States Reduction on Markov Processes

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Abstract

A Markov process is a stochastic process that satisfies the Markov property, in which the future is independent of the past given the present. We first consider a Markov process over the real line with values on a finite set, where the law is defined by exponentially distributed jumps and a transition measure according to which the location of the process at the jump time is chosen; or indistinctly by the generator matrix. We also study Piecewise deterministic Markov processes, a more complex process that consists on two sub-processes: one on a continuous-space and the other on a discrete-space, and together are a Markov process involving a deterministic motion punctuated by random jumps. In the case when there are multiple weakly irreducible classes and the generator matrix can be rewritten as a double scales generator for a small parameter ϵ, we present a method to approximate the process to a two-scales process: a slow-process on a reduced state space and fast-process inside each new class, and we prove an approximation error of the laws of order ϵ. We present simulation examples and an application to the sodium channel in the Hodgkin and Huxley model, where we separate the voltage-gates of type h and m into two different time scales.

Contents

Chapter 1

Markov process in discrete space and continuous time

A Markov process is a stochastic process that satisfies the Markov property, in which the future is independent of the past given the present. We consider a Markov process $(X_t)_{t\geq 0}$ defined over the real line with values on a finite set, where the law is defined by exponentially distributed jumps with rate λ and a transition measure Π according to which the location of the process at the jump time is chosen; or indistinctly by the generator matrix Q.
In the case when there are multiple weakly irreducible classes and the generator can be rewritten as a double scale generator for a small parameter ϵ, we present a method to approximate the process to a two-scales process: a slow-process on a reduced state space and fast-process inside each new class.

1.1 Definition Markov process

Let consider a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_t, \mathbb{P})$ and let $(X_t)_{t\geq 0}$ be a Markov process with values on $I = \{1, \dots, N\}$ for some $N \in \mathbb{N}$. Constructing the Markov process as in [1], we define $(T_n)_{n\in\mathbb{N}}$ the series of jumps time of X_t such that $T_0 = 0$ and

$$T_n = \inf\{t \geq T_{n-1} | X_t \neq X_{t-}\}$$

for all $n \geq 1$; then $(X_t)_{t\geq 0}$ is defined by:

$$\begin{cases} 0 = T_0 < T_1 < T_2 < \cdots < \infty \quad \text{as } \lim_{n\to\infty} T_n = \infty \\ X_t = \sum_{n\geq 0} X_{T_n} \mathbb{1}_{\{T_n < t \leq T_{n+1}\}} \end{cases}$$

and initial condition $i \in I$. The law of the process $(X_t)_t$ is defined by the its *local characteristics* (λ, Π), where

- $\lambda : I \to \mathbb{R}_+$ is the jump rate,

- $\Pi : I \times I \to [0,1]$ is a Markov transition kernel such that $\Pi(i, \cdot) \in \mathcal{P}(I)$, a probability measure in I for all $i \in I$,

then, the process follows

$$\mathbb{P}\left[T_{n+1} - T_n > t \middle| X_{T_n} = i\right] = \mathrm{e}^{-t\lambda(i)}, \text{ for all } n \geq 0$$

and

$$\mathbb{P}\left[X_{T_{n+1}} = j \middle| X_{T_n} = i\right] = \Pi(i,j), \text{ for all } n \geq 0$$

Theorem 1.1.1 (Markov property) *The process $(X_t)_t$ is a Markov process; i.e. for all $n \geq 0$ and $0 \leq s_1 < \ldots < s_n < t$ and $s \geq 0$ and $i_1, \ldots, i_n, i, j$ in I,*

$$\mathbb{P}\left[X_{t+s} = j \middle| X_{s_1} = i_1, \ldots, X_{s_n} = i_n, X_t = i\right] = \mathbb{P}\left[X_{t+s} = j \middle| X_s = i\right]$$

And it is homogeneous in time; i.e. for all $t, s \geq 0$

$$\mathbb{P}\left[X_{t+s} = j \middle| X_s = i\right] = \mathbb{P}\left[X_t = j \middle| X_0 = i\right]$$

The proof can be found in [3] (Theorem 31). In order to better understand the process, we present the following operator:

Definition The semigroup P_t of the process $(X_t)_t$ is an operator defined by

$$P_t f(i) = \mathbb{E}_i\left[f\left(X_t\right)\right]$$

for all bounded function $f : I \to \mathbb{R}$ and where $\mathbb{E}_i\left[f(X_t)\right]$ stands for the expectation value of process $f(X_t)$ with starting condition $X_0 = i$.

Definition The infinitesimal generator L of the process $(X_t)_t$ is an operator defined by

$$Lf(i) = \lim_{h \to 0} \frac{P_h f(i) - f(i)}{h}$$

for all bounded function $f : I \to \mathbb{R}$.

Theorem 1.1.2 (Markov Process characterization) *The law of the Markov process $(X_t)_t$ with states in I is either characterized by*

A. the local characteristics (λ, Π),

B. the infinitesimal generator L, which is identified by the generator matrix $Q = (Q_{ij})_{i,j \in I}$ such that

$$Lf(i) = \sum_{j \in I} Q_{ij} f(j)$$

and it satisfies

$$\begin{aligned} &Q_{ij} \geq 0, \text{ for all } i, j \in I \text{ and } i \neq j \\ &Q_{ii} = -\sum_{i \neq j} Q_{ij}, \text{ for all } i \in I \end{aligned}$$

The generator and the local characteristics *are related according to*

$$Q_{ij} = -\lambda(i)\mathbb{1}_{\{i=j\}} + \lambda(i)\Pi(i,j)\mathbb{1}_{\{i\neq j\}}$$

for all $i, j \in I$.

Proof (*B.* $\Rightarrow$ *A.*) As presented in [1] (Theorem 5.3), let consider the process $\left(X^{\delta}_{k\delta}\right)_k$, equal to $(X_t)_t$ over a discretized grid-mesh of size $\delta > 0$ and T_1 the first jump-time of X_t, then it follows that

$$\{T_1 > \lceil t/\delta\rceil\,\delta\} \subset \left\{X^{\delta}_0 = X^{\delta}_{\delta} = \cdots = X^{\delta}_{\lceil t/\delta\rceil\delta} = x\right\} \subset \{T_1 > \lceil t/\delta\rceil\,\delta\} \cup \{T_2 - T_1 < \delta\}$$

where $\{T_2 - T_1 < \delta\}$ control the event of jumps between the mesh-grid, and as $\lim\limits_{\delta\to 0} \mathbb{P}_i\left(T_2 - T_1 < \delta\right) = 0$ by right continuity, we have

$$\mathbb{P}_i\left(T_1 > t\right) = \lim_{\delta\to 0} \mathbb{P}\left[X^{\delta}_0 = X^{\delta}_{\delta} = \cdots = X^{\delta}_{\lceil t/\delta\rceil\delta} = i\right]$$

