Certified Algorithms for proving the structural stability of two dimensional systems possibly with parameters

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Certified Algorithms for proving the structural stability of two-dimensional systems possibly with parameters

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Abstract— In [1], a new method for testing the structural stability of multidimensional systems has been presented. The key idea of this method is to reduce the problem of testing the structural stability to that of deciding if an algebraic set has real points. Following the same idea, we consider in this work the specific case of two-dimensional systems and focus on the practical efficiency aspect. For such systems, the problem of testing the stability is reduced to that of deciding if a bivariate algebraic system with finitely many solutions has real ones. Our first contribution is an algorithm that answers this question while achieving practical efficiency. Our second contribution concerns the stability of two dimensional systems with parameters. More precisely, given a two-dimensional system depending on a set of parameters, we present a new algorithm that computes *regions* of the parameter space in which the considered system is structurally stable.

I. INTRODUCTION

Two-dimensional systems have wide applications in several areas such as signal and image processing or iterative algorithm design. An important question in the study of such systems concerns their stability which is a necessary condition for them to work properly. In this paper, we are interested in testing the *structural stability* of two-dimensional discrete linear systems.

Given a two-dimensional discrete linear system described within the frequency domain by the transfer function

$$G(z_1, z_2) := \frac{N(z_1, z_2)}{D(z_1, z_2)},$$

where N and D are polynomials in the variables z_1, z_2 with real coefficients such that $N \wedge D = 1$. This system is said to be *structurally stable* if the denominator of its transfer function is devoid from zeros in the complex unit bi-disk $\mathbb{D}^2 := \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1| \leq 1 \text{ and } |z_2| \leq 1\}$, or in other words, if:

$$D(z_1, z_2) \neq 0 \text{ for } |z_1| \leq 1, \ |z_2| \leq 1. \tag{1}$$

In this work, we consider the two following problems:

Problem 1 (non parametric stability): *Given a two-dimensional system defined by a transfer function*

$$G(z_1, z_2) := \frac{N(z_1, z_2)}{D(z_1, z_2)},$$

with $N, D \in \mathbb{R}[z_1, z_2]$. Check if this system is stable, that is, if Condition (1) is satisfied.

Problem 2 (parametric stability): *Given a two-dimensional system defined by a transfer function*

$$G(z_1, z_2, U) := \frac{N(z_1, z_2, U)}{D(z_1, z_2, U)},$$

where $N, D \in \mathbb{R}[U][z_1, z_2]$ and $U = \{U_1, \ldots, U_k\}$ is a set of real parameters. Compute regions in the parameter space $\mathbb{R}^k$ in which the underlying system (after substitution of the parameters) is either stable or unstable. In other words, the goal is to compute an union of cells $\mathcal{C}_i$ in $\mathbb{R}^k$ such that, $\forall (u_1, \ldots, u_k) \in \mathcal{C}_i$, the system defined by $G(z_1, z_2, u_1, \ldots, u_k)$ is either stable or unstable.

There exist numerous tests for solving Problem 1. One can mention for instance the work in [2], [3], [4], [5], [6] and the references therein, where this problem is solved using purely algebraic methods. Common to all these methods is that they proceed recursively on the number of variables, reducing the computations with a 2-D polynomial to computations with a set of 1-D polynomials using algebraic tools like *resultants* and *sub-resultants* [7]. Another set of methods as for instance the one in [8] use the sum of square techniques for testing the stability condition. Such methods show better practical behavior compared to purely algebraic method, but are in general conservative i.e., provide only sufficient stability condition. For a complete overview on two-dimensional stability tests, the reader may refer to [9].

For Problem 2, to the best of the author's knowledge, there does not exist any general implemented method for solving it. In [5], the authors propose a two-dimensional stability test and apply the latter to example of systems with parameters. However, due to the simplicity of these systems, the computation of the desired regions is performed by hand and no indication is given on how to obtain them in an automatic way.

Our contribution in this paper is twofold. We firstly present a non conservative and practically efficient method for solving Problem 1. This method starts from the following set of conditions which has been shown in [10] to be equivalent to Condition (1):

$$\begin{cases} D(z_1, 1) \neq 0 \mid |z_1| \leq 1, \\ D(1, z_2) \neq 0 \mid |z_1| \leq 1, \\ D(z_1, z_2) \neq 0 \mid |z_1| = |z_2| = 1. \end{cases} \tag{2}$$

A first remark is that the first two conditions of (2) involve only univariate polynomials, and can thus be easily checked using classical one-dimensional stability tests (see for instance [11], [2]). The main difficulty is then to check the last condition of (2), i.e.:

$$D(z_1, z_2) \neq 0 \mid |z_1| = |z_2| = 1. \tag{3}$$

It has been shown in [1] that testing Condition (3) for multidimensional systems is equivalent, via a particular *Möbius* transformation, to deciding if an algebraic system of equations admits real zeros. Adapted to the two-dimensional system under consideration, this allows to reduce the problem to that of deciding if a bivariate algebraic system with finitely many complex solutions admits real solutions. For this problem, we present a method based on the computation of the so-called *separating form* for the solutions which is a classical notion in solving systems algorithms.

Our second contribution is a new method for solving Problem 2. Similarly as above, this method starts from the set of conditions (2) which now depends on a set of parameters $\{U_1, \ldots, U_k\}$. For the two first conditions, we extend a classical univariate stability test so that it can handle parameters, which allows to derive a stability condition as a sign condition on some polynomials depending only on the parameters. For the last condition of (2), we perform the same *Möbius* transformation as above and then make use of the the concept of *Discriminant variety* of a polynomial system which is a generalization of the classical notion of discriminant of a univariate polynomial. Such a *Discriminant variety* allows to partitioning the parameter space $\mathbb{R}^k$ into regions in which a given system of equations has a constant number of real solutions.

This paper is organized as follows. In Section II, some results obtained in [1] are stated in the case of two-dimensional systems. In Section III, we describe an algorithm for deciding if a bivariate algebraic system with real coefficients has or not real solutions. In Section IV, we address the problem of testing the stability of two-dimensional systems depending on parameters. Finally, in Section V, we illustrate our algorithm through a set of examples, both in the parametric and the non-parametric case.

II. ALGEBRAIC TRANSFORMATION

Notation: Throughout this paper. For a given set of polynomials $f_1, \ldots, f_s$ in $\mathbb{Q}[x_1, \ldots, x_n]$, $I := \langle f_1, \ldots, f_s \rangle$ denotes the ideal generated by $f_1, \ldots, f_s$, $V_{\mathbb{C}}(I) := \{\alpha \in \mathbb{C}^n \mid f_1(\alpha) = \cdots = f_s(\alpha) = 0\}$ the complex variety (the set of complex zeros) of I and $V_{\mathbb{R}}(I) := \{\alpha \in \mathbb{R}^n \mid f_1(\alpha) = \cdots = f_s(\alpha) = 0\}$ its real variety (the set of real zeros).

