothesis, $G(m)$ contains G' as a subgraph. This implies that $G(m)$ contains H as a subgraph. Putting x and y on v_1 and v_2 respectively and taking $B(xy) = \{v_3, \dots, v_{\frac{2k-2m}{3}}, x_1, \dots, x_{\frac{k-m}{3}}\}$, we can see that $G(k)$ contains G as a subgraph.

Case 2. $k \equiv 0 \pmod{3}$ and $m \equiv 1 \pmod{3}$.

Let

$$M = G(k) - \{v_1, \dots, v_{\frac{2k-2m-4}{3}}, v_{\frac{2k}{3}}\} - \{x_1, \dots, x_{\frac{k-m-2}{3}}, x_{\frac{k}{3}}\}.$$

Then $M = G(m)$. By the induction hypothesis, $G(m)$ contains G' as a subgraph, and hence contains H as a subgraph. Putting x and y on v_1 and v_2 respectively and taking

$$B(xy) = \{v_3, \dots, v_{\frac{2k-2m-4}{3}}, v_{\frac{2k}{3}}, x_1, \dots, x_{\frac{k-m-2}{3}}, x_{\frac{k}{3}}\},$$

we can see that $G(k)$ contains G as a subgraph.

Case 3. $k \equiv 0 \pmod{3}$ and $m \equiv 2 \pmod{3}$.

Let

$$M = G(k) - \{v_1, \dots, v_{\frac{2k-2m-2}{3}}\} - \{x_1, \dots, x_{\frac{k-m-1}{3}}, x_{\frac{k}{3}}\}.$$

Then $M = G(m)$. By the induction hypothesis, $G(m)$ contains H as a subgraph. Clearly, $G(k)$ contains G as a subgraph.

Case 4. $k \equiv 1 \pmod{3}$ and $m \equiv 0 \pmod{3}$.

Let

$$M = G(k) - \{v_1, \dots, v_{\frac{2k-2m-2}{3}}, v_{\frac{2k+1}{3}}\} - \{x_1, \dots, x_{\frac{k-m-1}{3}}\}.$$

Then $M = G(m)$. By the induction hypothesis, $G(m)$ contains H as a subgraph. Clearly, $G(k)$ contains G as a subgraph.

Case 5. $k \equiv 1(\text{mod } 3)$ and $m \equiv 1(\text{mod } 3)$.

Let

$$M = G(k) - \{v_1, \dots, v_{\frac{2k-2m}{3}}\} - \{x_1, \dots, x_{\frac{k-m}{3}}\}.$$

Then $M = G(m)$. By the induction hypothesis, $G(m)$ contains H as a subgraph. Clearly, $G(k)$ contains G as a subgraph.

Case 6. $k \equiv 1(\text{mod } 3)$ and $m \equiv 2(\text{mod } 3)$.

Let

$$M = G(k) - \{v_1, \dots, v_{\frac{2k-2m-4}{3}}, v_{\frac{2k+1}{3}}\} - \{x_1, \dots, x_{\frac{k-m-2}{3}}, x_{\frac{k-1}{3}}\}.$$

Then $M = G(m)$. By the induction hypothesis, $G(m)$ contains H as a subgraph. Clearly, $G(k)$ contains G as a subgraph.

Case 7. $k \equiv 2(\text{mod } 3)$ and $m \equiv 0(\text{mod } 3)$.

Let

$$M = G(k) - \{v_1, \dots, v_{\frac{2k-2m-4}{3}}, v_{\frac{2k-1}{3}}, v_{\frac{2k+2}{3}}\} - \{x_1, \dots, x_{\frac{k-m-2}{3}}\}.$$

Then $M = G(m)$. By the induction hypothesis, $G(m)$ contains H as a subgraph. Clearly, $G(k)$ contains G as a subgraph.

Case 8. $k \equiv 2(\text{mod } 3)$ and $m \equiv 1(\text{mod } 3)$.

Let

$$M = G(k) - \{v_1, \dots, v_{\frac{2k-2m-2}{3}}, v_{\frac{2k+2}{3}}\} - \{x_1, \dots, x_{\frac{k-m-1}{3}}\}.$$

Then $M = G(m)$. By the induction hypothesis, $G(m)$ contains H as a subgraph. Clearly, $G(k)$ contains G as a subgraph.

Case 9. $k \equiv 2(\text{mod } 3)$ and $m \equiv 2(\text{mod } 3)$.

Let

$$M = G(k) - \{v_1, \dots, v_{\frac{2k-2m}{3}}\} - \{x_1, \dots, x_{\frac{k-m}{3}}\}.$$

Then $M = G(m)$. By the induction hypothesis, $G(m)$ contains H as a subgraph. Clearly, $G(k)$ contains G as a subgraph. \square

We now define sequence $\pi_0, \pi_1, \dots, \pi_k$ as follows. Let $\pi_0 = \pi$. We define the sequence

$$\pi_1 = (d_2^{(1)}, \dots, d_k^{(1)}, d_{k+1}^{(1)}, \dots, d_n^{(1)})$$

from π_0 by deleting d_1 , decreasing the first d_1 remaining nonzero terms each by one unity, and then reordering the last $n - k$ terms to be non-increasing. Note that the definition of the residual sequence obtained from π by laying off d_k is to reorder all the remaining terms to be non-increasing.

