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# The initial-boundary value problem for general non-local scalar conservation laws in one space dimension

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## Abstract

We prove global well-posedness results for weak entropy solutions of bounded variation (BV) of scalar conservation laws with non-local flux on bounded domains, under suitable regularity assumptions on the flux function. In particular, existence is obtained by proving the convergence of an adapted Lax-Friedrichs algorithm. Lipschitz continuous dependence from initial and boundary data is derived applying Kružhkov's doubling of variable technique.

**Key words:** Scalar conservation laws, Non-local flux, Initial-boundary value problem, Lax-Friedrichs scheme.

## 1 Introduction

Given a bounded open interval  $I = ]a, b[ \subset \mathbb{R}$ , we consider the following initial-boundary value problem

$$\partial_t \rho + \partial_x f(t, x, \rho, \rho * \eta) = 0, \quad (t, x) \in \mathbb{R}^+ \times I, \quad (1.1a)$$

$$\rho(0, x) = \rho_0(x), \quad x \in I, \quad (1.1b)$$

$$\rho(t, a) = \rho_a(t), \quad t \in \mathbb{R}^+, \quad (1.1c)$$

$$\rho(t, b) = \rho_b(t), \quad t \in \mathbb{R}^+, \quad (1.1d)$$

where  $f \in \mathbf{C}^2(\mathbb{R}^+ \times \bar{I} \times \mathbb{R} \times \mathbb{R}; \mathbb{R})$  satisfies

$$f(t, x, 0, R) = 0 \quad \forall t, x, R, \quad (1.2a)$$

$$\sup_{t, x, \rho, R} |\partial_\rho f(t, x, \rho, R)| < L, \quad (1.2b)$$

$$\sup_{t, x, R} |\partial_x f(t, x, \rho, R)| < C|\rho|, \quad \sup_{t, x, R} |\partial_R f(t, x, \rho, R)| < C|\rho|, \quad (1.2c)$$

$$\sup_{t, x, R} |\partial_{xx}^2 f(t, x, \rho, R)| < C|\rho|, \quad \sup_{t, x, R} |\partial_{xR}^2 f(t, x, \rho, R)| < C|\rho|, \quad \sup_{t, x, R} |\partial_{RR}^2 f(t, x, \rho, R)| < C|\rho|, \quad (1.2d)$$

for some constants  $L > 0$  and  $C > 0$ , and  $\eta \in (\mathbf{C}^1 \cap \mathbf{W}^{1, \infty})(\mathbb{R}; \mathbb{R})$  is a convolution kernel (not necessarily with compact support) such that

$$\int_{\mathbb{R}} \eta(x) dx = 1.$$

Equations of type (1.1a) arise in several applications, and have made the object of a large literature in recent years. Space-integral terms appear for example in models for granular flows

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[3], sedimentation [7], supply chains [19], conveyor belts [18], weakly coupled oscillators [2], structured populations dynamics [24], or more general problems like gradient constrained equations [4]. Equations with non-local flux have been recently introduced also in traffic flow modeling to account for the reaction of drivers or pedestrians to the surrounding density of other individuals, see [8, 10, 11, 26].

General analytical results on non-local conservation laws, proving existence and eventually uniqueness of solutions of the Cauchy problem for (1.1a), can be found in [5] for scalar equations in one space dimension, in [12] for scalar equations in several space dimensions and in [1, 13, 14] for multi-dimensional systems of conservation laws. Besides, specific finite volume numerical methods have been developed recently in [1, 17, 21]. To our knowledge, initial-boundary value problems of the form (1.1) have not been rigorously studied yet, the difficulties lying in the presence of the non-local term, which may exceed the boundaries of the space domain. Nonetheless, real applications (confined environments, networks, etc.) and numerical computations require a precise account for boundary conditions.

The scope of the present article is to propose an approach for a rigorous treatment of boundary conditions, in the case of one space-dimensional problems. The strategies we employ are inspired by classical results on scalar conservation laws with boundary conditions. In particular, we refer to [6, 9, 27]. Our results are based on the extension of the solution outside the domain, set to be constantly equal to the corresponding boundary condition values. It is far from obvious to generalize this technique to problem in several space-dimensions.

As in the classical case, we assume that boundary conditions can not generally be satisfied in strong sense. Therefore, we introduce the following notion of weak entropy solution for (1.1), which extends to problems with boundaries the definition of solution given in [5] for the corresponding Cauchy problem. This formulation, based on semi Kruzhkov entropies [23, 27], has the advantage of not using explicitly the traces of the solution at the boundaries of the domain, which turns particularly useful in the existence proof, provided in Section 2.

**Definition 1** *Let  $\rho_0 \in \mathbf{L}^\infty(I; \mathbb{R})$  and  $\rho_a, \rho_b \in \mathbf{L}^\infty(\mathbb{R}^+; \mathbb{R})$ . A map  $\rho \in \mathbf{L}^\infty(\mathbb{R}^+ \times I; \mathbb{R})$  is a weak entropy solution to (1.1) if for every test function  $\varphi \in \mathbf{C}_c^1(\mathbb{R}^2; \mathbb{R}^+)$  and for every  $\kappa \in \mathbb{R}$*

$$\begin{aligned} & \int_0^{+\infty} \int_a^b \left( (\rho - \kappa)^\pm \partial_t \varphi + \operatorname{sgn}(\rho - \kappa)^\pm (f(t, x, \rho, R(t, x)) - f(t, x, \kappa, R(t, x))) \partial_x \varphi \right. \\ & \quad \left. - \operatorname{sgn}(\rho - \kappa)^\pm \frac{d}{dx} f(t, x, \kappa, R(t, x)) \varphi \right) dx dt + \int_a^b (\rho_0 - \kappa)^\pm \varphi(0, x) dx \\ & \quad + \operatorname{Lip}(f) \int_0^{+\infty} (\rho_a(t) - \kappa)^\pm \varphi(t, a) dt + \operatorname{Lip}(f) \int_0^{+\infty} (\rho_b(t) - \kappa)^\pm \varphi(t, b) dt \geq 0, \end{aligned} \quad (1.3)$$

$$+ \operatorname{Lip}(f) \int_0^{+\infty} (\rho_a(t) - \kappa)^\pm \varphi(t, a) dt + \operatorname{Lip}(f) \int_0^{+\infty} (\rho_b(t) - \kappa)^\pm \varphi(t, b) dt \geq 0, \quad (1.4)$$

where

$$\begin{aligned} R(t, x) & := (\rho(t, \cdot) * \eta)(x) = \int_{\mathbb{R}} \rho(t, y) \eta(x - y) dy \\ & = \int_a^b \rho(t, y) \eta(x - y) dy + \rho_a(t) \int_{-\infty}^a \eta(x - y) dy + \rho_b(t) \int_b^{+\infty} \eta(x - y) dy \\ & = \int_a^b \rho(t, y) \eta(x - y) dy + \rho_a(t) \int_{x-a}^{+\infty} \eta(y) dy + \rho_b(t) \int_{-\infty}^{x-b} \eta(y) dy. \end{aligned} \quad (1.5)$$

Above, we have noted  $\operatorname{sgn}^+(s) := \max\{s/|s|, 0\}$ ,  $\operatorname{sgn}^-(s) := -\operatorname{sgn}^+(-s)$ ,  $s^+ := s \operatorname{sgn}^+(s)$  and  $s^- := (-s)^+$  for  $s \in \mathbb{R}$ . In the paper, we will also denote  $\mathcal{I}(r, s) := [\min\{r, s\}, \max\{r, s\}]$  for any  $r, s \in \mathbb{R}$ .

The Definition 1 is equivalent to the one provided in [6] (for a proof of equivalence we refer the reader to [22, Theorem 7.31]). This second definition will be used in Section 3 to prove Lipschitz continuous dependence of solution with respect to initial and boundary data.

**Definition 2** Let  $\rho_0 \in \mathbf{L}^\infty(I; \mathbb{R})$  and  $\rho_a, \rho_b \in \mathbf{L}^\infty(\mathbb{R}^+; \mathbb{R})$ . A map  $\rho \in BV(\mathbb{R}^+ \times I; \mathbb{R})$  is a weak entropy solution to (1.1) if for every test function  $\varphi \in \mathbf{C}_c^1(\mathbb{R}^2; \mathbb{R}^+)$  and for every  $\kappa \in \mathbb{R}$

$$\begin{aligned} & \int_0^{+\infty} \int_a^b \left( |\rho - \kappa| \partial_t \varphi + \operatorname{sgn}(\rho - \kappa) [f(t, x, \rho, R(t, x)) - f(t, x, \kappa, R(t, x))] \partial_x \varphi \right. \\ & \quad \left. - \operatorname{sgn}(\rho - \kappa) \frac{d}{dx} f(t, x, \kappa, R(t, x)) \varphi \right) dx dt + \int_a^b |\rho_0 - \kappa| \varphi(0, x) dx \\ & + \int_0^{+\infty} \operatorname{sgn}(\rho_a - \kappa) [f(t, a, \rho(t, a+), R(t, a)) - f(t, a, \kappa, R(t, a))] \varphi(t, a) dt \\ & + \int_0^{+\infty} \operatorname{sgn}(\rho_b - \kappa) [f(t, b, \rho(t, b-), R(t, b)) - f(t, b, \kappa, R(t, b))] \varphi(t, b) dt \geq 0. \end{aligned} \quad (1.6)$$

We remark that to ensure that the traces of  $\rho$  at  $x = a, b$ , are well defined, we need to assume that the solutions have bounded variation, see [6, Lemma 1]. Moreover, following [6, 15], we recall that the entropy condition (1.6) implies that the traces of the solution at the boundary satisfy

- On the left boundary  $x = a$ : for all  $\kappa \in \mathbb{R}$

$$\left( \operatorname{sgn}(\rho(t, a+) - \kappa) - \operatorname{sgn}(\rho_a(t) - \kappa) \right) (f(t, a, \rho(t, a+), R(t, a)) - f(t, a, \kappa, R(t, a))) \leq 0, \quad (1.7)$$

- On the right boundary  $x = b$ : for all  $\kappa \in \mathbb{R}$

$$\left( \operatorname{sgn}(\rho(t, b-) - \kappa) - \operatorname{sgn}(\rho_b(t) - \kappa) \right) (f(t, b, \rho(t, b-), R(t, b)) - f(t, b, \kappa, R(t, b))) \geq 0. \quad (1.8)$$

Our main result states the global well-posedness of (1.1).

**Theorem 1** Let hypotheses (1.2) hold. If  $\rho_0 \in (\mathbf{L}^\infty \cap BV)(I; \mathbb{R}^+)$  and  $\rho_a, \rho_b \in (\mathbf{L}^\infty \cap BV)(\mathbb{R}^+; \mathbb{R}^+)$ , then for all  $T > 0$  problem (1.1) has a unique weak entropy solution  $\rho \in BV([0, T] \times I; \mathbb{R}^+)$  in the sense of Definitions 1, 2. Moreover, the following estimates hold:

$$\|\rho(T, \cdot)\|_{\mathbf{L}^1(I)} \leq \|\rho_0\|_{\mathbf{L}^1(I)} + \alpha \left( \|\rho_a\|_{\mathbf{L}^1([0, T])} + \|\rho_b\|_{\mathbf{L}^1([0, T])} \right), \quad (1.9)$$

$$\|\rho(T, \cdot)\|_{\mathbf{L}^\infty(I)} \leq e^{\mathcal{L}T} \|\rho_0\|_{\mathbf{L}^\infty(I)}, \quad (1.10)$$

$$TV(\rho(T, \cdot); I) \leq e^{\mathcal{K}_1 T} TV(\rho_0; I) + \frac{\mathcal{K}_2}{\mathcal{K}_1} \left( e^{\mathcal{K}_1 T} - 1 \right) + TV(\rho_a; [0, T]) + TV(\rho_b; [0, T]), \quad (1.11)$$

$$\|\rho(T, \cdot) - \rho(T - \tau, \cdot)\|_{\mathbf{L}^1(I)} \leq \mathcal{C}_t(T) \tau, \quad \tau > 0, \quad (1.12)$$

with  $\mathcal{L}$  as in (2.12),  $\mathcal{K}_{1,2}$  as in (2.18) and (2.24), and  $\mathcal{C}_t$  as in (2.28).

Finally, let  $\rho, \sigma \in \mathbf{C}^0(\mathbb{R}^+; \mathbf{L}^1(I; \mathbb{R}^+)) \cap BV_\infty([0, T] \times I; \mathbb{R}^+)$  be two weak entropy solutions to (1.1), with initial data  $\rho_0, \sigma_0 \in \mathbf{L}^\infty(I, \mathbb{R}^+)$  and boundary data  $\rho_a, \rho_b, \sigma_a, \sigma_b \in \mathbf{L}^\infty(\mathbb{R}^+; \mathbb{R}^+)$  respectively. Then the following estimate holds:

$$\|\rho(T, \cdot) - \sigma(T, \cdot)\|_{\mathbf{L}^1(I)} \leq e^{\mathcal{S}T} \left[ \|\rho_0 - \sigma_0\|_{\mathbf{L}^1(I)} + (L + \mathcal{S}') \left( \|\rho_a - \sigma_a\|_{\mathbf{L}^1([0, T])} + \|\rho_b - \sigma_b\|_{\mathbf{L}^1([0, T])} \right) \right],$$

where the constants  $\mathcal{S}, \mathcal{S}'$  are defined by (3.10).

The above result can be easily generalized to unbounded domains  $I = ]a, b[$ , with  $a = -\infty$  or  $b = +\infty$ , under the assumption that the initial datum also belongs to  $\mathbf{L}^1(I)$ .

## 2 Existence of weak entropy solutions

The proof of existence is based on the following strategy: we construct a sequence of approximate solutions using a finite volume algorithm, we prove the convergence of a subsequence and, finally, we show that the limit is indeed a weak entropy solution in the sense of Definition 1. The procedure follows closely [1, 5].

Let us fix a space grid in  $[a, b]$  of size  $\Delta x = (b - a)/N$ ,  $N \in \mathbb{N}$ , and choose a time step  $\Delta t$  (satisfying some stability conditions which will be detailed later). We introduce the usual notation

$$t^n = n \Delta t, \quad n \in \mathbb{N}; \quad x_j = a + \left(j - \frac{1}{2}\right) \Delta x, \quad x_{j+1/2} = a + j \Delta x, \quad j = 1, \dots, N; \quad \lambda = \frac{\Delta t}{\Delta x}.$$

Throughout, an initial datum  $\rho_0 \in (\mathbf{L}^\infty \cap \mathbf{BV})(\mathbb{R}; \mathbb{R})$  is fixed and we denote

$$\rho_j^0 = \frac{1}{\Delta x} \int_{x_{j-1/2}}^{x_{j+1/2}} \rho_0(x) \, dx \, dy \quad \text{for } j = 1, \dots, N.$$

We define a piecewise constant approximate solution  $\rho_\Delta$  to (1.1) as

$$\rho_\Delta(t, x) = \rho_j^n \quad \text{for} \quad \begin{cases} t \in [t^n, t^{n+1}[ , \\ x \in [x_{j-1/2}, x_{j+1/2}[ , \end{cases} \quad \text{where} \quad \begin{matrix} n \in \mathbb{N}, \\ j = 1, \dots, N, \end{matrix}$$

through the following adapted Lax-Friedrichs scheme

$$\rho_j^{n+1} = \rho_j^n - \lambda \left[ F_{j+1/2}^n(\rho_j^n, \rho_{j+1}^n) - F_{j-1/2}^n(\rho_{j-1}^n, \rho_j^n) \right], \quad (2.1)$$

where

$$F_{j+1/2}^n(\rho_j^n, \rho_{j+1}^n) := \frac{1}{2} \left[ f(t^n, x_j, \rho_j^n, R_j^n) + f(t^n, x_{j+1}, \rho_{j+1}^n, R_{j+1}^n) + \alpha(\rho_j^n - \rho_{j+1}^n) \right] \quad (2.2)$$

is the numerical flux (for some  $\alpha \in \mathbb{R}$ ,  $\alpha > 0$ ) and

$$R_j^n := \Delta x \sum_{k \in \mathbb{Z}} \eta(x_{j-k}) \rho_k^n, \quad j = 1, \dots, N,$$

are the quadrature formulae approximating the convolution terms. Remark that, due to the boundedness of the domain  $]a, b[$ , we can set

$$\begin{aligned} R_j^n &:= \Delta x \sum_{k=1}^N \eta(x_{j-k}) \rho_k^n + \rho_a^n \Delta x \sum_{k \leq 0} \eta(x_{j-k}) + \rho_b^n \Delta x \sum_{k > N} \eta(x_{j-k}) \\ &= \Delta x \sum_{k=1}^N \eta(x_{j-k}) \rho_k^n + \rho_a^n \Delta x \sum_{k \geq j} \eta(x_k) + \rho_b^n \Delta x \sum_{k < j-N} \eta(x_k). \end{aligned}$$

The proof of the convergence of approximate solutions is divided in several steps, which are intended to show that the sequence verifies the hypotheses of Helly's compactness theorem.