In the following, we recall and adapt the approach presented in [1] to the specific case of two-dimensional system.

As mentioned in the introduction, the main step in checking conditions (2) is the test of Condition (3), i.e., $D(z_1, z_2) \neq 0, |z_1| = |z_2| = 1$, which resumes to decide the existence of complex zeros of $D(z_1, z_2)$ on the bi-circle

$$\mathbb{T}^2 := \{(\alpha_1, \alpha_2) \in \mathbb{C}^2 \mid |\alpha_1| = 1 \text{ and } |\alpha_2| = 1\}.$$

The first step in [1] consists in applying the *Möbius* substitution $z \mapsto \frac{x-i}{x+i}$ to each variable of $D(z_1, z_2)$ (such a transformation maps the real line $\overline{\mathbb{R}} := \mathbb{R} \cup \infty$ to the unit circle $\mathbb{T}$ deprived from the point 1, i.e., to $\mathbb{T} \backslash \{1\}$). This yields a rational fraction in $\mathbb{C}(x_1, x_2)$ whose numerator writes as $\mathcal{R}(x_1, x_2) + i \, \mathcal{C}(x_1, x_2)$. Accordingly, it follows:

Theorem 1: Let $D(z_1, z_2) \in \mathbb{R}[z_1, z_2]$ of degrees d_1, d_2 in z_1, z_2. One can compute two polynomials $\mathcal{R}(x_1, x_2)$ and $\mathcal{C}(x_1, x_2)$ of total degrees bounded by $d_1 + d_2$, such that $V_{\mathbb{C}}(D(z_1, z_2)) \cap [\mathbb{T} \backslash \{1\}]^2 = \emptyset \Longleftrightarrow V_{\mathbb{R}}(\mathcal{R}, \mathcal{C}) = \emptyset$.

As pointed out in [1], the condition stated in the above theorem is not equivalent to Condition (3) since it excludes the points of the unit bi-circle that have at least one of their coordinates equals to one. However, checking that $D(z_1, z_2)$ does not vanish at these points, i.e., $D(1, z_2) \neq 0 \mid |z_2| = 1$ and $D(z_1, 1) \neq 0 \mid |z_1| = 1$ is included in the test of the two first conditions of (2). Consequently, $D(z_1, z_2)$ satisfies Condition (1) if and only if

- $D(z_1, 1) \neq 0$ for $|z_1| \leq 1$.
- $D(1, z_2) \neq 0$ for $|z_2| \leq 1$.
- The polynomial system $\{\mathcal{R}(x_1, x_2) = \mathcal{C}(x_1, x_2) = 0\}$ does not have real solutions.

III. NON PARAMETRIC STABILITY

As shown in the previous section, testing the stability of a two-dimensional system is equivalent to testing the stability of two one-dimensional systems and deciding if an algebraic system $\{\mathcal{R}(x_1, x_2) = \mathcal{C}(x_1, x_2) = 0\}$ admits real solutions. Testing the stability of one-dimensional systems can be efficiently achieved using existing implementations (see for instance [2]). Therefore, in the following, we focus our attention on the second problem, that is, deciding if the polynomial system $\{\mathcal{R}(x_1, x_2) = \mathcal{C}(x_1, x_2) = 0\}$ has real solutions.

In the following, we assume without loss of generality that $\mathcal{R}(x_1, x_2)$ and $\mathcal{C}(x_1, x_2)$ are weakly coprime in $\mathbb{Q}[x_1, x_2]$, i.e. $\gcd(\mathcal{R}, \mathcal{C}) = 1$, which implies that the ideal $I := \langle \mathcal{R}, \mathcal{C} \rangle$ is zero-dimensional[1]. $V(I) := \{(\alpha_1, \alpha_2) \in \mathbb{C}^2 \mid \mathcal{R}(\alpha_1, \alpha_2) = \mathcal{C}(\alpha_1, \alpha_2) = 0\}$ denotes the set of its complex solutions.

Our idea is to reduce the problem of deciding the existence of real solutions of I to that of deciding the existence of real roots of a well chosen univariate polynomial. To do so, let start with the following result which stems from the fact that the quotient algebra $\mathcal{A} := \frac{\mathbb{Q}[x_1, x_2]}{I}$ is a finite dimensional $\mathbb{Q}$-vector space.

Theorem 2: Let $P \in \mathbb{Q}[x_1, x_2]$ and let M_P be the endomorphism of the multiplication by P in $\mathcal{A}$

$$M_P : \quad \mathcal{A} \quad \longrightarrow \quad \mathcal{A}$$
$$f \quad \longmapsto \quad \overline{Pf}.$$

The eigenvalues of M_P are $P(\alpha_1, \alpha_2)$, with $(\alpha_1, \alpha_2) \in V(I)$. The multiplicity of $P(\alpha_1, \alpha_2)$ as an eigenvalue of M_P is equals to the multiplicity of (α_1, α_2) as a zero of I [13].

Hence, if C_P denotes the characteristic polynomial of M_P, then $C_P(t) = \prod_{(\alpha_1, \alpha_2) \in V} (t - P(\alpha_1, \alpha_2))^{\mu(\alpha_1, \alpha_2)}$, where $\mu(\alpha_1, \alpha_2)$ denotes the multiplicity of the zero (α_1, α_2) in I. Moreover, a bijection exists between the solutions of $V(I)$

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[1] If $\mathcal{R}$ and $\mathcal{C}$ are not coprime, it is sufficient to compute their gcd in $\mathbb{Q}[x_1, x_2]$, $\mathcal{G}(x_1, x_2)$, and to consider the two systems $\{\frac{\mathcal{R}}{\mathcal{G}}, \frac{\mathcal{C}}{\mathcal{G}}\}$ and $\{\mathcal{G}, \frac{\partial \mathcal{G}}{\partial x_1} \mathcal{G}\}$ which are known to be zero-dimensional and whose the union of real solutions are the real solutions of $\{\mathcal{R}, \mathcal{C}\}$ [12].

and the roots of $C_P(t)$ providing that the polynomial P is a *separating element* for $V(I)$.

Definition 1: $P \in \mathbb{Q}[x_1, x_2]$ is a separating element for $V(I)$ if and only if the map $(\alpha_1, \alpha_2) \in V(I) \longmapsto P(\alpha_1, \alpha_2)$ is injective.

The fact that P is a separating element for $V(I)$ yields an important property regarding to the existence of real solutions of $V(I)$. The following result can straightforwardly be proved considering a parameterization of the solutions, for example using [14].