For $2 \leq i \leq k$, we define the sequence

$$\pi_i = (d_{i+1}^{(i)}, \dots, d_k^{(i)}, d_{k+1}^{(i)}, \dots, d_n^{(i)})$$

from

$$\pi_{i-1} = (d_i^{(i-1)}, \dots, d_k^{(i-1)}, d_{k+1}^{(i-1)}, \dots, d_n^{(i-1)})$$

by deleting $d_i^{(i-1)}$, decreasing the first $d_i^{(i-1)}$ remaining nonzero terms each by one unity, and then reordering the last $n - k$ terms to be non-increasing.

Lemma 2.6 Let $k \geq 6$, $n \geq k$ and $\pi = (d_1, \dots, d_{\lceil \frac{2k}{3} \rceil}, d_{\lceil \frac{2k}{3} \rceil + 1}, \dots, d_k, d_{k+1}, \dots, d_n) \in GS_n$ satisfy $d_i \geq k - \lceil \frac{i}{2} \rceil$ for $i = 1, \dots, \lceil \frac{2k}{3} \rceil$. If π_k is graphic, then π has a realization containing $G(k)$ as a subgraph.

Proof: Suppose that π_k is realized by graph G_k with vertex set $V(G_k) = \{v_{k+1}, \dots, v_n\}$ such that $d_{G_k}(v_i) = d_i^{(k)}$ for $k+1 \leq i \leq n$. For $i = k, \dots, 1$ in turn, form G_{i-1} from G_i by adding a new vertex v_i that is adjacent to the vertices of G_i whose degrees are reduced by one in going from π_{i-1} to π_i . Then, for each i , G_i has degrees given by π_i . In particular, G_0 has degrees given by π . Since π satisfies $d_i \geq k - \lceil \frac{i}{2} \rceil$ for $i = 1, \dots, \lceil \frac{2k}{3} \rceil$, by the definition of π_i for $i = 1, \dots, k$ in turn, we can see that $G_0[\{v_1, \dots, v_k\}]$ contains $G(k)$ as a subgraph. \square

Lemma 2.7 Let $k \geq 6$, $n \geq k$ and $\pi = (d_1, \dots, d_{\lceil \frac{2k}{3} \rceil}, d_{\lceil \frac{2k}{3} \rceil + 1}, \dots, d_k, d_{k+1}, \dots, d_n) \in GS_n$. Let $\pi'_1 = (d'_1, \dots, d'_{n-1})$ be the residual sequence obtained from π by laying off d_1 and $\rho = (\rho_1, \dots, \rho_{n-2})$ be the residual sequence obtained from π'_1 by laying off the term $d_2 - 1$. If π satisfies one of (a)–(c), where

- (a) $d_1 = d_2 = n - 1$,
- (b) $d_1 = n - 1$, $d_2 \leq n - 2$ and $d_k > d_{d_2+2}$,
- (c) $d_1 \leq n - 2$, $d_k > d_{d_2+2}$ and $d_k - d_{d_1+2} \geq 2$,

then $\rho_1 = d_3 - 2$, $\rho_2 = d_4 - 2, \dots, \rho_{k-2} = d_k - 2$.

Proof: If π satisfies (a), then $\rho = (d_3 - 2, \dots, d_n - 2)$, and so $\rho_1 = d_3 - 2$, $\rho_2 = d_4 - 2, \dots, \rho_{k-2} = d_k - 2$. If π satisfies (b), then $\pi'_1 = (d_2 - 1, d_3 - 1, \dots, d_n - 1)$. By $d_k - 2 \geq d_{d_2+2} - 1$, we further have that $\rho_1 = d_3 - 2$, $\rho_2 = d_4 - 2, \dots, \rho_{k-2} = d_k - 2$. Assume that π satisfies (c). If $d_{d_2+2} > d_{d_1+2}$, then $d_{d_2+2} - 1 \geq d_{d_1+2}$, and hence $d'_1 = d_2 - 1, \dots, d'_{d_2+1} = d_{d_2+2} - 1$. By $d_k > d_{d_2+2}$, we have $d_k - 2 \geq d_{d_2+2} - 1$, implying that $\rho_1 = d_3 - 2$, $\rho_2 = d_4 - 2, \dots, \rho_{k-2} = d_k - 2$. If $d_{d_2+2} = \dots = d_{d_1+2}$, then $d_{d_2+2} - 1 < d_{d_1+2}$. By $d_k - d_{d_1+2} \geq 2$, we have $d'_1 = d_2 - 1, \dots, d'_{k-1} = d_k - 1$ and $d'_{d_2+1} \leq d_{d_1+2}$, implying that $\rho_1 = d_3 - 2$, $\rho_2 = d_4 - 2, \dots, \rho_{k-2} = d_k - 2$. \square

Lemma 2.8 Let $k \geq 6$, $n \geq k$ and $\pi = (d_1, \dots, d_{\lceil \frac{2k}{3} \rceil}, d_{\lceil \frac{2k}{3} \rceil + 1}, \dots, d_k, d_{k+1}, \dots, d_n) \in GS_n$. For each $\pi_i = (d_{i+1}^{(i)}, \dots, d_k^{(i)}, d_{k+1}^{(i)}, \dots, d_n^{(i)})$, let $t_i = \max\{j \mid d_{k+1}^{(i)} - d_{k+j}^{(i)} \leq 1\}$.