by the Markov property, it holds that

$$\begin{aligned}
&\mathbb{P}\left[X^{\delta}_0 = X^{\delta}_{\delta} = \cdots = X^{\delta}_{\lceil t/\delta\rceil\delta} = i\right]\\
&= \mathbb{P}\left[X^{\delta}_{\lceil t/\delta\rceil\delta} = i | X^{\delta}_0 = \cdots = X^{\delta}_{(\lceil t/\delta\rceil-1)} = i\right]\mathbb{P}\left[X^{\delta}_0 = \cdots = X^{\delta}_{(\lceil t/\delta\rceil-1)\delta} = i\right]\\
&\vdots\\
&= \prod_{k=1}^{\lceil t/\delta\rceil} \mathbb{P}\left[X^{\delta}_{k\delta} = i | X^{\delta}_0 = \cdots = X^{\delta}_{(k-1)\delta} = i\right] = \prod_{k=1}^{\lceil t/\delta\rceil} \mathbb{P}\left[X^{\delta}_{k\delta} = i | X^{\delta}_{(k-1)\delta} = i\right]\\
&= \mathbb{P}\left[X^{\delta}_{\delta} = i | X^{\delta}_0 = i\right]^{\lceil t/\delta\rceil} = \mathrm{e}^{\left(\lceil t/\delta\rceil \log\left(\mathbb{P}\left[X^{\delta}_{\delta} = i | X^{\delta}_0 = i\right]\right)\right.}
\end{aligned}$$

and as around $x \approx 1$ we have $\log(x) \approx x - 1$, it follows that

$$\begin{aligned}
\lim_{\delta\to 0} \frac{\log\left(\mathbb{P}\left[X^{\delta}_{\delta} = i | X^{\delta}_0 = i\right]\right)}{\delta} &= \lim_{\delta\to 0} \frac{\mathbb{P}\left[X^{\delta}_{\delta} = i | X^{\delta}_0 = i\right] - 1}{\delta}\\
&= \lim_{\delta\to 0} \frac{\mathbb{E}_i\left[\mathbb{1}_{\{i\}}(X^{\delta}_{\delta})\right] - \mathbb{1}_{\{i\}}(i)}{\delta}\\
&= L\mathbb{1}_{\{i\}}(i) = Q_{ii}
\end{aligned}$$

and we conclude

$$\mathbb{P}_i\left[T_1 > t\right] = \lim_{\delta\to 0} \mathrm{e}^{\left(\lceil t/\delta\rceil \log\left(\mathbb{P}\left[X^{\delta}_{\delta} = i | X^{\delta}_0 = i\right]\right)\right.} = \mathrm{e}^{tQ_{ii}}$$

We deduce that $\lambda(i) := -Q_{ii}$ corresponds to the jump rate. On the other hand, for all $j \neq i$,

$$\begin{aligned}
&\bigcup_{k=1}^{\lceil t/\delta\rceil} \left\{X^{\delta}_0 = X^{\delta}_{\delta} = \cdots = X^{\delta}_{(k-1)\delta} = i, X^{\delta}_{k\delta} = j\right\}\\
&\subset \{T_2 - T_1 < \delta\} \cup \left\{T_1 \leq \lceil t/\delta\rceil\,\delta,\ X_0 = i,\ X^{\delta}_{T_1} = j\right\}\\
&\subset \{T_2 - T_1 < \delta\} \cup \bigcup_{k=1}^{\lceil t/\delta\rceil} \left\{X^{\delta}_0 = X^{\delta}_{\delta} = \cdots = X^{\delta}_{(k-1)\delta} = i, X^{\delta}_{k\delta} = j\right\}
\end{aligned}$$

and hence the limit

$$\mathbb{P}_i\left[T_1 \le t,\, X_{T_1} = i\right] = \lim_{\delta\to 0}\sum_{k=1}^{\lceil t/\delta\rceil} \mathbb{P}\left[X_0^\delta = X_\delta^\delta = \cdots = X_{(k-1)\delta}^\delta = i, X_{k\delta}^\delta = j\right]$$

by the Markov property, it holds that

$$\begin{aligned}
&\sum_{k=1}^{\lceil t/\delta\rceil} \mathbb{P}\left[X_0^\delta = X_\delta^\delta = \cdots = X_{(k-1)\delta}^\delta = i, X_{k\delta}^\delta = j\right] \\
&= \sum_{k=1}^{\lceil t/\delta\rceil} \mathbb{P}\left[X_0^\delta = X_\delta^\delta = \cdots = X_{(k-1)\delta}^\delta = i\right] \mathbb{P}\left[X_{k\delta}^\delta = j | X_0^\delta = X_\delta^\delta = \cdots = X_{(k-1)\delta}^\delta = i\right] \\
&= \sum_{k=1}^{\lceil t/\delta\rceil} \mathbb{P}_i\left[T_1 > (k-1)\delta\right] \frac{\mathbb{P}\left[X_\delta^\delta = j | X_0^\delta = i\right]}{\delta}\delta
\end{aligned}$$

we observe that

$$\begin{aligned}
\lim_{\delta\to 0}\frac{\mathbb{P}\left[X_\delta^\delta = j | X_0^\delta = i\right]}{\delta} &= \lim_{\delta\to 0}\frac{\mathbb{E}_i\left[\mathbb{1}_{\{j\}}(X_\delta^\delta)\right] - \mathbb{1}_{\{j\}}(i)}{\delta} \\
&= L\mathbb{1}_{\{j\}}(i) = Q_{ij}
\end{aligned}$$

and we conclude

$$\begin{aligned}
\mathbb{P}_i\left[T_1 \le t, X_{T_1} = j\right] &= \lim_{\delta\to 0}\sum_{k=1}^{\lceil t/\delta\rceil} \mathbb{P}_i\left[T_1 > (k-1)\delta\right] \frac{\mathbb{P}\left[X_\delta^\delta = j | X_0^\delta = i\right]}{\delta}\delta \\
&= \int_0^t \mathrm{e}^{Q_{ii}s} Q_{ij}\, ds \\
&= -\mathbb{P}_i\left[T_1 \le t\right]\frac{Q_{ij}}{Q_{ii}}
\end{aligned}$$

and it follows that

$$\mathbb{P}_i\left[X_{T_1} = j | \, T_1 \le t\right] = \frac{\mathbb{P}_i\left[T_1 \le t, X_{T_1} = j\right]}{\mathbb{P}_i\left[T_1 \le t\right]} = -\frac{Q_{ij}}{Q_{ii}}$$

We conclude that $\mathbb{P}_i\left[X_{T_1} = j\right] = -Q_{ij}/Q_{ii}$; then $-Q_{i\cdot}/Q_{ii}$ is a probability measure in $I \setminus \{i\}$ for all $i \in I$, and immediately we have that

$$\begin{aligned}
&Q_{ij} \ge 0, \text{ for all } i, j \in I \text{ and } i \ne j \\
&Q_{ii} = -\sum_{i\ne j} Q_{ij}, \text{ for all } i \in I
\end{aligned}$$