Theorem 3: Let $P \in \mathbb{Q}[x_1, x_2]$ be a separating element for $V(I)$. Then, the polynomial $C_P(t)$ has real roots if and only if $V(I)$ has real solutions.

Consequently, computing a separating element of $V(I)$ as well as the corresponding polynomial $C_P(t)$ allows to reduce the problem of searching for real solutions of $V(I)$ to that of searching for real roots of $C_P(t)$.

For the computation of a separating element of $V(I)$, an important remark is that the number of non separating elements is bounded by $\frac{n(n-1)}{2}$ where n denotes the cardinal of $V(I)$ [13]. Thus, a separating element can always be found among the set $\left\{ x_1 + a\, x_2, a = 0, \ldots, \frac{n(n-1)}{2} \right\}$. On the other hand, we know by Bezout that, for any polynomials $\mathcal{R}$ and $\mathcal{C}$ of total degree d, the cardinal of $V(I)$ is bounded by d^2. Hence one strategy for computing a separating element for $V(I)$ is to loop over $1 + \frac{d^2(d^2-1)}{2}$ different integers a, compute for each a the number of distinct roots of $C_{x_1 + a\, x_2}(t)$ (the degree of its squarefree part $\overline{C_{x_1 + a\, x_2}(t)} := \frac{C_{x_1 + a\, x_2}}{\gcd(C_{x_1 + a\, x_2}, C'_{x_1 + a\, x_2})}$), and finally select an a for which this number is maximal. This ensures that the degree of $\overline{C_{x_1 + a\, x_2}(t)}$ is equals to the cardinal of $V(I)$, and thus that the roots of $\overline{C_{x_1 + a\, x_2}(t)}$ are in bijection with the points of $V(I)$. However from the computation point of view this strategy is not recommended since it requires the computations of $\frac{d^2(d^2-1)}{2}$ characteristic polynomials along with their squarefree parts. The latter calculation requires in addition the computation of a Gröbner basis of I (to get the description of $\frac{\mathbb{Q}[x_1, x_2]}{I}$) which can be costly in practice.

Alternatively, we propose below a method that avoids the computation of a Gröbner basis of I, and searches adaptively for a separating element. More precisely, this method makes use of a separation test which allows to stop the algorithm as soon as a separating element is found.

The following first result shows that for a given $x_1 + a\, x_2$, the polynomial $C_{x_1 + a\, x_2}(t)$ is equals, up to a factor in $\mathbb{Q}$, to the resultant of two polynomials resulting from $\mathcal{R}$ and $\mathcal{C}$ after a change of variable.

Theorem 4: [12] Let $\mathcal{R}(x_1, x_2), \mathcal{C}(x_1, x_2) \in \mathbb{Q}[x_1, x_2]$. Define $\mathcal{R}'(t, x_2) := \mathcal{R}(t - a\, x_2, x_2)$ and $\mathcal{C}'(t, x_2) := \mathcal{C}(t - a\, x_2, x_2)$ where $a \in \mathbb{Z}$ is such that the leading coefficient of $\mathcal{R}'$ and $\mathcal{C}'$ with respect to x_2 are coprime. Then, the resultant of $\mathcal{R}'$ and $\mathcal{C}'$ with respect to x_2, denoted $Res_{x_2}(\mathcal{R}', \mathcal{C}')$, is equal to $c \prod_{(\alpha_1, \alpha_2) \in V} (t - \alpha_1 - a\, \alpha_2)^{\mu(\alpha_1, \alpha_2)}$, with $c \in \mathbb{Q}$.

On the other hand, given a linear form $x_1 + a\, x_2$, it is well known that the latter is separating for $V(I)$ if and only if for each root α of $Res_{x_2}(\mathcal{R}', \mathcal{C}')$ (where $\mathcal{R}'$ and $\mathcal{C}'$ are defined as in Theorem 4), the gcd of $\mathcal{R}'(\alpha, x_2)$ and $\mathcal{C}'(\alpha, x_2)$ has exactly one root.

In order to check this separation condition for a given $x_1 + a\, x_2$, we first perform a *triangular decomposition* of $\{\mathcal{R}'(t, x_2), \mathcal{C}'(t, x_2)\}$ (see Section (A) of Appendix for details). This yields a set of triangular systems of the form $\{r_k(t), \mathrm{Sres}_k(t, x_2)\}$, $k = 1 \ldots \min(\deg_{x_2}(\mathcal{R}), \deg_{x_2}(\mathcal{C}))$, such that $\mathrm{Sres}_k(\alpha, x_2)$ is the gcd of $\mathcal{R}'(\alpha, x_2)$ and $\mathcal{C}'(\alpha, x_2)$ for any root α of $r_k(t)$. Then, we use the following result (which proof can be found in [15]).

Theorem 5: Let $\mathcal{R}(x_1, x_2), \mathcal{C}(x_1, x_2) \in \mathbb{Q}[x_1, x_2]$. Define the polynomials $\mathcal{R}'(t, x_2)$, $\mathcal{C}'(t, x_2)$ as in Theorem 4, and let $\{r_k(t), \mathrm{Sres}_k(t, x_2)\}$, $k = 1 \ldots m$ be the triangular decomposition of $\{\mathcal{R}', \mathcal{C}'\}$. Then $x_1 + a\, x_2$ is a separating element for $V(I)$ if and only if $\forall k \in \{1, \ldots, m\}$ and $\forall i \in \{0, \ldots, k-1\}$,

$$k\,(k-i)\mathrm{sr}_{\{k,i\}}(t)\mathrm{sr}_{\{k,k\}}(t) - (i+1)\mathrm{sr}_{\{k,k-1\}}(t)\,\mathrm{sr}_{\{k,i+1\}}(t)$$

is zero modulo $r_k(t)$.

Finally, our algorithm for checking if $V(I)$ has real solutions consists in computing for arbitrary linear forms $x_1 + a\, x_2$ the above triangular decomposition and stop as soon as the condition of Theorem 5 is satisfied, which implies that the form $x_1 + a\, x_2$ is separating. Then, it remains to check if the resultant of $\mathcal{R}'$ and $\mathcal{C}'$ with respect to x_2 has real roots which can be done using for example Sturm sequence [7].

Remark: In practice, several strategies are used in order to reduce the running time of the above algorithm. For instance, the computation is stopped as soon as a resultant, computed for an arbitrary form $x_1 + a\, x_2$, is square free, which implies that the form $x_1 + a\, x_2$ is separating according to Theorem 4. The computation is also stopped when the computed resultant does not have real zeros, since it implies that the system does not have real zeros as well. Another example is the way we choose the candidate forms $x_1 + a\, x_2$. Indeed, in order to increase the probability of the form to be separating, a first computation is performed modulo a prime number ν (coefficients are considered in $\frac{\mathbb{Z}}{\nu\mathbb{Z}}$). Such a computation turns out to be very fast since it avoids coefficient swell in the algorithm. Providing that a linear form is separating modulo the prime ν, the latter has then a high probability to be separating in $\mathbb{Z}$ and one can choose it as a candidate for the algorithm in $\mathbb{Z}$.

