Vertex-Coloring Edge-Weighting of Bipartite Graphs with Two Edge Weights

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Let G be a graph and $\mathcal{S}$ be a subset of Z. A vertex-coloring $\mathcal{S}$-edge-weighting of G is an assignment of weights by the elements of $\mathcal{S}$ to each edge of G so that adjacent vertices have different sums of incident edges weights.

It was proved that every 3-connected bipartite graph admits a vertex-coloring $\mathcal{S}$-edge-weighting for $\mathcal{S} = \{1, 2\}$ (H. Lu, Q. Yu and C. Zhang, Vertex-coloring 2-edge-weighting of graphs, European J. Combin., **32** (2011), 22-27). In this paper, we show that every 2-connected and 3-edge-connected bipartite graph admits a vertex-coloring $\mathcal{S}$-edge-weighting for $\mathcal{S} \in \{\{0, 1\}, \{1, 2\}\}$. These bounds we obtain are tight, since there exists a family of infinite bipartite graphs which are 2-connected and do not admit vertex-coloring $\mathcal{S}$-edge-weightings for $\mathcal{S} \in \{\{0, 1\}, \{1, 2\}\}$.

Keywords: edge-weighting, vertex-coloring, 2-connected, bipartite graph

1 Introduction

In this paper, we consider only finite, undirected and simple connected graphs. For a vertex v of graph $G = (V, E)$, $N_G(v)$ denotes the set of vertices which are adjacent to v and $d_G(v) = |N_G(v)|$ is called the *degree* of vertex v. Let $\delta(G)$ and $\triangle(G)$ denote the minimum degree and maximum degree of graph G, respectively. For $v \in V(G)$ and $r \in Z^+$, let $N_G^r(v) = \{u \in N(v) \mid d_G(u) = r\}$. If $v \in V(G)$ and $e \in E(G)$, we use $v \sim e$ to denote that v is an end-vertex of e. For two disjoint subsets S, T of $V(G)$, let $E_G(S, T)$ denote the subset of edges of $E(G)$ with one end in S and other end in T and let $e_G(S, T) = |E_G(S, T)|$. Let $G = (U, W, E)$ denote a bipartite graph with bipartition (U, W) and edge set E.

Let $\mathcal{S}$ be a subset of Z. An *$\mathcal{S}$-edge-weighting* of a graph G is an assignment $w : E(G) \to S$. An $\mathcal{S}$-edge-weighting w of a graph G induces a coloring of the vertices of G, where the color of vertex v, denoted by $c(v)$, is $\sum_{e\sim v} w(e)$. An $\mathcal{S}$-edge-weighting of a graph G is a *vertex-coloring* if for every edge $e = uv$, $c(u) \neq c(v)$ and we say that G admits a *vertex-coloring $\mathcal{S}$-edge-weighting*. If $\mathcal{S} = \{1, 2, \ldots, k\}$, then a vertex-coloring $\mathcal{S}$-edge-weighting of a graph G is usually called a *vertex-coloring k-edge-weighting*.

For vertex-coloring edge-weighting, Karoński et al. (2004) posed the following conjecture:

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Conjecture 1.1 *Every graph without isolated edges admits a vertex-coloring* 3*-edge-weighting.*

This conjecture is still wide open. Karoński et al. (2004) showed that Conjecture 1.1 is true for 3-colorable graphs. Recently, Kalkowski et al. (2010) showed that every graph without isolated edges admits a vertex-coloring 5-edge-weighting. This result is an improvement on the previous bounds on k established by Addario-Berry et al. (2007), Addario-Berry et al. (2008), and Wang and Yu (2008), who obtained the bounds $k = 30, k = 16$, and $k = 13$, respectively.

Many graphs actually admit a vertex-coloring 2-edge-weighting (in fact, experiments suggest (see Addario-Berry et al. (2008)) that almost all graphs admit a vertex-coloring 2-edge-weighting), however it is not known which ones do not. Khatirinejad et al. (2012) explored the problem of classifying those graphs which admit a vertex-coloring 2-edge-weighting. Chang et al. (2011) had made some progress in determining which classes of graphs admit vertex-coloring 2-edge-weighting, and proved that there exists a family of infinite bipartite graphs (e.g., the generalized θ-graphs) which are 2-connected and admit a vertex-coloring 3-edge-weighting but not vertex-coloring 2-edge-weightings. Lu et al. (2011) showed that every 3-connected bipartite graph admits a vertex-coloring 2-edge-weighting.