We deduce that the Markov kernel Π of the process is equal to

$$\Pi(i,j) := \begin{cases} -\dfrac{Q_{ij}}{Q_{ii}}\mathbb{1}_{\{Q_{ii}\ne 0\}} & \text{if } i \ne j \\ \mathbb{1}_{\{Q_{ii}=0\}} & \text{if } j = i \end{cases}$$

($A. \Rightarrow B.$) As shown in [3] (Theorem 31), when $h \downarrow 0$

$$\begin{aligned}\mathbb{P}_i\left[X_h = i\right] &\geq \mathbb{P}_i\left[T_1 \geq h\right]\\ &= \mathrm{e}^{-\lambda(i)h}\\ &= 1-\lambda(i)h+o(h)\end{aligned}$$

and for all $j \neq i$

$$\begin{aligned}\mathbb{P}_i\left[X_h = j\right] &\geq \mathbb{P}_i\left[T_1 \leq h, X_{T_1} = y, T_2 - T_1 \geq h\right]\\ &= \left(1-\mathrm{e}^{-\lambda(i)h}\right)\Pi(i,j)\mathrm{e}^{-\lambda(j)h}\\ &= \lambda(i)\Pi(i,j)h+o(h)\end{aligned}$$

We observe that as the sum over y on the left side has to be 1, last inequalities have to be equalities. To prove this let suppose that one of the quantities $\left(\mathbb{P}_i\left[X_h = i\right] - 1 + \lambda(i)h\right)/h$ or $\left(\mathbb{P}_i\left[X_h = j\right] - \lambda(i)\Pi(i,j)h\right)/h$ does not tend to 0. Let suppose it is the second one, then there exists $\delta > 0$ and a sequence $(h_k)_{k\geq 0}$ such that $h_k \downarrow 0$ and $\left|\mathbb{P}_i\left[X_{h_k} = j\right] - \lambda(i)\Pi(i,j)h_k\right|/h_k > \delta$; it follows that exists K such that for all $k \geq K$,

$$\frac{\mathbb{P}_i\left[X_{h_k} = j\right] - \lambda(i)\Pi(i,j)h_k}{h_k} > \delta$$

Then, as $\sum_{j\in I}\mathbb{P}_i\left[X_h = j\right] = 1$ and $\sum_{j\neq i}\lambda(i)\Pi(i,j) - \lambda(i) = 0$, for all $k > K$ it holds that

$$0 = \frac{\mathbb{P}_i\left[X_{h_k} = i\right] - 1 + \lambda(i)h_k}{h_k} + \sum_{j\neq i}\frac{\mathbb{P}_i\left[X_{h_k} = j\right] - \lambda(i)\Pi(i,j)h_k}{h_k} > o(1) + (N-1)\delta$$

where $o(1) \downarrow 0$ when $k \uparrow \infty$; and we arrive to the contradiction $\delta < 0$. We conclude that

$$\begin{aligned}\mathbb{P}_i\left[X_h = i\right] &= 1-\lambda(i)h+o(h)\\ \mathbb{P}_i\left[X_h = j\right] &= \lambda(i)\Pi(i,j)h+o(h),\ \text{for all } j\neq i\end{aligned}$$

Now, for all bounded f and $h \downarrow 0$, we have

$$\begin{aligned}\mathbb{E}_i\left[f(X_h)\right] &= f(i)\mathbb{P}_i\left[X_h = i\right] + \sum_{j\in I\setminus\{i\}} f(j)\mathbb{P}_i\left[X_h = j\right]\\ &= f(i) - f(i)\lambda(i)h + h\sum_{j\in I\setminus\{i\}} f(j)\lambda(i)\Pi(i,j) + o(h)\end{aligned}$$

and in order to compute the infinitesimal generator we have that

$$\frac{\mathbb{E}_i\left[f(X_h)\right] - f(i)}{h} = -\lambda(i)f(i) + \sum_{j\in I\setminus\{i\}} f(j)\lambda(i)\Pi(i,j) + o(1)$$

taking the limit and $o(1) \downarrow 0$ when $h \downarrow 0$, it follows that

$$Lf(x) = \lim_{h\to 0}\frac{\mathbb{E}_x\left[f(X_h)\right] - f(i)}{h} = \sum_{j\in I} f(j)Q_{ij}$$

where $Q_{ij} = -\lambda(i)\mathbb{1}_{\{j=i\}} + \lambda(i)\Pi(i,j)\mathbb{1}_{\{j\neq i\}}$. ∎

Remark Theorem 1.1.2 allows us to use indistinctly the *local characteristics* or the matrix generator when referring to the characterization of a Markov process.

An important property of the generator is that it satisfies the *Dynkin formula*,

$$P_t f(i) = f(z) + \int_0^t P_s L f(i) ds \tag{1.1}$$

And if we define the distribution $p(\cdot, t)$ for all $t \geq 0$, such that

$$p(i,t) = \mathbb{P}_{p_0}\left[X_t = i\right]$$

for all $i \in I$ and initial distribution p_0, we can rewrite the expectation as the sum over the probability in equation (1.1), we have

$$\frac{d}{dt} \sum_{z \in I} f(z) p(z,t) = \sum_{z \in I} \sum_{j \in I} f(j) p(z,t) Q_{zj}$$

and setting $f(z) = \delta_i$, we derive the forward equation,

$$\begin{cases} \dfrac{d}{dt} p(i,t) & = & \displaystyle\sum_{j \in I} p(j,t) Q_{ji} \\ p(i,0) & = & p_0(i) \end{cases} \tag{1.2}$$

for all $i \in I$. Equation (1.2) can be solved directly, with a solution given by

$$p(\cdot, t) = p_0(\cdot) \exp\left(Qt\right)$$

in the sense that $p(\cdot,t) = (p(1,t), \ldots, p(N,t))$ and $\exp\left(tQ\right) = \displaystyle\sum_{n=0}^{\infty} \frac{(tQ)^n}{n!}$.

1.1.1 Simulation

In order to simulate a Markov process $(X_t)_{t \in [0,T]}$ with values in $I = \{1, \ldots, N\}$, initial distribution p_0 and *local characteristics* (λ, Π), the idea is:

- draw a sample i_0 from p_0 for the initial value,
- then iteratively, if the process is in state i,
    1. draw a sample from $\mathcal{E}(\lambda(i))$ exponential law for the duration at i,
    2. draw the next state $j \neq i$ from the law $\Pi(i, \cdot)$.

If $\lambda(i) = 0$ then the process is absorbed at i. To achieve this, we consider the following lemmas that allows us to draw samples of the distributions from the uniform distribution.