IV. PARAMETRIC STABILITY

In this section, we consider a two-dimensional system defined by a polynomial $D(z_1, z_2, U)$ where $U = \{U_1, \ldots, U_k\}$ is a set of parameters. As mentioned in the introduction, our goal is to study the stability of this system (the truth of Condition (1)) depending on the values of the parameters. Starting from the polynomial $D(z_1, z_2, U)$, our approach consists roughly in computing a set of polynomials $\{p_1, \ldots, p_s\}$ in $\mathbb{Q}[U_1, \ldots, U_k]$ satisfying the property

that the stability of the system defined by $D(z_1, z_2, U)$ does not change, provided that the sign of the sequence $\{p_1(U), \ldots, p_s(U)\}$ does not change. Then $\mathbb{R}^k$ is decomposed into cells in which the signs of $\{p_1, \ldots, p_s\}$ remain invariants, and cells for which the system is stable are kept.

Considering $D(z_1, z_2, U)$ as a polynomial in the variables z_1, z_2 with coefficients are polynomials in $\mathbb{Q}[U_1, \ldots, U_k]$, we still have an equivalence between Condition (1) and the set of conditions (2) and we can apply to the last condition of (2) the transformation given in Section II which yields the following set of conditions depending on the parameters U.

$$\begin{cases} D(z_1, 1, U) \neq 0 \mid |z_1| \leq 1, \\ D(1, z_2, U) \neq 0 \mid |z_2| \leq 1, \\ V(\langle \mathcal{R}(x_1, x_2, U), \mathcal{C}(x_1, x_2, U) \rangle) \cap \mathbb{R}^2 = \emptyset. \end{cases} \quad (4)$$

We shall start with the study of the first two conditions involving univariate polynomials with parameters. Our first step consists in transforming these conditions so that continuous stability tests can be used. More precisely, we apply the change of variable $s_1 = \frac{1-z_1}{1+z_1}$ (resp. $s_2 = \frac{1-z_2}{1+z_2}$) to the polynomial $D(z_1, 1, U)$ (resp. $D(1, z_2, U)$). These conditions then write as $D_1(s_1, 1, U) \neq 0, Re(s_1) \geq 0$ and $D_2(1, s_2, U) \neq 0, Re(s_1) \geq 0$, where $D_1(s_1, 1, U)$ (resp. $D_2(1, s_2, U)$) is the numerator of $D(\frac{1-s_1}{1+s_1}, 1, U)$ (resp. $D(1, \frac{1-s_2}{1+s_2}, U)$). In a second step, we use a classical result of Linard and Chipart [7] that expresses the stability condition of a continuous polynomial $D(s)$ as a positivity condition of its coefficients as well as some *signed principal subresultant* sequence of two polynomials $F(s)$ and $G(s)$ satisfying $D(s) = F(s^2) + s\,G(s^2)$ (see [7, Thm. 9.30]). Using the specialization property of subresultants (see Section (A)), we can generalize this result to the case of univariate polynomials depending on parameters. In particular, applying this test to the polynomials $D_1(s_1, U)$ and $D_2(s_2, U)$ yields a set of polynomials depending only on parameters U, and the stability of $D_1(s_1, U)$ and $D_2(s_2, U)$ (resp. $D(z_1, 1, U)$ and $D(1, z_2, U)$) is then satisfied providing that these polynomials are positive. In the sequel, we shall denote these polynomials by $\phi_i(U)$.

The next question of interest is to decide if the system

$$\mathcal{S} := \begin{cases} \mathcal{R}(x_1, x_2, U) = 0 \\ \mathcal{C}(x_1, x_2, U) = 0 \end{cases} \quad (5)$$

where $U = [U_1, \ldots, U_k]$, admits real solutions.

In the following, we shall assume that the system $\mathcal{S}$ is *generically zero-dimensional*, that is, for almost any values of the parameters $u \in \mathbb{C}^k$, the underlying system, after substituting the parameters, $\{\mathcal{R}(x_1, x_2, u) = 0, \mathcal{C}(x_1, x_2, u) = 0\}$ admits a finite number of complex solutions.

A. Discriminant varieties: definition and properties

Our approach for answering the above question makes use of the concept of *discriminant variety* of a polynomial system depending on parameters [16]. Loosely speaking, a discriminant variety, denoted by W_D, is an algebraic variety in the parameter space defining a partition of the latter that consists of:

- The discriminant variety W_D itself, and
- Disjoint open connected subsets $\mathcal{U}_1, \cdots, \mathcal{U}_r$ of the parameter space which do not insect the discriminant variety and such that any solution of (5) with parameters lying in some $\mathcal{U}_i$, belongs to the image of an analytic function of $\mathcal{U}_i$ into the solutions of (5).

Remark. An important property of discriminant varieties is that, if u and v are two vectors of parameters which belong to the same $\mathcal{U}_i$, the specialized systems $\mathcal{S}_{U=u}$ and $\mathcal{S}_{U=v}$ have exactly the same number of real roots.

It should be stressed that the concept of *discriminant variety* (as well as its computation) is defined for general systems of equations (with n variables). However, in the sequel, for the sake of simplicity, we restrict our description to the case of systems of two equations in two variables. For a complete description, one may refer to the work in [16].

Since W_D belongs to the parameter space, we introduce the projection mapping $\Pi_U : (\alpha_1, \alpha_2, u) \in \mathbb{C}^{(2+k)} \longmapsto u \in \mathbb{C}^k$. We also introduce the inverse projection on the parameter space $\Pi_U^{-1} : u \longmapsto (u, \alpha_1, \alpha_2) \subset V(\mathcal{S})$.

It has been shown that if one considers the set of all $u \in \mathbb{C}^k$ such that there exists no neighborhood $\mathcal{U}$ of u such that $\Pi_U^{-1}(\mathcal{U}) \cap V(\mathcal{S})$ is an analytic covering of $\mathcal{U}$, this set defines a variety named *minimal discriminant variety* of $V(\mathcal{S})$ associated with Π_U, and a key remark is that this minimal variety is defined independently of any algorithm.

