Interval Observers for Linear Impulsive Systems

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Abstract The problem of interval observer design is studied for a class of linear hybrid systems. Several observers are designed oriented on different conditions of positivity and stability for estimation error dynamics. Efficiency of the proposed approach is demonstrated by computer experiments for academic and bouncing ball systems.

Keywords: Interval observers, Hybrid systems, Stability analysis

1. INTRODUCTION

There are many approaches dealing with the design techniques for state observers Besançon (2007); Meurer et al. (2005). Frequently, these methods are based on (partial) linearity of the observed system, since analysis and design of stability and performance for linear systems are more developed. If it comes to take into account the presence of a disturbance or uncertain parameters, then synthesis of a conventional estimator (whose estimates are converging to the true values of the state) may be complicated Efimov et al. (2013a); Besançon (2007); Degue et al. (2016). In such a case the problem of pointwise estimation can be substituted by the interval one, then using input-output measurements an observer has to estimate the set of admissible values (interval) for the state at each instant of time Gouzé et al. (2000). An advantage of interval observer is that it allows many types of uncertainties to be taken into account in the system. The interval observer design techniques have been developed for many types of models: continuous-time Mazenc and Bernard (2011); Raïssi et al. (2012), discrete-time Efimov et al. (2013a); Mazenc et al. (2013); Efimov et al. (2013c); Mazenc et al. (2014), time-delay Mazenc et al. (2012); Efimov et al. (2013b, 2015b) and algebraic-differential Efimov et al. (2015a) ones.

Continuing this line, the problem of design of interval observers for a class of linear hybrid systems Branicky (2005); Goebel et al. (2012) is studied in this paper. Impulsive systems are an important class of hybrid systems that includes both continuous and discrete event dynamics Briat (2013). The continuous dynamics are generally represented by differential equations and the discrete one by switching laws, which govern discontinuous jumps of continuous states Goebel et al. (2012); Fichera et al. (2013); Kim et al. (2014). The instants of these jumps can be time-dependent or state-dependent Branicky (2005); Goebel et al. (2012); Kim et al. (2014). The main peculiarity of interval observation is that it is necessary to ensure positivity of the estimation error dynamics in addition to their stability. Since two types of dynamics (continuous and discrete) are present in the hybrid systems, then the conditions of positivity for these two cases (see Efimov and Raïssi (2015) for examples) have to be combined, which leads to variety of the applicability conditions and design structures proposed in this work. Only linear systems where impulse instants can be inferred from the measured output, or by using a sensor that detects mode transitions are considered.

The outline of the paper is as follows. Some basic facts from the theories of interval estimation and hybrid systems are given in Section 2. In Section 3 the main results are described and proven. In Section 4 these results are applied to some examples of linear impulsive systems, including a bouncing ball model.

2. PRELIMINARIES

2.1 Notation

In this work, the real and integer numbers are denoted by $\mathbb{R}$ and $\mathbb{Z}$ respectively, $\mathbb{R}_+ = \{\tau \in \mathbb{R} : \tau \geq 0\}$ and $\mathbb{Z}_+ = \mathbb{Z} \cap \mathbb{R}_+$, $|x|$ is stated for the Euclidean norm of a vector $x \in \mathbb{R}^n$. For a measurable and locally essentially bounded input $u : \mathbb{R}_+ \to \mathbb{R}$ the symbol $||u||_{[t_0,t_1]}$ denotes its L_∞ norm:

$$||u||_{[t_0,t_1]} = ess \sup_{t\in[t_0,t_1]} |u(t)|,$$

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if $t_1 = +\infty$ then we will simply write $||u||$. We will denote as $\mathcal{L}_\infty$ the set of all inputs u with the property $||u|| < \infty$. We will denote the sequence of integers $1, ..., n$ as $\overline{1,n}$. $E_{n\times m}$ denotes the matrix with all entries equal 1 (with dimensions $n \times m$). For a matrix $A \in \mathbb{R}^{n\times n}$ the vector of its eigenvalues is denoted as $\lambda(A)$. The relation $P \succ 0$ ($P \succeq 0$) for a symmetric matrix $P \in \mathbb{R}^{n\times n}$ means that it is positive (nonnegative) definite, the set of such $n \times n$ matrices will be denoted by $S^n_{\succ 0}$.

2.2 Interval analysis

For two vectors $x_1, x_2 \in \mathbb{R}^n$ or matrices $A_1, A_2 \in \mathbb{R}^{n\times n}$, the relations $x_1 \le x_2$ and $A_1 \le A_2$ are understood elementwise. Given a matrix $A \in \mathbb{R}^{m\times n}$, define $A^+ = \max\{0, A\}$, $A^- = A^+ - A$ (similarly for vectors) and denote the matrix of absolute values of all elements by $|A| = A^+ + A^-$.