(1) If π satisfies (d) or (e), where

- (d) $d_1 \leq n - 2$, $d_k > d_{d_2+2}$ and $d_k - d_{d_1+2} \leq 1$,
- (e) $d_1 \leq n - 2$, $d_k = d_{d_2+2}$ and $d_{d_2+2} = d_{d_1+2}$,

then $d_{k+r}^{(k)} = d_{k+r}$ for $r > t_k$.

(2) If π satisfies (f) or (g), where

- (f) $d_1 = n - 1$, $d_2 \leq n - 2$ and $d_k = d_{d_2+2}$,
- (g) $d_1 \leq n - 2$, $d_k = d_{d_2+2}$ and $d_{d_2+2} > d_{d_1+2}$,

then $d_{k+r}^{(k)} = d_{k+r}^{(1)}$ for $r > t_k$.

Proof: (1) If π satisfies (d) or (e), then $k + t_0 \geq d_1 + 2$. Since $d_{k+1}^{(i-1)} - d_{k+t_{i-1}}^{(i-1)} \leq 1$ implies that $d_{k+1}^{(i)} - d_{k+t_{i-1}}^{(i)} \leq 1$ for $1 \leq i \leq k$, we have that $t_k \geq t_{k-1} \geq \dots \geq t_0 \geq d_1 + 2 - k$. By $\min\{d_{k+1}^{(i-1)} - 1, \dots, d_{d_i+1}^{(i-1)} - 1, d_{d_i+2}^{(i-1)}, \dots, d_{k+t_{i-1}}^{(i-1)}\} \geq d_{k+1}^{(i-1)} - 2 \geq d_{k+t_{i-1}+1}^{(i-1)} \geq \dots \geq d_n^{(i-1)}$,

we have that $d_{k+t_{i-1}+m}^{(i)} = d_{k+t_{i-1}+m}^{(i-1)}$ for $m \geq 1$. Thus, $d_{k+r}^{(i)} = d_{k+r}^{(i-1)}$ for $r > t_i$. This implies that $d_{k+r}^{(k)} = d_{k+r}$ for $r > t_k$.

(2) If π satisfies (f) or (g), then $t_k \geq t_{k-1} \geq \dots \geq t_1 \geq t_0 \geq d_2 + 2 - k$. Since $\min\{d_{k+1}^{(i-1)} - 1, \dots, d_{d_i+1}^{(i-1)} - 1, d_{d_i+2}^{(i-1)}, \dots, d_{k+t_{i-1}}^{(i-1)}\} \geq d_{k+1}^{(i-1)} - 2 \geq d_{k+t_{i-1}+1}^{(i-1)} \geq \dots \geq d_n^{(i-1)}$ for $i \geq 2$, we have that $d_{k+t_{i-1}+m}^{(i)} = d_{k+t_{i-1}+m}^{(i-1)}$ for $i \geq 2$ and $m \geq 1$. Thus, $d_{k+r}^{(i)} = d_{k+r}^{(i-1)}$ for $i \geq 2$ and $r > t_i$. This implies that $d_{k+r}^{(k)} = d_{k+r}^{(1)}$ for $r > t_k$. □

If $\pi = (d_1, \dots, d_n) \in GS_n$ has a realization containing every 2-tree on k vertices as a subgraph, then π is *potentially $A'(k)$ -graphic*. If π has a realization in which the subgraph induced by the k vertices of largest degrees contains every 2-tree on k vertices as a subgraph, then π is *potentially $A''(k)$ -graphic*. It is easy to see that if π is potentially $A''(k)$ -graphic, then π is potentially $A'(k)$ -graphic.