In our setting, the ideal $\mathcal{S} = \langle \mathcal{R}, \mathcal{C} \rangle$ is equidimensional and the minimal discriminant variety W_D of $V(\mathcal{S})$ associated with Π_U is the union of two subsets:

- O_∞: the set of $\alpha \in \mathbb{C}^k$ such that $\Pi_U^{-1}(\mathcal{U}) \cap V(\mathcal{S})$ is not compact for any compact neighborhood $\mathcal{U}$ of α in $\mathbb{C}^k$
- O_c: the set of the critical values of Π_U union the projection of the singular points of $V(\mathcal{S})$

Intuitively, O_∞ represents parameter's values such that there exist either vertical leafs of solutions or leafs that are going to infinity above some of their neighborhood, while O_c represents parameter's values such that above some of their neighborhood, the number of leafs varies. Roughly speaking, W_D represents parameter's values over which the number of solutions changes.

In our case, $O_c = \Pi_U(V(\langle \mathcal{R}, \mathcal{C}, \mathrm{Jac}_{x_1, x_2}(\mathcal{R}, \mathcal{C}) \rangle))$ where $\mathrm{Jac}_{x_1, x_2}(\mathcal{R}, \mathcal{C})$ denotes the determinant of the Jacobian matrix with respect to the variables x_1 and x_2.

B. Discriminant varieties: computations

An important remark for the computation of the discriminant variety W_D of $\mathcal{S}$ is that both O_∞ and O_c are algebraic sets (for general systems, this is not always the case for O_c). W_D can thus be described as the union of two algebraic sets that can be computed independently.

O_∞ and O_c are projections of some algebraic varieties and computing them remains to eliminating variables in the systems of equations corresponding to these varieties, that is, for a given $I = \langle f_1, \ldots, f_l \rangle \subset \mathbb{K}[x_1, x_2, U]$, computing $\Pi_U(V(I)) = V(I_U)$ where $I_U \subset \mathbb{K}[U]$ is defined by $I_U = I \cap \mathbb{K}[U]$. Algorithmically, I_U can be computed by means

of Gröbner basis for any elimination ordering $<$ such that $U < x_1, x_2$ (see Appendix for details). More precisely, it suffices to compute a Gröbner basis for such an ordering and to keep the polynomials that belong to $\mathbb{K}[U]$.

Hence, computing an ideal I_c such that $O_c = V(I_c)$ remains to computing the determinant $\text{Jac}_{x_1,x_2}(\mathcal{R}, \mathcal{C})$ and a Gröbner basis of the ideal $\langle \mathcal{R}, \mathcal{C}, \text{Jac}_{x_1,x_2}(\mathcal{R}, \mathcal{C}) \rangle$ for any elimination ordering $<$ such that $U < x_1, x_2$.

In [16], it has been also remarked that such elimination orderings allow to compute an ideal $I_\infty \subset \mathbb{Q}[U]$ such that $O_\infty = V(I_\infty)$. Precisely, Suppose that G is a reduced Gröbner basis of $\langle \mathcal{R}, \mathcal{C} \rangle$ for a monomial ordering $<_{U,x_1,x_2}$, that is, the product of two degree reverse lexicographic orderings $<_U$ for the parameters and $<_{x_1,x_2}$ for the variables. We define the set $\mathcal{E}_i^\infty = \{\text{LM}_{<_{x_1,x_2}}(g) \mid g \in G, \exists m \geq 0, \text{LM}_{<_{x_1,x_2}}(g) = x_i^m\}$, where $LM_<$ denotes the leading monomial of a polynomial for an admissible monomial ordering $<$, then:

- $\mathcal{E}_i^\infty$ is the Gröbner basis of some ideal $I_i^\infty \subset \mathbb{K}[U]$ for $<_U$;
- $O_\infty = \bigcup_{i=k+1}^n V(I_i^\infty)$.

C. Discussing the number of real solutions

Once a discriminant variety W_D of $\mathcal{S}$ computed, we get a partition of the parameter space made of the discriminant variety and of the connected components of its complementary with the property that over any neighborhood $\mathcal{U}$ that does not meet W_D, $\Pi_U^{-1}(\mathcal{U})$ is an analytic covering of $\mathcal{U}$. In particular, the number of solutions of $\mathcal{S}$ is constant over any connected set that do not intersect the discriminant variety.

Also, for computing the (constant) number of solutions over each connected component that do not meet the discriminant variety, it suffices to take one vector u of parameter values in each of these components and to solve the zero-dimensional system $\mathcal{S}_{U=u}$.

Remark: Note that the structure of the solutions is not known above the discriminant variety itself. As it is a set of null measure, it is useless here to study what is going on for such parameter values. However, the discriminant variety is defined by a polynomial system that can be merged to the original system to follow the study recursively.

The discriminant variety has been defined with respect to complex solutions. For real solutions, two cases occur : either $\Pi_U(V(\mathcal{S}) \cap \mathbb{R}^{k+2}) \subset W_D$ and one needs to study $V(\langle S \rangle) \cap \Pi_U^{-1}(W_D)$ instead of $V(\mathcal{S})$ or $\Pi_U(V(\mathcal{S}) \cap \mathbb{R}^{k+2}) \not\subseteq W_D$ and then $W_D \cap \mathbb{R}^k$ is a discriminant variety for $V(\mathcal{S}) \cap \mathbb{R}^{k+2}$, which is the usual situation. Note that in the second case, if W_d is minimal for $V(\mathcal{S})$, then $W_d \cap \mathbb{R}^k$ it not necessarily minimal for $V(\mathcal{S}) \cap \mathbb{R}^{k+2}$.

D. Computing regions of stability

We now go back to our initial problem which is the computation of regions in the parameter space, such that the set of Conditions (4) is satisfied (and thus the system is stable). As mentioned at the beginning of this section, we can compute a set of polynomials $\{\phi_i(U)\}_{i=1,\ldots,l_1}$ such that, the two first conditions of (4) are satisfied if and only

if $\phi_i(U) > 0$. On the other hand, according to above, we can also compute a set of polynomials $\{f_i(U)\}_{i=1,\ldots,l_2}$ that defines a partition of the parameter space in which the number of real solutions of $\mathcal{S}$ is constant. Now, considering the global set of polynomials $F := \{\phi_i(U), f_i(U)\}$, we can compute a *Cylindrical Algebraic Decomposition* (CAD) adapted to F [17]. This yields a disjoint union of cells in $\mathbb{R}^k$ in which the signs of the polynomials in F are constant. In particular, by definition, inside each of these cells, both the sign of the polynomial $\phi_i(U)$ and the number of real solutions of S are constant. This implies that the system is either stable or unstable. To determine the cells for which the system is stable, it suffices to select a simple point (vector of parameter values) in each cell and to test the Conditions (4) after substitution of the parameters.

In practice, we compute a partial CAD since we are only interested in computing cells of maximal dimension.

