Localization Bounds for the Graph Translation

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Context: Signal Processing over Graphs

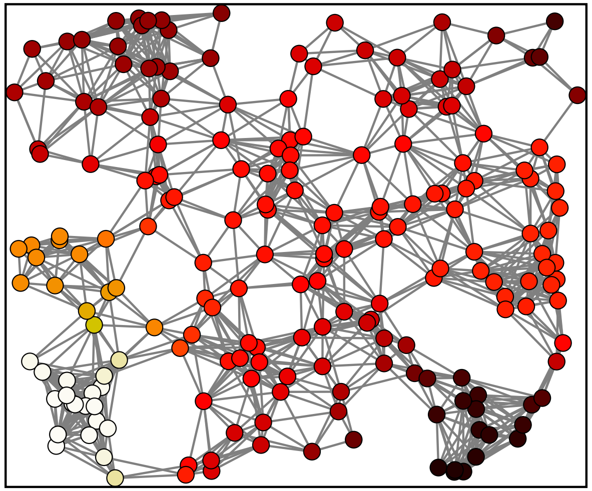
Grand Goal: Interpretation of variations over a discrete structure.

Structure: Vertices linked by edges.

Signal: Values carried by vertices.

Assumption: The structure explains variations.

Tools: Fourier transform, wavelet transform, filtering, sampling...



Fundamental Question: Time Shift Equivalent?

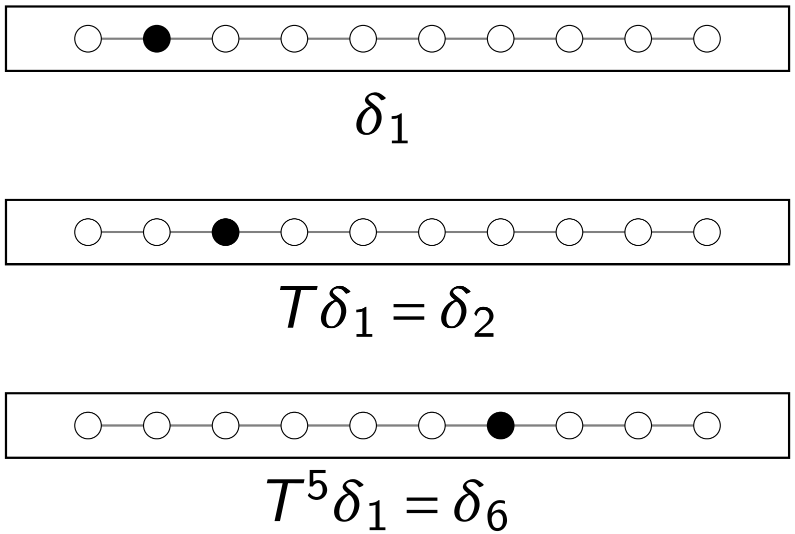
Observation: Time shift is at the core of Temporal Signal Processing.

Examples: Fourier, Wavelets, Time-Frequency, Stationarity...

Time Shift properties:

Linear Operator

Delta signal mapped to a delta signal



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Time Shift properties:

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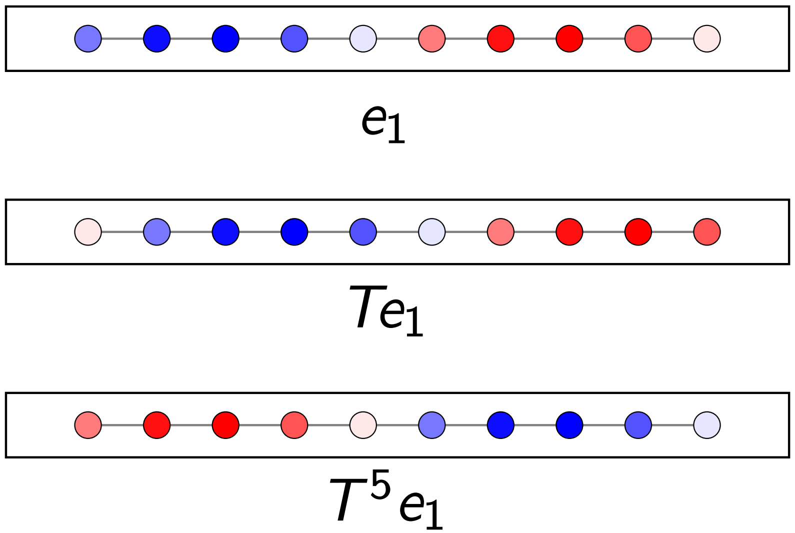
Fourier mode: phase shifted
⇒ Convolutive operator