Lemma 1.1.3 *Let $\lambda > 0$ and $U \sim U[0,1]$, then the random variable $\xi = -\frac{1}{\lambda} \log\left(1 - U\right) \sim \mathcal{E}(\lambda)$.*

Proof

$$\begin{aligned}\mathbb{P}[\xi \leq t] &= \mathbb{P}\left[-\frac{1}{\lambda}\log(1-U) \leq t\right] \\ &= \mathbb{P}[U \leq 1-\exp(-\lambda t)] \\ &= 1-\exp(-\lambda t)\end{aligned}$$

Remark As $U \sim 1-U$, we have also that $\tilde{\xi} = -\frac{1}{\lambda}\log(U) \sim \mathcal{E}(\lambda)$

Lemma 1.1.4 *Let $U \sim U[0,1]$ and $i \in I$, and we consider the random variable $Y \in I$ defined by*

$$Y = \begin{cases} 1 & \text{if} \quad U \leq \Pi(i,1) \\ i & \text{if} \quad \sum_{w=1}^{i-1}\Pi(i,w) < U \leq \sum_{w=1}^{i}\Pi(i,w), \quad i = 2,\dots,N \end{cases}$$

Then, it follows that $Y \sim \Pi(i,\cdot)$.

Proof It holds immediately that $\mathbb{P}[Y = z] = \Pi(x,z)$, for all $z \in I$.

1.2 Multiple weakly irreducible classes

With the objective to perform a state reduction approximation for processes with multiple weakly irreducible classes, let consider a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_t, \mathbb{P})$ and let $(X^\epsilon)_{t\geq 0}$ be a Markov process with values on $I = \{1,\dots,N\}$ for some $N \in \mathbb{N}$, initial condition $x \in I$ and characterized by a generator $Q^\epsilon \in \mathbb{R}^{N\times N}$. A process is considered to have multiple weakly irreducible classes if its generator depends on a small parameter $\epsilon > 0$ and two generators $\tilde{Q}$ and $\hat{Q} \in \mathbb{R}^{N\times N}$ such that

$$Q^\epsilon = \frac{1}{\epsilon}\tilde{Q} + \hat{Q} \tag{1.3}$$

where $\tilde{Q}$ governs the rapidly changing part and $\hat{Q}$ describes the slowly changing components. We consider that $\tilde{Q}$ can be put into block-diagonal form,

$$\tilde{Q} = \begin{pmatrix} \tilde{Q}^1 & 0 & & 0 \\ 0 & \tilde{Q}^2 & & \\ & & \ddots & 0 \\ 0 & & 0 & \tilde{Q}^{\bar{N}} \end{pmatrix}$$

where each $\tilde{Q}^k \in \mathbb{R}^{m_k\times m_k}$ is a generator for some integer m_k, such that $\sum_{k=1}^{\bar{N}} m_k = N$, and it defines the class

$$\bar{s}_k = \{i_{k1},\dots,i_{km_k}\} \subset I$$

that denotes the states associated with $\tilde{Q}^k$. We define the new set

$$\bar{S} = \{\bar{s}_1,\dots,\bar{s}_{\bar{N}}\}$$

that will be space of the slow process; where we notice that $\bigcup_{k=1}^{\bar{N}} \bar{s}_k = I$.

Assumption 1.2.1 *For all $k = 1, \ldots, \bar{N}$, the generator $\tilde{Q}^k$ is weakly irreducible; that is the system of equations*

$$\left\{ \begin{array}{rcl} \sum_{j=1}^{m_k} \nu_j^k \tilde{Q}_{ji}^k & = & 0, \quad \textit{for } i = 1, \ldots, m_k \\ \sum_{i=1}^{m_k} \nu_i^k & = & 1 \end{array} \right.$$

has a unique non-negative solution.

We consider for all $t \geq 0$ the probability measure $p^\epsilon(\cdot, t) \in \mathcal{P}(E, \mathcal{E})$, such that

$$p^\epsilon(i, t) = \mathbb{P}_{p_0}\left[X_t^\epsilon = i\right] \tag{1.4}$$

for all $i \in I$ and initial probability $p(\cdot, 0) = p_0$. It holds that p^ϵ is solution of (1.2), the forward equation

$$\left\{ \begin{array}{rcl} \frac{d}{dt} p^\epsilon(i, t) & = & \sum_{j \in I} p^\epsilon(j, t) \left(\frac{1}{\epsilon} \tilde{Q}_{ji}^\epsilon + \hat{Q}_{ji}^\epsilon \right) \\ p^\epsilon(i, 0) & = & p_0(i) \end{array} \right. \tag{1.5}$$

for all $i \in I$. The slow and fast components are coupled through weak and strong interactions in the sense that the underlying Markov chain fluctuates rapidly within a single group $\bar{s}_k$ and jumps less frequently between groups $\bar{s}_k$ and $\bar{s}_p$ for $k \neq p$. The states in $\bar{s}_k$, $k = 1, \ldots, \bar{N}$, are not isolated or independent of each other; more precisely, if we consider the states in $\bar{s}_k$ as a single state, then these states are coupled through the matrix $\hat{Q}$, and transitions from $\bar{s}_k$ to $\bar{s}_p$, $k \neq p$ are possible. In fact $\hat{Q}$, together with the quasi-stationary distributions ν^k of $\tilde{Q}^k$, determines the transition rates among states in $\bar{s}_k$, for $k = 1, ..., \bar{N}$.

1.2.1 Asymptotic expansion

To start we present the following lemmas.

Lemma 1.2.2 *Consider the matrix differential equation*

$$\left\{ \begin{array}{rcl} \frac{d}{ds} P(s) & = & P(s) Q \\ P(0) & = & I \end{array} \right. \tag{1.6}$$

where $P(s) \in \mathbb{R}^{N \times N}$. Suppose $Q \in \mathbb{R}^{N \times N}$ is a generator of a (homogeneous or stationary) finite-state Markov chain and is weakly irreducible. Then $P(s) \to \bar{P}$ as $s \to \infty$ and

$$\left| \exp(Qs) - \bar{P} \right| \leq K \exp\left(-\tilde{k} s\right) \ \textit{for some } \tilde{k} > 0,$$

where $\bar{P} = \mathbb{1}_N (\nu_1, \ldots, \nu_N) \in \mathbb{R}^{N \times N}$ and $(\nu_1, \ldots, \nu_N)$ is the stationary distribution of the Markov process with generator Q.