Partial CAD: Given a set of polynomials $\{P_1, \ldots, P_k\} \in \mathbb{Q}[x_1, .., x_{n-1}][x_n]$, consider $\text{Proj}(\{P_1, \ldots, P_d\}, x_n) \subset \mathbb{Q}[x_1, .., x_{n-1}] = \{\text{LeadingCoeff}_{x_n}(P_i), \text{Discriminant}_{x_n}(P_i), i = 1..d\} \cup \{\text{Res}_{x_n}(P_i, P_j), i \neq j; i,j = 1..d\}$. Then $\cup_{i=1..d} V(P_i)$ is an analytic covering of each open connected set of $\mathbb{R}^{n-1}$ that do not meet $V(\text{Proj}(P_1, \ldots, P_d, x_n))$. Roughly speaking, $\cup_{i=1..d} V(P_i)$ decomposes the cylinder over any connected open set $\mathcal{U} \subset \mathbb{R}^{n-1}$ that do not meet $V(\text{Proj}(P_1, \ldots, P_d, x_n))$ into the union of leafs (of dimension $n - 1$) of $\cup_{i=1..d} V(P_i)$ and bands (of dimension n) between two of these leafs with the property that $[\text{sign}(P_1), \ldots, \text{sign}(P_d)]$ defines a constant sequence in each band. Now, given $\text{Proj}(P_1, \ldots, P_d, x_n)$ one can then compute recursively $\text{Proj}(\ldots \text{Proj}(P_1, .., P_d, x_n), \ldots, x_{n-1})$ until getting points and then obtain recursively a partition of $\mathbb{R}^n$ into some algebraic proper set $\mathcal{D}$ (of dimension at most $n - 1$) and some cells of dimension n in which $P_1, \ldots, P_d$ have all a constant sign.

This process is a partial CAD adapted to $P_1, \ldots, P_d$. The difference with the classical CAD is that $P_1, \ldots, P_d$ have not necessarily a constant sign on the algebraic set $\mathcal{D}$. Note that $\mathcal{D}$ can be decomposed itself using the same process.

V. EXPERIMENTATION

A. Non parametric case

As mentioned in Section III, the algorithm described in the present article is a set of optimizations for the two-dimensional case of a general algorithm we proposed in [1].

Roughly speaking, we mainly propose a dedicated method for deciding if a system of two equations in two variables admits real solutions, keeping track of the shape of the systems linked to the stability problem we want to study.

In order to measure the gain we obtain, we compare against the general method Isolate partially developed by the same authors and available in the Maple software RootFinding. This function first computes a Rational Univariate Representation ([14]) from a Gröbner basis computed with F_4 algorithm ([18]) and then makes use of

a variant of Descartes' algorithm [19] as well as multi-precision interval arithmetic [20] to isolate the real roots of the system.

For the present experiments we re-use some black boxes developed for the algorithms described in [21] or [22] which are using exactly the same technical base as the above function to design the algorithm's component that computes the univariate polynomial $C_{x_1 + a\, x_2}$ and performs the separation check. All the other components are shared with the `RootFinding[Isolate]` function from Maple.

For dense polynomials with coefficients that can be encoded on 23 bits (such as if there were coming from floating point numbers), the results we obtained on a `core i7 3.5 Ghz` with 32 GB of memory are summarized in the following table in which $Degree$ denotes the degree of the polynomial $D(z_1, z_2)$ to be studied, $\sharp V(I)$ the number of complex roots of the bivariate system to be solved to decide the stability, $RootFinding$ the computation time of the general function `RootFinding[Isolate]` and $Dedicated$ our new (dedicated) algorithm.

Degree	$\sharp V(I)$	RootFinding	Dedicated
10	200	2.3	< 1
15	450	29.8	< 1
20	800	223.4	< 1
25	1280	866.9	1.42
30	1800	3348.2	2.79
35	2450	> 1 hour	7.81
40	3200	> 1 hour	15.51

Note that on these examples, we did not report the computation times required by the two other conditions (stability of $D(1, z_2)$ and $D(z_1, 1)$) since they are small compared to the resolution of the full bivariate system.

As an interesting fact, it is worth mentioning that using a naive implementation of the Möbius transforms makes the latter step the most dominant part in the dedicated algorithm.

B. Parametric case

Let now consider a $2D$ transfer function depending on two parameters and whose denominator is:
$D(z_1, z_2) = (4u_1 + 2u_2 + 3)z_1 + (-2u_1 + 1)z_2 + (4u_1 - 2u_2 - 2)z_1 z_2 + (2u_1 - 2u_2 + 4)z_1^2 z_2 + (-u_1 - u_2 + 1)z_1 z_2^2$.

We first apply the algebraic transformation from Section II to $D(z_1, z_2)$, the resulting bivariate parametric system we have to study is $\{\mathcal{R}(x_1, x_2) = \mathcal{C}(x_1, x_2) = 0\}$ with $\mathcal{R}(x, y) = 7\, u_1\, x^2 y^2 - 3\, u_2\, x^2 y^2 + 7\, x^2 y^2 + u_1\, x^2 + 7\, y^2 u_1 - 5\, u_2\, x^2 + y^2 u_2 - x^2 - 3\, y^2 + u_1 - u_2 - 11$,
$\mathcal{C}(x, y) = 10\, u_1\, x^2 y - 8\, u_1\, xy^2 + 6\, u_2\, x^2 y + 4\, u_2\, xy^2 + 4\, x^2 y - 6\, xy^2 - 8\, u_1\, x + 10\, u_1\, y + 4\, u_2\, x + 6\, yu_2 - 6\, x + 4\, y]$

The minimal discriminant variety of this bivariate system with respect to the projection onto (u_1, u_2) can be obtained by running the Maple function `RootFinding[Parametric][Discriminant Variety]` which gives an union of 10 lines, 2 quadrics and one curve of degree 6.

Computing the conditions on the parameters that discriminate the situations where $D(z_1, 1)$ (resp. $D(1, z_2)$) has (or
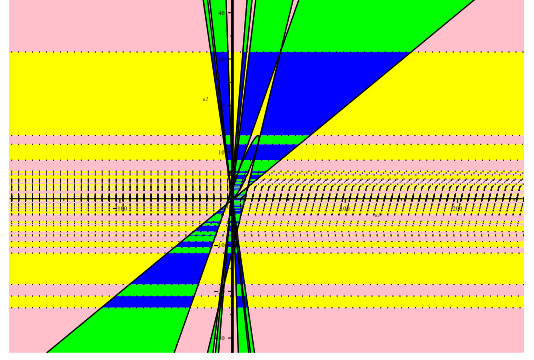

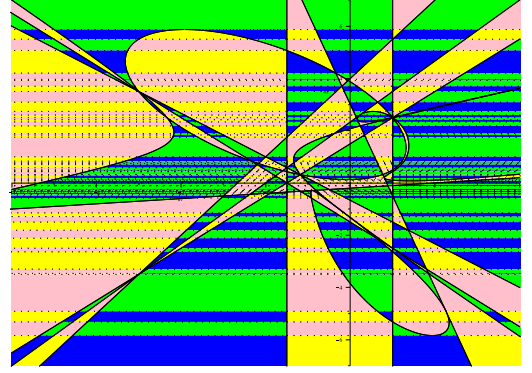Fig. 1. Global view - Parameter space decomposition

Fig. 2. Zoom $u_1 = -4\ldots2, u_2 = -7\ldots7$ - Parameter space decomposition

not) roots in the unit disk lead to a list of 6 lines with 3 of them already in the discriminant variety.