## 2.1 Positivity

The following lemma ensures the positivity of approximate solutions corresponding to positive initial and boundary data.

**Lemma 1** *Let  $\rho_0 \in \mathbf{L}^\infty(I; \mathbb{R}^+)$  and  $\rho_a, \rho_b \in \mathbf{L}^\infty(\mathbb{R}^+; \mathbb{R}^+)$ . Moreover, assume that*

$$\alpha \geq L, \quad \lambda \leq \frac{1}{3} \min \left\{ \frac{1}{\alpha}, \frac{1}{L(1 + \Delta x)} \right\}. \quad (2.3)$$

Then  $\rho_\Delta(t, x) \geq 0$  for all  $x \in I$ ,  $t > 0$ .

*Proof.* We rearrange (2.1) as

$$\begin{aligned} \rho_j^{n+1} &= \rho_j^n - \lambda \left[ F_{j+1/2}^n(\rho_j^n, \rho_{j+1}^n) \pm F_{j+1/2}^n(\rho_j^n, \rho_j^n) \pm F_{j-1/2}^n(\rho_j^n, \rho_j^n) - F_{j-1/2}^n(\rho_{j-1}^n, \rho_j^n) \right] \\ &= \left( 1 - \alpha_j^n - \beta_j^n \right) \rho_j^n + \alpha_j^n \rho_{j-1}^n + \beta_j^n \rho_{j+1}^n - \lambda \left( F_{j+1/2}^n(\rho_j^n, \rho_j^n) - F_{j-1/2}^n(\rho_j^n, \rho_j^n) \right), \end{aligned}$$

where, for  $j \in \{1, \dots, N\}$ ,

$$\alpha_j^n := \begin{cases} \lambda \frac{F_{j-1/2}^n(\rho_j^n, \rho_j^n) - F_{j-1/2}^n(\rho_{j-1}^n, \rho_j^n)}{\rho_j^n - \rho_{j-1}^n} & \text{if } \rho_j^n \neq \rho_{j-1}^n, \\ 0 & \text{if } \rho_j^n = \rho_{j-1}^n, \end{cases} \quad (2.4)$$

and

$$\beta_j^n := \begin{cases} -\lambda \frac{F_{j+1/2}^n(\rho_j^n, \rho_{j+1}^n) - F_{j+1/2}^n(\rho_j^n, \rho_j^n)}{\rho_{j+1}^n - \rho_j^n} & \text{if } \rho_{j+1}^n \neq \rho_j^n, \\ 0 & \text{if } \rho_{j+1}^n = \rho_j^n. \end{cases} \quad (2.5)$$

We consider the following estimates:

$$\begin{aligned} & \left| F_{j+1/2}^n(\rho_j^n, \rho_j^n) - F_{j-1/2}^n(\rho_j^n, \rho_j^n) \right| = \\ &= \frac{1}{2} \left| f(t^n, x_{j+1}, \rho_j^n, R_{j+1}^n) - f(t^n, x_{j-1}, \rho_j^n, R_{j-1}^n) \right| \\ &= \frac{1}{2} \left| f(t^n, x_{j+1}, \rho_j^n, R_{j+1}^n) \pm f(t^n, x_{j-1}, \rho_j^n, R_{j+1}^n) - f(t^n, x_{j-1}, \rho_j^n, R_{j-1}^n) \right| \\ &\leq \frac{1}{2} \left| f(t^n, x_{j+1}, \rho_j^n, R_{j+1}^n) - f(t^n, x_{j-1}, \rho_j^n, R_{j+1}^n) \right| \\ &\quad + \frac{1}{2} \left| f(t^n, x_{j-1}, \rho_j^n, R_{j+1}^n) \right| + \frac{1}{2} \left| f(t^n, x_{j-1}, \rho_j^n, R_{j-1}^n) \right| \\ &\leq L \left| \rho_j^n \right| \Delta x + \frac{1}{2} \left| f(t^n, x_{j-1}, \rho_j^n, R_{j+1}^n) - f(t^n, x_{j-1}, 0, R_{j+1}^n) \right| \\ &\quad + \frac{1}{2} \left| f(t^n, x_{j-1}, \rho_j^n, R_{j-1}^n) - f(t^n, x_{j-1}, 0, R_{j-1}^n) \right| \\ &\leq L \left| \rho_j^n \right| (1 + \Delta x). \end{aligned}$$

Moreover, we observe that, whenever  $\rho_j^n \neq \rho_{j-1}^n$  and  $\rho_{j+1}^n \neq \rho_j^n$ ,

$$\alpha_j^n = \frac{\lambda}{2(\rho_j^n - \rho_{j-1}^n)} \left[ f(t^n, x_{j-1}, \rho_j^n, R_{j-1}^n) - f(t^n, x_{j-1}, \rho_{j-1}^n, R_{j-1}^n) + \alpha(\rho_j^n - \rho_{j-1}^n) \right]$$

$$= \frac{\lambda}{2} \left( \partial_\rho f(t^n, x_{j-1}, \xi_{j-1/2}^n, R_{j-1}^n) + \alpha \right) \quad (2.6)$$

and

$$\begin{aligned} \beta_j^n &= -\frac{\lambda}{2(\rho_{j+1}^n - \rho_j^n)} \left[ f(t^n, x_{j+1}, \rho_{j+1}^n, R_{j+1}^n) + \alpha(\rho_j^n - \rho_{j+1}^n) - f(t^n, x_{j+1}, \rho_j^n, R_{j+1}^n) \right] \\ &= \frac{\lambda}{2} \left( \alpha - \partial_\rho f(t^n, x_{j+1}, \xi_{j+1/2}^n, R_{j+1}^n) \right), \end{aligned} \quad (2.7)$$

for some  $\xi_{j-1/2}^n \in \mathcal{I}(\rho_{j-1}^n, \rho_j^n)$  and  $\xi_{j+1/2}^n \in \mathcal{I}(\rho_j^n, \rho_{j+1}^n)$ . Assuming that

$$\alpha \geq L, \quad \lambda\alpha \leq \frac{1}{3}, \quad \lambda L(1 + \Delta x) \leq \frac{1}{3},$$

we get

$$\alpha_j^n, \beta_j^n \in \left[ 0, \frac{1}{3} \right], \quad \left( 1 - \alpha_j^n - \beta_j^n \right) \in \left[ \frac{1}{3}, 1 \right], \quad \lambda \left( F_{j+1/2}^n(\rho_j^n, \rho_j^n) - F_{j-1/2}^n(\rho_j^n, \rho_j^n) \right) \leq \frac{1}{3} |\rho_j^n|,$$

which allow us to recover the sought estimate

$$\begin{aligned} \rho_j^{n+1} &\geq \left( 1 - \alpha_j^n - \beta_j^n \right) \rho_j^n + \alpha_j^n \rho_{j-1}^n + \beta_j^n \rho_{j+1}^n - \frac{1}{3} |\rho_j^n|, \\ &\geq \left( \frac{2}{3} - \alpha_j^n - \beta_j^n \right) \rho_j^n + \alpha_j^n \rho_{j-1}^n + \beta_j^n \rho_{j+1}^n \\ &\geq 0. \end{aligned}$$

□

## 2.2 $L^1$ bound

**Lemma 2** *Let hypotheses (1.2) and conditions (2.3) hold. If  $\rho_0 \in \mathbf{L}^\infty(I; \mathbb{R}^+)$  and  $\rho_a, \rho_b \in \mathbf{L}^\infty(\mathbb{R}^+; \mathbb{R}^+)$ , then for all  $T > 0$*

$$\|\rho_\Delta(T, \cdot)\|_{\mathbf{L}^1(I)} \leq \|\rho_0\|_{\mathbf{L}^1(I)} + \alpha \left( \|\rho_a\|_{\mathbf{L}^1([0, T])} + \|\rho_b\|_{\mathbf{L}^1([0, T])} \right) =: \mathcal{C}_1(T). \quad (2.8)$$

*Proof.* Thanks to the positivity of the discrete solution, using the definition of the scheme, we compute

$$\begin{aligned} \|\rho^{n+1}\|_{L^1(a, b)} &= \Delta x \sum_{j=1}^N \rho_j^{n+1} \\ &= \Delta x \sum_{j=1}^N \left[ \rho_j^n - \lambda \left( F_{j+1/2}^n(\rho_j^n, \rho_{j+1}^n) - F_{j-1/2}^n(\rho_{j-1}^n, \rho_j^n) \right) \right] \\ &= \Delta x \sum_{j=1}^N \rho_j^n - \lambda \Delta x \left( F_{N+1/2}^n(\rho_N^n, \rho_b^n) - F_{1/2}^n(\rho_a^n, \rho_1^n) \right) \\ &= \Delta x \sum_{j=1}^N \rho_j^n - \frac{\Delta t}{2} \left( f(t^n, x_N, \rho_N^n, R_N^n) + f(t^n, x_{N+1}, \rho_b^n, R_{N+1}^n) + \alpha(\rho_N^n - \rho_b^n) \right) \end{aligned}$$

$$\begin{aligned}
& + \frac{\Delta t}{2} \left( f(t^n, x_0, \rho_a^n, R_0^n) + f(t^n, x_1, \rho_1^n, R_1^n) + \alpha (\rho_a^n - \rho_1^n) \right) \\
& = \Delta x \sum_{j=1}^N \rho_j^n \\
& \quad - \frac{\Delta t}{2} \left( \partial_\rho f(t^n, x_N, \xi_{N,0}^n, R_N^n) \rho_N^n + \partial_\rho f(t^n, x_{N+1}, \xi_{N+1,0}^n, R_{N+1}^n) \rho_b^n + \alpha (\rho_N^n - \rho_b^n) \right) \\
& \quad + \frac{\Delta t}{2} \left( \partial_\rho f(t^n, x_0, \xi_{a,0}^n, R_0^n) \rho_a^n + \partial_\rho f(t^n, x_1, \xi_1^n, R_1^n) \rho_1^n + \alpha (\rho_a^n - \rho_1^n) \right) \\
& = \Delta x \sum_{j=1}^N \rho_j^n + \frac{\Delta t}{2} \left( \partial_\rho f(t^n, x_0, \xi_{a,0}^n, R_0^n) + \alpha \right) \rho_a^n \\
& \quad + \frac{\Delta t}{2} \left( -\partial_\rho f(t^n, x_{N+1}, \xi_{N+1,0}^n, R_{N+1}^n) + \alpha \right) \rho_b^n \\
& \quad + \frac{\Delta t}{2} \left( -\partial_\rho f(t^n, x_N, \xi_{N,0}^n, R_N^n) - \alpha \right) \rho_N^n \\
& \quad + \frac{\Delta t}{2} \left( \partial_\rho f(t^n, x_1, \xi_1^n, R_1^n) - \alpha \right) \rho_1^n.
\end{aligned}$$

Being the last two coefficients in the previous estimate non positive, we can conclude that

$$\Delta x \sum_{j=1}^N \rho_j^{n+1} \leq \Delta x \sum_{j=1}^N \rho_j^n + \alpha \Delta t (\rho_a^n + \rho_b^n),$$

thus ending the proof.  $\square$

### 2.3 $L^\infty$ bound

**Lemma 3** *Let hypotheses (1.2) and conditions (2.3) hold. If  $\rho_0 \in \mathbf{L}^\infty(I; \mathbb{R}^+)$  and  $\rho_a, \rho_b \in \mathbf{L}^\infty(\mathbb{R}^+; \mathbb{R}^+)$ , then for all  $T > 0$*

$$\|\rho_\Delta(T, \cdot)\|_{\mathbf{L}^\infty(I)} \leq e^{\mathcal{L}T} \|\rho_0\|_{\mathbf{L}^\infty(I)}, \quad (2.9)$$

where  $\mathcal{L}$  is given by (2.12).

*Proof.* We observe that

$$\begin{aligned}
& \left| R_{j+1}^n - R_{j-1}^n \right| \leq \\
& \leq \Delta x \rho_a^n \sum_{k \leq 0} |\eta(x_{j+1-k}) - \eta(x_{j-1-k})| + \Delta x \sum_{k=1}^N \rho_k^n |\eta(x_{j+1-k}) - \eta(x_{j-1-k})| \\
& \quad + \Delta x \rho_b^n \sum_{k \geq N+1} |\eta(x_{j+1-k}) - \eta(x_{j-1-k})| \\
& = \Delta x \rho_a^n \sum_{k \leq 0} \left| \int_{x_{j-1-k}}^{j+1-k} \eta'(s) ds \right| + \Delta x \sum_{k=1}^N \rho_k^n \left| \int_{x_{j-1-k}}^{j+1-k} \eta'(s) ds \right| + \Delta x \rho_b^n \sum_{k \leq 0} \left| \int_{x_{j-1-k}}^{j+1-k} \eta'(s) ds \right| \\
& \leq 2\Delta x^2 \|\eta'\|_{\mathbf{L}^\infty(\mathbb{R})} \left( \rho_a^n + \sum_{j=1}^N \rho_j^n + \rho_b^n \right)
\end{aligned}$$



$$\begin{aligned}
&\leq 2\Delta x \|\eta'\|_{\mathbf{L}^\infty(\mathbb{R})} \left( \|\rho_0\|_{\mathbf{L}^1(I)} + \alpha \left( \|\rho_a\|_{\mathbf{L}^1([0,T])} + \|\rho_b\|_{\mathbf{L}^1([0,T])} \right) \right) \\
&\leq 2\mathcal{T}\Delta x,
\end{aligned} \tag{2.10}$$

where we have set

$$\mathcal{T} := \|\eta'\|_{\mathbf{L}^\infty(\mathbb{R})} \left( \|\rho_0\|_{\mathbf{L}^1(I)} + \alpha \left( \|\rho_a\|_{\mathbf{L}^1([0,T])} + \|\rho_b\|_{\mathbf{L}^1([0,T])} \right) \right) \tag{2.11}$$

and for the latest bound we have applied Lemma 2.