Energy invariance

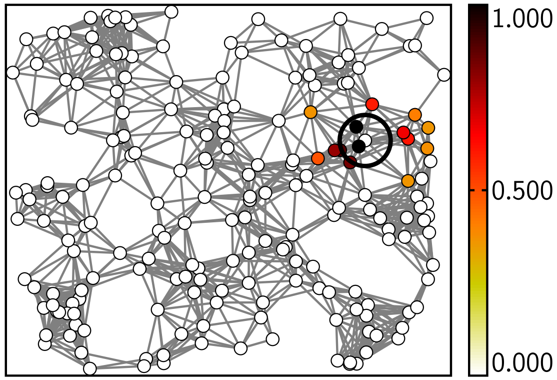


Fundamental Question: Time Shift Equivalent? (cont'd)

<u>State of the Art</u> of its equivalent Graph operator:

Graph Shift

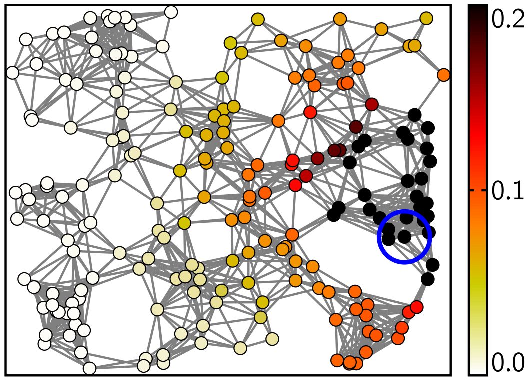
[Sandryhaila & Moura TSP'13]



$|A\delta_1|$

Generalized Translation

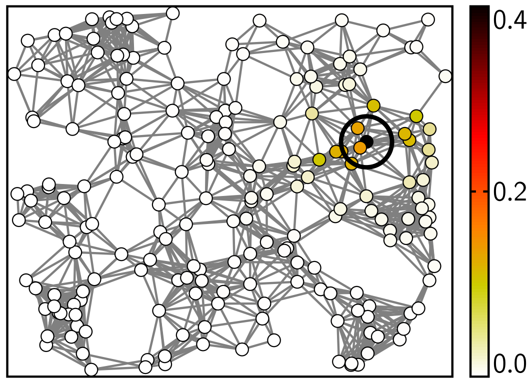
[Shuman et al. SPM'13]



$|T_2 h|$

Graph Translation

[Girault et al. SPL'15]



$|T_{\mathscr{G}}\delta_1|$

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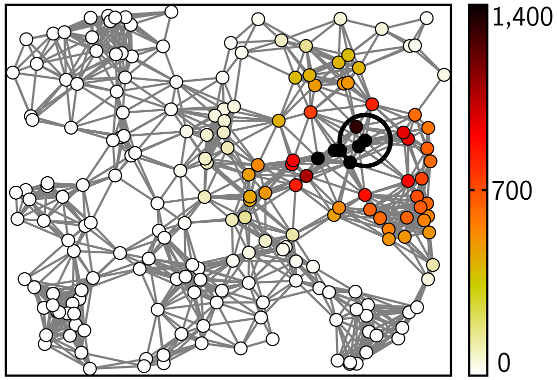
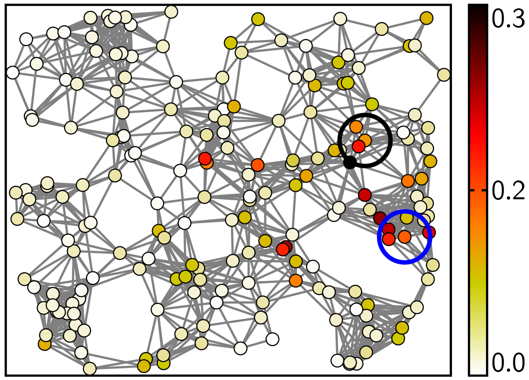


$|A^5 \delta_1|$

Generalized Translation

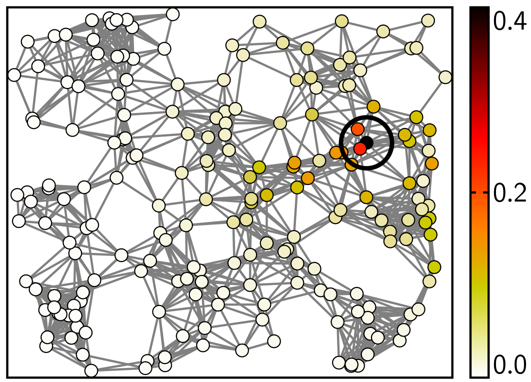
[Shuman et al. SPM'13]



$|T_2 \delta_1|$

Graph Translation

[Girault et al. SPL'15]



$|T_{\mathcal{G}}^5 \delta_1|$

$\Rightarrow$ Time Shift **transposes** to Diffusion.