Lemma 1. Efimov et al. (2012) Let $x \in \mathbb{R}^n$ be a vector variable, $\underline{x} \le x \le \overline{x}$ for some $\underline{x}, \overline{x} \in \mathbb{R}^n$, and $A \in \mathbb{R}^{m\times n}$ be a constant matrix, then

$$A^+\underline{x} - A^-\overline{x} \le Ax \le A^+\overline{x} - A^-\underline{x}. \quad (1)$$

2.3 Nonnegative continuous-time linear systems

A matrix $A \in \mathbb{R}^{n\times n}$ is called Hurwitz if all its eigenvalues have negative real parts, it is called Metzler if all its elements outside the main diagonal are nonnegative, *i.e.* $A_{i,j} \ge 0$ for $1 \le i \ne j \le n$. Any solution of the linear system

$$\dot{x} = Ax + B\omega(t),\ \omega : \mathbb{R}_+ \to \mathbb{R}^q_+, \quad (2)$$
$$y = Cx + D\omega(t),$$

with $x \in \mathbb{R}^n$, $y \in \mathbb{R}^p$ and a Metzler matrix $A \in \mathbb{R}^{n\times n}$, is elementwise nonnegative for all $t \ge 0$ provided that $x(0) \ge 0$ and $B \in \mathbb{R}^{n\times q}_+$ Farina and Rinaldi (2000); Smith (1995). The output solution $y(t)$ is nonnegative if $C \in \mathbb{R}^{p\times n}_+$ and $D \in \mathbb{R}^{p\times q}_+$. Such dynamical systems are called cooperative (monotone) or nonnegative if only initial conditions in $\mathbb{R}^n_+$ are considered Farina and Rinaldi (2000); Smith (1995).

For a Metzler matrix $A \in \mathbb{R}^{n\times n}$ its stability can be checked verifying a Linear Programming (LP) problem

$$A^{\mathrm{T}}\lambda < 0$$

for some $\lambda \in \mathbb{R}^n_+ \setminus \{0\}$.

2.4 Nonnegative discrete-time linear systems

A matrix $A \in \mathbb{R}^{n\times n}$ is called Schur stable if all its eigenvalues have absolute value less than one, it is called nonnegative if all its elements are nonnegative (*i.e.* $A \ge 0$). Any solution of the system

$$x_{t+1} = Ax_t + B\omega_t,\ \omega : \mathbb{Z}_+ \to \mathbb{R}^m_+,\ t \in \mathbb{Z}_+$$

with $x_t \in \mathbb{R}^n$ and nonnegative matrices $A \in \mathbb{R}^{n\times n}_+$ and $B \in \mathbb{R}^{n\times m}_+$, is elementwise nonnegative for all $t \in \mathbb{Z}_+$ provided that $x(0) \ge 0$ Hirsch and Smith (2005). Such a system is called cooperative (monotone) or nonnegative Hirsch and Smith (2005).

Lemma 2. Farina and Rinaldi (2000) A matrix $A \in \mathbb{R}^{n\times n}_+$ is Schur stable iff there exists a diagonal matrix $P \in S^n_{\succ 0}$ such that $A^{\mathrm{T}}PA - P \prec 0$.

2.5 Stability of hybrid systems under ranged dwell-time

Consider a hybrid (impulsive) linear system

$$\dot{x}(t) = Ax(t) + b(t) \quad \forall t \in [t_i, t_{i+1}),\ i \in \mathbb{Z}_+, \quad (3)$$
$$x(t_{i+1}) = Gx(t^-_{i+1}) + d(t_{i+1}) \quad \forall i \ge 1,$$
$$y(t) = Cx(t) + v(t),$$

where $x(t) \in \mathbb{R}^n$ is the state vector and $x(t^-_{i+1})$ is the left-sided limit of $x(t)$ for $t \to t_{i+1}$; $A, G \in \mathbb{R}^{n\times n}$; $b : \mathbb{R}_+ \to \mathbb{R}^n$, $b \in \mathcal{L}_\infty$ is the input $\forall t \in [t_i, t_{i+1})$; $d : \mathbb{R}_+ \to \mathbb{R}^n$, $d \in \mathcal{L}_\infty$ is the input at time instants $t_{i+1}\,\forall i \ge 1$; $y(t) \in \mathbb{R}^p$ is the output signal available for measurements; $v \in \mathcal{L}_\infty$ is the measurement noise; $C \in \mathbb{R}^{p\times n}$. The sequence of impulse events t_i with $i \in \mathbb{Z}_+$ is assumed to be positively incremental, *i.e.* $T_i = t_{i+1} - t_i > 0$ and $t_0 = 0$.