Proceeding as in Lemma 1, we can rearrange (2.1) as

$$\rho_j^{n+1} = \left(1 - \alpha_j^n - \beta_j^n\right) \rho_j^n + \alpha_j^n \rho_{j-1}^n + \beta_j^n \rho_{j+1}^n - \lambda \left( F_{j+1/2}^n(\rho_j^n, \rho_j^n) - F_{j-1/2}^n(\rho_j^n, \rho_j^n) \right),$$

where  $\alpha_j^n$  and  $\beta_j^n$  are as in (2.6) and (2.7) respectively. Using (2.10), we get

$$\begin{aligned}
&\left| F_{j+1/2}^n(\rho_j^n, \rho_j^n) - F_{j-1/2}^n(\rho_j^n, \rho_j^n) \right| = \\
&= \frac{1}{2} \left| f(t^n, x_{j+1}, \rho_j^n, R_{j+1}^n) - f(t^n, x_{j-1}, \rho_j^n, R_{j-1}^n) \right| \\
&= \frac{1}{2} \left( \left| \partial_x f(t^n, \tilde{x}_j, \rho_j^n, \tilde{R}_j^n) \right| |x_{j+1} - x_{j-1}| + \left| \partial_x f(t^n, \tilde{x}_j, \rho_j^n, \tilde{R}_j^n) \right| \left| R_{j+1}^n - R_{j-1}^n \right| \right) \\
&\leq 2C \left| \rho_j^n \right| \Delta x + C \left| \rho_j^n \right| \left| R_{j+1}^n - R_{j-1}^n \right| \\
&\leq 2C\Delta x \left| \rho_j^n \right| (1 + \mathcal{T}).
\end{aligned}$$

Thus,

$$\begin{aligned}
\rho_j^{n+1} &\leq \left(1 - \alpha_j^n - \beta_j^n\right) \rho_j^n + \alpha_j^n \rho_{j-1}^n + \beta_j^n \rho_{j+1}^n + \lambda \left| F_{j+1/2}^n(\rho_j^n, \rho_j^n) - F_{j-1/2}^n(\rho_j^n, \rho_j^n) \right| \\
&\leq \left(1 - \alpha_j^n - \beta_j^n\right) \|\rho^n\|_{\mathbf{L}^\infty(I)} + \alpha_j^n \|\rho^n\|_{\mathbf{L}^\infty(I)} + \beta_j^n \|\rho^n\|_{\mathbf{L}^\infty(I)} + 2C\lambda\Delta x \|\rho^n\|_{\mathbf{L}^\infty(I)} (1 + \mathcal{T}) \\
&= \|\rho^n\|_{\mathbf{L}^\infty(I)} (1 + \Delta t\mathcal{L}) \\
&\leq e^{\mathcal{L}\Delta t} \|\rho^n\|_{\mathbf{L}^\infty(I)},
\end{aligned}$$

for  $j = 1, \dots, N$ , being

$$\mathcal{L} := 2C \left( 1 + \|\eta'\|_{\mathbf{L}^\infty(\mathbb{R})} \left( \|\rho_0\|_{\mathbf{L}^1(I)} + \alpha \left( \|\rho_a\|_{\mathbf{L}^1([0,T])} + \|\rho_b\|_{\mathbf{L}^1([0,T])} \right) \right) \right). \tag{2.12}$$

A standard iterative argument completes the proof.  $\square$

## 2.4 BV estimates

**Proposition 1 (BV estimate in space)** *Let hypotheses (1.2) and conditions (2.3) hold. If  $\rho_0 \in \mathbf{L}^\infty(I; \mathbb{R}^+)$  and  $\rho_a, \rho_b \in \mathbf{L}^\infty(\mathbb{R}^+; \mathbb{R}^+)$ , then  $\rho_\Delta$  satisfies the following Total Variation estimate*

$$\sum_{j=0}^N \left| \rho_{j+1}^n - \rho_j^n \right| \leq \mathcal{C}_x(t^n), \tag{2.13}$$

for all  $n \in \mathbb{N}$ , where

$$C_x(t^n) := e^{\mathcal{K}_1 t^n} \sum_{j=0}^N \left| \rho_{j+1}^0 - \rho_j^0 \right| + \frac{\mathcal{K}_2}{\mathcal{K}_1} \left( e^{\mathcal{K}_1 t^n} - 1 \right) + \sum_{m=1}^n \left| \rho_a^m - \rho_a^{m-1} \right| + \sum_{m=1}^n \left| \rho_b^m - \rho_b^{m-1} \right|,$$

with  $\mathcal{K}_1$  and  $\mathcal{K}_2$  positive constants defined in (2.18), (2.19), (2.24).

*Proof.* We consider separately the central and boundary terms.

For  $j = 1, \dots, N-1$ ,  $n \in \mathbb{N}$ , we consider the difference

$$\begin{aligned} \rho_{j+1}^{n+1} - \rho_j^{n+1} &= \rho_{j+1}^n - \rho_j^n - \lambda \left[ F_{j+3/2}^n(\rho_{j+1}^n, \rho_{j+2}^n) - F_{j-1/2}^n(\rho_{j-1}^n, \rho_j^n) \pm F_{j+1/2}^n(\rho_j^n, \rho_{j+1}^n) \right] \\ &\mp \lambda \left( F_{j+3/2}^n(\rho_j^n, \rho_{j+1}^n) + F_{j+1/2}^n(\rho_{j-1}^n, \rho_j^n) \right) \\ &= \rho_{j+1}^n - \rho_j^n \\ &\quad - \lambda \left( F_{j+3/2}^n(\rho_{j+1}^n, \rho_{j+2}^n) - F_{j+1/2}^n(\rho_j^n, \rho_{j+1}^n) - F_{j+3/2}^n(\rho_j^n, \rho_{j+1}^n) + F_{j+1/2}^n(\rho_{j-1}^n, \rho_j^n) \right) \\ &\quad - \lambda \left( F_{j+3/2}^n(\rho_j^n, \rho_{j+1}^n) - F_{j+1/2}^n(\rho_j^n, \rho_{j+1}^n) + F_{j-1/2}^n(\rho_{j-1}^n, \rho_j^n) - F_{j+1/2}^n(\rho_{j-1}^n, \rho_j^n) \right) \\ &= \mathcal{A}_j^n - \lambda \mathcal{B}_j^n, \end{aligned} \tag{2.14}$$

where we have set

$$\begin{aligned} \mathcal{A}_j^n &:= \rho_{j+1}^n - \rho_j^n - \lambda \left( F_{j+3/2}^n(\rho_{j+1}^n, \rho_{j+2}^n) - F_{j+1/2}^n(\rho_j^n, \rho_{j+1}^n) - F_{j+3/2}^n(\rho_j^n, \rho_{j+1}^n) + F_{j+1/2}^n(\rho_{j-1}^n, \rho_j^n) \right), \\ \mathcal{B}_j^n &:= F_{j+3/2}^n(\rho_j^n, \rho_{j+1}^n) - F_{j+1/2}^n(\rho_j^n, \rho_{j+1}^n) + F_{j-1/2}^n(\rho_{j-1}^n, \rho_j^n) - F_{j+1/2}^n(\rho_{j-1}^n, \rho_j^n). \end{aligned}$$

Concerning the first term  $\mathcal{A}_j^n$  and recalling (2.4) and (2.5), after suitable rearrangements we get

$$\begin{aligned} \mathcal{A}_j^n &= \left( \rho_{j+1}^n - \rho_j^n \right) \\ &\quad \times \left[ 1 + \lambda \frac{F_{j+1/2}^n(\rho_j^n, \rho_{j+1}^n) - F_{j+1/2}^n(\rho_j^n, \rho_j^n)}{\rho_{j+1}^n - \rho_j^n} - \lambda \frac{F_{j+3/2}^n(\rho_{j+1}^n, \rho_{j+1}^n) - F_{j+3/2}^n(\rho_j^n, \rho_{j+1}^n)}{\rho_{j+1}^n - \rho_j^n} \right] \\ &\quad + \left( \rho_{j+2}^n - \rho_{j+1}^n \right) \left( -\lambda \frac{F_{j+3/2}^n(\rho_{j+1}^n, \rho_{j+2}^n) - F_{j+3/2}^n(\rho_{j+1}^n, \rho_{j+1}^n)}{\rho_{j+2}^n - \rho_{j+1}^n} \right) \\ &\quad + \left( \rho_j^n - \rho_{j-1}^n \right) \left( \lambda \frac{F_{j+1/2}^n(\rho_j^n, \rho_j^n) - F_{j+1/2}^n(\rho_{j-1}^n, \rho_j^n)}{\rho_j^n - \rho_{j-1}^n} \right) \\ &= \left( 1 - \beta_j^n - \gamma_{j+1}^n \right) \left( \rho_{j+1}^n - \rho_j^n \right) + \beta_{j+1}^n \left( \rho_{j+2}^n - \rho_{j+1}^n \right) + \gamma_j^n \left( \rho_j^n - \rho_{j-1}^n \right), \end{aligned} \tag{2.15}$$

where

$$\gamma_j^n := \lambda \frac{F_{j+1/2}^n(\rho_j^n, \rho_j^n) - F_{j+1/2}^n(\rho_{j-1}^n, \rho_j^n)}{\rho_j^n - \rho_{j-1}^n},$$

and the bounds  $\gamma_j^n \in [0, 1/3]$  can be proved exactly as it has been done for  $\alpha_j^n$ , thanks to (2.3).

From (2.15) we recover

$$\sum_{j=1}^{N-1} |\mathcal{A}_j^n| \leq \sum_{j=1}^{N-1} \left[ \left( 1 - \beta_j^n - \gamma_{j+1}^n \right) \left| \rho_{j+1}^n - \rho_j^n \right| + \beta_{j+1}^n \left| \rho_{j+2}^n - \rho_{j+1}^n \right| + \gamma_j^n \left| \rho_j^n - \rho_{j-1}^n \right| \right]$$

$$= \sum_{j=1}^{N-1} |\rho_{j+1}^n - \rho_j^n| - \beta_1^n |\rho_2^n - \rho_1^n| + \gamma_1^n |\rho_1^n - \rho_a^n| + \beta_N^n |\rho_b^n - \rho_N^n| - \gamma_N^n |\rho_N^n - \rho_{N-1}^n|. \quad (2.16)$$

We now focus on the term  $\mathcal{B}_j^n$  in (2.14):

$$\begin{aligned} \mathcal{B}_j^n &= \frac{1}{2} \left( f(t^n, x_{j+2}, \rho_{j+1}^n, R_{j+2}^n) - f(t^n, x_{j+1}, \rho_{j+1}^n, R_{j+1}^n) \right) \\ &\quad - \frac{1}{2} \left( f(t^n, x_j, \rho_{j-1}^n, R_j^n) - f(t^n, x_{j-1}, \rho_{j-1}^n, R_{j-1}^n) \right) \\ &= \left( \partial_x f(t^n, \tilde{x}_{j+3/2}, \rho_{j+1}^n, \tilde{R}_{j+3/2}^n) \Delta x + \partial_R f(t^n, \tilde{x}_{j+3/2}, \rho_{j+1}^n, \tilde{R}_{j+3/2}^n) (R_{j+2}^n - R_{j+1}^n) \right) \\ &\quad - \frac{1}{2} \left( \partial_x f(t^n, \tilde{x}_{j-1/2}, \rho_{j-1}^n, \tilde{R}_{j-1/2}^n) \Delta x + \partial_R f(t^n, \tilde{x}_{j-1/2}, \rho_{j-1}^n, \tilde{R}_{j-1/2}^n) (R_j^n - R_{j-1}^n) \right) \\ &= \frac{\Delta x}{2} \left( \partial_{xx}^2 f(t^n, \hat{x}_j, \hat{\rho}_j^n, \hat{R}_j^n) (\tilde{x}_{j+3/2} - \tilde{x}_{j-1/2}) + \partial_{\rho x}^2 f(t^n, \hat{x}_j, \hat{\rho}_j^n, \hat{R}_j^n) (\rho_{j+1}^n - \rho_{j-1}^n) \right. \\ &\quad \left. + \partial_{R x}^2 f(t^n, \hat{x}_j, \hat{\rho}_j^n, \hat{R}_j^n) (\tilde{R}_{j+3/2}^n - \tilde{R}_{j-1/2}^n) \right) \\ &\quad + \frac{1}{2} \left( \partial_{xR}^2 f(t^n, \bar{x}_j, \bar{\rho}_j^n, \bar{R}_j^n) \check{R}_{j+1/2}^n (\tilde{x}_{j+3/2} - \tilde{x}_{j-1/2}) + \partial_{\rho R}^2 f(t^n, \bar{x}_j, \bar{\rho}_j^n, \bar{R}_j^n) \check{R}_{j+1/2}^n (\rho_{j+1}^n - \rho_{j-1}^n) \right. \\ &\quad \left. + \partial_{RR}^2 f(t^n, \bar{x}_j, \bar{\rho}_j^n, \bar{R}_j^n) \check{R}_{j+1/2}^n (\tilde{R}_{j+3/2}^n - \tilde{R}_{j-1/2}^n) \right. \\ &\quad \left. + \partial_R f(t^n, \bar{x}_j, \bar{\rho}_j^n, \bar{R}_j^n) (R_{j+2}^n - R_{j+1}^n - R_j^n + R_{j-1}^n) \right), \end{aligned}$$

with  $\tilde{x}_{j-1/2} \in (x_{j-1}, x_j)$ ,  $\hat{\rho}_j^n, \bar{\rho}_j^n \in \mathcal{I}(\rho_{j-1}^n, \rho_{j+1}^n)$ ,  $\tilde{R}_{j-1/2}^n \in \mathcal{I}(R_{j-1}^n, R_j^n)$ ,  $\hat{R}_j^n, \bar{R}_j^n \in \mathcal{I}(\tilde{R}_{j-1/2}^n, \tilde{R}_{j+3/2}^n)$ ,  $\hat{x}_j, \bar{x}_j \in \mathcal{I}(\tilde{x}_{j-1/2}, \tilde{x}_{j+3/2})$ ,  $\check{R}_{j+1/2}^n \in \mathcal{I}(R_j^n - R_{j-1}^n, R_{j+2}^n - R_{j+1}^n)$ .

Notice that as in (2.10) we can estimate

$$\begin{aligned} |R_{j+2}^n - R_{j+1}^n| &\leq \mathcal{T} \Delta x, \\ |R_j^n - R_{j-1}^n| &\leq \mathcal{T} \Delta x, \\ |R_{j+2}^n - R_j^n| &\leq 2\mathcal{T} \Delta x. \end{aligned}$$

Moreover, by their very definition,

$$\begin{aligned} |\tilde{R}_{j+3/2}^n - \tilde{R}_{j-1/2}^n| &= \left| \lambda_{j+3/2}^n R_{j+2}^n + (1 - \lambda_{j+3/2}^n) R_{j+1}^n - \mu_{j-1/2}^n R_{j-1}^n - (1 - \mu_{j-1/2}^n) R_j^n \right| \\ &\leq |R_{j+2}^n - R_{j+1}^n| + |R_j^n - R_{j-1}^n| + |R_{j+1}^n - R_j^n| \\ &\leq 3\mathcal{T} \Delta x, \end{aligned}$$

and

$$\begin{aligned} |\check{R}_{j+1/2}^n| &= \left| \delta_{j+1/2}^n (R_j^n - R_{j-1}^n) + (1 - \delta_{j+1/2}^n) (R_{j+2}^n - R_{j+1}^n) \right| \\ &\leq \delta_{j+1/2}^n |R_j^n - R_{j-1}^n| + (1 - \delta_{j+1/2}^n) |R_{j+2}^n - R_{j+1}^n| \\ &\leq \mathcal{T} \Delta x, \end{aligned}$$

for some  $\delta_{j+1/2}^n, \lambda_{j+3/2}^n, \mu_{j-1/2}^n \in [0, 1]$ . Finally,

$$|\hat{\rho}_j^n| = \left| \hat{\epsilon}_j^n \rho_{j-1}^n + (1 - \hat{\epsilon}_j^n) \rho_{j+1}^n \right| \leq \hat{\epsilon}_j^n |\rho_{j-1}^n| + (1 - \hat{\epsilon}_j^n) |\rho_{j+1}^n|,$$

$$|\bar{\rho}_j^n| = \left| \bar{\epsilon}_j^n \rho_{j-1}^n + (1 - \bar{\epsilon}_j^n) \rho_{j+1}^n \right| \leq \bar{\epsilon}_j^n \left| \rho_{j-1}^n \right| + (1 - \bar{\epsilon}_j^n) \left| \rho_{j+1}^n \right|,$$

with  $\bar{\epsilon}_j^n, \bar{\epsilon}_j^n \in [0, 1]$ . Therefore

$$\begin{aligned} |\mathcal{B}_j^n| &\leq \frac{3}{2} C (\Delta x)^2 |\hat{\rho}_j^n| + \frac{1}{2} \Delta x \|\partial_{\rho x}^2 f\|_{\mathbf{L}^\infty} \left| \rho_{j+1}^n - \rho_{j-1}^n \right| \\ &\quad + \frac{3}{2} C \mathcal{T} (\Delta x)^2 |\hat{\rho}_j^n| + \frac{3}{2} C \mathcal{T} (\Delta x)^2 |\bar{\rho}_j^n| + \frac{1}{2} \mathcal{T} \|\partial_{\rho R}^2 f\|_{\mathbf{L}^\infty} \Delta x \left| \rho_{j+1}^n - \rho_{j-1}^n \right| \\ &\quad + \frac{3}{2} C \mathcal{T}^2 (\Delta x)^2 |\bar{\rho}_j^n| + C \mathcal{T} (\Delta x)^2 |\bar{\rho}_j^n| \\ &\leq \frac{1}{2} \Delta x \left| \rho_{j+1}^n - \rho_{j-1}^n \right| \left( \|\partial_{\rho x}^2 f\|_{\mathbf{L}^\infty(\mathbb{R})} + \mathcal{T} \|\partial_{\rho R}^2 f\|_{\mathbf{L}^\infty} \right) \\ &\quad + \frac{3}{2} C (1 + \mathcal{T}) (\Delta x)^2 |\hat{\rho}_j^n| + \frac{1}{2} C \mathcal{T} (5 + 3\mathcal{T}) (\Delta x)^2 |\bar{\rho}_j^n|. \end{aligned}$$