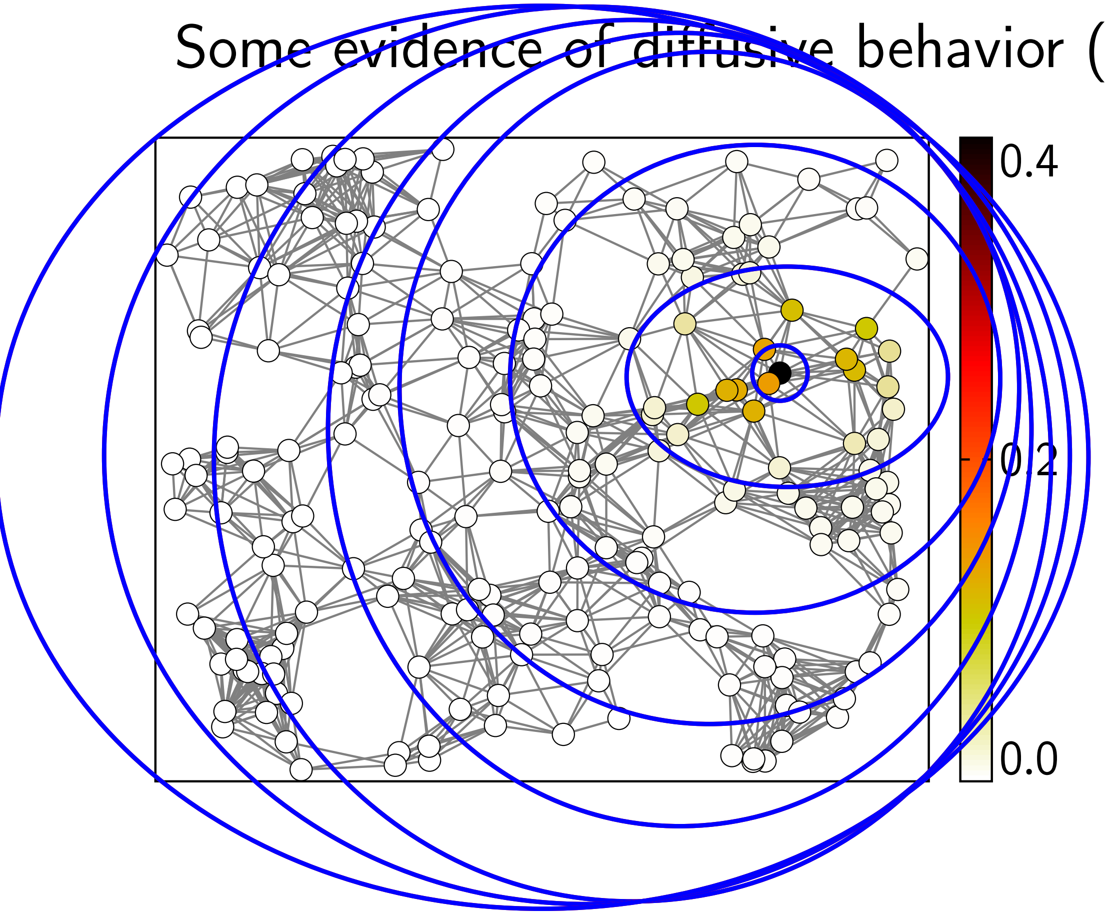
Is the Graph Translation formally a diffusion operator?

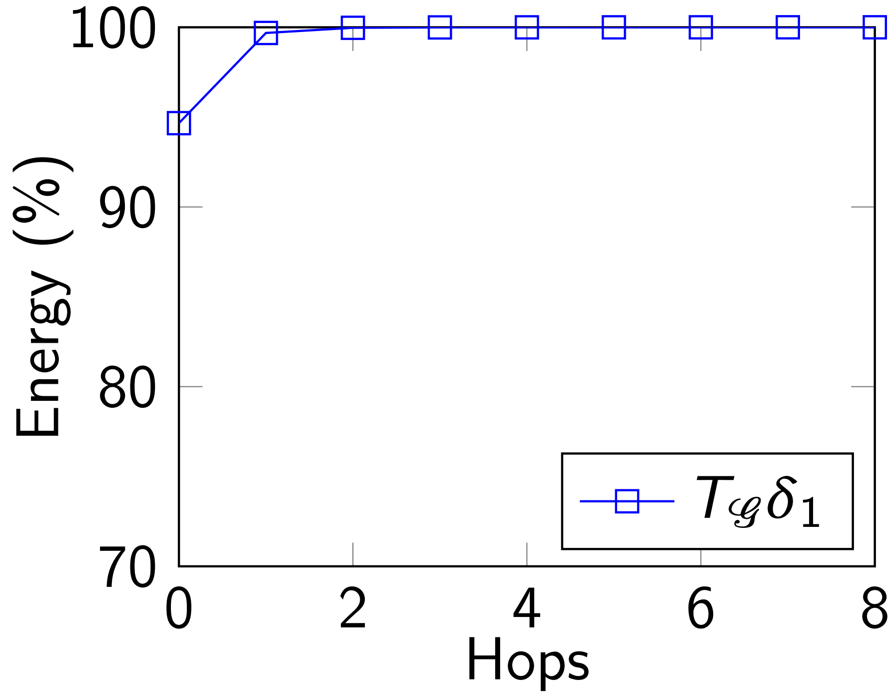
Graph Translation: well defined isometric operator in the Fourier domain.