Lemma 2.9 *Let $k \geq 3, n \geq 6k$ and $\pi = (d_1, \dots, d_n) \in GS_n$ with $d_n \geq \frac{2k}{3} - 2$ and $\sigma(\pi) > \frac{4kn}{3} - \frac{5n}{3}$. Then π is potentially $A''(k)$ -graphic.*

Proof: We use induction on k . If $k = 3$, then by $\sigma(\pi) > \frac{4kn}{3} - \frac{5n}{3} \geq 2n$ and Theorem 2.7 (1), π has a realization containing K_3 . By Theorem 2.4, π is potentially $A''(3)$ -graphic. If $k = 4$, then by $\sigma(\pi) > \frac{4kn}{3} - \frac{5n}{3} \geq 3n - 1$ and Theorem 2.7 (2), π has a realization containing $K_4 - e$. By Theorem 2.4, π is potentially $A''(4)$ -graphic. If $k = 5$, then by $\sigma(\pi) > \frac{4kn}{3} - \frac{5n}{3} \geq 5n - 6$ and Theorem 2.7 (3), π has a realization containing $K_5 - e$. Since $K_5 - e$ contains every 2-tree on 5 vertices, by Theorem 2.4, π is potentially $A''(5)$ -graphic. Assume $k \geq 6$. We only need to prove that $\pi = (d_1, \dots, d_n)$ has a realization in which the subgraph induced by the vertices with degrees d_1, \dots, d_k contains every 2-tree on k vertices. Let $\pi'_1 = (d'_1, \dots, d'_{n-1})$ be the residual sequence obtained from π by laying off d_1 and $\rho = (\rho_1, \dots, \rho_{n-2})$ be the residual sequence obtained from π'_1 by laying off the term $d_2 - 1$. Then $n - 2 \geq 6(k - 3), \rho_{n-2} \geq (\frac{2k}{3} - 2) - 2 = \frac{2(k-3)}{3} - 2$ and $\sigma(\rho) = \sigma(\pi) - 2d_1 - 2d_2 + 2 > \frac{4kn}{3} - \frac{5n}{3} - 4(n - 1) + 2 > \frac{4(k-3)(n-2)}{3} - \frac{5(n-2)}{3}$. By the induction hypothesis, ρ has a realization G_1 in which the subgraph induced by the vertices with degrees $\rho_1, \dots, \rho_{k-3}$ contains every 2-tree on $k - 3$ vertices. Denote F to be the subgraph induced by the vertices with degrees $\rho_1, \dots, \rho_{k-3}$ in G_1 , and let F' be the graph obtained from F by adding three new vertices x, y, u such that x, y are adjacent to each vertex of F and $xy, xu, yu \in E(F')$.

Claim F' contains every 2-tree on k vertices.

Proof of Claim. Let G be any one 2-tree on k vertices. Take $xy \in C(G)$ and $u \in B(xy)$, and denote $H = G \setminus \{x, y, u\}$. By Lemma 2.2, it is easy to get that H is a spanning subgraph of some 2-tree on $k - 3$ vertices. Since F contains every 2-tree on $k - 3$ vertices, we have that F contains H as a subgraph. By the definition of F' , we can see that F' contains G as a subgraph. By the arbitrary of G , F' contains every 2-tree on k vertices. This proves Claim.

If π satisfies one of (a)–(c), by Lemma 2.7, then $\rho_1 = d_3 - 2, \rho_2 = d_4 - 2, \dots, \rho_{k-2} = d_k - 2$. Now by the definitions of ρ and π'_1 , it is easy to get that π has a realization G' in which the subgraph induced by the vertices with degrees d_1, \dots, d_k contains F' as a subgraph. Thus by Claim, π has a realization in which the subgraph induced by the vertices with degrees d_1, \dots, d_k contains every 2-tree on k vertices.

We now assume that π satisfies one of (d)–(g). If $d_k \geq 2k - 3$, then by Theorem 2.5, π has a realization containing K_k , and hence π is potentially $A''(k)$ -graphic by Theorem 2.4. Assume that $d_k \leq 2k - 4$. By $\sigma(\pi) > \frac{4kn}{3} - \frac{5n}{3}$ and Lemmas 2.3 and 2.4, we have that $d_i \geq k - \lceil \frac{i}{2} \rceil$ for $i = 1, \dots, \lceil \frac{2k}{3} \rceil$ and $d_{\lceil \frac{2k}{3} \rceil + 1} \geq 2\lfloor \frac{k}{3} \rfloor$. It is enough to prove that π_k is graphic by Theorem 2.4 and Lemmas 2.5 and 2.6. If π satisfies (d) or (e), by Lemma 2.8 (1), then

$$\pi_k = (d_{k+1}^{(k)}, \dots, d_{k+t_k}^{(k)}, d_{k+t_k+1}, \dots, d_n).$$

If π satisfies (f) or (g), by Lemma 2.8 (2), then

$$\pi_k = (d_{k+1}^{(k)}, \dots, d_{k+t_k}^{(k)}, d_{k+t_k+1}^{(1)}, \dots, d_n^{(1)}).$$