Proof As it is presented on [2] (Lemma A.2 p.374), we first notice that the solution of (1.6) is $P(s) = \exp(Qs)$. By virtue of Theorem II.10.1 of [11], $\lim_{s\to\infty} P(s)$ exists and is equal to a constant matrix $\bar{P}$. Then we observe that

$$\lim_{s\to\infty} \exp(Qs) = \lim_{s\to\infty} P(s) = \bar{P}$$

and so $\lim\limits_{s\to\infty} \dfrac{d}{ds}\exp(Qs) = 0$ and by system (1.6),

$$0 = \lim_{s\to\infty} \frac{d}{ds}P(s) = \lim_{s\to\infty} P(s)Q = \bar{P}Q$$

For each $i = 1, \ldots, m$, denote the ith row of $\bar{P}$ by $\bar{P}_i$. The weak irreducibility of Q then implies that the system of equations

$$\bar{P}_i Q = 0, \quad \bar{P}_i \mathbb{1} = 1$$

has a unique solution. Since $\bar{P}$ is the limit of the transition matrix, $\bar{P}_i \geq 0$. As a result, $\bar{P}_i$ is the quasi-stationary distribution ν and $\bar{P}$ has identical rows with $\bar{P} = \mathbb{1}(\nu_1, \ldots, \nu_m)$.
Using the Jordan canonical form, there is a nonsingular matrix U such that

$$\exp(Qs) = U\mathrm{diag}\left(\exp(J_0 s), \exp(J_1 s), \ldots, \exp(J_q s)\right) U^{-1}$$

where $J_0, J_1, \ldots, J_q$ are the Jordan blocks satisfying that J_0 is a diagonal matrix having appropriate dimension (if λ_i is a simple eigenvalue of Q, it appears in the block J_0), and that $J_k \in R^{m_k \times m_k}$, $k = 1, \ldots, q$. Since $\lim_{s\to\infty} \exp(Qs)$ exists, all the nonzero eigenvalues λ_i, for $1 \leq i \leq m-1$, must have negative real parts. Moreover, in view of the weak irreducibility of Q, the eigenvalue zero is a simple eigenvalue (having multiplicity 1). Then it is easily seen that

$$\left|\exp(Qs) - \bar{P}\right| \leq K \exp(-\tilde{\kappa} s)$$

where $\tilde{\kappa} = (1/2)\max_{1\leq i\leq m1} \mathrm{Re}(\lambda_i)$. ∎

Lemma 1.2.3 (Gronwall's lemma) *If f is a positive locally bounded Borel function on $\mathbb{R}_+$ such that*

$$f(t) \leq a + b\int_0^t f(s)ds$$

for every t and two constants a and b, then $f(t) \leq a\exp(bt)$.

Proof We have

$$\begin{aligned} f(t) &\leq a + b\left(\int_0^t \left(a + b\int_0^s f(u)du\right) ds\right) \\ &= a + abt + b^2\int_0^t (t-u)f(u)du \leq a + abt + b^2 t\int_0^t f(u)du \end{aligned}$$

Proceeding inductively we get

$$f(t) \leq a + abt + \ldots + ab^n\frac{t^n}{n!} + \frac{b^{n+1}t^n}{n!}\int_0^t f(u)du$$

Since f is locally bounded, the last term on the right converges as n tend to infinity and the result follows. ∎

Now we present the most important theorem of the section, that presents a characterization for the probability measure of the process $(X_t^\epsilon)_t$, based on the results given in [2].

Theorem 1.2.4 (Asymptotic Expansion) *The probability measure $p^\epsilon(\cdot,t)$ (1.4) of the process $(X_t^\epsilon)_t$ can be expanded in the form:*

$$p^\epsilon(i,t) = \varphi(i,t) + \gamma\left(i, \frac{t}{\epsilon}\right) + e^\epsilon(i,t) \tag{1.7}$$

In this approach φ is set to be an approximation on the slow-scale t away from 0, γ approximate the fast-scale $\tau = t/\epsilon$ and e^ϵ corresponds to the error of the expansion. The functions φ, γ and e^ϵ are such that

- *$\varphi(i,t)$ is differentiable for all $t \in [0,T]$*
- *there is a $\kappa_0 > 0$ such that*
$$|\gamma(i,\tau)| \leq K \exp(-\kappa_0 \tau)$$
uniformly for all $i \in I$
- *and the following estimate holds*
$$\sup_{t\in[0,T]} |e^\epsilon(i,t)| \leq K\epsilon$$
uniformly for all $i \in I$

To proceed, we define an operator $\mathcal{L}^\epsilon$ by

$$\mathcal{L}^\epsilon f = \epsilon \frac{df}{dt} - f\left(\tilde{Q} + \epsilon \hat{Q}\right) \tag{1.8}$$

for any smooth row-vector-valued function f; then $\mathcal{L}^\epsilon f = 0$ iff it is a solution to the forward differential equation (1.2). We set that both φ and γ are solution to the forward equation, then they satisfy

$$\mathcal{L}^\epsilon \varphi(t) = 0 \text{ and } \mathcal{L}^\epsilon \gamma\left(\frac{t}{\epsilon}\right) = 0$$

that is,

$$\begin{aligned} \epsilon \frac{d}{dt}\varphi(i,t) &= \sum_{j\in I} \varphi(j,t)\left(\tilde{Q}_{ji} + \epsilon \hat{Q}_{ji}\right) \\ \epsilon \frac{d}{dt}\gamma\left(i, \frac{t}{\epsilon}\right) &= \sum_{j\in I} \gamma\left(y, \frac{t}{\epsilon}\right)\left(\tilde{Q}_{ji} + \epsilon \hat{Q}_{ji}\right) \end{aligned} \tag{1.9}$$

and we write a new time scale $\tau = t/\epsilon$ for the second equation

$$\frac{d}{d\tau}\gamma(i,\tau) = \sum_{j\in I} \gamma(j,\tau)\left(\tilde{Q}_{ji} + \epsilon \hat{Q}_{ji}\right)$$

And if we identify the terms over ϵ^0 and ϵ^1 in the time-scale t and setting that $\gamma(\tau)$ must not depends on ϵ, we have the following set of equations

$$\begin{cases} \dfrac{d}{dt}\varphi(i,t) = \sum_{j\in I} \varphi(j,t)\hat{Q}_{ji} \\ \sum_{j\in I} \varphi(j,t)\tilde{Q}_{ji} = 0 \end{cases}$$

and

$$\begin{cases} \dfrac{d}{d\tau}\gamma(i,\tau) = \sum_{j\in I} \gamma(j,\tau)\tilde{Q}_{ji} \end{cases}$$

Remark From the last term comes the approximation error of this expansion, and due the fact that $\hat{Q}$ is not weakly irreducible.