$$T_{\mathscr{G}} = \exp\left(-\imath\pi\sqrt{\frac{L}{\rho_{\mathscr{G}}}}\right) \qquad \Rightarrow \qquad T_{\mathscr{G}}\chi_l = e^{-\imath\pi\sqrt{\lambda_l/\rho_{\mathscr{G}}}}\chi_l.$$

Question: vertex domain behavior?

Some evidence of diffusive behavior ($|T_{\mathscr{G}}\delta_1|$):





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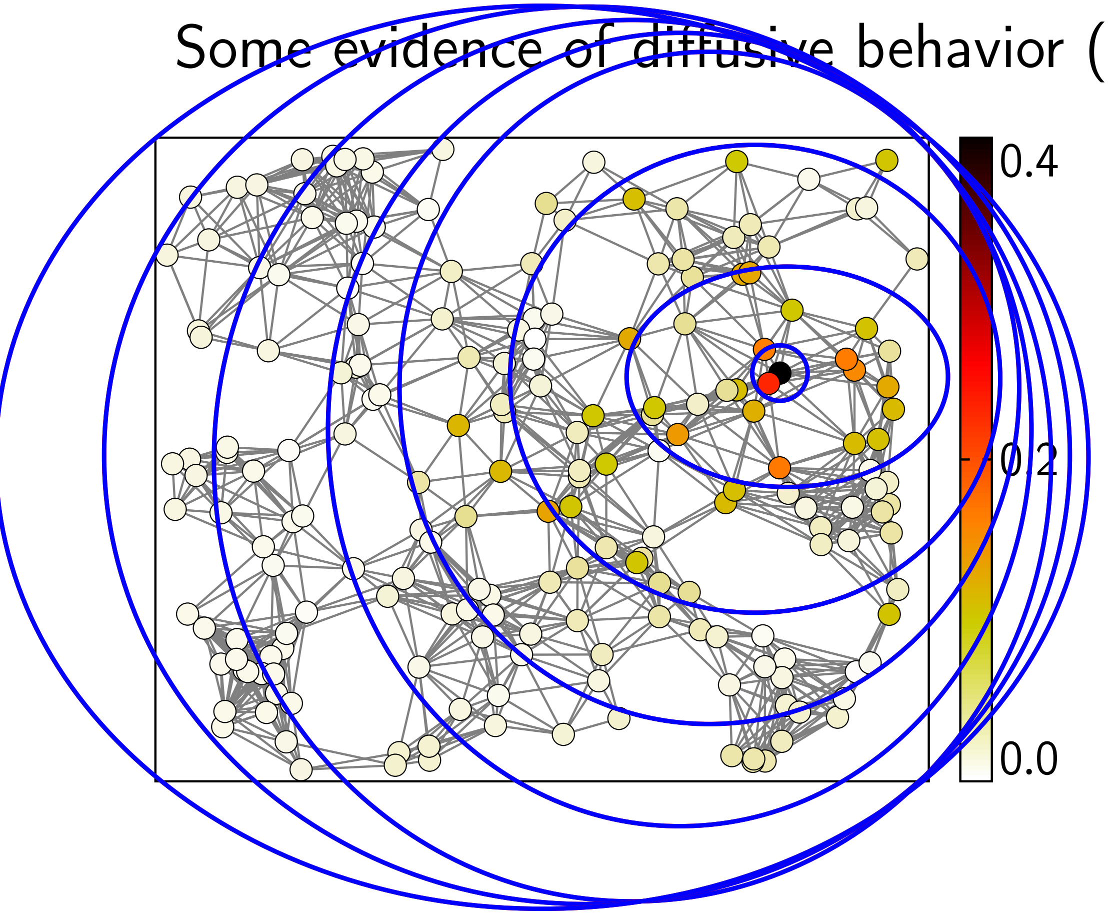
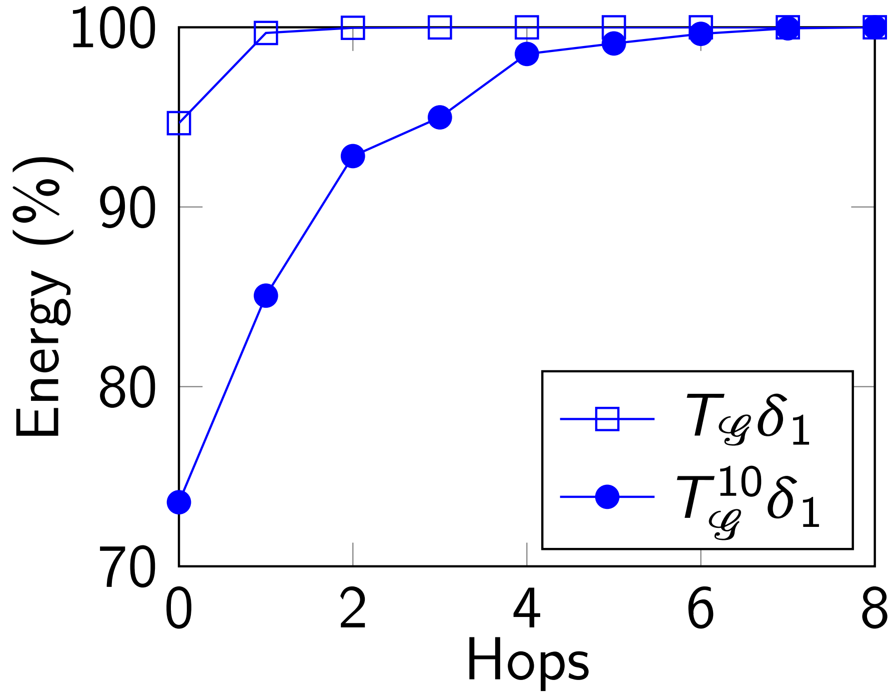
Some evidence of diffusive behavior ($|T_{\mathcal{G}}^{10}\delta_1|$):