We write

$$\begin{aligned}
\mathscr{G}_{12} &= \{G \mid G \text{ admits a vertex-coloring } \{1,2\}\text{-edge-weighting}\};\\
\mathscr{G}_{01} &= \{G \mid G \text{ admits a vertex-coloring } \{0,1\}\text{-edge-weighting}\};\\
\mathscr{G}^*_{12} &= \{G \mid G \text{ is bipartite and admits a vertex-coloring } \{1,2\}\text{-edge-weighting}\};\\
\mathscr{G}^*_{01} &= \{G \mid G \text{ is bipartite and admits a vertex-coloring } \{0,1\}\text{-edge-weighting}\}.
\end{aligned}$$

Dudek and Wajc (2011) showed that determining whether a graph belongs to $\mathscr{G}_{12}$ or $\mathscr{G}_{01}$ is NP-complete. Moreover, they showed that $\mathscr{G}_{12} \neq \mathscr{G}_{01}$. The counterexamples constructed by Dudek and Wajc (2011) are non-bipartite.

Now we construct a bipartite graph, which admits a vertex-coloring 2-edge-weighting but not vertex-coloring $\{0,1\}$-edge-weightings. Let C_6 be a cycle of length six and Γ be a graph obtained by connecting an isolated vertex to one of the vertices of C_6. Take two disjoint copies of Γ. Connect two vertices of degree one of the two copies and this gives a connected bipartite graph G. It is easy to prove that G admits a vertex-coloring 2-edge-weighting but not vertex-coloring $\{0,1\}$-edge-weighting. Hence $\mathscr{G}^*_{01} \neq \mathscr{G}^*_{12}$. Next we would like to propose the following problem.

Problem 1 *Determining whether a graph* $G \in \mathscr{G}^*_{12}$ *or* $G \in \mathscr{G}^*_{01}$ *is polynomial?*

In this paper, we characterize bipartite graphs which admit a vertex-coloring $\mathcal{S}$-edge-weighting for $\mathcal{S} \in \{\{0,1\},\{1,2\}\}$, and obtain the following result.

Theorem 1.2 *Let* G *be a 3-edge-connected bipartite graph* $G = (U, W, E)$ *with minimum degree* $\delta(G)$*. If* G *contains a vertex* u *of degree* $\delta(G)$ *such that* $G - u$ *is connected, then* G *admits a vertex-coloring* $\mathcal{S}$*-edge-weighting for* $\mathcal{S} \in \{\{0,1\},\{1,2\}\}$.

By Theorem 1.2, it is easy to obtain the following result, which improves and extends the result obtained by Lu et al. (2011).

Theorem 1.3 *Every 2-connected and 3-edge-connected bipartite graph* $G = (U, W, E)$ *admits a vertex-coloring* $\mathcal{S}$*-edge-weighting for* $\mathcal{S} \in \{\{0,1\},\{1,2\}\}$.

So far, all known counterexamples of bipartite graphs, which do not have vertex-coloring $\{0,1\}$-edge-weightings or vertex-coloring $\{1,2\}$-edge-weightings are graphs with minimum degree 2. So we would like to propose the following problem.

Problem 2 *Does every bipartite graph with $\delta(G) \geq 3$ admit a vertex-coloring $\mathcal{S}$-edge-weighting, where $\mathcal{S} \in \{\{0,1\},\{1,2\}\}$.*

A *factor* of a graph G is a spanning subgraph. For a graph G, there is a close relationship between 2-edge-weighting and graph factors. Namely, a 2-edge-weighting problem is equivalent to finding special factors of graphs (see Addario-Berry et al. (2007) and Addario-Berry et al. (2008)). So to find factors with pre-specified degree is an important part of edge-weighting.