Thus

$$\begin{aligned} \sum_{j=1}^{N-1} \lambda |\mathcal{B}_j^n| &\leq \frac{1}{2} \Delta t \left( \|\partial_{\rho x}^2 f\|_{\mathbf{L}^\infty} + \mathcal{T} \|\partial_{\rho R}^2 f\|_{\mathbf{L}^\infty} \right) \sum_{j=1}^{N-1} \left| \rho_{j+1}^n - \rho_{j-1}^n \right| \\ &\quad + \frac{1}{2} C \Delta t (3(1 + \mathcal{T}) + \mathcal{T}(5 + 3\mathcal{T})) \Delta x \sum_{j=1}^{N-1} \left[ (\bar{\epsilon}_j^n + \bar{\epsilon}_j^n) \left| \rho_{j-1}^n \right| + (2 - \bar{\epsilon}_j^n - \bar{\epsilon}_j^n) \left| \rho_{j+1}^n \right| \right] \\ &\leq \mathcal{K}_1 \Delta t \sum_{j=0}^N \left| \rho_{j+1}^n - \rho_j^n \right| + \tilde{\mathcal{K}}_2 \Delta t \Delta x \sum_{j=1}^N \left| \rho_{j+1}^n \right|, \end{aligned} \quad (2.17)$$

where we have set

$$\mathcal{K}_1 = \|\partial_{\rho x}^2 f\|_{\mathbf{L}^\infty} + \mathcal{T} \|\partial_{\rho R}^2 f\|_{\mathbf{L}^\infty}, \quad (2.18)$$

$$\tilde{\mathcal{K}}_2 = C (3 + 8\mathcal{T} + 3\mathcal{T}^2). \quad (2.19)$$

Collecting (2.16) and (2.17), we conclude that

$$\begin{aligned} \sum_{j=1}^N \left| \rho_{j+1}^{n+1} - \rho_j^{n+1} \right| &\leq \sum_{j=1}^N |\mathcal{A}_j^n| + \lambda \sum_{j=1}^N |\mathcal{B}_j^n| \\ &\leq (1 + \mathcal{K}_1 \Delta t) \sum_{j=1}^N \left| \rho_{j+1}^n - \rho_j^n \right| + \tilde{\mathcal{K}}_2 \Delta t \Delta x \sum_{j=1}^N \left| \rho_{j+1}^n \right| + \frac{1}{2} \mathcal{K}_1 \Delta t \left| \rho_1^n - \rho_a^n \right| \\ &\leq (1 + \mathcal{K}_1 \Delta t) \sum_{j=1}^N \left| \rho_{j+1}^n - \rho_j^n \right| + \tilde{\mathcal{K}}_2 \mathcal{C}_1(t^n) \Delta t + \frac{1}{2} \mathcal{K}_1 \Delta t \left| \rho_1^n - \rho_a^n \right|. \end{aligned} \quad (2.20)$$

We now take into account the boundary terms. From the definition of the scheme we know that

$$\begin{aligned} \rho_1^{n+1} - \rho_a^{n+1} &= (1 - \alpha_1^n - \beta_1^n) \rho_1^n + \alpha_1^n \rho_a^n + \beta_1^n \rho_2^n - \lambda \left( F_{3/2}^n(\rho_1^n, \rho_1^n) - F_{1/2}^n(\rho_1^n, \rho_1^n) \right) - \rho_a^{n+1} \pm \rho_a^n \\ &= \beta_1^n (\rho_2^n - \rho_1^n) + (1 - \alpha_1^n) (\rho_1^n - \rho_a^n) + (\rho_a^n - \rho_a^{n+1}) \end{aligned}$$

$$\begin{aligned}
& -\lambda \left( F_{3/2}^n(\rho_1^n, \rho_1^n) - F_{1/2}^n(\rho_1^n - \rho_1^n) \right) \\
& = \beta_1^n (\rho_2^n - \rho_1^n) + (1 - \gamma_1^n) (\rho_1^n - \rho_a^n) + (\rho_a^n - \rho_a^{n+1}) \\
& \quad - \lambda \left( F_{3/2}^n(\rho_a^n, \rho_1^n) - F_{1/2}^n(\rho_a^n, \rho_1^n) \right).
\end{aligned}$$

Indeed, by definition of  $\alpha_1^n$  and of  $\gamma_1^n$  we get

$$\begin{aligned}
& \alpha_1^n (\rho_1^n - \rho_a^n) + \lambda \left( F_{3/2}^n(\rho_1^n, \rho_1^n) - F_{1/2}^n(\rho_1^n, \rho_1^n) \right) = \\
& = \lambda \left( F_{3/2}^n(\rho_1^n, \rho_1^n) - F_{1/2}^n(\rho_a^n, \rho_1^n) \pm F_{3/2}^n(\rho_a^n, \rho_1^n) \right) \\
& = \lambda \frac{F_{3/2}^n(\rho_1^n, \rho_1^n) - F_{3/2}^n(\rho_a^n, \rho_1^n)}{\rho_1^n - \rho_a^n} (\rho_1^n - \rho_a^n) + \lambda \left( F_{3/2}^n(\rho_a^n, \rho_1^n) - F_{1/2}^n(\rho_a^n, \rho_1^n) \right) \\
& = \gamma_1^n (\rho_1^n - \rho_a^n) + \lambda \left( F_{3/2}^n(\rho_a^n, \rho_1^n) - F_{1/2}^n(\rho_a^n, \rho_1^n) \right).
\end{aligned}$$

Since

$$\begin{aligned}
& \lambda \left( F_{3/2}^n(\rho_a^n, \rho_1^n) - F_{1/2}^n(\rho_a^n, \rho_1^n) \right) = \\
& = \frac{\lambda}{2} \left( f(t^n, x_1, \rho_a^n, R_1^n) - f(t^n, x_0, \rho_a^n, R_0^n) \right) + \frac{\lambda}{2} \left( f(t^n, x_2, \rho_1^n, R_2^n) - f(t^n, x_1, \rho_1^n, R_1^n) \right) \\
& = \frac{\lambda}{2} \left( \partial_x f(t^n, \tilde{x}_{1/2}, \rho_a^n, \tilde{R}_{1/2}^n) \Delta x + \partial_R f(t^n, \tilde{x}_{1/2}, \rho_a^n, \tilde{R}_{1/2}^n) (R_1^n - R_0^n) \right) \\
& \quad + \frac{\lambda}{2} \left( \partial_x f(t^n, \tilde{x}_{3/2}, \rho_1^n, \tilde{R}_{3/2}^n) \Delta x + \partial_R f(t^n, \tilde{x}_{3/2}, \rho_1^n, \tilde{R}_{3/2}^n) (R_2^n - R_1^n) \right)
\end{aligned}$$

(where we used obvious notations for  $\tilde{x}_{1/2}$ ,  $\tilde{x}_{3/2}$ ,  $\tilde{R}_{1/2}^n$  and  $\tilde{R}_{3/2}^n$ ), we can conclude that

$$\lambda \left| F_{3/2}^n(\rho_a^n, \rho_1^n) - F_{1/2}^n(\rho_a^n, \rho_1^n) \right| \leq \Delta t \frac{C}{2} (1 + \mathcal{T}) (|\rho_a^n| + |\rho_1^n|).$$

Thus, because of the positivity of the coefficients involved, we get

$$\begin{aligned}
\left| \rho_1^{n+1} - \rho_a^{n+1} \right| & \leq \beta_1^n |\rho_2^n - \rho_1^n| + (1 - \gamma_1^n) |\rho_1^n - \rho_a^n| + \left| \rho_a^n - \rho_a^{n+1} \right| \\
& \quad + \Delta t \frac{C}{2} (1 + \mathcal{T}) (|\rho_a^n| + |\rho_1^n|).
\end{aligned} \tag{2.21}$$

Concerning the right boundary data, we have

$$\begin{aligned}
\rho_N^{n+1} - \rho_b^{n+1} & = (1 - \alpha_N^n - \beta_N^n) \rho_N^n + \alpha_N^n \rho_{N-1}^n + \beta_N^n \rho_b^n - \lambda \left( F_{N+1/2}^n(\rho_N^n, \rho_N^n) - F_{N-1/2}^n(\rho_N^n, \rho_N^n) \right) \\
& \quad - \rho_b^{n+1} \pm \rho_b^n \\
& = -\alpha_N^n (\rho_N^n - \rho_{N-1}^n) + (1 - \beta_N^n) (\rho_N^n - \rho_b^n) + (\rho_b^n - \rho_b^{n+1}) \\
& \quad + \alpha_N^n (\rho_N^n - \rho_{N-1}^n) - \gamma_N^n (\rho_N^n - \rho_{N-1}^n) \\
& \quad - \lambda \left( F_{N+1/2}^n(\rho_{N-1}^n, \rho_N^n) - F_{N-1/2}^n(\rho_{N-1}^n, \rho_N^n) \right) \\
& = (1 - \beta_N^n) (\rho_N^n - \rho_b^n) + (\rho_b^n - \rho_b^{n+1}) - \gamma_N^n (\rho_N^n - \rho_{N-1}^n) \\
& \quad - \lambda \left( F_{N+1/2}^n(\rho_{N-1}^n, \rho_N^n) - F_{N-1/2}^n(\rho_{N-1}^n, \rho_N^n) \right).
\end{aligned} \tag{2.22}$$

We can justify the above equalities as follows. Taking into account of the expressions of  $\alpha_N^n$  and of  $\gamma_N^n$ , we can rearrange

$$\begin{aligned}
& -\lambda \left( F_{N+1/2}^n(\rho_N^n, \rho_N^n) - F_{N-1/2}^n(\rho_N^n, \rho_N^n) \right) = \\
& = -\lambda \left( F_{N+1/2}^n(\rho_N^n, \rho_N^n) - F_{N-1/2}^n(\rho_N^n, \rho_N^n) \pm F_{N-1/2}^n(\rho_{N-1}^n, \rho_N^n) \pm F_{N+1/2}^n(\rho_{N-1}^n, \rho_N^n) \right) \\
& = -\lambda \left( F_{N-1/2}^n(\rho_{N-1}^n, \rho_N^n) - F_{N-1/2}^n(\rho_N^n, \rho_N^n) \right) - \lambda \left( F_{N+1/2}^n(\rho_N^n, \rho_N^n) - F_{N+1/2}^n(\rho_{N-1}^n, \rho_N^n) \right) \\
& \quad - \lambda \left( F_{N+1/2}^n(\rho_{N-1}^n, \rho_N^n) - F_{N-1/2}^n(\rho_{N-1}^n, \rho_N^n) \right) \\
& = \alpha_N^n (\rho_N^n - \rho_{N-1}^n) - \gamma_N^n (\rho_N^n - \rho_{N-1}^n) - \lambda \left( F_{N+1/2}^n(\rho_{N-1}^n, \rho_N^n) - F_{N-1/2}^n(\rho_{N-1}^n, \rho_N^n) \right).
\end{aligned}$$

Let us estimate the last term in (2.22):

$$\begin{aligned}
& \lambda \left| F_{N+1/2}^n(\rho_{N-1}^n, \rho_N^n) - F_{N-1/2}^n(\rho_{N-1}^n, \rho_N^n) \right| = \\
& = \frac{\lambda}{2} \left| f(t^n, x_N, \rho_{N-1}^n, R_N^n) - f(t^n, x_{N-1}, \rho_{N-1}^n, R_{N-1}^n) + f(t^n, x_{N+1}, \rho_N^n, R_{N+1}^n) - f(t^n, x_N, \rho_N^n, R_N^n) \right| \\
& = \frac{\lambda}{2} \left| \partial_x f(t^n, \tilde{x}_{N-1/2}, \rho_{N-1}^n, \tilde{R}_{N-1/2}^n) \Delta x + \partial_R f(t^n, \tilde{x}_{N-1/2}, \rho_{N-1}^n, \tilde{R}_{N-1/2}^n) (R_N^n - R_{N-1}^n) \right. \\
& \quad \left. + \partial_x f(t^n, \tilde{x}_{N+1/2}, \rho_N^n, \tilde{R}_{N+1/2}^n) \Delta x + \partial_R f(t^n, \tilde{x}_{N+1/2}, \rho_N^n, \tilde{R}_{N+1/2}^n) (R_{N+1}^n - R_N^n) \right| \\
& \leq \Delta t \frac{C}{2} (1 + \mathcal{T}) (|\rho_{N-1}^n| + |\rho_N^n|).
\end{aligned}$$

Therefore we get

$$\begin{aligned}
\left| \rho_N^{n+1} - \rho_b^{n+1} \right| & \leq (1 - \beta_N^n) |\rho_N^n - \rho_b^n| + \gamma_N^n |\rho_N^n - \rho_{N-1}^n| + \left| \rho_b^n - \rho_b^{n+1} \right| \\
& \quad + \Delta t \frac{C}{2} (1 + \mathcal{T}) (|\rho_{N-1}^n| + |\rho_N^n|).
\end{aligned} \tag{2.23}$$

Collecting estimates (2.20), (2.21) and (2.23), we obtain

$$\begin{aligned}
& \sum_{j=0}^N \left| \rho_{j+1}^{n+1} - \rho_j^{n+1} \right| \\
& = \left| \rho_1^{n+1} - \rho_a^{n+1} \right| + \left| \rho_b^{n+1} - \rho_N^{n+1} \right| + \sum_{j=1}^{N-1} \left| \mathcal{A}_j^n - \lambda \mathcal{B}_j^n \right| \\
& \leq \beta_1^n |\rho_2^n - \rho_1^n| + (1 - \gamma_1^n) |\rho_1^n - \rho_a^n| + \left| \rho_a^n - \rho_a^{n+1} \right| + \Delta t \frac{C}{2} (1 + \mathcal{T}) (|\rho_a^n| + |\rho_1^n|) \\
& \quad + (1 - \beta_N^n) |\rho_b^n - \rho_N^n| + \gamma_N^n |\rho_N^n - \rho_{N-1}^n| + \left| \rho_b^n - \rho_b^{n+1} \right| + \Delta t \frac{C}{2} (1 + \mathcal{T}) (|\rho_{N-1}^n| + |\rho_N^n|) \\
& \quad + \sum_{j=1}^{N-1} \left| \rho_{j+1}^n - \rho_j^n \right| - \beta_1^n |\rho_2^n - \rho_1^n| + \gamma_1^n |\rho_1^n - \rho_a^n| + \beta_N^n |\rho_b^n - \rho_N^n| - \gamma_N^n |\rho_N^n - \rho_{N-1}^n| \\
& \quad + \lambda \sum_{j=1}^{N-1} |\mathcal{B}_j^n|
\end{aligned}$$

$$\begin{aligned}
&\leq \left| \rho_a^n - \rho_a^{n+1} \right| + \left| \rho_b^n - \rho_b^{n+1} \right| + \sum_{j=0}^N \left| \rho_{j+1}^n - \rho_j^n \right| \\
&\quad + \Delta t \frac{C}{2} (1 + \mathcal{T}) (|\rho_a^n| + |\rho_1^n| + |\rho_{N-1}^n| + |\rho_N^n|) \\
&\quad + \mathcal{K}_1 \Delta t \sum_{j=0}^N \left| \rho_{j+1}^n - \rho_j^n \right| + \tilde{\mathcal{K}}_2 \Delta t \Delta x \sum_{j=1}^N \left| \rho_{j+1}^n \right| \\
&\leq \left| \rho_a^n - \rho_a^{n+1} \right| + \left| \rho_b^n - \rho_b^{n+1} \right| + (1 + \mathcal{K}_1 \Delta t) \sum_{j=0}^N \left| \rho_{j+1}^n - \rho_j^n \right| + \tilde{\mathcal{K}}_2 \mathcal{C}_1(t^n) \Delta t \\
&\quad + C (1 + \mathcal{T}) \left( \frac{3}{2} e^{\mathcal{L}t^n} \|\rho_0\|_{\mathbf{L}^\infty(I)} + \frac{1}{2} \|\rho_a\|_{\mathbf{L}^\infty([0,T])} \right) \Delta t.
\end{aligned}$$