In order to match asymptotic expansion, we have necessarily at $t = 0$ that

$$p_0(i) = \varphi(i,0) + \gamma(i,0)$$

Sending $\epsilon \to 0$ in the asymptotic expansion (1.9), the fast-scale and error disappear and only remains the slow-scale, then as $\sum_{i\in I} p^\epsilon(i,t) = 1$ for all $\epsilon > 0$, we have

$$\lim_{\epsilon\to 0} \sum_{i\in I} p^\epsilon(i,t) = \sum_{i\in I} \varphi(i,t) = 1$$

Remark What it is presented here is a zero level asymptotic expansion for a generator that does not depend on time. On a different case as shown in [2], when $Q^\epsilon(t)$ depends on time we would do an n-level asymptotic expansion on the form

$$p^\epsilon(i,t) = \Phi_n(i,t) + \Gamma_n\left(\frac{t}{\epsilon}\right) + e_n^\epsilon(i,t) = \sum_{i=0}^{n} \epsilon^i \varphi_n(i,t) + \sum_{i=0}^{n} \epsilon^i \gamma_n\left(i, \frac{t}{\epsilon}\right) + e_n^\epsilon(i,t)$$

with a remainder of order ϵ^n. It would be necessary to add condition to $Q^\epsilon(t)$, such that it is n-times continuously differentiable on $[0, T]$ with each derivative Lipschitz continuous.

Determining φ

We need to determine $\varphi(i,t)$ for $i \in I$ and $t \in [0, T]$ such that

$$\begin{cases} \sum_{j\in I} \varphi(j,t)\tilde{Q}_{ji} = 0 \\ \dfrac{d}{dt}\varphi(i,t) = \sum_{j\in I} \varphi(j,t)\hat{Q}_{ji} \\ \sum_{i\in I} \varphi(i,t) = 1 \end{cases} \tag{1.10}$$

Since $\tilde{Q} = \operatorname{diag}\left(\tilde{Q}^1, \ldots, \tilde{Q}^{\bar{N}}\right)$ where each $\tilde{Q}^k$ is weakly irreducible, therefore if we consider $\varphi^k(i,t)$, the function $\varphi(i,t)$ restricted on $i \in \bar{s}_k$, we have that it satisfies

$$\sum_{j \in \bar{s}_k} \varphi^k(j,t) \tilde{Q}^k_{ji} = 0$$

which solution is $\varphi^k(i,t) = \theta(k,t)\nu^k_i$, the product of the invariant measure ν^k of $\tilde{Q}^k$ and a scalar multiplier $\theta(k,t)$, a function defined for $k = 1, \ldots, \bar{N}$ and $t \in [0,T]$. We observe that as ν^k is a distribution in $\bar{s}_k$, it holds that

$$\begin{aligned} \sum_{i \in \bar{s}_k} \varphi^k(i,t) &= \sum_{i \in \bar{s}_k} \theta(k,t)\nu^k_i \\ &= \theta(k,t) \end{aligned}$$

and by consequence of the third equation in (1.10), it also holds that $\sum_{k=1}^{\bar{N}} \theta(k,t) = 1$. Setting all this with the second equation in (1.10), it follows that

$$\begin{aligned} \frac{d}{dt}\theta(k,t) &= \sum_{i \in \bar{s}_k} \frac{d}{dt}\varphi(i,t) = \sum_{i \in \bar{s}_k} \sum_{j \in I} \varphi(j,t)\hat{Q}_{ij} \\ &= \sum_{i \in \bar{s}_k} \sum_{p=1}^{\bar{N}} \sum_{j \in \bar{s}_p} \theta(p,t)\nu^p_j \hat{Q}_{ji} \\ &= \sum_{p=1}^{\bar{N}} \theta(p,t) \left(\sum_{j \in \bar{s}_p} \sum_{x \in \bar{s}_k} \nu^p_y \hat{Q}_{ji} \right) \end{aligned} \tag{1.11}$$

where we observe the emergence of a new generator $\bar{Q} \in \mathbb{R}^{\bar{N} \times \bar{N}}$ such that

$$\bar{Q}_{pk} = \sum_{j \in \bar{s}_p} \sum_{i \in \bar{s}_k} \nu^p_j \hat{Q}_{ji}, \tag{1.12}$$

for all $p, k = 1, \ldots, \bar{N}$. To determine the initial condition $\theta(k,0)$, we first observe that in the asymptotic expansion it has to hold that

$$\sum_{i \in \bar{s}_k} \varphi(i,0) = \lim_{\delta \to 0} \lim_{\epsilon \to 0} \sum_{i \in \bar{s}_k} p^\epsilon(i,\delta) \tag{1.13}$$

moreover, in view of the forward equation (1.5) and that $\sum_{i \in \bar{s}_k} \tilde{Q}^k_{ji} = 0$, which comes from the property of generators, we have

$$\sum_{i \in \bar{s}_k} p^\epsilon(i,t) = \sum_{i \in \bar{s}_k} p_0(i) + \int_0^\delta \sum_{i \in \bar{s}_k} \sum_{j \in \bar{s}_k} p^\epsilon(i,s)\hat{Q}_{ji} ds$$

and since $p^\epsilon(i,t)$ is bounded it follows that

$$\lim_{\delta \to 0} \left(\limsup_{\epsilon \to 0} \int_0^\delta \sum_{i \in \bar{s}_k} \sum_{j \in \bar{s}_k} p^\epsilon(i,s)\hat{Q}_{ji} ds \right) = 0$$

therefore by (1.13) it yields

$$\sum_{i\in\bar{s}_k}\varphi(i,0)=\lim_{\delta\to 0}\left(\lim_{\epsilon\to 0}\sum_{i\in\bar{s}_k}p^{\epsilon}(i,\delta)\right)=\sum_{i\in\bar{s}_k}p_0(i)$$

and we finally have

$$\theta(k,0)=\sum_{x\in\bar{s}_k}p_0(i)$$

To conclude this section we present the following Corollary, that allows us to understand more clearly the solution to φ.

Corollary 1.2.5 *The system* (1.10) *for $\varphi(i,t)$, is equivalent to the system*

$$\begin{cases}\displaystyle\sum_{j\in\bar{s}_k}\varphi(j,t)\tilde{Q}_{ji}=0,\ \textit{for } i\in\bar{s}_k\\ \displaystyle\sum_{i\in\bar{s}_k}\varphi(i,t)=\theta(k,t)\\ \displaystyle\frac{d}{dt}\theta(k,t)=\sum_{p=1}^{\bar{N}}\theta(p,t)\bar{Q}_{pk}\\ \displaystyle\theta(k,0)=\sum_{i\in\bar{s}_k}p_0(i)\end{cases}$$

for $k=1,\dots,\bar{N}$; where $\bar{Q}$ is the generator defined in (1.12).

Remark In Corollary 1.2.5 appears the function $\theta(k,t)$, which can be interpreted as the probability of a new Markov process $(Z_t)_{t\geq 0}$ defined over the aggregate states space $\bar{S}=\{\bar{s}_1,\dots,\bar{s}_{\bar{N}}\}$ with infinitesimal generator $\bar{Q}$.