Decomposing the parameter space cylindrically with respect to these 16 curves gives 1161 cells (see Figure 1).

Among each cell, the system is either stable or unstable. It is then sufficient to pick up one couple of parameter values in each cell and to count the number of real solutions of the (non parametric) zero-dimensional system $\{\mathcal{R}, \mathcal{C}\}$ and perform the stability test of $D(z_1, 1)$ and $D(1, z_2)$.

It should be noticed that in some regions of the parameter space, some cells are very small.

Finally, it turns out that 118 of these regions correspond to unstable systems. For example the cell containing the couple $(u_1 = -.4717912847, u_2 = -.5389591122)$ contains parameters values that all correspond to unstable systems while the cell containing the couple $(u_1 = -.6152602220, u_2 = -.5389591122)$ contains parameters values that all correspond to stable systems (see Figure 3).

APPENDIX

A. Resultant and Subresultants

A key tool, related to the study of solutions of algebraic systems, is the Subresultant sequence. We provide below its definition and some of its properties that are needed for the description of our algorithms. For a complete overview, the reader may refer to [7].
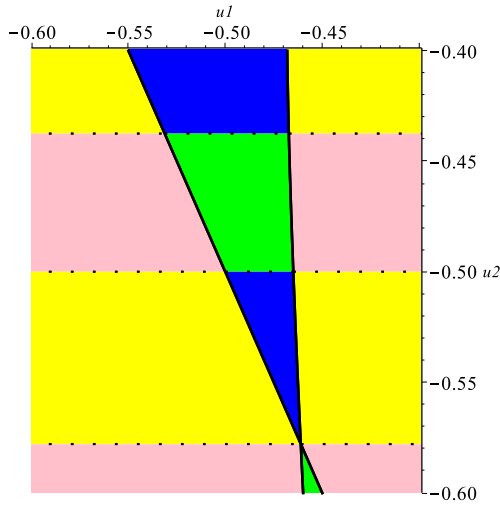
Fig. 3. Zoom around a non stable region : $u_1 = -0.4 \ldots -0.6, u_2 = -0.4 \ldots -0.6$ - Parameter space decomposition

$\mathbb{D}$ denotes a unique factorization domain and $\mathbb{F}$ its fraction field. Let $f = a_0 + \ldots + a_n x^n$ and $g = b_0 + \ldots + b_m x^m$ be two polynomials with coefficients in $\mathbb{D}$. We shall always assume in the following that $a_n \neq 0$, $b_n \neq 0$ and $n \geq m$.

For an integer k such that $0 \leq k \leq m$, we define the following $\mathbb{D}$-linear map

$$\varphi_k : (u, v) \longmapsto uf + vg$$

such that $u, v \in \mathbb{D}[x]$ are polynomials with degrees respectively less or equal than $n - k - 1$ and $m - k - 1$ whose the corresponding matrix is given as:

$$S_k = \begin{pmatrix} a_n & a_{n-1} & \cdots & & a_0 & & & \\ & \ddots & & & & \ddots & & \\ & & a_n & a_{n-1} & \cdots & & a_0 & \\ b_m & b_{m-1} & \cdots & & b_0 & & & \\ & \ddots & & & & \ddots & & \\ & & b_m & b_{m-1} & \cdots & & b_0 & \end{pmatrix}$$

The matrix S_0 is the classical Sylvester matrix associated to f and g. To be coherent with the degree of polynomials, we will attach index $i-1$ to the i-th column of S_k, so that the indexes of the columns go from 0 to $n + m - k - 1$

Definition 2: For $j \leq n + m - k - 1$ and $0 \leq k \leq m$, let $\mathrm{sr}_{k,j}$ be the determinant of the submatrix of S_k formed by the last $n + m - 2k - 1$ columns, the column of index j and all the $n + m - 2k$ rows. The polynomial $\mathrm{Sres}_k(f, g) = \mathrm{sr}_{k,k} x^k + \ldots + \mathrm{sr}_{k,0}$ is the k-th polynomial subresultant of f and g, and its leading term $\mathrm{sr}_{k,k}$ (or simply sr_k) is the k-th principal subresultant of f and g. The polynomial $\mathrm{Res}(f, g) = \mathrm{sr}_0$ is the resultant of f and g.

Proposition 1: The following properties are equivalent:
- f, g have a common root in $\overline{\mathbb{F}}$, the alg. closure of $\mathbb{F}$,
- $\mathrm{Sres}_0(f, g) = 0$,
- f, g have a non trivial gcd which is proportional to the non-zero polynomial subresultant of minimal index.

In addition, the subresultant sequence bears an important specialization property.

Proposition 2: Let $\mathbb{D}$ and $\mathbb{D}'$ be unique factorization domains and $\phi : \mathbb{D} \to \mathbb{D}'$ be a morphism. Let $f, g \in \mathbb{D}[x]$ and suppose that $\mathrm{degree}(\phi(f)) = \mathrm{degree}(f) > \mathrm{degree}(g) = \mathrm{degree}(\phi(g))$. Then $\phi(\mathrm{Sres}_i(f, g)) = \mathrm{Sres}_i(\phi(f), \phi(g)), \forall i = 0 \ldots \mathrm{degree}(g)$.

Consider now two polynomials $f = \sum_{i=0}^n a_i(x_1) x_2^i$ and $g = \sum_{i=0}^m b_i(x_1) x_2^i$ in $\mathbb{Q}[x_1, x_2]$. The two above properties leads to the following result.

Proposition 3: For any α such that $a_n(\alpha)$ and $b_m(\alpha)$ do not both vanish. The first polynomial $\mathrm{Sres}_{x_2,k}(\alpha, x_2)$ (for k increasing) that does not identically vanish is of degree k and it is the gcd of $f(\alpha, x_1)$ and $g(\alpha, x_2)$(up to a nonzero constant in the fraction field of $\mathbb{D}(\alpha)$).