Theorem 3. Briat (2013) Consider system (3) with $||b|| = ||d|| = 0$ and a ranged dwell-time $T_i \in [T_{\min}, T_{\max}]$ for all $i \in \mathbb{Z}_+$, where $0 \le T_{\min} \le T_{\max} < +\infty$ are given constants. Then it is asymptotically stable provided that there exists a matrix $P \in S^n_{\succ 0}$ such that for all $\theta \in [T_{\min}, T_{\max}]$

$$G^{\mathrm{T}}e^{A^{\mathrm{T}}\theta}Pe^{A\theta}G - P \prec 0. \quad (4)$$

The proof of the above theorem is based on the fact that in this case $W(x) = x^{\mathrm{T}}Px$ is a Lyapunov function for (3) at discrete instants of time t_i. Following Hespanha et al. (2005); Dashkovskiy and Mironchenko (2013), robustness with respect to the inputs b and d can be proven (see the definition of the input-to-state stability (ISS) given in those works):

Corollary 4. Consider system (3) with a ranged dwell-time $T_i \in [T_{\min}, T_{\max}]$ for all $i \in \mathbb{Z}_+$, where $0 \le T_{\min} \le T_{\max} < +\infty$ are given constants. Then it is ISS provided that there exists a matrix $P \in S^n_{\succ 0}$ such that for all $\theta \in [T_{\min}, T_{\max}]$ the LMI (4) is satisfied.

This result implies that (3) has bounded solutions for any bounded inputs b and d if the LMI (4) is valid.

3. MAIN RESULTS

We will need the following assumptions for the system (3):

Assumption 1. The state $x(t)$ is bounded, *i.e.* $x \in \mathcal{L}_\infty$, and $T_i = t_{i+1} - t_i \in [T_{\min}, T_{\max}]$ for all $i \in \mathbb{Z}_+$, where $0 \le T_{\min} \le T_{\max} < +\infty$ are given constants.

Assumption 2. There exist matrices $L \in \mathbb{R}^{n\times p}$, $M \in \mathbb{R}^{n\times p}$, $P \in S^n_{\succ 0}$ such that:

i) the LMI

$$(G - MC)^{\mathrm{T}}e^{(A-LC)^{\mathrm{T}}\theta}Pe^{(A-LC)\theta}(G - MC) - P \prec 0 \quad (5)$$

holds for all $\theta \in [T_{\min}, T_{\max}]$;

ii) the matrix $(A - LC)$ is Metzler;

iii) the matrix $(G - MC)$ is nonnegative.

When Assumption 2.i holds, the quadratic form $W(x) = x^{\mathrm{T}}Px$ is a discrete-time Lyapunov function for the LTI discrete-time system $z_{i+1} = e^{(A-LC)\theta}(G - MC)z_i$ for all $\theta \in [T_{\min}, T_{\max}]$ and $i \in \mathbb{Z}_+$ by Theorem 3.

Assumption 3. Let

i) two functions $\underline{b}$, $\bar{b} : \mathbb{R}_+ \to \mathbb{R}^n$, $\underline{b}$, $\bar{b} \in \mathcal{L}_\infty$ are given such that

$$\underline{b}(t) \le b(t) \le \bar{b}(t) \quad \forall t \in \mathbb{R}_+;$$

ii) two functions $\underline{d}$, $\bar{d} : \mathbb{R}_+ \to \mathbb{R}^n$, $\underline{d}, \bar{d} \in \mathcal{L}_\infty$ are given such that

$$\underline{d}(t) \le d(t) \le \bar{d}(t) \quad \forall t \in \mathbb{R}_+;$$

iii) the constant $0 \le V \le +\infty$ is given such that $||v|| < V$.

Assumption 1 is introduced since the problem of control design is not considered in this work. Furthermore this assumption is common in the existing literature concerning observer design. Assumptions 2.ii and 2.iii are essential for the approach but are rather restrictive. They will be relaxed later. Assumptions 3.i and 3.ii state that the inputs of the hybrid system (3) are known up to some interval errors $\bar{b}(t) - \underline{b}(t)$ and $\bar{d}(t) - \underline{d}(t)$. Assumption 3.iii suggests an upper bound V for the noise v amplitude.

Under the introduced assumptions an interval observer equations for (3) take the form:

$$\begin{aligned}
\dot{\underline{x}}(t) &= (A - LC)\underline{x}(t) + Ly(t) + \underline{b}(t) \\
&\quad -\overline{L}V \; \forall t \in [t_i, t_{i+1}), \\
\underline{x}(t_{i+1}) &= (G - MC)\underline{x}(t_{i+1}^-) + My(t_{i+1}) \\
&\quad +\underline{d}(t_{i+1}) - \overline{M}V, \\
\dot{\overline{x}}(t) &= (A - LC)\overline{x}(t) + Ly(t) + \bar{b}(t) \\
&\quad +\overline{L}V \; \forall t \in [t_i, t_{i+1}), \\
\overline{x}(t_{i+1}) &= (G - MC)\overline{x}(t_{i+1}^-) + My(t_{i+1}) \\
&\quad +\bar{d}(t_{i+1}) + \overline{M}V,
\end{aligned} \tag{6}$$

$\forall i \in \mathbb{Z}_+$, where $\underline{x}(t) \in \mathbb{R}^n$ and $\overline{x}(t) \in \mathbb{R}^n$ are respectively the lower and the upper interval estimates for the state $x(t)$, $\overline{L} = |L|E_{p\times 1}$ and $\overline{M} = |M|E_{p\times 1}$.