Let $g, f : V(G) \to Z$ be two integer-valued functions such that $g(v) \leq f(v)$ and $g(v) \equiv f(v) \pmod 2$ for all $v \in V(G)$. A factor F of G is called (g,f)*-parity factor* if $g(v) \leq d_F(v) \leq f(v)$ and $d_F(v) \equiv f(v) \pmod 2$ for all $v \in V(G)$. For $X \subseteq V(G)$, we write $g(X) = \sum_{x\in X} g(x)$ and $f(X)$ is defined similarly. For (g,f)-parity factors, Lovász obtained a sufficient and necessary condition.

Theorem 1.4 (Lovász (1972)) *A graph G contains a (g,f)-parity factor if and only if for any two disjoint subsets S and T of $V(G)$, it follows that*

$$\eta(S,T) = f(S) - g(T) + \sum_{x\in T} d_{G-S}(x) - \tau(S,T) \geq 0,$$

where $\tau(S,T)$ denotes the number of components C, called g-odd components of $G-S-T$ such that $g(V(C)) + e_G(V(C),T) \equiv 1 \pmod 2$.

In the proof of the main theorems, we also need the following three lemmas.

Theorem 1.5 (Chang et al. (2011)) *Every non-trivial connected bipartite graph $G = (A,B,E)$ with $|A|$ even, admits a vertex-coloring 2-edge-weighting w such that $c(u)$ is odd for $u \in A$ and $c(v)$ is even for $v \in B$.*

Theorem 1.6 (Chang et al. (2011)) *Let $r \geq 3$ be an integer. Every r-regular bipartite graph G admits a vertex-coloring 2-edge-weighting.*

Theorem 1.7 (Khatirinejad et al. (2012)) *Every r-regular graph G admits a vertex-coloring 2-edge-weighting if and only if it admits a vertex-coloring $\{0,1\}$-edge-weighting.*

2 Proof of Theorem 1.2

Corollary 2.1 *Every non-trivial connected bipartite graph $G = (A,B,E)$ with $|A|$ even admits a vertex-coloring $\{0,1\}$-edge-weighting.*

Proof: By Theorem 1.5, G admits a vertex-coloring 2-edge-weighting w such that $c(u)$ is odd for $u \in A$ and $c(v)$ is even for $v \in B$. Let $w'(e) = 0$ if $w(e) = 2$ and $w'(e) = 1$ if $w(e) = 1$. Then w' is a vertex-coloring $\{0,1\}$-edge-weighting of graph G. $\square$

For completing the proof of Theorem 1.2, we need the following two technical lemmas.

Lemma 2.2 *Let G be a bipartite graph with bipartition (U,W), where $|U| \equiv |W| \equiv 1 \pmod 2$. Let $\delta(G) = \delta$ and $u \in U$ such that $d_G(u) = \delta$. If one of the following two conditions holds, then G contains a factor F such that $d_F(u) = \delta$, $d_F(x) \equiv \delta + 1 \pmod 2$ for all $x \in U - u$, $d_F(y) \equiv \delta \pmod 2$ for all $y \in W$ and $d_F(y) \leq \delta - 2$ for all $y \in N_G^{\delta}(u)$.*

(i) *$\delta(G) \geq 4$, G is 3-edge-connected and $G - u$ is connected.*

(ii) *$\delta(G) = 3$, G is 3-edge-connected and $|N_G^{\delta}(u)| \leq 2$.*

Proof: Let M be an integer such that $M \geq \triangle(G)$ and $M \equiv \delta \pmod 2$. Let $m \in \{0, -1\}$ such that $m \equiv \delta \pmod 2$. Let $g, f : V(G) \to Z$ such that

$$g(x) = \begin{cases} \delta & \text{if } x = u, \\ m - 1 & \text{if } x \in U - u, \\ m & \text{if } x \in W, \end{cases}$$

and

$$f(x) = \begin{cases} M + 1 & \text{if } x \in U - u, \\ M & \text{if } x \in (W \cup \{u\}) - N_G^{\delta}(u), \\ \delta - 2 & \text{if } x \in N_G^{\delta}(u). \end{cases}$$