Determining γ

We consider $\gamma(i,\tau)$, for all $i\in I$ and $t\in[0,T]$, solution to

$$\begin{cases}\displaystyle\frac{d}{d\tau}\gamma(i,\tau)=\sum_{j\in I}\gamma(j,\tau)\tilde{Q}_{ji}\end{cases}\tag{1.14}$$

To match the asymptotic expansion, we have at $t=0$ that

$$p_0(i)=\varphi(i,0)+\gamma(i,0)$$

and as $\tilde{Q}$ is constant, we can solve (1.14) directly and together with the above initial condition, we obtain

$$\gamma(\cdot,\tau)=(p_0(\cdot)-\varphi(\cdot,0))\exp\left(\tilde{Q}\tau\right)\tag{1.15}$$

Considering that for each $k = 1, \ldots, \bar{N}$, $\tilde{Q}^k$ is weakly irreducible, we need to prove that $\gamma(i, \tau)$ can be obtain by equation (1.15), and there is a positive number κ_0 such that

$$|\gamma(i, \tau)| \leq K \exp\left(-\kappa_0 \tau\right)$$

uniformly for $i \in I$. To prove this, let ν^k be the stationary distribution corresponding to the generator $\tilde{Q}^k$. We define the column vector $\mathbb{1}_m = (1, 1, \ldots, 1)' \in \mathbb{R}^{m \times 1}$ and the matrix

$$\pi \quad = \quad \begin{pmatrix} \mathbb{1}_{m_1}\nu^1 & 0 & & 0 \\ 0 & \mathbb{1}_{m_2}\nu^2 & & \\ & & \ddots & 0 \\ 0 & & 0 & \mathbb{1}_{m_{\bar{N}}}\nu^{\bar{N}} \end{pmatrix}$$

where

$$\mathbb{1}_{m_k}\nu^k = \begin{pmatrix} \nu_1^k & \cdots & \nu_{m_k}^k \\ & \vdots & \\ \nu_1^k & \cdots & \nu_{m_k}^k \end{pmatrix}$$

Noting the block-diagonal structure of $\tilde{Q}$, we have

$$\exp\left(\tilde{Q}\tau\right) = \begin{pmatrix} \exp\left(\tilde{Q}^1\tau\right) & 0 & & 0 \\ 0 & \exp\left(\tilde{Q}^2\tau\right) & & \\ & & \ddots & 0 \\ 0 & & 0 & \exp\left(\tilde{Q}^{\bar{N}}\tau\right) \end{pmatrix}$$

Furthermore, we see that for $k = 1, \ldots, \bar{N}$ it holds

$$\sum_{i \in \bar{s}_k} \left(p_0(i) - \varphi(i, 0)\right) = \sum_{i \in \bar{s}_k} p_0(i) - \sum_{i \in \bar{s}_k} \varphi(i, 0) = \sum_{i \in \bar{s}_k} p_0(i) - \theta(k, 0) = 0$$

we conclude that the initial condition $(p_0(\cdot) - \varphi(\cdot, 0))$ is orthogonal to π, and by virtue of Lemma 1.2.2, for each $k = 1, \ldots, \bar{N}$ there exists $\kappa_k > 0$ such that

$$\left|\exp\left(\tilde{Q}^k\tau\right) - \mathbb{1}_{m_k}\nu^k\right| \leq K \exp\left(-\kappa_k \tau\right)$$

then we have

$$\begin{aligned} |\gamma(\cdot, \tau)| &= \left|\left(p_0(\cdot) - \varphi(\cdot, 0)\right)\left(\exp\left(\tilde{Q}\tau\right) - \pi\right)\right| \\ &\leq K \sup_{k \leq \bar{N}} \left|\exp\left(\tilde{Q}^k\tau\right) - \mathbb{1}_{m_k}\nu^k\right| \\ &\leq K \exp\left(-\kappa_0 \tau\right) \end{aligned}$$

where $\kappa_0 = \min_{k \leq \bar{N}} \kappa_k$.

Analysis of remainder

The remainder of the asymptotic expansion (1.7) corresponds to

$$e^{\epsilon}(i,t) = p^{\epsilon}(i,t) - \varphi(i,t) - \gamma\left(i, \frac{t}{\epsilon}\right)$$

where $e^{\epsilon}(0) = 0$, and if we consider the operator $\mathcal{L}^{\epsilon}$ as in (1.8),

$$\mathcal{L}^{\epsilon} f = \epsilon \frac{df}{dt} - f\left(\tilde{Q} + \epsilon \hat{Q}\right)$$

it holds that $\mathcal{L}^{\epsilon} p^{\epsilon}(t) = 0$ and then

$$\mathcal{L}^{\epsilon} e^{\epsilon}(i,t) = -\epsilon\left(\frac{d}{dt}\varphi(i,t) + \frac{d}{dt}\gamma\left(i, \frac{t}{\epsilon}\right)\right) + \sum_{j\in I}\left(\varphi(j,t) + \gamma\left(j, \frac{t}{\epsilon}\right)\right)\left(\tilde{Q}_{ji} + \epsilon \hat{Q}_{ji}\right)$$

and from equations (1.10) and (1.14), we have

$$\mathcal{L}^{\epsilon} e^{\epsilon}(t) = \epsilon\gamma\left(i, \frac{t}{\epsilon}\right)\hat{Q}$$

expanding the operator and as $Q^{\epsilon} = \tilde{Q}/\epsilon + \hat{Q}$, we have

$$\frac{d}{dt} e^{\epsilon}(x,t) = \sum_{j\in I} e^{\epsilon}(j,t) Q^{\epsilon}_{ji} + \sum_{j\in I}\gamma\left(j, \frac{t}{\epsilon}\right)\hat{Q}_{ji}$$

taking the norm, integrating and making use of the exponential decay property of γ, it holds that

$$\begin{aligned}
|e^{\epsilon}(i,t)| &\leq |e^{\epsilon}(i,0)| + \left|\int_0^t \sum_{j\in I} e^{\epsilon}(j,t) Q^{\epsilon}_{ji} ds\right| + \left|\int_0^t \sum_{j\in I}\gamma\left(j, \frac{t}{\epsilon}\right)\hat{Q}_{ji} ds\right| \\
&\leq C\int_0^t |e^{\epsilon}(i,s)|\, ds + K\int_0^t \exp\left(-\kappa_0 \frac{s}{\epsilon}\right) ds \\
&\leq C\int_0^t |e^{\epsilon}(i,s)|\, ds + K\frac{\epsilon}{\kappa_0}\left(1 - \exp\left(-\kappa_0\frac{t}{\epsilon}\right)\right) \\
&\leq C\int_0^t |e^{\epsilon}(i,s)|\, ds + K\epsilon
\end{aligned}$$

and by Gronwall's lemma (1.2.3), we conclude

$$|e^{\epsilon}(i,t)| \leq K\epsilon \exp(Ct)$$

and taking the supreme on time, the remainder satisfies

$$\sup_{t\in[0,T]}\left|p^{\epsilon}(i,t) - \varphi(i,t) - \gamma\left(i, \frac{t}{\epsilon}\right)\right| = K\epsilon$$

where K is a positive constant that depends on T.