If $t_k < n - k$, then $k + t_k < n$. By $d_{k+1}^{(k)} \leq d_{k+1} \leq d_k \leq 2k - 4$ and $d_n \geq d_n^{(1)} \geq d_n - 1 \geq \frac{2k}{3} - 3 \geq 1$, we have that $d_{k+1}^{(k)} \leq 2k - 4$ and $d_n \geq d_n^{(1)} \geq \lceil \frac{2k}{3} - 3 \rceil \geq 1$. Since $\frac{(2k-3+x)^2}{4x}$ is a monotone decreasing function of x on the interval $(0, 2k - 3]$, by $\lceil \frac{2k}{3} - 3 \rceil \geq \frac{2k}{3} - 3$, we have that

$$\begin{aligned} \frac{1}{\lceil \frac{2k}{3} - 3 \rceil} \lfloor \frac{(2k-4+\lceil \frac{2k}{3} - 3 \rceil + 1)^2}{4} \rfloor &\leq \frac{(2k-3+\lceil \frac{2k}{3} - 3 \rceil)^2}{4\lceil \frac{2k}{3} - 3 \rceil} \\ &\leq \frac{(2k-3+\frac{2k}{3}-3)^2}{4(\frac{2k}{3}-3)} \\ &= \frac{\frac{16k^2}{9} - 24k + 27}{2k-9} \\ &= \frac{\frac{8}{3}k(2k-9) + 27}{2k-9} \\ &\leq \frac{8k}{3} + 9 \leq n - k. \end{aligned}$$

By Theorem 2.3, π_k is graphic. If $t_k = n - k$, then $d_{k+1}^{(k)} - d_n^{(k)} \leq 1$. Denote $d_n^{(k)} = m$. If $m = 0$, then by $d_{k+1}^{(k)} \leq 1$ and $\sigma(\pi_k)$ being even, π_k is clearly graphic. If $m \geq 1$, then $d_{k+1}^{(k)} \leq m + 1$, and hence

$$\frac{1}{m} \lfloor \frac{(m+1+m+1)^2}{4} \rfloor = \frac{(m+1)^2}{m} \leq m+3 \leq 2k-4+3 \leq n-k.$$

By Theorem 2.3, π_k is also graphic. □

Lemma 2.10 Let $k \geq 6$, $n = 6k$ and $\pi = (d_1, \dots, d_n) \in GS_n$ with $\sigma(\pi) > \frac{4kn}{3} - \frac{5n}{3} + 4k^2 - 14k$. Then π is potentially $A''(k)$ -graphic.

Proof: By $\sigma(\pi) \geq \frac{4kn}{3} - \frac{5n}{3} + 4k^2 - 14k + 2 = 2n(k-2) + 2$ and Theorem 2.6, π has a realization containing K_k . By Theorem 2.4, π is potentially $A''(k)$ -graphic. □

Lemma 2.11 Let $k \geq 6$ and $n = 6k + t$, where $0 \leq t \leq 2k^2 - 7k$. If $\pi = (d_1, \dots, d_n) \in GS_n$ with $\sigma(\pi) > \frac{4kn}{3} - \frac{5n}{3} + 4k^2 - 14k - 2t$, then π is potentially $A'(k)$ -graphic.

Proof: We use induction on t . It is known from Lemma 2.10 that Lemma 2.11 holds for $t = 0$. Suppose now that $1 \leq t \leq 2k^2 - 7k$. Then $\sigma(\pi) > \frac{4kn}{3} - \frac{5n}{3}$. If $d_n \geq \frac{2k}{3} - 2$, then π is potentially $A''(k)$ -graphic by Lemma 2.9. If $d_n < \frac{2k}{3} - 2$, then the residual sequence $\pi'_n = (d'_1, \dots, d'_{n-1})$ obtained by

laying off d_n from π satisfies $\sigma(\pi'_n) = \sigma(\pi) - 2d_n > \frac{4kn}{3} - \frac{5n}{3} + 4k^2 - 14k - 2t - 2(\frac{2k}{3} - 2) > \frac{4k(n-1)}{3} - \frac{5(n-1)}{3} + 4k^2 - 14k - 2(t-1)$. By the induction hypothesis, π'_n is potentially $A'(k)$ -graphic, and hence so is π . \square

We now prove Theorem 1.2.

Proof of Theorem 1.2: Let $k \geq 3$, $n \geq 2k^2 - k$ and $\pi = (d_1, \dots, d_n) \in GS_n$ with $\sigma(\pi) > \frac{4kn}{3} - \frac{5n}{3}$. We only need to prove that π is potentially $A'(k)$ -graphic. If $k = 3$, then by $\sigma(\pi) > \frac{4kn}{3} - \frac{5n}{3} \geq 2n$ and Theorem 2.7 (1), π has a realization containing K_3 , and hence π is potentially $A'(3)$ -graphic. If $k = 4$, then by $\sigma(\pi) > \frac{4kn}{3} - \frac{5n}{3} \geq 3n - 1$ and Theorem 2.7 (2), π has a realization containing $K_4 - e$, and hence π is potentially $A'(4)$ -graphic. If $k = 5$, then by $\sigma(\pi) > \frac{4kn}{3} - \frac{5n}{3} \geq 5n - 6$ and Theorem 2.7 (3), π has a realization containing $K_5 - e$. Since $K_5 - e$ contains every 2-tree on 5 vertices, π is potentially $A'(5)$ -graphic. Assume that $k \geq 6$. We now use induction on n . If $n = 2k^2 - k$, then by Lemma 2.11 ($t = 2k^2 - 7k$), π is potentially $A'(k)$ -graphic. Assume that $n \geq 2k^2 - k + 1$. If $d_n \geq \frac{2k}{3} - 2$, then by Lemma 2.9, π is potentially $A'(k)$ -graphic. If $d_n < \frac{2k}{3} - 2$, then the residual sequence $\pi'_n = (d'_1, \dots, d'_{n-1})$ obtained from π by laying off d_n satisfies $\sigma(\pi'_n) = \sigma(\pi) - 2d_n > \frac{4kn}{3} - \frac{5n}{3} - 2(\frac{2k}{3} - 2) > \frac{4k(n-1)}{3} - \frac{5(n-1)}{3}$. By the induction hypothesis, π'_n is potentially $A'(k)$ -graphic, and hence so is π . \square