Setting

$$\mathcal{K}_2 := \tilde{\mathcal{K}}_2 \mathcal{C}_1(t^n) + C (1 + \mathcal{T}) \left( \frac{3}{2} e^{\mathcal{L}t^n} \|\rho_0\|_{\mathbf{L}^\infty(I)} + \frac{1}{2} \|\rho_a\|_{\mathbf{L}^\infty[0,T]} \right), \quad (2.24)$$

we deduce from the previous estimate

$$\begin{aligned}
\sum_{j=0}^N \left| \rho_{j+1}^n - \rho_j^n \right| &\leq (1 + \mathcal{K}_1 \Delta t)^{t^n/\Delta t} \sum_{j=0}^N \left| \rho_{j+1}^0 - \rho_j^0 \right| + \mathcal{K}_2 \frac{(1 + \mathcal{K}_1 \Delta t)^{t^n/\Delta t} - 1}{\mathcal{K}_1} \\
&\quad + \sum_{m=1}^n \left| \rho_a^m - \rho_a^{m-1} \right| + \sum_{m=1}^n \left| \rho_b^m - \rho_b^{m-1} \right| \\
&\leq e^{\mathcal{K}_1 t^n} \sum_{j=0}^N \left| \rho_{j+1}^0 - \rho_j^0 \right| + \frac{\mathcal{K}_2}{\mathcal{K}_1} (e^{\mathcal{K}_1 t^n} - 1) + \sum_{m=1}^n \left| \rho_a^m - \rho_a^{m-1} \right| + \sum_{m=1}^n \left| \rho_b^m - \rho_b^{m-1} \right|,
\end{aligned}$$

thus concluding the proof.  $\square$

**Corollary 2 (BV estimate in space and time)** *Let hypotheses (1.2) and conditions (2.3) hold. If  $\rho_0 \in \mathbf{L}^\infty(I; \mathbb{R}^+)$  and  $\rho_a, \rho_b \in \mathbf{L}^\infty(\mathbb{R}^+; \mathbb{R}^+)$ , then  $\rho_\Delta$  satisfies the following Total Variation estimate in space and time*

$$\sum_{m=0}^{n-1} \sum_{j=0}^N \Delta t \left| \rho_{j+1}^m - \rho_j^m \right| + \sum_{m=0}^{n-1} \sum_{j=0}^N \Delta x \left| \rho_j^{m+1} - \rho_j^m \right| \leq \mathcal{C}_{xt}(n\Delta t), \quad (2.25)$$

with  $\mathcal{C}_{xt}(n\Delta t)$  given by (2.30).

*Proof.* The spatial BV estimate (2.13) yields

$$\sum_{m=0}^{n-1} \sum_{j=0}^N \Delta t \left| \rho_{j+1}^m - \rho_j^m \right| \leq n\Delta t \mathcal{C}_x(n\Delta t). \quad (2.26)$$

In order to bound the second term in (2.25), we make use of the definition of the numerical scheme (2.1), (2.2). In fact, by (1.2) and (2.10) we have the following estimate

$$\left| \rho_j^{m+1} - \rho_j^m \right| \leq \frac{\lambda\alpha}{2} \left( \left| \rho_j^m - \rho_{j-1}^m \right| + \left| \rho_{j+1}^m - \rho_j^m \right| \right)$$

$$\begin{aligned}
& + \frac{\lambda}{2} \left[ 2 \left| \partial_x f(t^m, \tilde{x}_j, \xi_j^m, \tilde{R}_j^m) \right| \Delta x + \left| \partial_\rho f(t^m, \tilde{x}_j, \xi_j^m, \tilde{R}_j^m) \left( \rho_{j+1}^m - \rho_{j-1}^m \right) \right| \right. \\
& \left. + \left| \partial_{R_j} f(t^m, \tilde{x}_j, \xi_j^m, \tilde{R}_j^m) \left( R_{j+1}^m - R_{j-1}^m \right) \right| \right] \\
& \leq \frac{\lambda}{2} (\alpha + L) \left( \left| \rho_j^m - \rho_{j-1}^m \right| + \left| \rho_{j+1}^m - \rho_j^m \right| \right) \\
& \quad + C \Delta t \left| \epsilon \rho_{j+1}^m + (1 - \epsilon) \rho_{j-1}^m \right| + \mathcal{T} \Delta t \left| \epsilon \rho_{j+1}^m + (1 - \epsilon) \rho_{j-1}^m \right| \\
& \leq \frac{\lambda}{2} (\alpha + L) \left( \left| \rho_j^m - \rho_{j-1}^m \right| + \left| \rho_{j+1}^m - \rho_j^m \right| \right) + \Delta t (C + \mathcal{T}) \left( \epsilon \left| \rho_{j+1}^m \right| + (1 - \epsilon) \left| \rho_{j-1}^m \right| \right)
\end{aligned}$$

for  $j = 1, \dots, N - 1$ , where  $\xi_j^m = \epsilon \rho_{j+1}^m + (1 - \epsilon) \rho_{j-1}^m \in \mathcal{I}(\rho_{j-1}^m, \rho_{j+1}^m)$ ,  $\epsilon \in [0, 1]$ ,  $\tilde{x}_j \in [x_{j-1}, x_{j+1}]$  and  $\tilde{R}_j^m \in \mathcal{I}(R_{j-1}^m, R_{j+1}^m)$ . Therefore,

$$\begin{aligned}
\sum_{j=0}^N \Delta x \left| \rho_j^{m+1} - \rho_j^m \right| & = \Delta x \left( \left| \rho_a^{m+1} - \rho_a^m \right| + \left| \rho_b^{m+1} - \rho_b^m \right| \right) + \sum_{j=1}^{N-1} \Delta x \left| \rho_j^{m+1} - \rho_j^m \right| \\
& \leq \Delta x \left( \left| \rho_a^{m+1} - \rho_a^m \right| + \left| \rho_b^{m+1} - \rho_b^m \right| \right) \\
& \quad + \Delta t (\alpha + L) \sum_{j=0}^{N-1} \left| \rho_{j+1}^m - \rho_j^m \right| + 2 \Delta x \Delta t (C + \mathcal{T}) \sum_{j=0}^N \left| \rho_j^m \right| \\
& \leq \Delta x \left( \left| \rho_a^{m+1} - \rho_a^m \right| + \left| \rho_b^{m+1} - \rho_b^m \right| \right) \\
& \quad + \Delta t (\alpha + L) \mathcal{C}_x(m \Delta t) + 2 \Delta x \Delta t (C + \mathcal{T}) \mathcal{C}_1(m \Delta t), \tag{2.27}
\end{aligned}$$

due to estimates (2.13) and (2.8). In particular,

$$\sum_{j=0}^N \Delta x \left| \rho_j^{m+1} - \rho_j^m \right| \leq \Delta t \mathcal{C}_t(m \Delta t),$$

where

$$\mathcal{C}_t(m \Delta t) := \lambda \left( \left| \rho_a^{m+1} - \rho_a^m \right| + \left| \rho_b^{m+1} - \rho_b^m \right| \right) + (\alpha + L) \mathcal{C}_x(m \Delta t) + 2 \Delta x (C + \mathcal{T}) \mathcal{C}_1(m \Delta t), \tag{2.28}$$

which allows to derive the  $\mathbf{L}^1$  Lipschitz continuity in time (1.12).

Summing over  $j = 0, \dots, n - 1$ , we get

$$\begin{aligned}
\sum_{m=0}^{n-1} \sum_{j=0}^N \Delta x \left| \rho_j^{m+1} - \rho_j^m \right| & \leq \Delta x \sum_{m=0}^{n-1} \left( \left| \rho_a^{m+1} - \rho_a^m \right| + \left| \rho_b^{m+1} - \rho_b^m \right| \right) \\
& \quad + n \Delta t (\alpha + L) \mathcal{C}_x(n \Delta t) + 2 n \Delta t \Delta x (C + \mathcal{T}) \mathcal{C}_1(n \Delta t). \tag{2.29}
\end{aligned}$$

Summing (2.26) and (2.29) we get (2.25) with

$$\mathcal{C}_{xt}(n \Delta t) := n \Delta t (\alpha + L + 1) \mathcal{C}_x(n \Delta t)$$



$$+ \Delta x \left[ 2\Delta t (C + \mathcal{T}) \mathcal{C}_1(n\Delta t) + \sum_{m=0}^{n-1} \left( \left| \rho_a^{m+1} - \rho_a^m \right| + \left| \rho_b^{m+1} - \rho_b^m \right| \right) \right], \quad (2.30)$$

thus completing the proof.  $\square$

## 2.5 Discrete entropy inequalities

We adopt the following notation

$$\begin{aligned} F_{j+1/2}^n(u, v) &:= \frac{1}{2} \left( f(t^n, x_j, u, R_j^n) + f(t^n, x_{j+1}, v, R_{j+1}^n) + \alpha(u - v) \right), \\ H_j(u, v, z) &:= v - \lambda \left( F_{j+1/2}^n(v, z) - F_{j-1/2}^n(u, v) \right), \\ G_{j+1/2}^\kappa &:= F_{j+1/2}^n(u \wedge \kappa, v \wedge \kappa) - F_{j+1/2}^n(\kappa, \kappa). \end{aligned}$$

The approximate solution  $\rho_\Delta$  satisfies the following inequalities.

**Lemma 4** *Under the hypotheses (1.2) and the conditions (2.3), the following discrete entropy inequalities hold for  $j = 1, \dots, N$ ,  $n \in \mathbb{N}$ ,  $\kappa \in \mathbb{R}$ :*

$$\begin{aligned} & \left( \rho_j^{n+1} - \kappa \right)^+ - \left( \rho_j^n - \kappa \right)^+ + \lambda \left( G_{j+1/2}^\kappa(\rho_j^n, \rho_{j+1}^n) - G_{j-1/2}^\kappa(\rho_{j-1}^n, \rho_j^n) \right) \\ & + \frac{\lambda}{2} \operatorname{sgn}^+ \left( \rho_j^{n+1} - \kappa \right) \left( f(t^n, x_{j+1}, \kappa, R_{j+1}^n) - f(t^n, x_{j-1}, \kappa, R_{j-1}^n) \right) \leq 0. \end{aligned} \quad (2.31)$$

*Proof.* Let us consider the map  $(u, v, z) \mapsto H_j(u, v, z)$ . By (2.3), it holds

$$\begin{aligned} \frac{\partial H}{\partial u}(u, v, z) &= -\frac{\lambda}{2} \left( \partial_\rho f(t^n, x_{j-1}, u, R_{j-1}^n) - \alpha \right) \geq 0, \\ \frac{\partial H}{\partial v}(u, v, z) &= 1 - \frac{\alpha\lambda}{2} \geq 0, \\ \frac{\partial H}{\partial z}(u, v, z) &= -\frac{\lambda}{2} \left( \partial_\rho f(t^n, x_{j+1}, z, R_{j+1}^n) - \alpha \right) \geq 0. \end{aligned}$$

Notice that

$$H_j(\kappa, \kappa, \kappa) = \kappa - \frac{\lambda}{2} \left( f(t^n, x_{j+1}, \kappa, R_{j+1}^n) - f(t^n, x_{j-1}, \kappa, R_{j-1}^n) \right).$$

For  $\kappa \in \mathbb{R}$ ,  $j = 1, \dots, N$ , noticing that  $\rho_j^{n+1} = H_j(\rho_{j-1}^n, \rho_j^n, \rho_{j+1}^n)$  and using the monotonicity above, we get

$$\begin{aligned} & H_j(\rho_{j-1}^n \wedge \kappa, \rho_j^n \wedge \kappa, \rho_{j+1}^n \wedge \kappa) - H_j(\kappa, \kappa, \kappa) \geq \\ & \geq H_j(\rho_{j-1}^n, \rho_j^n, \rho_{j+1}^n) \wedge H_j(\kappa, \kappa, \kappa) - H_j(\kappa, \kappa, \kappa) \\ & = \left( H_j(\rho_{j-1}^n, \rho_j^n, \rho_{j+1}^n) - H_j(\kappa, \kappa, \kappa) \right)^+ \\ & = \left( \rho_j^{n+1} - \kappa + \frac{\lambda}{2} \left( f(t^n, x_{j+1}, \kappa, R_{j+1}^n) - f(t^n, x_{j-1}, \kappa, R_{j-1}^n) \right) \right)^+. \end{aligned}$$

On the other hand we obtain

$$H_j(\rho_{j-1}^n \wedge \kappa, \rho_j^n \wedge \kappa, \rho_{j+1}^n \wedge \kappa) - H_j(\kappa, \kappa, \kappa) =$$

$$\begin{aligned}
&= (\rho_j^n - \kappa)^+ \\
&\quad - \lambda \left( F_{j+1/2}^n(\rho_j^n \wedge \kappa, \rho_{j+1}^n \wedge \kappa) - F_{j-1/2}^n(\rho_{j-1}^n \wedge \kappa, \rho_j^n \wedge \kappa) - F_{j+1/2}^n(\kappa, \kappa) + F_{j-1/2}^n(\kappa, \kappa) \right) \\
&= (\rho_j^n - \kappa)^+ - \lambda \left( G_{j+1/2}^\kappa(\rho_j^n, \rho_{j+1}^n) - G_{j-1/2}^\kappa(\rho_{j-1}^n, \rho_j^n) \right).
\end{aligned}$$

Thus,

$$\begin{aligned}
&(\rho_j^n - \kappa)^+ - \lambda \left( G_{j+1/2}^\kappa(\rho_j^n, \rho_{j+1}^n) - G_{j-1/2}^\kappa(\rho_{j-1}^n, \rho_j^n) \right) \geq \\
&\geq \left( \rho_j^{n+1} - \kappa + \frac{\lambda}{2} \left( f(t^n, x_{j+1}, \kappa, R_{j+1}^n) - f(t^n, x_{j-1}, \kappa, R_{j-1}^n) \right) \right)^+ \\
&= \operatorname{sgn}^+ \left( \rho_j^{n+1} - \kappa + \frac{\lambda}{2} \left( f(t^n, x_{j+1}, \kappa, R_{j+1}^n) - f(t^n, x_{j-1}, \kappa, R_{j-1}^n) \right) \right) \\
&\quad \times \left( \rho_j^{n+1} - \kappa + \frac{\lambda}{2} \left( f(t^n, x_{j+1}, \kappa, R_{j+1}^n) - f(t^n, x_{j-1}, \kappa, R_{j-1}^n) \right) \right) \\
&\geq \operatorname{sgn}^+ \left( \rho_j^{n+1} - \kappa \right) \left( \rho_j^{n+1} - \kappa + \frac{\lambda}{2} \left( f(t^n, x_{j+1}, \kappa, R_{j+1}^n) - f(t^n, x_{j-1}, \kappa, R_{j-1}^n) \right) \right) \\
&= (\rho_j^{n+1} - \kappa)^+ + \frac{\lambda}{2} \operatorname{sgn}^+ \left( \rho_j^{n+1} - \kappa \right) \left( f(t^n, x_{j+1}, \kappa, R_{j+1}^n) - f(t^n, x_{j-1}, \kappa, R_{j-1}^n) \right),
\end{aligned}$$

which proves (2.31).  $\square$

## 2.6 Convergence towards a weak entropy solution

The estimates given by Lemmas 3 and Corollary 2 allow to apply Helly's compactness theorem, ensuring the existence of a subsequence, still denoted  $\{\rho_\Delta\}$  converging to a function  $\rho \in \mathbf{L}^\infty([0, T] \times I)$  in the  $\mathbf{L}^1$ -norm, for all  $T > 0$  (see for example [16, Section 5.3.5]). We need now to prove that the limit of approximate solutions is indeed a weak entropy solution, in the sense of Definition 1.