By definition, we have $g(v) \equiv f(v) \pmod 2$ for all $v \in V(G)$. It is sufficient for us to show that G contains a (g, f)-parity factor. Indirectly, suppose that G contains no (g, f)-parity factors. By Theorem 1.4, there exist two disjoint subsets S and T such that

$$\eta(S,T) = f(S) - g(T) + \sum_{x \in T} d_{G-S}(x) - \tau(S,T) < 0,$$

where $\tau(S,T)$ denotes the number of g-odd components of $G - S - T$. Since $f(V(G))$ is even, by parity, we have

$$\eta(S,T) = f(S) - g(T) + \sum_{x \in T} d_{G-S}(x) - \tau(S,T) \leq -2. \tag{1}$$

Hence $S \cup T \neq \emptyset$. We choose S and T such that $S \cup T$ is minimal. Let $A = V(G) - S - T$.

Claim 1. $T \subseteq \{u\}$.

Otherwise, let $v \in T - u$ and $T' = T - v$. We have

$$\begin{aligned}
\eta(S,T') &= f(S) - g(T') + \sum_{x \in T'} d_{G-S}(x) - \tau(S,T') \\
&= f(S) - (g(T) - g(v)) + \left(\sum_{x \in T} d_{G-S}(x) - d_{G-S}(v) \right) - \tau(S,T') \\
&\le f(S) - g(T) + \left(\sum_{x \in T} d_{G-S}(x) - d_{G-S}(v) \right) - (\tau(S,T) - e_G(v,A)) + g(v) \\
&\le f(S) - g(T) + \sum_{x \in T} d_{G-S}(x) - \tau(S,T) + (g(v) - d_{G-S}(v) + e_G(v,A)) \\
&= \eta(S,T) + (g(v) - d_{G-S}(v) + e_G(v,A)) \\
&\le \eta(S,T) - (d_{G-S}(v) - e_G(v,A)) \\
&\le \eta(S,T) \le -2,
\end{aligned}$$

contradicting the choice of S and T.

Claim 2. $S \subseteq N_G^{\delta}(u)$.

Otherwise, suppose that $S - N_G^{\delta}(u) \neq \emptyset$ and let $v \in S - N_G^{\delta}(u)$. Let $S' = S - v$. We have

$$\begin{aligned}
\eta(S',T) &= f(S') - g(T) + \sum_{x \in T} d_{G-S'}(x) - \tau(S',T) \\
&= (f(S) - f(v)) - g(T) + \left(\sum_{x \in T} d_{G-S}(x) + e_G(v,T) \right) - \tau(S',T) \\
&\le f(S) - g(T) + \sum_{x \in T} d_{G-S}(x) - (\tau(S,T) - e_G(v,A)) - f(v) + e_G(v,T) \\
&= f(S) - g(T) + \sum_{x \in T} d_{G-S}(x) - \tau(S,T) + (e_G(v,T) + e_G(v,A) - f(v)) \\
&\le \eta(S,T) + (d_G(v) - f(v)) \\
&\le \eta(S,T) \le -2,
\end{aligned}$$

contradicting the choice of S and T again.

We write $\tau(S,T) = \tau$. By Claims 1 and 2, we have

$$\begin{aligned}
\eta(S,T) &= f(S) - g(T) + \sum_{x \in T} d_{G-S}(x) - \tau \\
&= (\delta - 2)|S| - \delta|T| + |T|(\delta - |S|) - \tau \quad \text{(by Claims 1 and 2)} \\
&= (\delta - 2 - |T|)|S| - \tau \quad \text{(by Claim 1)} \\
&\le -2,
\end{aligned}$$

i.e.,

$$(\delta - 2 - |T|)|S| + 2 \leq \tau, \tag{2}$$

which implies $\tau \geq 2$ since $|T| \leq 1$.