3 Proof of Theorem 1.3

In order to prove Theorem 1.3, we recursively define a new graph $F(k)$ on $k \geq 3$ vertices as follows. Let $F(3) = K_3$, and let $V(F(k-1)) = \{x_1, \dots, x_{k-1}\}$ for $k \geq 4$. Define $F(k)$ be the graph obtained from $F(k-1)$ by adding a new vertex x_k and joining x_k to x_{k-2}, x_{k-1} . Clearly, $F(k)$ is a 2-tree on k vertices. Let $\alpha(G)$ denote the independence number of G . We need the following Lemma 3.1.

Lemma 3.1 *Let $k \geq 3$ and $e \in E(F(k))$. Then*

- (1) $\alpha(F(k)) \leq \lceil \frac{k}{3} \rceil$;
- (2) *If $k \equiv 1 \pmod{3}$, then $\alpha(F(k) - e) \leq \lceil \frac{k}{3} \rceil$.*

Proof: (1) We use induction on k . It is easy to check that Lemma 3.1(1) holds for $k = 3, 4, 5$. Assume that $k \geq 6$. Let $V(F(k)) = \{x_1, \dots, x_k\}$. By the construction of $F(k)$, we have that the subgraph induced by $\{x_{k-2}, x_{k-1}, x_k\}$ in $F(k)$ is K_3 . Let X be a maximum independent set of $F(k)$. Then $|\{x_{k-2}, x_{k-1}, x_k\} \cap X| \leq 1$. If $|\{x_{k-2}, x_{k-1}, x_k\} \cap X| = 0$, then X is an independent set of $F(k) - \{x_{k-2}, x_{k-1}, x_k\} = F(k-3)$. By the induction hypothesis, we have that $\alpha(F(k)) = |X| \leq \alpha(F(k-3)) \leq \lceil \frac{k-3}{3} \rceil \leq \lceil \frac{k}{3} \rceil$. If $|\{x_{k-2}, x_{k-1}, x_k\} \cap X| = 1$, let $\{x_{k-2}, x_{k-1}, x_k\} \cap X = \{x\}$, then $X \setminus \{x\}$ is an independent set of $F(k) - \{x_{k-2}, x_{k-1}, x_k\} = F(k-3)$. By the induction hypothesis, we have that $\alpha(F(k)) - 1 = |X \setminus \{x\}| \leq \alpha(F(k-3)) \leq \lceil \frac{k-3}{3} \rceil$, i.e., $\alpha(F(k)) \leq \lceil \frac{k-3}{3} \rceil + 1 = \lceil \frac{k}{3} \rceil$.

(2) Clearly, Lemma 3.1(2) holds for $k = 4$. Assume that $k \geq 7$. By the construction of $F(k)$, we have that $e = x_i x_{i+1}$ for $1 \leq i \leq k-1$ or $e = x_j x_{j+2}$ for $1 \leq j \leq k-2$. Let X be a maximum independent set of $F(k) - e$.

Firstly, we assume that $e = x_i x_{i+1}$ for $1 \leq i \leq k-1$. If $|\{x_i, x_{i+1}\} \cap X| \leq 1$, then X is an independent set of $F(k)$, and hence $\alpha(F(k) - e) = |X| \leq \alpha(F(k)) \leq \lceil \frac{k}{3} \rceil$. Assume that $|\{x_i, x_{i+1}\} \cap X| = 2$, i.e., $\{x_i, x_{i+1}\} \subseteq X$. If $i = 1$ (or $i = k-1$), then $X \setminus \{x_1, x_2\}$ (or $X \setminus \{x_{k-1}, x_k\}$) is an independent set of