Aims and Outline

- **Aim:** Study the diffusion properties through the impulse response
- **Context:** Operators verifying $H = f(M)$ (M a local operator)
- **Premise:** If f is a polynomial of degree K, the energy of the impulse response is located within K-hops of the impulse
- **Method:** Approximation using a truncation of the analytical form of f

Fundamental result (Theorem 1): If p_K is a polynomial of degree K and $|f - p_K| \leq \kappa(K)$, then the energy of $f(M)\delta_i$ outside the K-hops neighborhood of i is at most $\kappa(K)$.

Outline

1. **Case Study #1:** Simple analytical form of f
2. **Case Study #2:** More complex analytical form involving composition

Adjacency-Based Translation Operator

(GFT based on [Sandryhaila & Moura 2013].)

Adjacency matrix $A = U\Gamma U^*$, with $\Gamma = \mathrm{diag}(\gamma_0, \ldots, \gamma_{N-1})$

Graph frequencies $\pi(1 - \gamma_l/\gamma_{\max}) \in [0, 2\pi]$, Fourier modes U_l.

Definition (Adjacency-Based Isometric Translation Operator)

$$\mathscr{A} = \exp\Big(-\imath\pi(I - A/\gamma_{\max})\Big)$$

With $M = I - A/\gamma_{\max}$, we obtain $\mathscr{A} = f(M)$ and:

$$f(x) = \exp(-\imath\pi x) = \cos(\pi x) - \imath\sin(\pi x).$$

Question: Polynomial approximation of f?

Polynomial Approximation

Analytical form of f:

$$f(x) = \sum_{k=0}^{\infty} (-1)^k \left[\frac{\pi^{2k}}{(2k)!} x^{2k} - \imath \frac{\pi^{2k+1}}{(2k+1)!} x^{2k+1} \right]$$

Lemma (Alternating Series Approximation (Lemma 2))