Theorem 5. Let assumptions 1–3 be satisfied. Then for all $t \in \mathbb{R}_+$ the estimates $\underline{x}(t)$ and $\overline{x}(t)$ given by (6) are bounded and

$$\underline{x}(t) \le x(t) \le \bar{x}(t) \tag{7}$$

provided that $\underline{x}(0) \le x(0) \le \bar{x}(0)$.

All proofs are skipped due to the space limitation.

Remark 6. The matrices A and G may be uncertain time-varying but this work is devoted to linear impulsive systems where A and G are constant matrices. Only the presence of bounded uncertain time-varying perturbations $b(t)$, $g(t)$ and $v(t)$ are considered in this work.

The imposed requirement that the matrices $A - LC$ and $G - MC$ are Metzler and nonnegative, respectively, is rather restrictive. In order to relax assumptions 2.ii and 2.iii, let us suggest the following.

Assumption 4. There exist a Metzler matrix R, a matrix $T \in \mathbb{R}_+^{n\times n}$ and a matrix $P \in S_{\succ 0}^n$ such that the LMI

$$T^{\mathrm{T}} e^{R^{\mathrm{T}}\theta} P e^{R\theta} T - P \prec 0 \tag{8}$$

is satisfied for all $\theta \in [T_{\min}, T_{\max}]$.

There exist a matrix $L \in \mathbb{R}^{n\times p}$ and a matrix $M \in \mathbb{R}^{n\times p}$ such that $\lambda(A-LC) = \lambda(R)$, $\lambda(G-MC) = \lambda(T)$, the pairs $(A-LC, e_1)$, (R, e_2), $(G-MC, e_3)$, (T, e_4) are observable for some $e_j \in \mathbb{R}^{1\times n}$ with $j = \overline{1,4}$.

When Assumption 4 holds, the quadratic form $W(x) = x^{\mathrm{T}}Px$ is a Lyapunov function for linear discrete-time system $z_{i+1} = e^{R\theta}Tz_i$ for all $\theta \in [T_{\min}, T_{\max}]$ and $i \in \mathbb{Z}_+$ by Theorem 3. In addition, comparing with assumptions 2.ii and 2.iii, in Assumption 4 it is proposed that the matrices $A - LC$ and $G - MC$ are similar to given Metzler and nonnegative matrices R and T respectively Raïssi et al. (2012), with differing similarity transformation matrices $S_1 \in \mathbb{R}^{n\times n}$ and $S_2 \in \mathbb{R}^{n\times n}$ (*i.e.* $S_1^{-1}(A-LC)S_1 = R$ and $S_2^{-1}(G-MC)S_2 = T$). The key idea of the following design of an interval observer is how to combine these different transformations of coordinate S_1 and S_2 (denote $S = (S_1^{-1}S_2)^{-1}$), without introducing an auxiliary restriction.

Theorem 7. Let assumptions 1, 3 and 4 be satisfied. Then for all $t \in \mathbb{R}_+$ the estimates $\underline{x}(t)$ and $\bar{x}(t)$ are bounded and

$$\underline{x}(t) \le x(t) \le \bar{x}(t)$$

provided that $\underline{x}(0) \le x(0) \le \bar{x}(0)$, where for all $i \in \mathbb{Z}_+$:

$$\begin{aligned}
\underline{x}(t) &= S_1^+\underline{z_1}(t) - S_1^-\overline{z_1}(t), \\
\bar{x}(t) &= S_1^+\overline{z_1}(t) - S_1^-\underline{z_1}(t), \\
\dot{\underline{z_1}}(t) &= R\underline{z_1}(t) + F_1y(t) - \overline{F_1}V + (S_1^{-1})^+\underline{b}(t) \\
&\quad -(S_1^{-1})^-\bar{b}(t) \; \forall t \in [t_i, t_{i+1}), \\
\underline{z_2}(t_{i+1}^-) &= S^+\underline{z_1}(t_{i+1}^-) - S^-\overline{z_1}(t_{i+1}^-), \\
\underline{z_2}(t_{i+1}) &= T\underline{z_2}(t_{i+1}^-) + F_2y(t_{i+1}) - \overline{F_2}V \\
&\quad +(S_2^{-1})^+\underline{d}(t_{i+1}) - (S_2^{-1})^-\bar{d}(t_{i+1}), \\
\underline{z_1}(t_{i+1}) &= (S^{-1})^+\underline{z_2}(t_{i+1}) - (S^{-1})^+\overline{z_2}(t_{i+1}), \\
\dot{\overline{z_1}}(t) &= R\overline{z_1}(t) + F_1y(t) + \overline{F_1}V + (S_1^{-1})^+\bar{b}(t) \\
&\quad -(S_1^{-1})^-\underline{b}(t) \; \forall t \in [t_i, t_{i+1}), \\
\overline{z_2}(t_{i+1}^-) &= S^+\overline{z_1}(t_{i+1}^-) - S^-\underline{z_1}(t_{i+1}^-), \\
\overline{z_2}(t_{i+1}) &= T\overline{z_2}(t_{i+1}^-) + F_2y(t_{i+1}) + \overline{F_2}V \\
&\quad +(S_2^{-1})^+\bar{d}(t_{i+1}) - (S_2^{-1})^-\underline{d}(t_{i+1}), \\
\overline{z_1}(t_{i+1}) &= (S^{-1})^+\overline{z_2}(t_{i+1}) - (S^{-1})^+\underline{z_2}(t_{i+1}), \\
\underline{z_1}(0) &= (S_1^{-1})^+\underline{x}(0) - (S_1^{-1})^-\overline{x}(0), \\
\overline{z_1}(0) &= (S_1^{-1})^+\overline{x}(0) - (S_1^{-1})^-\underline{x}(0), \\
\underline{z_2}(0) &= (S_2^{-1})^+\underline{x}(0) - (S_2^{-1})^-\overline{x}(0), \\
\overline{z_1}(0) &= (S_2^{-1})^+\overline{x}(0) - (S_2^{-1})^-\underline{x}(0),
\end{aligned} \tag{9}$$

where $F_1 = S_1^{-1}L$, $\overline{F_1} = |F_1|E_{p\times 1}$, $F_2 = S_2^{-1}M$ and $\overline{F_2} = |F_2|E_{p\times 1}$.

There is another possibility for an interval observer construction avoiding the restrictions of Assumption 2, but with more conservative stability conditions. To this end, consider the following assumption.

Assumption 5. There exist matrices $L \in \mathbb{R}^{n\times p}$, $M \in \mathbb{R}^{n\times p}$ and $P \in S_{\succ 0}^n$ such that the LMI

$$J^{\mathrm{T}} e^{U^{\mathrm{T}}\theta} P e^{U\theta} J - P \prec 0 \tag{10}$$

is satisfied for all $\theta \in [T_{\min}, T_{\max}]$ and $U = \begin{bmatrix} D_0 & D_1 \\ D_1 & D_0 \end{bmatrix}$, $J = \begin{bmatrix} (G-MC)_p & (G-MC)_n \\ (G-MC)_p & (G-MC)_n \end{bmatrix}$ for $A - LC = D_0 - D_1$ where D_0 is Metzler and $D_1, (G-MC)_p, (G-MC)_n \in \mathbb{R}_+^{n\times n}$.

Comparing with Assumption 4, here by construction the matrices U and J are Metzler and nonnegative respectively, *i.e.* these matrices can always be constructed satisfying these properties for any $A - LC$ and $G - MC$ (a possible but not unique choice is $(G-MC)_p = (G-MC)^+$ and $(G - MC)_n = (G - MC)^-$, for example), then there is no need in transformations of coordinates S_1 and S_2. However, the main restriction is on the stability of such U and J, and the conditions of stability are formulated by LMI (10) following Theorem 3. The following result can be proven.

Theorem 8. Let assumptions 1, 3 and 5 be satisfied. Then for all $t \in \mathbb{R}_+$ the estimates $\underline{x}(\mathrm{t})$ and $\bar{x}(\mathrm{t})$ are bounded and

$$\underline{x}(t) \leq x(t) \leq \bar{x}(t)$$

provided that $\underline{x}(0) \leq x(0) \leq \bar{x}(0)$, where for all $i \in \mathbb{Z}_+$:

$$\begin{aligned}
\dot{\underline{x}}(t) &= D_0\underline{x}(t) - D_1\overline{x}(t) + Ly(t) + \underline{b}(t) \\
&\quad -\overline{L}V \ \forall t \in [t_i, t_{i+1}), \\
\underline{x}(t_{i+1}) &= (G - MC)_p\underline{x}(t_{i+1}^-) - (G - MC)_n\overline{x}(t_{i+1}^-) \\
&\quad +My(t_{i+1}) + \underline{d}(t_{i+1}) - \overline{M}V, \qquad (11)\\
\dot{\overline{x}}(t) &= D_0\overline{x}(t) - D_1\underline{x}(t) + Ly(t) + \overline{b}(t) \\
&\quad +\overline{L}V \ \forall t \in [t_i, t_{i+1}), \\
\overline{x}(t_{i+1}) &= (G - MC)_p\overline{x}(t_{i+1}^-) - (G - MC)_n\underline{x}(t_{i+1}^-) \\
&\quad +My(t_{i+1}) + \overline{d}(t_{i+1}) + \overline{M}V,
\end{aligned}$$

where $\overline{L} = |L|E_{p\times 1}$ and $\overline{M} = |M|E_{p\times 1}$.