$F(k) - \{x_1, x_2, x_3, x_4\} = F(k-4)$ (or $F(k) - \{x_k, x_{k-1}, x_{k-2}, x_{k-3}\} = F(k-4)$). This implies that $\alpha(F(k) - e) - 2 = |X| - 2 \leq \alpha(F(k-4)) \leq \lceil \frac{k-4}{3} \rceil$, i.e., $\alpha(F(k) - e) \leq \lceil \frac{k-4}{3} \rceil + 2 = \lceil \frac{k+2}{3} \rceil = \lceil \frac{k}{3} \rceil$ (as $k \equiv 1 \pmod{3}$). If $i = 2$ (or $i = k-2$), then $X \setminus \{x_2, x_3\}$ (or $X \setminus \{x_{k-2}, x_{k-1}\}$) is an independent set of $F(k) - \{x_1, x_2, x_3, x_4, x_5\} = F(k-5)$ (or $F(k) - \{x_k, x_{k-1}, x_{k-2}, x_{k-3}, x_{k-4}\} = F(k-5)$). This implies that $\alpha(F(k) - e) - 2 = |X| - 2 \leq \alpha(F(k-5)) \leq \lceil \frac{k-5}{3} \rceil$, i.e., $\alpha(F(k) - e) \leq \lceil \frac{k-5}{3} \rceil + 2 = \lceil \frac{k+1}{3} \rceil = \lceil \frac{k}{3} \rceil$ (as $k \equiv 1 \pmod{3}$). If $3 \leq i \leq k-3$, then $X \setminus \{x_i, x_{i+1}\}$ is an independent set of $F(k) - \{x_{i-2}, x_{i-1}, x_i, x_{i+1}, x_{i+2}, x_{i+3}\}$. For convenience, we denote $F(i) = K_i$ for $i = 1, 2$. Clearly, $\alpha(F(i)) \leq \lceil \frac{i}{3} \rceil$ for $i = 1, 2$. Since $F(k) - \{x_{i-2}, x_{i-1}, x_i, x_{i+1}, x_{i+2}, x_{i+3}\}$ is the disjoint union of $F(i-3)$ and $F(k-i-3)$, we have that $\alpha(F(k) - e) - 2 = |X| - 2 \leq \alpha(F(k) - \{x_{i-2}, x_{i-1}, x_i, x_{i+1}, x_{i+2}, x_{i+3}\}) = \alpha(F(i-3)) + \alpha(F(k-i-3)) \leq \lceil \frac{i-3}{3} \rceil + \lceil \frac{k-i-3}{3} \rceil = \lceil \frac{i}{3} \rceil + \lceil \frac{k-i}{3} \rceil - 2$. Hence $\alpha(F(k) - e) \leq \lceil \frac{i}{3} \rceil + \lceil \frac{k-i}{3} \rceil = \frac{k+2}{3} = \lceil \frac{k}{3} \rceil$ (as $k \equiv 1 \pmod{3}$).

We now assume that $e = x_j x_{j+2}$ for $1 \leq j \leq k-2$. If $|\{x_j, x_{j+2}\} \cap X| \leq 1$, then X is an independent set of $F(k)$, and hence $\alpha(F(k) - e) = |X| \leq \alpha(F(k)) \leq \lceil \frac{k}{3} \rceil$. Assume that $|\{x_j, x_{j+2}\} \cap X| = 2$, i.e., $\{x_j, x_{j+2}\} \subseteq X$. If $j = 1$ (or $j = k-2$), then $X \setminus \{x_1, x_3\}$ (or $X \setminus \{x_{k-2}, x_k\}$) is an independent set of $F(k) - \{x_1, x_2, x_3, x_4, x_5\} = F(k-5)$ (or $F(k) - \{x_k, x_{k-1}, x_{k-2}, x_{k-3}, x_{k-4}\} = F(k-5)$). This implies that $\alpha(F(k) - e) - 2 = |X| - 2 \leq \alpha(F(k-5)) \leq \lceil \frac{k-5}{3} \rceil$, i.e., $\alpha(F(k) - e) \leq \lceil \frac{k-5}{3} \rceil + 2 = \lceil \frac{k+1}{3} \rceil = \lceil \frac{k}{3} \rceil$ (as $k \equiv 1 \pmod{3}$). If $j = 2$ (or $j = k-3$), then $X \setminus \{x_2, x_4\}$ (or $X \setminus \{x_{k-3}, x_{k-1}\}$) is an independent set of $F(k) - \{x_1, x_2, x_3, x_4, x_5, x_6\} = F(k-6)$ (or $F(k) - \{x_k, x_{k-1}, x_{k-2}, x_{k-3}, x_{k-4}, x_{k-5}\} = F(k-6)$). This implies that $\alpha(F(k) - e) - 2 = |X| - 2 \leq \alpha(F(k-6)) \leq \lceil \frac{k-6}{3} \rceil$, i.e., $\alpha(F(k) - e) \leq \lceil \frac{k-6}{3} \rceil + 2 = \lceil \frac{k}{3} \rceil$. If $3 \leq j \leq k-4$, then $X \setminus \{x_j, x_{j+2}\}$ is an independent set of $F(k) - \{x_{j-2}, x_{j-1}, x_j, x_{j+1}, x_{j+2}, x_{j+3}, x_{j+4}\}$. Since $F(k) - \{x_{j-2}, x_{j-1}, x_j, x_{j+1}, x_{j+2}, x_{j+3}, x_{j+4}\}$ is the disjoint union of $F(j-3)$ and $F(k-j-4)$, we have that $\alpha(F(k) - e) - 2 = |X| - 2 \leq \alpha(F(k) - \{x_{j-2}, x_{j-1}, x_j, x_{j+1}, x_{j+2}, x_{j+3}, x_{j+4}\}) = \alpha(F(j-3)) + \alpha(F(k-j-4)) \leq \lceil \frac{j-3}{3} \rceil + \lceil \frac{k-j-4}{3} \rceil = \lceil \frac{j}{3} \rceil + \lceil \frac{k-j-1}{3} \rceil - 2$. Hence $\alpha(F(k) - e) \leq \lceil \frac{j}{3} \rceil + \lceil \frac{k-j-1}{3} \rceil \leq \lceil \frac{k}{3} \rceil$ (as $k \equiv 1 \pmod{3}$). \square