Since $G - u$ is connected, we may see that $S \neq \emptyset$. Note that G is 3-edge-connected, by Claims 1 and 2, we have

$$\begin{aligned}
3\tau &\leq e_G(A, S \cup T) \\
&= e_G(A, S) + e_G(A, T) \\
&\leq (\delta - |T|)|S| + |T|(\delta - |S|) \quad \text{(by Claims 1 and 2)} \\
&= (\delta - 2|T|)|S| + |T|\delta,
\end{aligned}$$

i.e.,

$$3\tau \leq (\delta - 2|T|)|S| + |T|\delta. \tag{3}$$

Combining (2) and (3), we may see that

$$\delta|T| \geq (2\delta - |T| - 6)|S| + 6. \tag{4}$$

If $\delta \geq 4$, then we have

$$\begin{aligned}
\delta &\geq \delta|T| \quad \text{(since } |T| \leq 1) \\
&\geq (2\delta - |T| - 6)|S| + 6 \quad \text{(since } |S| \geq 1) \\
&\geq 2\delta - |T| \\
&\geq 2\delta - 1,
\end{aligned}$$

a contradiction. So we may assume that $\delta = 3$. Note that $|S| \leq |N_G^{\delta}(u)| \leq 2$. By (4), we have

$$3 = \delta \geq \delta|T| \geq -|T||S| + 6 \geq 4, \tag{5}$$

a contradiction again.

This completes the proof. $\square$

Lemma 2.3 *Let G be a bipartite graph with bipartition (U, W), where $|U| \equiv |W| \equiv 1 \pmod 2$. Let $\delta(G) = \delta$ and $u \in U$ such that $d_G(u) = \delta$. If one of the following two conditions holds, then G contains a factor F such that $d_F(u) = 0$, $d_F(x) \equiv 1 \pmod 2$ for all $x \in U - u$, $d_F(y) \equiv 0 \pmod 2$ for all $x \in W$ and $d_F(y) \geq 2$ for all $y \in N_G(u)$.*

(i) $\delta(G) \geq 4$, G is 3-edge-connected and $G - u$ is connected.

(ii) $\delta(G) = 3$, G is 3-edge-connected and there exists a vertex $v \in N_G(u)$ such that $d_G(v) > 3$.

Proof: Let M be an even integer such that $M \geq \triangle(G)$. Let $g, f : V(G) \to Z$ such that

$$g(x) = \begin{cases} 0 & \text{if } x \in (\{u\} \cup W) - N_G(u), \\ 2 & \text{if } x \in N_G(u), \\ -1 & \text{if } x \in U - u, \end{cases}$$

and

$$f(x) = \begin{cases} M & \text{if } x \in W \\ 0 & \text{if } x = u, \\ M+1 & \text{if } x \in U - u. \end{cases}$$

Clearly, $g(v) \equiv f(v) \pmod 2$ for all $v \in V(G)$ and $g(V(G))$ is even. It is also sufficient for us to show that G contains a (g, f)-parity factor.

Indirectly, suppose that G contains no (g, f)-parity factors. By Theorem 1.4, there exist two disjoint subsets S and T such that

$$\eta(S,T) = f(S) - g(T) + \sum_{x \in T} d_{G-S}(x) - \tau(S,T) \leq -2, \tag{6}$$

where $\tau(S,T)$ denotes the number of g-odd components of $G - S - T$. We choose S and T such that $S \cup T$ is minimal. Let $A = V(G) - S - T$.

Claim 1. $S \subseteq \{u\}$.

Otherwise, suppose that there exists a vertex $v \in S - u$. Let $S' = S - v$. Then we have

$$\begin{aligned}
\eta(S',T) &= f(S') - g(T) + \sum_{y \in T} d_{G-S'}(y) - \tau(S',T) \\
&= (f(S) - f(v)) - g(T) + \left(\sum_{y \in T} d_{G-S}(y) + e_G(v,T) \right) - \tau(S',T) \\
&\leq f(S) - g(T) + \sum_{y \in T} d_{G-S}(y) - f(v) + e_G(v,T) - (\tau(S,T) - e_G(v,A)) \\
&= \eta(S,T) - (f(v) - e_G(v,T) - e_G(v,A)) \\
&\leq \eta(S,T) - (f(v) - d_G(v)) \\
&\leq \eta(S,T) \leq -2,
\end{aligned}$$

contradicting the the choice of $S \cup T$.

Claim 2. $T \subseteq N_G(u)$.