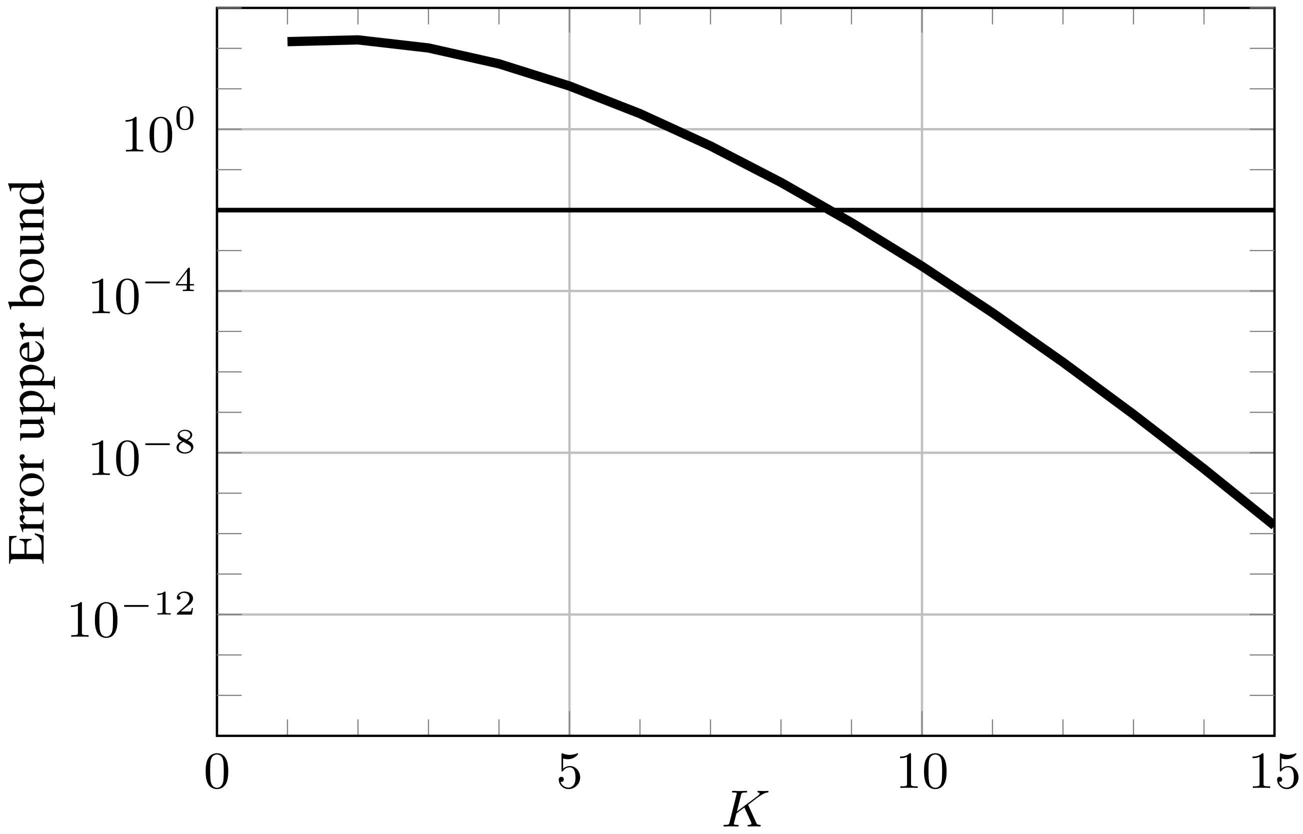
$$f(x) = \sum_k (-1)^k f_k x^k, \qquad f_k \geq 0$$

truncated at K leads to the polynomial p_K verifying

$$|f(x) - p_K(x)| \leq \kappa(K) = |f_{K+1} x^{K+1}|.$$

Approximation Curve

$$\kappa(K) = \frac{(2\pi)^{2K+2}}{(2K+2)!}\left(1 + \frac{2\pi}{2K+3}\right).$$



Important Remark: $\kappa(K)$ does not depend on the graph.

Laplacian-Based Translation Operator

Laplacian matrix $L = D - A = \chi\Lambda\chi^*$, with $\Lambda = \mathrm{diag}(\lambda_0, \ldots, \lambda_{N-1})$

Graph frequencies $\pi\sqrt{\lambda_l/\rho_{\mathscr{G}}} \in [0, \pi]$, Fourier modes χ_l.

Definition (Laplacian-Based Isometric Translation Operator)

$$T_{\mathscr{G}} = \exp\left(-\imath\pi\sqrt{\frac{L}{\rho_{\mathscr{G}}}}\right)$$

With $M = L/\rho_{\mathscr{G}}$, we obtain $T_{\mathscr{G}} = f(M)$ and:

$$f(x) = \exp\left(-\imath\pi\sqrt{x}\right) = \cos\left(\pi\sqrt{x}\right) - \imath\sin\left(\pi\sqrt{x}\right), \qquad x \in [0, 1].$$

Remark: Complexity due to the square root.

Dealing with the Square Root

Preliminaries: Rewrite f to obtain polynomial expansions:

$$f(x) = \cos\left(\pi\sqrt{x}\right) - \imath\sqrt{x}\frac{\sin\left(\pi\sqrt{x}\right)}{\sqrt{x}}$$

(cos(x): only even degree coeff. / sin(x): only odd degree)

⇒ We are left with approximating $\sqrt{x}$.

Idea: Use the Taylor expansion of $\sqrt{1+y}$ about 0 on $[-\epsilon, \epsilon]$ and rescale $\sqrt{x}$ to approximate it on $[\varrho, 1]$ (spectral gap of M)with $\epsilon \leq 1$ depending on the spectral gap.