**Lemma 5** *Let hypotheses (1.2) and conditions (2.3) hold. If  $\rho_0 \in (\mathbf{L}^\infty \cap BV)(I; \mathbb{R}^+)$  and  $\rho_a, \rho_b \in (\mathbf{L}^\infty \cap BV)(\mathbb{R}^+; \mathbb{R}^+)$ , then the piecewise constant approximate solutions  $\rho_\Delta$  resulting from the adapted Lax-Friedrichs scheme (2.1) converge, as  $\Delta x \searrow 0$ , towards a weak entropy solution of the initial boundary value problem (1.1).*

*Proof.* We follow closely [27]. Adding and subtracting  $G_{j+1/2}^\kappa(\rho_j^n, \rho_j^n)$ , we rearrange (2.31) as

$$\begin{aligned}
0 &\geq (\rho_j^{n+1} - \kappa)^+ - (\rho_j^n - \kappa)^+ + \lambda \left( G_{j+1/2}^\kappa(\rho_j^n, \rho_{j+1}^n) - G_{j+1/2}^\kappa(\rho_j^n, \rho_j^n) \right) \\
&\quad + \lambda \left( G_{j+1/2}^\kappa(\rho_j^n, \rho_j^n) - G_{j-1/2}^\kappa(\rho_{j-1}^n, \rho_j^n) \right) \\
&\quad + \frac{\lambda}{2} \operatorname{sgn}^+ \left( \rho_j^{n+1} - \kappa \right) \left( f(t^n, x_{j+1}, \kappa, R_{j+1}^n) - f(t^n, x_{j-1}, \kappa, R_{j-1}^n) \right). \tag{2.32}
\end{aligned}$$

Let  $\varphi \in C_c^1([0, T] \times [a, b], \mathbb{R}^+)$  for some  $T > 0$ . Multiplying (2.32) by  $\Delta x \varphi(t^n, x_j) \geq 0$ , and summing over  $j = 1, \dots, N$ ,  $n \in \mathbb{N}$ , we get the inequality

$$0 \geq \Delta x \sum_{n=0}^{+\infty} \sum_{j=1}^N \left( \eta_\kappa^+(\rho_j^{n+1}) - \eta_\kappa^+(\rho_j^n) \right) \varphi(t^n, x_j)$$

$$\begin{aligned}
& + \Delta t \sum_{n=0}^{+\infty} \sum_{j=1}^N \left( \left( G_{j+1/2}^{\kappa}(\rho_j^n, \rho_{j+1}^n) - G_{j+1/2}^{\kappa}(\rho_j^n, \rho_j^n) \right) - \left( G_{j-1/2}^{\kappa}(\rho_{j-1}^n, \rho_j^n) - G_{j+1/2}^{\kappa}(\rho_j^n, \rho_j^n) \right) \right) \varphi(t^n, x_j) \\
& + \frac{\Delta t}{2} \sum_{n=0}^{+\infty} \sum_{j=1}^N \operatorname{sgn}^+(\rho_j^{n+1} - \kappa) \left( f(t^n, x_{j+1}, \kappa, R_{j+1}^n) - f(t^n, x_{j-1}, \kappa, R_{j-1}^n) \right) \varphi(t^n, x_j). \quad (2.33)
\end{aligned}$$

Summing by parts in (2.33) we obtain

$$\begin{aligned}
T_1 & := \Delta x \sum_{n=0}^{+\infty} \sum_{j=1}^N \left( \eta_{\kappa}^+(\rho_j^{n+1}) - \eta_{\kappa}^+(\rho_j^n) \right) \varphi(t^n, x_j) \\
& = -\Delta x \sum_{j=1}^N \varphi(0, x_j) \eta_{\kappa}^+(\rho_j^0) - \Delta x \Delta t \sum_{n=1}^{+\infty} \sum_{j=1}^N \eta_{\kappa}^+(\rho_j^n) \frac{\varphi(t^n, x_j) - \varphi(t^{n-1}, x_j)}{\Delta t} \\
& \xrightarrow{\Delta x \searrow 0^+} - \int_a^b \varphi(0, x) \rho_0(x) dx - \int_0^{+\infty} \int_a^b \partial_t \varphi(t, x) \eta_{\kappa}^+(\rho(t, x)) dx dt,
\end{aligned}$$

and

$$\begin{aligned}
T_3 & := \frac{\Delta t}{2} \sum_{n=0}^{+\infty} \sum_{j=1}^N \operatorname{sgn}^{\pm}(\rho_j^{n+1} - \kappa) \left( f(t^n, x_{j+1}, \kappa, R_{j+1}^n) - f(t^n, x_{j-1}, \kappa, R_{j-1}^n) \right) \varphi(t^n, x_j) \\
& = \Delta t \Delta x \sum_{n=0}^{+\infty} \sum_{j=1}^N \operatorname{sgn}^+(\rho_j^{n+1} - \kappa) \frac{f(t^n, x_{j+1}, \kappa, R_{j+1}^n) - f(t^n, x_{j-1}, \kappa, R_{j-1}^n)}{2\Delta x} \\
& \xrightarrow{\Delta x \searrow 0^+} \int_0^{+\infty} \int_a^b \operatorname{sgn}^+(\rho_j^{n+1} - \kappa) \partial_x f(t, x, \kappa, R(t, x)) \varphi dx dt,
\end{aligned}$$

by the Dominated Convergence Theorem. Furthermore

$$\begin{aligned}
& \Delta t \sum_{n=0}^{+\infty} \sum_{j=1}^N \left[ \left( G_{j+1/2}^{\kappa}(\rho_j^n, \rho_{j+1}^n) - G_{j+1/2}^{\kappa}(\rho_j^n, \rho_j^n) \right) - \left( G_{j-1/2}^{\kappa}(\rho_{j-1}^n, \rho_j^n) - G_{j+1/2}^{\kappa}(\rho_j^n, \rho_j^n) \right) \right] \varphi(t^n, x_j) \\
& = \Delta t \sum_{n=0}^{+\infty} \sum_{j=1}^N \left( G_{j+1/2}^{\kappa}(\rho_j^n, \rho_{j+1}^n) - G_{j+1/2}^{\kappa}(\rho_j^n, \rho_j^n) \right) \varphi(t^n, x_j) \\
& \quad - \Delta t \sum_{n=0}^{+\infty} \sum_{j=0}^{N-1} \left( G_{j+1/2}^{\kappa}(\rho_j^n, \rho_{j+1}^n) - G_{j+3/2}^{\kappa}(\rho_{j+1}^n, \rho_{j+1}^n) \right) \varphi(t^n, x_{j+1}) \\
& = \Delta t \sum_{n=0}^{+\infty} \sum_{j=1}^{N-1} \left[ \left( G_{j+1/2}^{\kappa}(\rho_j^n, \rho_{j+1}^n) - G_{j+1/2}^{\kappa}(\rho_j^n, \rho_j^n) \right) \varphi(t^n, x_j) \right. \\
& \quad \left. - \left( G_{j+1/2}^{\kappa}(\rho_j^n, \rho_{j+1}^n) - G_{j+3/2}^{\kappa}(\rho_{j+1}^n, \rho_{j+1}^n) \right) \varphi(t^n, x_{j+1}) \right] \\
& \quad + \Delta t \sum_{n=0}^{+\infty} \left( G_{N+1/2}^{\kappa}(\rho_N^n, \rho_b^n) - G_{N+1/2}^{\kappa}(\rho_N^n, \rho_N^n) \right) \varphi(t^n, x_N) - \left( G_{1/2}^{\kappa}(\rho_a^n, \rho_1^n) - G_{3/2}^{\kappa}(\rho_1^n, \rho_1^n) \right) \varphi(t^n, x_1) \\
& =: T_2^{int} + T_2^b =: T_2. \quad (2.34)
\end{aligned}$$

Let us define

$$T_{20} = -\Delta t \Delta x \sum_{n \in \mathbb{N}} \sum_{j=1}^N G_{j+1/2}^\kappa(\rho_j^n, \rho_j^n) \frac{\varphi(t^n, x_{j+1}) - \varphi(t^n, x_j)}{\Delta x} \\ - \alpha \Delta t \sum_{n \in \mathbb{N}} \left( \varphi(t^n, b) \eta_\kappa^+(\rho_b^n) + \varphi(t^n, a) \eta_\kappa^+(\rho_a^n) \right).$$

Being

$$G_{j+1/2}^\kappa(\rho_j^n, \rho_j^n) = F_{j+1/2}^n(\rho_j^n \wedge \kappa, \rho_j^n \wedge \kappa) - F_{j+1/2}^n(\kappa, \kappa) \\ = \frac{1}{2} \left( f(t^n, x_j, \rho_j^n \wedge \kappa, R_j^n) - f(t^n, x_j, \kappa, R_j^n) \right) \\ + \frac{1}{2} \left( f(t^n, x_{j+1}, \rho_j^n \wedge \kappa, R_{j+1}^n) - f(t^n, x_{j+1}, \kappa, R_{j+1}^n) \right) \\ = \frac{1}{2} \operatorname{sgn}^+(\rho_j^n - \kappa) \left( f(t^n, x_j, \rho_j^n, R_j^n) - f(t^n, x_j, \kappa, R_j^n) \right) \\ + \frac{1}{2} \operatorname{sgn}^+(\rho_j^n - \kappa) \left( f(t^n, x_{j+1}, \rho_j^n, R_{j+1}^n) - f(t^n, x_{j+1}, \kappa, R_{j+1}^n) \right),$$

it is straightforward to see that

$$T_{20} \xrightarrow{\Delta x \searrow 0^+} - \int_0^{+\infty} \int_a^b \operatorname{sgn}^+(\rho - \kappa) (f(t, x, \rho, R(t, x)) - f(t, x, \kappa, R(t, x))) \partial_x \varphi(t, x) dx dt \\ - \operatorname{Lip}(f) \int_0^{+\infty} (\rho_a(t) - \kappa)^+ \varphi(t, a) dt - \operatorname{Lip}(f) \int_0^{+\infty} (\rho_b(t) - \kappa)^+ \varphi(t, b) dt.$$

We decompose  $T_{20}$  as

$$T_{20} = -\Delta t \sum_{n=0}^{+\infty} \sum_{j=1}^N G_{j+1/2}^\kappa(\rho_j^n, \rho_j^n) (\varphi(t^n, x_{j+1}) - \varphi(t^n, x_j)) \\ - \alpha \Delta t \sum_{n=0}^{+\infty} \left( \left( \varphi(t^n, b) \eta_\kappa^+(\rho_b^n) + \varphi(t^n, a) \eta_\kappa^+(\rho_a^n) \right) \right) \\ = -\Delta t \sum_{n=0}^{+\infty} \left( \sum_{j=1}^N G_{j+1/2}^\kappa(\rho_j^n, \rho_j^n) \varphi(t^n, x_{j+1}) - \sum_{j=0}^{N-1} G_{j+3/2}^\kappa(\rho_{j+1}^n, \rho_{j+1}^n) \varphi(t^n, x_{j+1}) \right) \\ - \alpha \Delta t \sum_{n=0}^{+\infty} \left( \left( \varphi(t^n, b) \eta_\kappa^+(\rho_b^n) + \varphi(t^n, a) \eta_\kappa^+(\rho_a^n) \right) \right) \\ = -\Delta t \sum_{n=0}^{+\infty} \sum_{j=1}^{N-1} \left( G_{j+1/2}^\kappa(\rho_j^n, \rho_j^n) - G_{j+3/2}^\kappa(\rho_{j+1}^n, \rho_{j+1}^n) \right) \varphi(t^n, x_{j+1}) \\ - \Delta t \sum_{n=0}^{+\infty} \left( G_{N+1/2}^\kappa(\rho_N^n, \rho_N^n) \varphi(t^n, x_{N+1}) - G_{3/2}^\kappa(\rho_1^n, \rho_1^n) \varphi(t^n, x_1) \right) \\ - \alpha \Delta t \sum_{n=0}^{+\infty} \left( \left( \varphi(t^n, b) \eta_\kappa^+(\rho_b^n) + \varphi(t^n, a) \eta_\kappa^+(\rho_a^n) \right) \right) \\ =: T_{20}^{int} + T_{20}^b.$$

We rewrite

$$\begin{aligned}
T_{20}^{int} &= -\Delta t \sum_{n=0}^{+\infty} \sum_{j=1}^{N-1} \left( G_{j+1/2}^\kappa(\rho_j^n, \rho_j^n) \mp G_{j+1/2}^\kappa(\rho_j^n, \rho_{j+1}^n) - G_{j+3/2}^\kappa(\rho_{j+1}^n, \rho_{j+1}^n) \right) \varphi(t^n, x_{j+1}) \\
&= \Delta t \sum_{n=0}^{+\infty} \sum_{j=1}^{N-1} \varphi(t^n, x_{j+1}) \left( G_{j+1/2}^\kappa(\rho_j^n, \rho_{j+1}^n) - G_{j+1/2}^\kappa(\rho_j^n, \rho_j^n) \right) \\
&\quad - \Delta t \sum_{n=0}^{+\infty} \sum_{j=1}^{N-1} \varphi(t^n, x_{j+1}) \left( G_{j+1/2}^\kappa(\rho_j^n, \rho_{j+1}^n) - G_{j+3/2}^\kappa(\rho_{j+1}^n, \rho_{j+1}^n) \right).
\end{aligned}$$

Hence

$$\left| T_2^{int} - T_{20}^{int} \right| \leq \Delta t \sum_{n=0}^{+\infty} \sum_{j=1}^{N-1} \left| \varphi(t^n, x_j) - \varphi(t^n, x_{j+1}) \right| \left| G_{j+1/2}^\kappa(\rho_j^n, \rho_{j+1}^n) - G_{j+1/2}^\kappa(\rho_j^n, \rho_j^n) \right|.$$

Notice that

$$\begin{aligned}
&\left| G_{j+1/2}^\kappa(\rho_j^n, \rho_{j+1}^n) - G_{j+1/2}^\kappa(\rho_j^n, \rho_j^n) \right| = \\
&= \frac{1}{2} \left| f(t^n, x_{j+1}, \rho_{j+1}^n \wedge \kappa, R_{j+1}^n) - f(t^n, x_{j+1}, \rho_j^n \wedge \kappa, R_{j+1}^n) + \alpha \left( \rho_j^n \wedge \kappa - \rho_{j+1}^n \wedge \kappa \right) \right| \\
&\leq \frac{L + \alpha}{2} \left| \rho_{j+1}^n - \rho_j^n \right| \\
&\leq \alpha \left| \rho_{j+1}^n - \rho_j^n \right|,
\end{aligned}$$

according to conditions (2.3). Therefore

$$\begin{aligned}
\left| T_2^{int} - T_{20}^{int} \right| &\leq \alpha \Delta x \Delta t \|\varphi_x\|_{\mathbf{L}^\infty} \sum_{n=0}^{+\infty} \sum_{j=1}^{N-1} \left| \rho_{j+1}^n - \rho_j^n \right| \\
&\leq \alpha \Delta x T \|\varphi_x\|_{\mathbf{L}^\infty} \max_{0 \leq n \leq T/\Delta t} \text{TV}(\rho_\Delta(t^n, \cdot)) = \mathcal{O}(\Delta x),
\end{aligned}$$

thanks to the uniform BV bound (2.13). We now compare the terms  $T_2^b$  and  $T_{20}^b$ :

$$\begin{aligned}
T_{20}^b - T_2^b &= -\Delta t \sum_{n=0}^{+\infty} \left( G_{N+1/2}^\kappa(\rho_N^n, \rho_N^n) \varphi(t^n, x_{N+1}) - G_{3/2}^\kappa(\rho_1^n, \rho_1^n) \varphi(t^n, x_1) \right) \\
&\quad - \alpha \Delta t \sum_{n=0}^{+\infty} \left( \varphi(t^n, b) \eta_\kappa^+(\rho_b^n) + \varphi(t^n, a) \eta_\kappa^+(\rho_a^n) \right) \\
&\quad - \Delta t \sum_{n=0}^{+\infty} \left( G_{N+1/2}^\kappa(\rho_N^n, \rho_b^n) - G_{N+1/2}^\kappa(\rho_N^n, \rho_N^n) \right) \varphi(t^n, x_N) \\
&\quad + \Delta t \sum_{n=0}^{+\infty} \left( G_{1/2}^\kappa(\rho_a^n, \rho_1^n) - G_{3/2}^\kappa(\rho_1^n, \rho_1^n) \right) \varphi(t^n, x_1) \\
&= \Delta t \sum_{n=0}^{+\infty} \left( G_{1/2}^\kappa(\rho_a^n, \rho_1^n) \varphi(t^n, x_1) - \alpha \eta_\kappa^+(\rho_a^n) \varphi(t^n, a) \right)
\end{aligned}$$