Otherwise, suppose that $T - N_G(u) \neq \emptyset$. Let $x \in T - N_G(u)$ and let $T' = T - x$. Then we have

$$\begin{aligned}
\eta(S,T') &= f(S) - g(T') + \sum_{y\in T'} d_{G-S}(y) - \tau(S,T') \\
&= f(S) - (g(T) - g(x)) + \left(\sum_{y\in T} d_{G-S}(y) - d_{G-S}(x)\right) - \tau(S,T') \\
&\leq f(S) - (g(T) - g(x)) + \left(\sum_{y\in T} d_{G-S}(y) - d_{G-S}(x)\right) - (\tau(S,T) - e_G(x,A)) \\
&= f(S) - g(T) + \sum_{y\in T} d_{G-S}(y) - \tau(S,T) - (d_{G-S}(x) - e_G(x,A)) + g(x) \\
&= \eta(S,T) - (d_{G-S}(x) - e_G(x,A)) + g(x) \\
&\leq \eta(S,T) \leq -2,
\end{aligned}$$

contradicting the choice of $S \cup T$.

By Claims 1 and 2, we may see that $f(S) = 0$ and $g(T) = 2|T|$. For simplicity, we write $\tau(S,T) = \tau$. By (6), we see that

$$\tau \geq \sum_{x\in T}(d_G(x) - |S|) - 2|T| + 2, \tag{7}$$

which implies

$$\tau \geq \sum_{x\in T}(\delta - 1) - 2|T| + 2 \geq 2. \tag{8}$$

Note that $G - u$ is connected, so we have $|T| \geq 1$. Since G is 3-edge-connected, we have

$$\begin{aligned}
3\tau &\leq \sum_{x\in T}(d_G(x) - |S|) + (\delta - |T|)|S| \\
&= \sum_{x\in T} d_G(x) + (\delta - 2|T|)|S|,
\end{aligned}$$

i.e.,

$$3\tau \leq \sum_{x\in T} d_G(x) + (\delta - 2|T|)|S|. \tag{9}$$

Inequalities (7) and (9) implies

$$\begin{aligned}
2\sum_{x\in T} d_G(x) + 6 &\leq |S||T| + 6|T| + \delta|S| \quad \text{(since } |S| \leq 1 \text{ and } |T| \geq 1) \\
&\leq 7|T| + \delta,
\end{aligned}$$

i.e.,

$$7|T| \geq 2\sum_{x\in T} d_G(x) + 6 - \delta. \tag{10}$$

If $\delta \geq 4$, by (10), it follows

$$7|T| \geq 6 + \delta(2|T| - 1) \geq 8|T| + 2,$$

a contradiction. So we may assume that $\delta = 3$. By condition (ii), $\sum_{x\in T} d_G(x) \geq 3|T| + 1$. Combining (10),

$$\begin{aligned} 7|T| &\geq 2\sum_{x\in T} d_G(x) + 6 - \delta \\ &\geq 2(3|T| + 1) + 3, \end{aligned}$$

which implies $|T| \geq 5$, a contradiction since $|T| \leq |N_G(u)| \leq 3$.

This completes the proof. □

Proof of Theorem 1.2: By Theorem 1.5 and Corollary 2.1, we can assume that both $|A|$ and $|B|$ are odd.

Firstly, we consider $\mathcal{S} = \{0, 1\}$. If G is 3-regular, by Theorem 1.6, then G admits a vertex-coloring 2-edge-weighting. By Theorem 1.7, G also admits a vertex-coloring $\{0, 1\}$-edge-weighting. So we can assume that $\delta(G) \geq 3$ and G is not 3-regular. If $\delta(G) = 3$, since G is 3-edge-connected, then $G - x$ is connected for every vertex x of G with degree three. Hence there exists a vertex v with degree three such that $N_G(v)$ contains a vertex with degree at least four. Let

$$u^* = \begin{cases} u & \text{if } \delta \geq 4, \\ v & \text{if } \delta = 3. \end{cases}$$

Without loss generality, we may assume that $u^* \in U$ and so it is a vertex satisfying the conditions of Lemma 2.3. Hence by Lemma 2.3, G contains a factor F, which satisfies the following three conditions.