The results of theorems 7 and 8 can be combined, *i.e.* only one transformation S_1 or S_2 can be used together with the decomposition from Assumption 5.

Remark 9. The conditions of theorems 5, 7 and 8 are infinitedimensional feasibility problems. In fact the LMIs 5, 8 and 10 are strongly nonlinear in the parameter θ, and for $\theta \in [T_{\min}, T_{\max}]$ these LMIs consist of an infinite number of LMIs. In order to solve them efficiently, we use Matlab YALMIP toolbox Löfberg (2004). The bisection method is used to find the interval $[T_{\min}, T_{\max}]$ where the LMIs 5, 8 and 10 are feasible.

4. EXAMPLES

In this section, we present three examples. The first and the third examples are academic linear impulsive systems and the second one is a bouncing ball.

4.1 Academic linear impulsive system

Consider the following system:

$$\begin{aligned}
\dot{x}(t) &= Ax(t) + b(t) \ \forall t \in [0,\, 5) \cup (5,\, 10) \cup (10, +\infty), \\
x(t) &= Gx(t^-) + d(t) \ \forall t \in \{5,\, 10\}, \\
y(t) &= Cx(t) + v(t),
\end{aligned}$$

where the matrices A, C and G are defined as follows Briat (2013):

$$A = \begin{bmatrix} -1 & 0 \\ 1 & -2 \end{bmatrix},\ C = [\,0\ \ 1\,],\ G = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix},$$

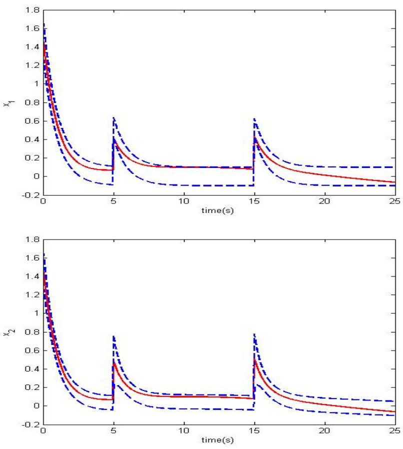

Figure 1. Results of the simulation for the academic linear impulsive system

and $x(t) \in \mathbb{R}^2$, $y(t) \in \mathbb{R}$ are the state and the output respectively. The signals $b(t)$, $d(t)$ and $v(t)$ are:

$$b(t) = \begin{bmatrix} \beta \sin(t) \\ \beta \sin(t) \end{bmatrix},\ d(t) = \begin{bmatrix} \delta \sin(t) \\ \delta \sin(t) \end{bmatrix},\ v(t) = V\sin(t),$$

where $\beta = 0.1$, $\delta = 0.3$ and $V = 0.03$ are known parameters. Thus,

$$\underline{b}(t) = \begin{bmatrix} -\beta \\ -\beta \end{bmatrix},\ \overline{b}(t) = \begin{bmatrix} \beta \\ \beta \end{bmatrix},$$
$$\underline{d}(t) = \begin{bmatrix} -\delta \\ -\delta \end{bmatrix},\ \overline{d}(t) = \begin{bmatrix} \delta \\ \delta \end{bmatrix}.$$

Assumption 3 is then satisfied. Assume that $||x|| < +\infty$ and Assumption 1 is valid. Assumption 2.ii is verified for $L = [\,0\ \ 1\,]^{\mathrm{T}}$: the matrix $A - LC = \begin{bmatrix} -1 & 0 \\ 1 & -3 \end{bmatrix}$ is Metzler. Assumption 2.iii is verified for $M = [\,1\ \ 2.8\,]^{\mathrm{T}}$: the matrix $G - MC = \begin{bmatrix} 2 & 0 \\ 1 & 0.2 \end{bmatrix}$ is nonnegative but not Schur stable. By applying Matlab YALMIP toolbox Löfberg (2004) to solve the LMI (5), we found that Assumption 2.i holds for all $\theta \in [0.6580, +\infty)$. Then the dynamics of the errors $\underline{e}(t) = x(t) - \underline{x}(t)$, $\overline{e}(t) = \overline{x}(t) - x(t)$ with ranged dwell-time $\theta \in [0.6580, +\infty)$ are ISS. Therefore all conditions of Theorem 5 are satisfied and the interval observer (6) solves the problem of interval state estimation. The results of simulation are shown in Fig 1, where the solid lines represent the states x_k, $k = 1, 2$ and the dash lines are used for the interval estimates $\underline{x_k}$ and $\overline{x_k}$.