$$\begin{aligned}
& - \Delta t \sum_{n=0}^{+\infty} \left( \alpha \eta_{\kappa}^{+}(\rho_b^n) \varphi(t^n, b) + G_{N+1/2}^{\kappa}(\rho_N^n, \rho_b^n) \varphi(t^n, x_N) \right) \\
& - \Delta t \sum_{n=0}^{+\infty} \left( G_{N+1/2}^{\kappa}(\rho_N^n, \rho_N^n) (\varphi(t^n, x_{N+1}) - \varphi(t^n, x_N)) \right) \\
& = \Delta t \sum_{n=0}^{+\infty} \left\{ \left( G_{1/2}^{\kappa}(\rho_a^n, \rho_1^n) - \alpha \eta_{\kappa}^{+}(\rho_a^n) \right) \varphi(t^n, a) - \left( \alpha \eta_{\kappa}^{+}(\rho_b^n) + G_{N+1/2}^{\kappa}(\rho_N^n, \rho_b^n) \right) \varphi(t^n, b) \right\} \\
& + \mathcal{O}(\Delta x),
\end{aligned}$$

owing to the regularity of  $\varphi$ . Indeed, we observe that

$$\begin{aligned}
& \Delta t \left| \sum_{n \in \mathbb{N}} \left( G_{N+1/2}^{\kappa}(\rho_N^n, \rho_N^n) (\varphi(t^n, x_{N+1}) - \varphi(t^n, x_N)) \right) \right| \\
& \leq \Delta t \Delta x \|\varphi_x\|_{\mathbf{L}^{\infty}} \sum_{n \in \mathbb{N}} \left| G_{N+1/2}^{\kappa}(\rho_N^n, \rho_N^n) \right| \\
& = \Delta t \Delta x \|\varphi_x\|_{\mathbf{L}^{\infty}} \sum_{n \in \mathbb{N}} \left| F_{N+1/2}^n(\rho_N^n \wedge \kappa, \rho_N^n \wedge \kappa) - F_{N+1/2}^n(\kappa, \kappa) \right| \\
& \leq \frac{\Delta t \Delta x \|\varphi_x\|_{\mathbf{L}^{\infty}}}{2} \sum_{n \in \mathbb{N}} \left( |f(t^n, x_N, \rho_N^n \wedge \kappa, R_N^n) - f(t^n, x_N, \kappa, R_N^n)| \right) \\
& \quad + \frac{\Delta t \Delta x \|\varphi_x\|_{\mathbf{L}^{\infty}}}{2} \sum_{n \in \mathbb{N}} \left( |f(t^n, x_{N+1}, \rho_N^n \wedge \kappa, R_{N+1}^n) - f(t^n, x_{N+1}, \kappa, R_{N+1}^n)| \right) \\
& \leq L \Delta t \Delta x \|\varphi_x\|_{\mathbf{L}^{\infty}} \sum_{n \in \mathbb{N}} (\rho_N^n - \kappa)^+ \\
& \leq LT \|\varphi_x\|_{\mathbf{L}^{\infty}} e^{\mathcal{L}T} \|\rho_0\|_{\mathbf{L}^{\infty}(I)} \Delta x = \mathcal{O}(\Delta x),
\end{aligned}$$

thanks to the  $\mathbf{L}^{\infty}$ -bound (2.9). Moreover, since

$$\begin{aligned}
G_{j+1/2}^{\kappa}(u, v) & = F_{j+1/2}^n(u \wedge \kappa, v \wedge \kappa) - F_{j+1/2}^n(\kappa, \kappa) \\
& \geq F_{j+1/2}^n(\kappa, v \wedge \kappa) - F_{j+1/2}^n(\kappa, \kappa) \\
& = \frac{1}{2} \left( f(t^n, x_{j+1}, v \wedge \kappa, R_{j+1}^n) - f(t^n, x_{j+1}, \kappa, R_{j+1}^n) - \alpha (v \wedge \kappa - \kappa) \right) \\
& \geq -\frac{1}{2} \left( L |v \wedge \kappa - \kappa| + \alpha (v - \kappa)^+ \right) \\
& = -\frac{1}{2} \left( L (v - \kappa)^+ + \alpha (v - \kappa)^+ \right) \\
& \geq -\alpha (v - \kappa)^+
\end{aligned}$$

and

$$\begin{aligned}
G_{j+1/2}^{\kappa}(u, v) & = F_{j+1/2}^n(u \wedge \kappa, v \wedge \kappa) - F_{j+1/2}^n(\kappa, \kappa) \\
& \leq F_{j+1/2}^n(u \wedge \kappa, \kappa) - F_{j+1/2}^n(\kappa, \kappa) \\
& = \frac{1}{2} \left( f(t^n, x_j, u \wedge \kappa, R_j^n) - f(t^n, x_j, \kappa, R_j^n) + \alpha (u \wedge \kappa - \kappa) \right) \\
& \leq \frac{1}{2} \left( L |u \wedge \kappa - \kappa| + \alpha (u - \kappa)^+ \right)
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2} \left( L(u - \kappa)^+ + \alpha(u - \kappa)^+ \right) \\
&\leq \alpha(u - \kappa)^+,
\end{aligned}$$

we conclude that

$$T_{20}^b - T_2^b \leq \mathcal{O}(\Delta x).$$

Therefore we get

$$\begin{aligned}
0 &\geq T_1 + T_3 + T_2 \\
&= T_1 + T_3 + T_2^{int} + T_2^b \\
&\geq T_1 + T_3 + T_2^{int} + T_{20}^b - \mathcal{O}(\Delta x) \\
&= T_1 + T_3 + T_{20} - \mathcal{O}(\Delta x),
\end{aligned}$$

thus concluding the proof.  $\square$

### 3 Stability

**Proposition 2** *Under hypotheses (1.2), let  $\rho, \sigma \in \mathbf{C}^0(\mathbb{R}^+; \mathbf{L}^1(I; \mathbb{R}^+)) \cap BV([0, T] \times I; \mathbb{R})$ ,  $T > 0$ , be two weak entropy solutions to (1.1), with initial data  $\rho_0, \sigma_0 \in \mathbf{L}^\infty(I, \mathbb{R}^+)$  and boundary data  $\rho_a, \rho_b, \sigma_a, \sigma_b \in \mathbf{L}^\infty(\mathbb{R}^+; \mathbb{R}^+)$  respectively. Then the following estimate holds:*

$$\|\rho(T, \cdot) - \sigma(T, \cdot)\|_{\mathbf{L}^1(I)} \leq e^{ST} \left[ \|\rho_0 - \sigma_0\|_{\mathbf{L}^1(I)} + (L + \mathcal{S}') \left( \|\rho_a - \sigma_a\|_{\mathbf{L}^1([0, T])} + \|\rho_b - \sigma_b\|_{\mathbf{L}^1([0, T])} \right) \right], \quad (3.1)$$

where the constants  $\mathcal{S}, \mathcal{S}'$  are defined by (3.6), (3.7) and (3.10), and  $L$  is as in (2.28).

*Proof.* Let  $\rho, \sigma$  be two weak entropy solutions to (1.1), with fluxes  $f(t, x, \rho, R(t, x))$  and  $f(t, x, \sigma, S(t, x))$  respectively, where

$$R(t, x) := \int_{\mathbb{R}} \rho(t, y) \eta(x - y) dy \quad \text{and} \quad S(t, x) := \int_{\mathbb{R}} \sigma(t, y) \eta(x - y) dy.$$

In particular from Definition 2, they satisfy

$$\begin{aligned}
\partial_t |\rho - \kappa| + \frac{d}{dx} \left[ \operatorname{sgn}(\rho - \kappa) (f(t, x, \rho, R(t, x)) - f(t, x, \kappa, R(t, x))) \right] \\
+ \operatorname{sgn}(\rho - \kappa) \frac{d}{dx} f(t, x, \kappa, R(t, x)) \leq 0,
\end{aligned} \quad (3.2)$$

$$\begin{aligned}
\partial_t |\sigma - \kappa| + \frac{d}{dx} \left[ \operatorname{sgn}(\sigma - \kappa) (f(t, x, \sigma, S(t, x)) - f(t, x, \kappa, S(t, x))) \right] \\
+ \operatorname{sgn}(\sigma - \kappa) \frac{d}{dx} f(t, x, \kappa, S(t, x)) \leq 0,
\end{aligned} \quad (3.3)$$

in distributional sense on  $\mathbb{R}^+ \times I$ . Rearranging (3.2) we get

$$0 \geq \partial_t |\rho - \kappa| + \operatorname{sgn}(\rho - \kappa) \frac{d}{dx} (f(t, x, \kappa, R(t, x)) \pm f(t, x, \kappa, S(t, x)))$$

$$\begin{aligned}
& + \frac{d}{dx} \left[ \operatorname{sgn}(\rho - \kappa) (f(t, x, \rho, R(t, x)) - f(t, x, \kappa, R(t, x)) \pm f(t, x, \rho, S(t, x)) \pm f(t, x, \kappa, S(t, x))) \right] \\
& = \partial_t |\rho - \kappa| + \frac{d}{dx} \left[ \operatorname{sgn}(\rho - \kappa) (f(t, x, \rho, S(t, x)) - f(t, x, \kappa, S(t, x))) \right] + \operatorname{sgn}(\rho - \kappa) \frac{d}{dx} f(t, x, \kappa, S(t, x)) \\
& + \frac{d}{dx} \left[ \operatorname{sgn}(\rho - \kappa) \left( (f(t, x, \rho, R(t, x)) - f(t, x, \rho, S(t, x))) - (f(t, x, \kappa, R(t, x)) - f(t, x, \kappa, S(t, x))) \right) \right] \\
& + \operatorname{sgn}(\rho - \kappa) \frac{d}{dx} [f(t, x, \kappa, R(t, x)) - f(t, x, \kappa, S(t, x))],
\end{aligned}$$

thus

$$\begin{aligned}
& \partial_t |\rho - \kappa| + \frac{d}{dx} \left[ \operatorname{sgn}(\rho - \kappa) (f(t, x, \rho, S(t, x)) - f(t, x, \kappa, S(t, x))) \right] + \operatorname{sgn}(\rho - \kappa) \frac{d}{dx} f(t, x, \kappa, S(t, x)) \\
& \leq \frac{d}{dx} \left\{ \operatorname{sgn}(\rho - \kappa) \left[ (f(t, x, \rho, S(t, x)) - f(t, x, \rho, R(t, x))) - (f(t, x, \kappa, S(t, x)) - f(t, x, \kappa, R(t, x))) \right] \right\} \\
& + \operatorname{sgn}(\rho - \kappa) \frac{d}{dx} [f(t, x, \kappa, S(t, x)) - f(t, x, \kappa, R(t, x))]. \tag{3.4}
\end{aligned}$$

Notice that from (1.5) we can bound

$$\begin{aligned}
|R(t, x) - S(t, x)| & \leq \|\eta\|_{\mathbf{L}^\infty} \int_a^b |\rho(t, x) - \sigma(t, x)| dx + |\rho_a(t) - \sigma_a(t)| + |\rho_b(t) - \sigma_b(t)|, \\
|\partial_x R(t, x) - \partial_x S(t, x)| & \leq \|\partial_x \eta\|_{\mathbf{L}^\infty} \int_a^b |\rho(t, x) - \sigma(t, x)| dx + \|\eta\|_{\mathbf{L}^\infty} (|\rho_a(t) - \sigma_a(t)| + |\rho_b(t) - \sigma_b(t)|).
\end{aligned}$$

Therefore we recover the following estimate:

$$\begin{aligned}
& \left| \frac{d}{dx} (f(t, x, \kappa, S(t, x)) - f(t, x, \kappa, R(t, x))) \right| \leq \\
& \leq |\partial_x f(t, x, \kappa, S(t, x)) - \partial_x f(t, x, \kappa, R(t, x))| \\
& + |\partial_R f(t, x, \kappa, S(t, x)) \partial_x S(t, x) - \partial_R f(t, x, \kappa, R(t, x)) \partial_x R(t, x) \pm \partial_R f(t, x, \kappa, S(t, x)) \partial_x R(t, x)| \\
& \leq \left| \partial_{xR}^2 f(t, x, \kappa, \tilde{R}_1(t, x)) \right| |S(t, x) - R(t, x)| \\
& + |\partial_R f(t, x, \kappa, S(t, x))| |\partial_x S(t, x) - \partial_x R(t, x)| \\
& + |\partial_R f(t, x, \kappa, S(t, x)) - \partial_R f(t, x, \kappa, R(t, x))| |\partial_x R(t, x)| \\
& \leq \left( \left| \partial_{xR}^2 f(t, x, \kappa, \tilde{R}_1(t, x)) \right| + \left| \partial_{RR}^2 f(t, x, \kappa, \tilde{R}_2(t, x)) \right| |\partial_x R(t, x)| \right) |S(t, x) - R(t, x)| \\
& + |\partial_R f(t, x, \kappa, S(t, x))| |\partial_x S(t, x) - \partial_x R(t, x)| \\
& \leq C|\kappa| \left( 1 + |\partial_x R(t, x)| \right) |S(t, x) - R(t, x)| + C|\kappa| |\partial_x S(t, x) - \partial_x R(t, x)| \\
& = C|\kappa| \left( 1 + \left| \int_{\mathbb{R}} \partial_x \eta(x-y) \rho(t, y) dy \right| \right) |S(t, x) - R(t, x)| + C|\kappa| |\partial_x S(t, x) - \partial_x R(t, x)| \\
& \leq C|\kappa| \left( 1 + \|\rho(t, \cdot)\|_{\mathbf{L}^\infty} \|\partial_x \eta\|_{\mathbf{L}^1} \right) |S(t, x) - R(t, x)| + C|\kappa| |\partial_x S(t, x) - \partial_x R(t, x)| \\
& \leq C|\kappa| \left( 1 + \|\rho(t, \cdot)\|_{\mathbf{L}^\infty} \|\partial_x \eta\|_{\mathbf{L}^1} \right) \left[ \|\eta\|_{\mathbf{L}^\infty} \int_a^b |\rho(t, x) - \sigma(t, x)| dx + |\rho_a(t) - \sigma_a(t)| + |\rho_b(t) - \sigma_b(t)| \right] \\
& + C|\kappa| \left[ \|\partial_x \eta\|_{\mathbf{L}^\infty} \int_a^b |\rho(t, x) - \sigma(t, x)| dx + \|\eta\|_{\mathbf{L}^\infty} (|\rho_a(t) - \sigma_a(t)| + |\rho_b(t) - \sigma_b(t)|) \right]
\end{aligned}$$



$$\begin{aligned}
&\leq C|\kappa| \left[ \|\eta\|_{\mathbf{L}^\infty} \left( 1 + \|\rho(t, \cdot)\|_{\mathbf{L}^\infty} \|\partial_x \eta\|_{\mathbf{L}^1} \right) + \|\partial_x \eta\|_{\mathbf{L}^\infty} \right] \int_a^b |\rho(t, x) - \sigma(t, x)| dx \\
&\quad + C|\kappa| \left( 1 + \|\rho(t, \cdot)\|_{\mathbf{L}^\infty} \|\partial_x \eta\|_{\mathbf{L}^1} + \|\eta\|_{\mathbf{L}^\infty} \right) \left( |\rho_a(t) - \sigma_a(t)| + |\rho_b(t) - \sigma_b(t)| \right) \\
&\leq \mathcal{S}_1 \int_a^b |\rho(t, y) - \sigma(t, y)| dy + \mathcal{S}'_1 \left( |\rho_a(t) - \sigma_a(t)| + |\rho_b(t) - \sigma_b(t)| \right), \tag{3.5}
\end{aligned}$$

where

$$\mathcal{S}_1 = C \sup_{t \in [0, T]} \left\{ \max \left\{ \|\rho(t, \cdot)\|_{\mathbf{L}^\infty}, \|\sigma(t, \cdot)\|_{\mathbf{L}^\infty} \right\} \left[ \|\eta\|_{\mathbf{L}^\infty} \left( 1 + \|\rho(t, \cdot)\|_{\mathbf{L}^\infty} \|\partial_x \eta\|_{\mathbf{L}^1} \right) + \|\partial_x \eta\|_{\mathbf{L}^\infty} \right] \right\}, \tag{3.6}$$