(i) $d_F(u^*) = 0$;

(ii) $d_F(x) \equiv 1 \pmod 2$ for all $x \in U - u^*$;

(iii) $d_F(y) \equiv 0 \pmod 2$ for all $x \in W$ and $d_F(y) \geq 2$ for all $y \in N_G(u^*)$.

Clearly, $d_F(x) \neq d_F(y)$ for all $xy \in E(G)$. We assign weight 1 for each edge of $E(F)$ and weight 0 for each edge of $E(G) - E(F)$. Then we obtain a vertex-coloring $\{0, 1\}$-edge-weighting of graph G.

Secondly, we show that G admits a vertex-coloring 2-edge-weighting. By Theorem 1.6, we may assume that G is not 3-regular. If $\delta = 3$, since G is 3-edge-connected, then G contains a vertex v' such that $d_G(v') = 3$, $G - v'$ is connected and $|N_G^{\delta}(v')| \leq 2$. Let

$$u^* = \begin{cases} u & \text{if } \delta \geq 4, \\ v' & \text{if } \delta = 3. \end{cases}$$

Then u^* is a vertex satisfying the conditions of Lemma 2.2. Hence by Lemma 2.2, G contains a factor F such that

(i) $d_F(u^*) = \delta$;
(ii) $d_F(x) \equiv \delta \pmod 2$ for all $x \in W$ and $d_F(x) \leq \delta - 2$ for all $x \in N_G^{\delta}(u^*)$;
(iii) $d_F(y) \equiv \delta + 1 \pmod 2$ for all $y \in U - u^*$.

Let $w : E(G) \to \{1, 2\}$ be a 2-edge-weighting such that $w(e) = 1$ for each $e \in E(F)$ and $w(e') = 2$ for each $e' \in E(G) - E(F)$. Clearly, $c(u^*) = \delta$. If $y \in N_G^{\delta}(u^*)$, since there exists an edge $e \sim y$ such that $e \notin E(F)$, then $c(y) = \sum_{e \sim y} w(e) > \delta$. If $y \in N_G(u^*) - N_G^{\delta}(u^*)$, then $c(y) \geq d_G(y) > \delta$. Hence $c(y) \neq c(u^*)$ for all $y \in N_G(u^*)$. For each $xy \in E(G)$, where $x \in U - u^*$ and $y \in W$, by the choice of F, we have $c(x) \equiv \delta + 1 \pmod 2$ and $c(y) \equiv \delta \pmod 2$. Hence w is a vertex-coloring $\{1, 2\}$-edge-weighting of the graph G.

This completes the proof. □

Corollary 2.4 *Let G be a 3-edge-connected bipartite graph. If $3 \leq \delta(G) \leq 5$, then G admits a vertex-coloring $\mathcal{S}$-edge-weighting for $\mathcal{S} \in \{\{0, 1\}, \{1, 2\}\}$.*

Proof: Since $3 \leq \delta \leq 5$ and G is 3-edge-connected, then for every vertex v of degree δ, $G - v$ is connected. By Lemma 2.2 and Theorem 1.2, with the same proof, G admits a vertex-coloring $\mathcal{S}$-edge-weighting for $\mathcal{S} \in \{\{0, 1\}, \{1, 2\}\}$. □

3 Conclusions

In this paper, we prove that every 2-connected and 3-edge-connected bipartite graph admits a vertex-coloring $\mathcal{S}$-edge-weighting for $\mathcal{S} \in \{\{0, 1\}, \{1, 2\}\}$. The generalized θ-graphs is 2-connected and has a vertex-coloring 3-edge-weighting but not vertex-coloring $\{0, 1\}$-edge-weighting or vertex-coloring 2-edge-weighting. So it is an interesting problem to classify all 2-connected bipartite graphs admitting a vertex-coloring $\mathcal{S}$-edge-weighting. Since the parity-factor problem is polynomial, then there exists a polynomial algorithm to find a vertex-coloring $\mathcal{S}$-edge-weighting of bipartite graphs satisfying the conditions of Theorem 1.2.

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