Dealing with the Square Root (cont'd)

Approximation bound for $\sqrt{x}$:

Lemma (Non-Alternating Series Approximation (Lemma 1))

Taylor's expansion $f(x) = \sum_k f_k x^k$ *leads to*

$$\kappa(K) \leq \frac{1}{(K+1)!} \max\left\{|f^{(K+1)}|\right\} \max\left\{|x|^{K+1}\right\}.$$

Approximation bound for the product $\sqrt{x}\frac{\sin(\pi\sqrt{x})}{\sqrt{x}}$:

Lemma (Functional Composition Approximation (Lemma 3))

If $f(x) = g(x)h(x)$ *then:*

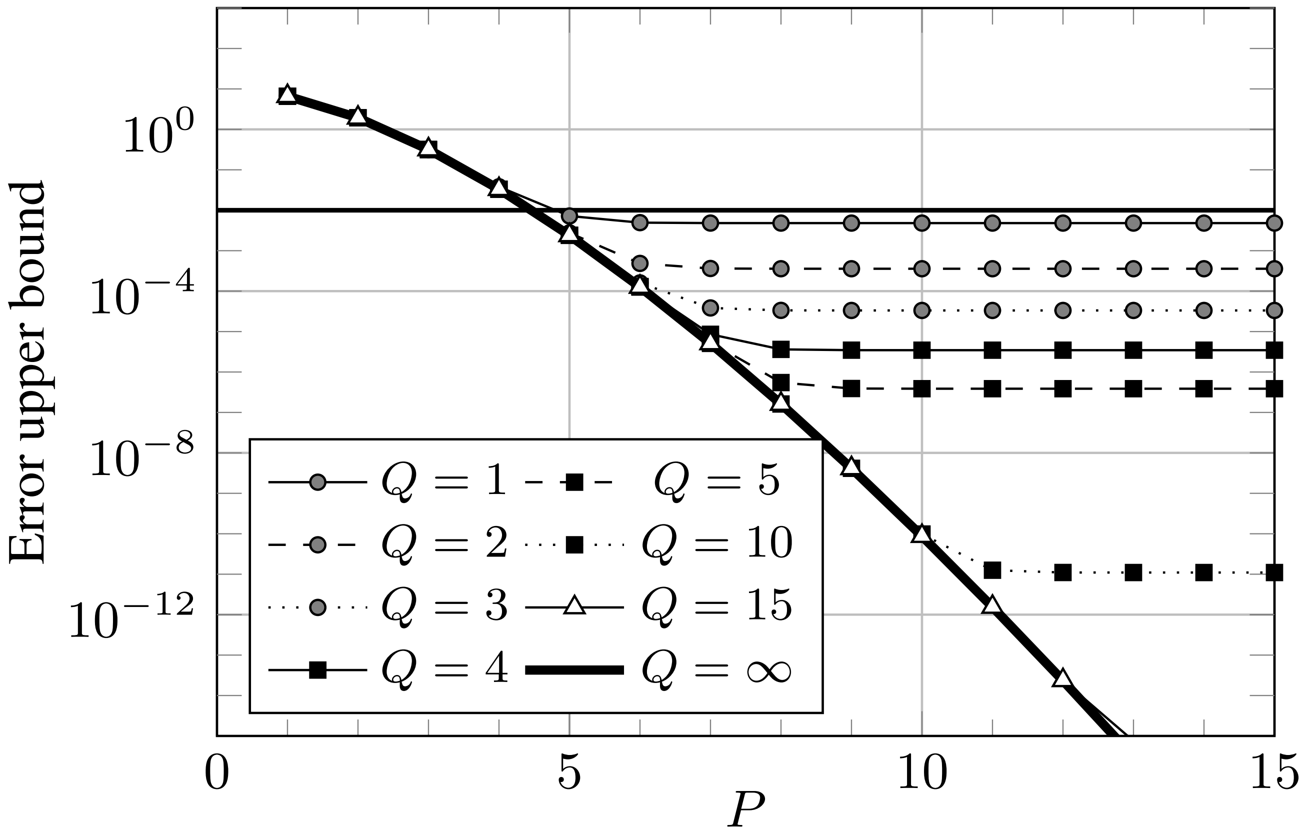
$$\kappa_f(P,Q) \leq \kappa_g(Q) \max|h| + \kappa_h(P)(\max|g| + \kappa_g(Q)).$$

Note: The resulting polynomial in Lemma 3 is of degree $P+Q$.