4.2 Bouncing ball

Consider the case of vertical motion of a ball under gravity with a constant acceleration g. The dynamics are given by

$$\dot{p}(t) = v(t);\ \dot{v}(t) = -g,$$

where $p(t) \in \mathbb{R}_+$ is the position of the ball and $v(t) \in \mathbb{R}$ is its velocity, which is assumed to be downward. Upon hitting the ground at instant of time $t' \geq 0$ with $p(t') = 0$, we instantly set $v(t')$ to -$\rho v(t'^-)$, where $\rho \in [0, 1]$ is the coefficient of restitution. In general, this model can be presented in the form of system (3):

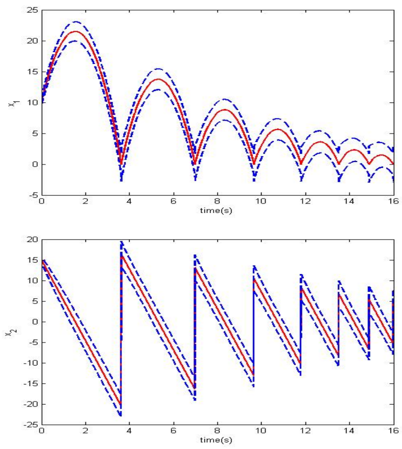

Figure 2. Results of the simulation for the bouncing ball model

$$
\begin{aligned}
& x(t) = [p(t)\ v(t)]^{\mathrm{T}}, \\
& \dot{x}(t) = Ax(t) + b(t) \text{ when } x_1(t) \neq 0, \\
& x(t) = Gx(t^-) + d(t) \text{ when } x_1(t) = 0, \\
& y(t) = Cx(t),
\end{aligned}
$$

where $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $C = [\,1\ 0\,]$, $G = \begin{bmatrix} 1 & 0 \\ 0 & -\rho \end{bmatrix}$; $x(t) \in \mathbb{R}^2$, $y(t) \in \mathbb{R}$ are respectively the state and the output; the signals $b(t)$ and $d(t)$ model some additional perturbing forces applied to the ball:

$$
b(t) = \begin{bmatrix} \beta \sin(t) \\ -g + \beta \sin(t) \end{bmatrix},\ d(t) = \begin{bmatrix} \delta \sin(t) \\ \delta \sin(t) \end{bmatrix},
$$

where $\beta = 0.5$ and $\delta = 0.5$ are known parameters. Thus,

$$
\underline{b}(t) = \begin{bmatrix} -\beta \\ -g - \beta \end{bmatrix},\ \overline{b}(t) = \begin{bmatrix} \beta \\ -g + \beta \end{bmatrix},
$$

$$
\underline{d}(t) = \begin{bmatrix} -\delta \\ -\delta \end{bmatrix},\ \overline{d}(t) = \begin{bmatrix} \delta \\ \delta \end{bmatrix}
$$

and Assumption 3 is then satisfied. Assume that $||x|| < +\infty$ (Assumption 1 is valid). Verifying the LMI (8) with Matlab YALMIP toolbox Löfberg (2004), we found that Assumption 4 holds for all ranged dwell-time $T_k > 0$. Therefore, all conditions of Theorem 7 are satisfied. Finally, the matrices

$$
R = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix},\ S_1 = \begin{bmatrix} -0.7071 & -0.4472 \\ -0.7071 & -0.8944 \end{bmatrix},
$$

$$
T = \begin{bmatrix} -0.8 & 0 \\ 0 & 0.9 \end{bmatrix},\ S_2 = \begin{bmatrix} 0 & 0.9594 \\ 1 & 0.2822 \end{bmatrix}
$$

satisfy all conditions of Theorem 7 and the interval observer (9) solves the problem of interval state estimation for bouncing ball. The results of simulation are shown in Fig 2, where the solid lines represent the states x_k, $k = 1, 2$ and the dash lines are used for the interval estimates.

Remark 10. In the example of the bouncing ball considered in this work, the measurement noise is equal to zero. This means the times of the jumps in the state are well estimated as the output signal is supposed to be perfect (without noise). In the real case, there is always a measurement noise in the output signal: the jumps times in the state are not known and need to be estimated. It introduces a time-delay in the estimated jumping time and causes some additional error in the state estimation.