Proof of Theorem 1.3: Let $k \geq 3$ with $k \equiv i \pmod{3}$. Denote $H = K_{\lfloor \frac{2k}{3} \rfloor - 1} + \overline{K_{n - \lfloor \frac{2k}{3} \rfloor + 1}}$. If H contains $F(k)$ on the vertices u_1, \dots, u_k , then $k - (\lfloor \frac{2k}{3} \rfloor - 1) \leq \alpha(H[\{u_1, \dots, u_k\}]) \leq \alpha(F(k)) \leq \lceil \frac{k}{3} \rceil$ (Lemma 3.1(1)). This is impossible as $k - (\lfloor \frac{2k}{3} \rfloor - 1) = \lceil \frac{k}{3} \rceil + 1$. Hence H contains no $F(k)$.

For $i = 0$ or 2 , we let $\pi = ((n-1)^{\lfloor \frac{2k}{3} \rfloor - 1}, (\lfloor \frac{2k}{3} \rfloor - 1)^{n - \lfloor \frac{2k}{3} \rfloor + 1})$. Then $\pi \in GS_n$, $\sigma(\pi) = 2\lfloor \frac{2k}{3} \rfloor n - 2n - \lfloor \frac{2k}{3} \rfloor^2 + \lfloor \frac{2k}{3} \rfloor$ and H is the unique realization of π . Since H contains no $F(k)$, we have that π has no realization containing $F(k)$. This implies that π has no realization containing every 2-tree on k vertices.

For $i = 1$, we let $\pi = ((n-1)^{\lfloor \frac{2k}{3} \rfloor - 1}, (\lfloor \frac{2k}{3} \rfloor)^2, (\lfloor \frac{2k}{3} \rfloor - 1)^{n - \lfloor \frac{2k}{3} \rfloor - 1})$. Then $\pi \in GS_n$, $\sigma(\pi) = 2\lfloor \frac{2k}{3} \rfloor n - 2n - \lfloor \frac{2k}{3} \rfloor^2 + \lfloor \frac{2k}{3} \rfloor + 2$ and $H + e$ (a simple graph is obtained from H by adding an edge e) is the unique realization of π . Assume that $H + e$ contains $F(k)$. Since H contains no $F(k)$, we have that H contains $F(k) - e$. If H contains $F(k) - e$ on the vertices u_1, \dots, u_k , then $k - (\lfloor \frac{2k}{3} \rfloor - 1) \leq \alpha(H[\{u_1, \dots, u_k\}]) \leq \alpha(F(k) - e) \leq \lceil \frac{k}{3} \rceil$ (Lemma 3.1(2)), a contradiction. Hence π has no realization containing $F(k)$. This proves Theorem 1.3. \square

Since

$$\lim_{n \rightarrow +\infty} \frac{\frac{4kn}{3} - \frac{5n}{3}}{2\lfloor \frac{2k}{3} \rfloor n - 2n - \lfloor \frac{2k}{3} \rfloor^2 + \lfloor \frac{2k}{3} \rfloor + 1 - (-1)^i} = \frac{\frac{4k}{3} - \frac{5}{3}}{2\lfloor \frac{2k}{3} \rfloor - 2} \approx 1,$$

we have that $\frac{4kn}{3} - \frac{5n}{3}$ is almost the best possible lower bound in Theorem 1.2.

For $k \equiv i \pmod{3}$, we feel that $2\lfloor \frac{2k}{3} \rfloor n - 2n - \lfloor \frac{2k}{3} \rfloor^2 + \lfloor \frac{2k}{3} \rfloor + 1 - (-1)^i$ is the best possible lower bound for sufficiently large n , thus we propose the following conjecture.

Conjecture *If $k \geq 3$ with $k \equiv i \pmod{3}$, n is sufficiently large, and $\pi \in GS_n$ with $\sigma(\pi) > 2\lfloor \frac{2k}{3} \rfloor n - 2n - \lfloor \frac{2k}{3} \rfloor^2 + \lfloor \frac{2k}{3} \rfloor + 1 - (-1)^i$, then π has a realization H containing every 2-tree on k vertices. Moreover, the lower bound $2\lfloor \frac{2k}{3} \rfloor n - 2n - \lfloor \frac{2k}{3} \rfloor^2 + \lfloor \frac{2k}{3} \rfloor + 1 - (-1)^i$ is the best possible.*

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