Approximation curve of the Graph Translation

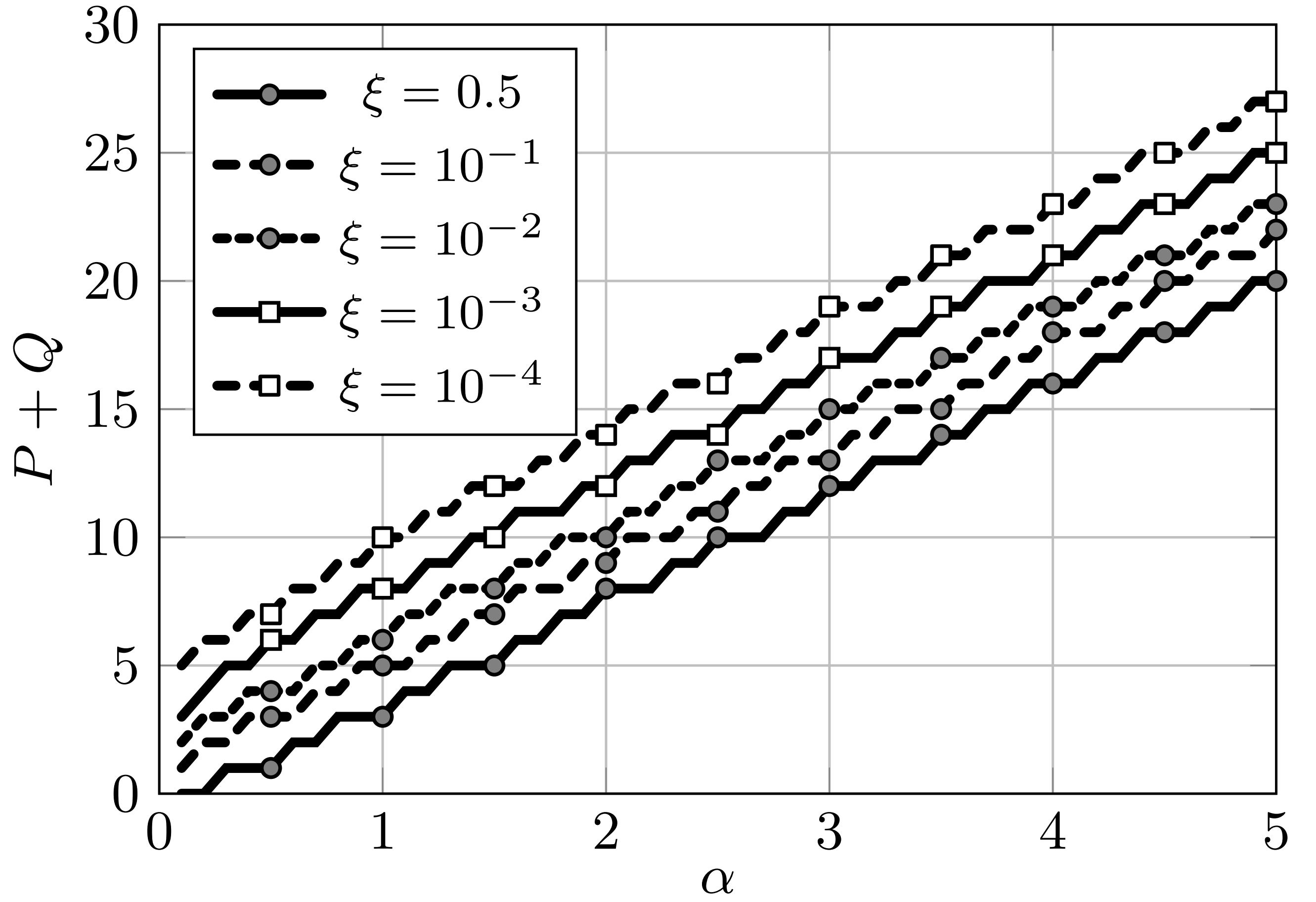
Application to $T_{\mathscr{G}}\delta_i = \cos(\pi\sqrt{M})\delta_i + \imath\sqrt{M}\frac{\sin(\pi\sqrt{M})}{\sqrt{M}}\left(\delta_i - \frac{1}{\sqrt{N}}\mathbf{1}\right)$

$$\kappa_{T_{\mathscr{G}}}(P,Q) = \kappa_C(P) + \kappa_S(P) + \kappa_R(Q)\Big(1+\kappa_S(P)\Big)$$



Important Remark: $\kappa_{T_{\mathscr{G}}}(P,Q)$ depends on the spectral gap of the graph.

Optimal $P + Q$ for a given error ξ to $T_{\mathcal{G}}^{\alpha}$



Summary and Perspectives

Summary

- **Graph Translation:** Approximately a diffusion operator
- **Tools developed:** Simple Lemmas to get polynomial approximations bounds on operators and composed operators

Perspectives

- **Loose bound:** use the weights to better characterize the bound
- **Link with the generalized translation:** $T_{\mathcal{G}}\delta_i = T_i t_{\mathcal{G}}$
- **Use this** diffusion characterization to interpret **stationary graph signals**