4.3 Academic linear impulsive system

Consider the following system:

$$
\begin{aligned}
& \dot{x}(t) = Ax(t) + b(t)\ \forall t \in [0,\, 5) \cup (5,\, 10) \cup (10, +\infty), \\
& x(t) = Gx(t^-) + d(t)\ \forall t \in \{5,\, 10\}, \\
& y(t) = Cx(t),
\end{aligned}
$$

where the matrices A, C and G are defined as follows:

$$
A = \begin{bmatrix} -2 & 0 \\ -4 & -3 \end{bmatrix},\ C = [\,0\ 1\,],\ G = \begin{bmatrix} 2 & 0 \\ 1 & -0.2 \end{bmatrix}
$$

and $x(t) \in \mathbb{R}^2$, $y(t) \in \mathbb{R}$ are respectively the state and the output. The signals $b(t)$ and $d(t)$ are:

$$
b(t) = \begin{bmatrix} \beta \sin(2t)\cos(t) \\ \beta \sin(2t)\cos(t) \end{bmatrix},\ d(t) = \begin{bmatrix} 0.2 + \delta \sin(t) \\ 0.2 + \delta \sin(t) \end{bmatrix}
$$

with known $\beta = 0.1$ and $\delta = 0.1$. Thus,

$$
\underline{b}(t) = \begin{bmatrix} -\beta \\ -\beta \end{bmatrix},\ \overline{b}(t) = \begin{bmatrix} \beta \\ \beta \end{bmatrix},
$$

$$
\underline{d}(t) = \begin{bmatrix} -\delta + 0.2 \\ -\delta + 0.2 \end{bmatrix},\ \overline{d}(t) = \begin{bmatrix} \delta + 0.2 \\ \delta + 0.2 \end{bmatrix}.
$$

Assumption 3 is then satisfied. Assume that $||x|| < +\infty$ and Assumption 1 is valid. There is no observer gain L such that the matrix $A - LC$ is Metzler. For $L = [\,0\ -2\,]^{\mathrm{T}}$ and $A - LC = \begin{bmatrix} -2 & 0 \\ -4 & -1 \end{bmatrix}$, we choose

$$
D_0 = \begin{bmatrix} -1.5 & 0 \\ 0 & -1 \end{bmatrix},\ D_1 = \begin{bmatrix} 0.5 & 0 \\ 4 & 0 \end{bmatrix},
$$

then D_0 is Metzler and $D_1 \in \mathbb{R}_+^{n\times n}$. For $M = [\,-1\ 2.8\,]^{\mathrm{T}}$ and $G - MC = \begin{bmatrix} 2 & 1 \\ 1 & -3 \end{bmatrix}$, we choose:

$$
(G - MC)_p = \begin{bmatrix} 2.5 & 1 \\ 1 & 0 \end{bmatrix},\ (G - MC)_n = \begin{bmatrix} 0.5 & 0 \\ 0 & 3 \end{bmatrix}.
$$

$(G - MC)_p \in \mathbb{R}_+^{n\times n}$ and $(G - MC)_n \in \mathbb{R}_+^{n\times n}$. Note that the matrix $G - MC$ is negative and is not Schur stable . By applying Matlab YALMIP toolbox Löfberg (2004) to solve the LMI (10), we found that Assumption 5 holds for all $T_k \in (2.7579, +\infty)$. Therefore, all conditions of Theorem 8 are satisfied and the interval observer (11) solves the problem of interval state estimation. The results of simulation are shown in Fig 3, where the solid lines represent the states x_k, $k = 1, 2$ and the dash lines are used for the interval estimates $\underline{x_k}$ and $\overline{x_k}$.

5. CONCLUSION

Interval state estimation for linear impulsive systems has been considered in this paper. The goal of the proposed approaches is to take into account the presence of disturbance or uncertain parameters during the synthesis of these interval observers. Two main techniques have been proposed. The first one is based on a static transformation of coordinates, which connects a linear impulsive system with its nonnegative representation when the system is asymptotically stable with a ranged dwell-time. The second technique uses a representation of impulsive system in a nonnegative form. The boundedness of the estimation error (ISS property) and the observer stability can be

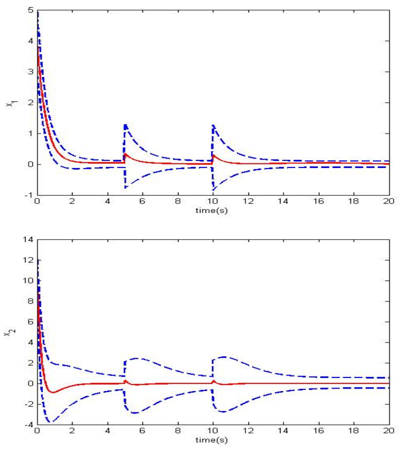

Figure 3. Results of the simulation for the academic linear impulsive system

checked using LMIs. The efficiency of these techniques is shown on examples of computer simulation for two academic systems and a bouncing ball. A future work can focus on nonlinear hybrid systems with parameter uncertainties, and control design based on interval estimates as in Efimov et al. (2013d).

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