$$\mathcal{S}'_1 = C \sup_{t \in [0, T]} \left\{ \max \left\{ \|\rho(t, \cdot)\|_{\mathbf{L}^\infty}, \|\sigma(t, \cdot)\|_{\mathbf{L}^\infty} \right\} \left( 1 + \|\rho(t, \cdot)\|_{\mathbf{L}^\infty} \|\partial_x \eta\|_{\mathbf{L}^1} + \|\eta\|_{\mathbf{L}^\infty} \right) \right\}, \tag{3.7}$$

which are bounded by assumption. Moreover

$$\begin{aligned}
&\left| \frac{d}{dx} \left\{ \operatorname{sgn}(\rho - \kappa) \left[ (f(t, x, \rho, S(t, x)) - f(t, x, \rho, R(t, x))) - (f(t, x, \kappa, S(t, x)) - f(t, x, \kappa, R(t, x))) \right] \right\} \right| \\
&= \left| \left\{ \partial_x f(t, x, \rho, S(t, x)) - \partial_x f(t, x, \rho, R(t, x)) + \partial_x f(t, x, \kappa, R(t, x)) - \partial_x f(t, x, \kappa, S(t, x)) \right. \right. \\
&\quad + \partial_R f(t, x, \rho, S(t, x)) (\partial_x S(t, x) - \partial_x R(t, x)) + \partial_x R(t, x) (\partial_x f(t, x, \rho, S(t, x)) - \partial_x f(t, x, \rho, R(t, x))) \\
&\quad \left. \left. + \partial_R f(t, x, \kappa, S(t, x)) (\partial_x R(t, x) - \partial_x S(t, x)) + \partial_x R(t, x) (\partial_x f(t, x, \kappa, R(t, x)) - \partial_x f(t, x, \kappa, S(t, x))) \right\} \right| \\
&= \left| \partial_{xR}^2 f(t, x, \rho, \tilde{R}_1) (S(t, x) - R(t, x)) + \partial_{xR}^2 f(t, x, \kappa, \tilde{R}_2) (R(t, x) - S(t, x)) \right. \\
&\quad + \partial_R f(t, x, \rho, S(t, x)) (\partial_x S(t, x) - \partial_x R(t, x)) + \partial_x R(t, x) \partial_{xR}^2 f(t, x, \rho, \tilde{R}_3) (S(t, x) - R(t, x)) \\
&\quad \left. + \partial_R f(t, x, \kappa, S(t, x)) (\partial_x R(t, x) - \partial_x S(t, x)) + \partial_x R(t, x) \partial_{xR}^2 f(t, x, \kappa, \tilde{R}_4) (R(t, x) - S(t, x)) \right| \\
&\leq C (\|\rho(t, \cdot)\|_{\mathbf{L}^\infty} + |\kappa|) (1 + \|\rho(t, \cdot)\|_{\mathbf{L}^\infty} \|\partial_x \eta\|_{\mathbf{L}^1}) |R(t, x) - S(t, x)| \\
&\quad + C (\|\rho(t, \cdot)\|_{\mathbf{L}^\infty} + |\kappa|) |\partial_x R(t, x) - \partial_x S(t, x)| \\
&= \mathcal{S}_2 \int_a^b |\rho(t, y) - \sigma(t, y)| dy + \mathcal{S}'_2 \left( |\rho_a(t) - \sigma_a(t)| + |\rho_b(t) - \sigma_b(t)| \right), \tag{3.8}
\end{aligned}$$

being  $\mathcal{S}_2 = 2\mathcal{S}_1$  and  $\mathcal{S}'_2 = 2\mathcal{S}'_1$ .

Inserting estimates (3.5) and (3.8) in (3.4) we get

$$\begin{aligned}
&\partial_t |\rho - \kappa| + \frac{d}{dx} \left[ \operatorname{sgn}(\rho - \kappa) (f(t, x, \rho, S(t, x)) - f(t, x, \kappa, S(t, x))) \right] + \operatorname{sgn}(\rho - \kappa) \frac{d}{dx} f(t, x, \kappa, S(t, x)) \\
&\leq \mathcal{S} \int_a^b |\rho(t, y) - \sigma(t, y)| dy + \mathcal{S}' \left( |\rho_a(t) - \sigma_a(t)| + |\rho_b(t) - \sigma_b(t)| \right), \tag{3.9}
\end{aligned}$$

with

$$\mathcal{S} = \mathcal{S}_1 + \mathcal{S}_2 = 3\mathcal{S}_1 \quad \text{and} \quad \mathcal{S}' = \mathcal{S}'_1 + \mathcal{S}'_2 = 3\mathcal{S}'_1. \tag{3.10}$$

Following [6, Theorem 2] and [25, Theorem 15.1.5], we apply the standard Kruzhkov doubling of variable technique [20, Section 3] to (3.9) and (3.3), with a test function  $\varphi \in \mathcal{C}_c^1(\mathbb{R} \times I; \mathbb{R}^+)$ . We obtain the following Kato inequality

$$\int_a^b |\rho_0(x) - \sigma_0(x)| \varphi(0, x) dx$$

$$\begin{aligned}
& + \int_0^{+\infty} \int_a^b \left( |\rho(t, x) - \sigma(t, x)| \partial_t \varphi + \operatorname{sgn}(\rho - \sigma) [f(t, x, \rho, S(t, x)) - f(t, x, \sigma, S(t, x))] \partial_x \varphi \right) dx dt \\
& + \mathcal{S} \|\varphi\|_{\mathbf{L}^\infty} \int_0^{+\infty} \left( \int_a^b |\rho(t, x) - \sigma(t, x)| dx \right) dt + \mathcal{S}' \|\varphi\|_{\mathbf{L}^\infty} \int_0^{+\infty} \left( |\rho_a(t) - \sigma_a(t)| + |\rho_b(t) - \sigma_b(t)| \right) dt \\
& \geq 0. \tag{3.11}
\end{aligned}$$

We now consider in (3.11) a test function of the form  $\varphi(t, x) = \psi(t)\theta_\delta(x)$ , where  $\theta_\delta \in \mathcal{C}^1([a, b])$  be such that

$$\begin{aligned}
\theta_\delta(a) &= 0, \quad \theta_\delta(b) = 0, \\
\|\theta'_\delta\|_\infty &\leq K/\delta, \\
\theta_\delta &\equiv 1 \text{ on } [a + \delta, b - \delta], \\
0 &\leq \theta_\delta(x) \leq 1 \text{ for all } x \in [a, b],
\end{aligned}$$

where  $C$  does not depend on  $\delta$ , and  $\psi \in \mathcal{C}_c^1([0, T])$ . In this case, (3.11) becomes

$$\begin{aligned}
& \int_a^b |\rho_0(x) - \sigma_0(x)| \psi(0) \theta_\delta(x) dx \\
& + \int_0^{+\infty} \int_a^b \left( |\rho(t, x) - \sigma(t, x)| \theta_\delta(x) \psi'(t) \right. \\
& \quad \left. + \psi(t) \theta'_\delta(x) \operatorname{sgn}(\rho - \sigma) [f(t, x, \rho, S(t, x)) - f(t, x, \sigma, S(t, x))] \right) dx dt \\
& + \mathcal{S} \|\varphi\|_{\mathbf{L}^\infty} \int_0^T \left( \int_a^b |\rho(t, x) - \sigma(t, x)| dx \right) dt + \mathcal{S}' \|\varphi\|_{\mathbf{L}^\infty} \int_0^T \left( |\rho_a(t) - \sigma_a(t)| + |\rho_b(t) - \sigma_b(t)| \right) dt \\
& \geq 0. \tag{3.12}
\end{aligned}$$

Integrating by parts in (3.12) and letting  $\delta \searrow 0$  we obtain

$$\begin{aligned}
& \int_a^b |\rho_0(x) - \sigma_0(x)| \psi(0) dx \\
& + \int_0^{+\infty} \int_a^b |\rho(t, x) - \sigma(t, x)| \psi'(t) dx dt + \mathcal{S} \|\varphi\|_{\mathbf{L}^\infty} \int_0^T \left( \int_a^b |\rho(t, x) - \sigma(t, x)| dx \right) dt \\
& + \mathcal{S}' \|\varphi\|_{\mathbf{L}^\infty} \int_0^T \left( |\rho_a(t) - \sigma_a(t)| + |\rho_b(t) - \sigma_b(t)| \right) dt \\
& + \int_0^{+\infty} \psi(t) \left\{ \operatorname{sgn}(\rho(t, a^+) - \sigma(t, a^+)) \left[ f(t, a, \rho(t, a^+), S(t, a)) - f(t, a, \sigma(t, a^+), S(t, a)) \right] \right. \\
& \quad \left. - \operatorname{sgn}(\rho(t, b^-) - \sigma(t, b^-)) \left[ f(t, b, \rho(t, b^-), S(t, b)) - f(t, b, \sigma(t, b^-), S(t, b)) \right] \right\} dt \geq 0. \tag{3.13}
\end{aligned}$$

From the weak boundary conditions (1.7) and (1.8), we earn

$$\begin{aligned}
& \operatorname{sgn}(\rho(t, a^+) - \sigma(t, a^+)) \left[ f(t, a, \rho(t, a^+), S(t, a)) - f(t, a, \sigma(t, a^+), S(t, a)) \right] \\
& = \frac{1}{2} \operatorname{sgn}(\rho(t, a^+) - \sigma(t, a^+)) \left[ f(t, a, \rho(t, a^+), S(t, a)) - f(t, a, \sigma(t, a^+), S(t, a)) \right]
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{2} \operatorname{sgn} \left( \rho(t, a^+) - \sigma(t, a^+) \right) \left[ f(t, a, \rho(t, a^+), S(t, a)) - f(t, a, \sigma(t, a^+), S(t, a)) \right] \\
& \leq \frac{1}{2} \operatorname{sgn} \left( \rho_a(t) - \sigma(t, a^+) \right) \left[ f(t, a, \rho(t, a^+), S(t, a)) - f(t, a, \sigma(t, a^+), S(t, a)) \right] \\
& \quad + \frac{1}{2} \operatorname{sgn} \left( \rho(t, a^+) - \sigma_a(t) \right) \left[ f(t, a, \rho(t, a^+), S(t, a)) - f(t, a, \sigma(t, a^+), S(t, a)) \right] \\
& = \frac{1}{2} \left( \operatorname{sgn} \left( \rho_a(t) - \sigma(t, a^+) \right) + \operatorname{sgn} \left( \rho(t, a^+) - \sigma_a(t) \right) \right) \left[ f(t, a, \rho(t, a^+), S(t, a)) - f(t, a, \sigma(t, a^+), S(t, a)) \right] \\
& \leq \sup_{s, r \in \mathcal{I}(\rho_a(t), \sigma_a(t))} |f(t, a, s, S(t, a)) - f(t, a, r, S(t, a))| \\
& \leq \sup_{s, r \in \mathcal{I}(\rho_a(t), \sigma_a(t))} L|s - r| \\
& \leq L|\rho_a(t) - \sigma_a(t)|, \tag{3.14}
\end{aligned}$$

and

$$\begin{aligned}
& \operatorname{sgn} \left( \rho(t, b^-) - \sigma(t, b^-) \right) \left[ f(t, b, \rho(t, b^-), S(t, b)) - f(t, b, \sigma(t, b^-), S(t, b)) \right] \\
& = \frac{1}{2} \operatorname{sgn} \left( \rho(t, b^-) - \sigma(t, b^-) \right) \left[ f(t, b, \rho(t, b^-), S(t, b)) - f(t, b, \sigma(t, b^-), S(t, b)) \right] \\
& \quad + \frac{1}{2} \operatorname{sgn} \left( \rho(t, b^-) - \sigma(t, b^-) \right) \left[ f(t, b, \rho(t, b^-), S(t, b)) - f(t, b, \sigma(t, b^-), S(t, b)) \right] \\
& \geq \frac{1}{2} \operatorname{sgn} \left( \rho_b(t) - \sigma(t, b^-) \right) \left[ f(t, b, \rho(t, b^-), S(t, b)) - f(t, b, \sigma(t, b^-), S(t, b)) \right] \\
& \quad + \frac{1}{2} \operatorname{sgn} \left( \rho(t, b^-) - \sigma_b(t) \right) \left[ f(t, b, \rho(t, b^-), S(t, b)) - f(t, b, \sigma(t, b^-), S(t, b)) \right] \\
& = \frac{1}{2} \left( \operatorname{sgn} \left( \rho_b(t) - \sigma(t, b^-) \right) + \operatorname{sgn} \left( \rho(t, b^-) - \sigma_b(t) \right) \right) \left[ f(t, b, \rho(t, b^-), S(t, b)) - f(t, b, \sigma(t, b^-), S(t, b)) \right] \\
& \geq - \sup_{s, r \in \mathcal{I}(\rho_b(t), \sigma_b(t))} |f(t, b, s, S(t, b)) - f(t, b, r, S(t, b))| \\
& \geq - \sup_{s, r \in \mathcal{I}(\rho_b(t), \sigma_b(t))} L|s - r| \\
& \geq -L|\rho_b(t) - \sigma_b(t)|. \tag{3.15}
\end{aligned}$$

Collecting (3.14) and (3.15) we conclude that

$$\begin{aligned}
& \int_0^{+\infty} \psi(t) \left\{ \operatorname{sgn} \left( \rho(t, a^+) - \sigma(t, a^+) \right) \left[ f(t, a, \rho(t, a^+), S(t, a)) - f(t, a, \sigma(t, a^+), S(t, a)) \right] \right. \\
& \quad \left. - \operatorname{sgn} \left( \rho(t, b^-) - \sigma(t, b^-) \right) \left[ f(t, b, \rho(t, b^-), S(t, b)) - f(t, b, \sigma(t, b^-), S(t, b)) \right] \right\} dt \\
& \leq L \int_0^{+\infty} \psi(t) \left( |\rho_a(t) - \sigma_a(t)| + |\rho_b(t) - \sigma_b(t)| \right) dt.
\end{aligned}$$

Thus (3.13) becomes

$$\begin{aligned}
& \int_a^b |\rho_0(x) - \sigma_0(x)| \psi(0) dx \\
& + \int_0^{+\infty} \int_a^b |\rho(t, x) - \sigma(t, x)| \psi'(t) dx dt + \mathcal{S} \|\varphi\|_{\mathbf{L}^\infty} \int_0^T \left( \int_a^b |\rho(t, x) - \sigma(t, x)| dx \right) dt
\end{aligned}$$

$$\begin{aligned}
& + \mathcal{S}' \|\varphi\|_{\mathbf{L}^\infty} \int_0^T \left( |\rho_a(t) - \sigma_a(t)| + |\rho_b(t) - \sigma_b(t)| \right) dt \\
& + L \int_0^{+\infty} \psi(t) \left( |\rho_a(t) - \sigma_a(t)| + |\rho_b(t) - \sigma_b(t)| \right) dt \geq 0.
\end{aligned}$$

We choose the test function  $\psi = \psi_\epsilon$  as

$$\psi_\epsilon(t) = \begin{cases} 1 & \text{if } t \in [0, T - \epsilon[ \\ \psi_\epsilon(t) \in [0, 1] & \text{for all } t \in [0, T] \\ |\psi'_\epsilon(t)| \leq K/\epsilon & \text{for all } t \in [0, T] \end{cases}.$$

As  $\epsilon \searrow 0$ , we get

$$\begin{aligned}
\|\rho(T, \cdot) - \sigma(T, \cdot)\|_{\mathbf{L}^1(I)} & \leq \|\rho_0 - \sigma_0\|_{\mathbf{L}^1(I)} + L \left( \|\rho_b - \sigma_b\|_{\mathbf{L}^1([0, T])} + \|\rho_a - \sigma_a\|_{\mathbf{L}^1([0, T])} \right) \\
& + \mathcal{S} \int_0^T \|\rho(t, \cdot) - \sigma(t, \cdot)\|_{\mathbf{L}^1(I)} dt + \mathcal{S}' \int_0^T \left( |\rho_a(t) - \sigma_a(t)| + |\rho_b(t) - \sigma_b(t)| \right) dt,
\end{aligned}$$

and Gronwall's lemma allows us to recover (3.1).  $\square$

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