Triangular decomposition: Given two bivariate polynomials $f, g \in \mathbb{Q}[x_1, x_2]$ such that $a_n(x_1)$ and $b_m(x_1)$ are coprime, one can use the above result in order to compute a triangular description of the solutions of the system $S := \{f = g = 0\}$. Indeed, starting from the resultant of f and g whose roots α are the x_1 coordinates of the common solutions of S, one can factorize the latter depending on the degree of the gcd of $f(\alpha, x_2)$ and $g(\alpha, x_2)$. For each root α of the resultant, the gcd of $f(\alpha, x_2)$ and $g(\alpha, x_2)$ is then given as the specialization at α of the first non vanishing polynomial subresultant according to Proposition 3. Consequently, the set of solutions of S, i,e., $\{(\alpha, \beta) \in \mathbb{C}^2 | f(\alpha, \beta) = g(\alpha, \beta) = 0\}$ is equal to the set $\bigcup_{i=1}^{m-1}\{(\alpha, \beta) \in \mathbb{C}^2 | h_i(\alpha) = \mathrm{Sres}_i(\alpha, \beta) = 0\}$, where the polynomial $h_i(x_1)$ is the factor of the resultant whose any root α satisfies the property that the degree of the gcd of $f(\alpha, x_2)$ and $g(\alpha, x_2)$ is equals to i. See [12] for more details about this triangular decomposition algorithm.

B. Gröbner bases

A Gröbner basis of an ideal $I \subset \mathbb{Q}[x_1, \ldots, x_n]$ is a computable generator set of I with good algorithmic properties. This generator set is defined with respect to a monomial ordering, say a total ordering on $\mathbb{N}^n$ which is compatible with the multiplication of monomials. The lexicographic ordering, denoted $<_{\mathrm{lex}}$, is the most well-known ordering:

$$\begin{aligned} & x_1^{\alpha_1} \cdots x_n^{\alpha_n} <_{\mathrm{lex}} x_1^{\beta_1} \cdots x_n^{\beta_n} \\ & \Leftrightarrow \exists i_0 \leq n \ , \ \begin{cases} \alpha_i = \beta_i, & \text{for } i = 1, \ldots, i_0 - 1, \\ \alpha_{i_0} < \beta_{i_0} \end{cases} \end{aligned} \quad (6)$$

However, for efficiency reasons, it is often preferable to consider the so-called degree orderings such as the *degree reverse lexicographic order (DRL)*:

$$\begin{aligned} & x_1^{\alpha_1} \cdot \ldots \cdot x_n^{\alpha_n} <_{\mathrm{DRL}} x_1^{\beta_1} \cdot \ldots \cdot x_n^{\beta_n} \Leftrightarrow \\ & x_0^{\sum_k \alpha_k} \cdot x_1^{-\alpha_n} \cdot \ldots \cdot x_n^{-\alpha_1} <_{\mathrm{lex}} x_0^{\sum_k \beta_k} \cdot x_1^{-\beta_n} \cdot \ldots \cdot x_n^{-\beta_1}. \end{aligned} \quad (7)$$

Once a monomial ordering $>$ is fixed it induces a natural notion of *leading monomial* for any polynomial p in $\mathbb{Q}[x_1, \ldots, x_n]$ which is the greatest monomial of p for $>$ denoted by $\mathrm{LM}(p, <)$ in the sequel.

Whatever the monomial ordering used, the key property of a Gröbner basis is to induce a canonical reduction function named *normalForm*:

Theorem 6: [13] Let G be a Gröbner basis of an ideal $I \subset \mathbb{Q}[x_1, \ldots, x_n]$ for a fixed ordering $<$. Then, a polynomial $p \in \mathbb{Q}[x_1, \ldots, x_n]$ belongs to I if and only if $NormalForm(p, G, <) = 0$. In particular, $NormalForm(p, G, <)$ is unique.

We pay a particular attention to Gröbner bases with respect to elimination or block orderings (defined below) since they provide a way of *eliminate* some variables from the system.

Definition 3: Given two monomial orderings $<_U$ (w.r.t. the variables $U_1, \ldots, U_d$) and $<_X$ (w.r.t. the variables $x_{d+1}, \ldots, x_n$), a *block ordering* $<_{U,X}$ is defined as follows : given two monomials m and m', then $m <_{U,X} m'$ if and only if either $m_{|U_1=1,\ldots,U_d=1} <_X m'_{|U_1=1,\ldots,U_d=1}$ or $(m_{|U_1=1,\ldots,U_d=1} = m'_{|U_1=1,\ldots,U_d=1}$ and $m_{|x_{d+1}=1,\ldots,x_n=1} <_U m'_{|x_{d+1}=1,\ldots,x_n=1})$. We say that such an ordering *eliminates* $x_{d+1}, \ldots, x_n$.

The lexicographical ordering such $x_1 < \ldots < x_n$ is a block ordering for any $1 < i < n$, which eliminates $x_{i+1}, \ldots, x_n$. However, this ordering is not recommended for elimination because the computation is usually much harder than with block orderings such both $<_U$ and $<_X$ are DRL orderings.

Two important applications of elimination orderings are the *projections* and *localizations*, which can be summarized in the following two propositions. To facilitate the illustration, the following notation is needed. Given any subset $\mathcal{V}$ of $\mathbb{C}^d$ (d is an arbitrary positive integer), $\overline{\mathcal{V}}$ is its Zariski closure which is the smallest subset of $\mathbb{C}^d$ containing $\mathcal{V}$. If $\mathcal{V}$ is a constructible set (i.e. it may be defined by equations and inequations), then $\overline{\mathcal{V}}$ is also the closure for the usual topology. This will be always the case in the following.

Proposition 4: [23] Let G be a Gröbner basis of an ideal $I \subset \mathbb{Q}[U, X]$ w.r.t. a block ordering $<_{U,X}$, then $G \bigcap \mathbb{Q}[U, X]$ is a Gröbner basis of $I \bigcap \mathbb{Q}[U, X]$ w.r.t. $<_U$. Moreover, if $\Pi_U : \mathbb{C}^n \longrightarrow \mathbb{C}^d$ denotes the canonical projection on the coordinates U, then $V(I \cap \mathbb{Q}[U]) = V(G \cap \mathbb{Q}[U]) = \overline{\Pi_U(V(I))}$.

Proposition 5: [24] Let $I \subset \mathbb{Q}[X]$ and T be a new indeterminate, then $\overline{V(I) \setminus V(f)} = V((I + \langle Tf - 1 \rangle) \bigcap \mathbb{Q}[X])$. If $G' \subset \mathbb{Q}[X, T]$ is a Gröbner basis of $I + \langle Tf - 1 \rangle$ w.r.t a block ordering $<_{X,T}$, then $G' \bigcap \mathbb{Q}[X]$ is a Gröbner basis of $I : f^\infty := (I + \langle Tf - 1 \rangle) \bigcap \mathbb{Q}[X]$ w.r.t. $<_X$. The variety $\overline{V(I) \setminus V(f)}$ and the ideal $I : f^\infty$ are usually called the localization of $V(I)$ and I by f.

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