1.2.2 Two-scales approximation

We now present a corollary that allows us, under the conditions already discussed, to represent the Markov process as a two-scales process: one on a slow-scale that goes over the state classes and a fast-scale that acknowledge the dynamic inside each class.

Corollary 1.2.6 *Let $(X^\epsilon)_{t\in[0,T]}$ be a Markov process over $I = \{1, \ldots, N\}$, initial distribution p_0 and with a two-scales generator $Q^\epsilon \in \mathbb{R}^{N\times N}$ that depends on $\epsilon > 0$ and two generators $\tilde{Q}$ and $\hat{Q}$ such that*

$$Q^\epsilon = \frac{1}{\epsilon}\tilde{Q} + \hat{Q}$$

where $\tilde{Q} = diag(\tilde{Q}^1, \ldots, \tilde{Q}^{\bar{N}})$ with each sub-generator $\tilde{Q}^k \in \mathbb{R}^{m_k\times m_k}$ weakly irreducible and determines the class $\bar{s}_k \subset I$. Then there exists positive constants K_T, K and κ_0 such that

$$\left|\mathbb{P}\left(X_t^\epsilon = i_{kx}\right) - \nu_x^k \theta(k,t)\right| \leq K_T\,\epsilon + K\exp\left(-\kappa_0 \frac{t}{\epsilon}\right)$$

for all $i_{kx} \in \bar{s}_k$, for $x = 1, \ldots, m_k$ and $k = 1, \ldots, \bar{N}$. Here ν^k is the stationary distribution for class $\bar{s}_k$ given by $\tilde{Q}^k$; and $\theta(k,t)$ is a function over $k = 1, \ldots, \bar{N}$ and $t \in [0,T]$, that satisfies

$$\begin{cases} \dfrac{d}{dt}\theta(k,t) &= \displaystyle\sum_{p=1}^{\bar{N}} \theta(p,t)\bar{Q}_{pk} \\ \theta(k,0) &= \displaystyle\sum_{i\in\bar{s}_k} p_0(i) \end{cases}$$

for the generator $\bar{Q} \in \mathbb{R}^{\bar{N}\times\bar{N}}$ defined by

$$\bar{Q}_{pk} = \sum_{j\in\bar{s}_p}\sum_{i\in\bar{s}_k} \nu_j^p \hat{Q}_{ji}$$

for $p, k = 1\ldots, \bar{N}$.

In this corollary, we can interpret $\theta(k,t)$ as the probability distribution of a Markov process $(Z_t)_{t\in[0,T]}$ defined over $\bar{S} = \{\bar{s}_1, \ldots, \bar{s}_{\bar{N}}\}$ with generator $\bar{Q}$, which holds the dynamics on the slow-scale. We can think of this process as

$$Z_t = \bar{s}_i \Longleftrightarrow X_t \in \bar{s}_i$$

and that $\theta(k,t) = \mathbb{P}\left(Z_t = \bar{s}_k\right)$ for $i = 1, \ldots, \bar{N}$. On the other hand, the dynamic in the fast-scale will be defined punctually on the position of Z_t by and random variable X_{fast} such that

$$\mathbb{P}\left[X_{\text{fast}} = i_{kx} | \, Z_t = \bar{s}_k\right] = \nu_x^k$$

for each $i_{kx} \in \bar{s}_k$. We notice that its value depends on the slow process for each time when we want to evaluate it.

Example 1

Consider a two machine flow-shop with machines that are subject to breakdown and repair. The production capacity of the machines is described by a finite-state Markov chain. If the machine is up, then it can produce parts with production rate $u(t)$; its production rate is zero if the machine is under repair. For simplicity, suppose each of the machines is either in operating condition (denoted by 1) or under repair (denoted by 0). Then the capacity of the workshop becomes a four-state Markov chain $(X_t)_{t\in[0,T]}$ with state space $I = \{(0,0),(1,0),(0,1),(1,1)\}$ and initial distribution p_0. Suppose that the first machine breaks down much more often than the second one. To reflect this situation, consider that $(X_t)_t$ is generated by Q^ϵ as (1.3) for a small $\epsilon > 0$, with $\tilde{Q}$ and $\hat{Q}$ given by

$$\tilde{Q} = \begin{pmatrix} -\lambda_1 & \lambda_1 & 0 & 0 \\ \mu_1 & -\mu_1 & 0 & 0 \\ 0 & 0 & -\lambda_1 & \lambda_1 \\ 0 & 0 & \mu_1 & -\mu_1 \end{pmatrix}, \quad \hat{Q} = \begin{pmatrix} -\lambda_2 & 0 & \lambda_2 & 0 \\ 0 & -\lambda_2 & 0 & \lambda_2 \\ \mu_2 & 0 & -\mu_2 & 0 \\ 0 & \mu_2 & 0 & -\mu_2 \end{pmatrix}$$

where λ_i and μ_i are the rates of repair and breakdown, respectively. We consider the probability

$$p^\epsilon(i,t) = \mathbb{P}_{p_0}(X_t = i)$$

that denote the probability distribution of the underlying chain at time t and it is solution of (1.2), the forward equation

$$\begin{cases} \dfrac{d}{dt}p^\epsilon(\cdot,t) = p^\epsilon(\cdot,t)\left(\dfrac{1}{\epsilon}\tilde{Q} + \hat{Q}\right) \\ p^\epsilon(\cdot,0) = p_0 \end{cases}$$

which can be solved directly; in particular for $\lambda_1 = \lambda_2 = \mu_1 = \mu_2 = 1$ and $p_0 = (1,0,0,0)$, the solution is

$$\begin{aligned} p^\epsilon(1,t) &= 1/4\,\mathrm{e}^{-2\,\frac{t}{\epsilon}} + 1/4\,\mathrm{e}^{-2\,t} + 1/4\,\mathrm{e}^{-2\,\frac{t(1+\epsilon)}{\epsilon}} + 1/4 \\ p^\epsilon(2,t) &= -1/4\,\mathrm{e}^{-2\,\frac{t}{\epsilon}} + 1/4\,\mathrm{e}^{-2\,t} - 1/4\,\mathrm{e}^{-2\,\frac{t(1+\epsilon)}{\epsilon}} + 1/4 \\ p^\epsilon(3,t) &= 1/4\,\mathrm{e}^{-2\,\frac{t}{\epsilon}} - 1/4\,\mathrm{e}^{-2\,t} - 1/4\,\mathrm{e}^{-2\,\frac{t(1+\epsilon)}{\epsilon}} + 1/4 \\ p^\epsilon(4,t) &= -1/4\,\mathrm{e}^{-2\,\frac{t}{\epsilon}} - 1/4\,\mathrm{e}^{-2\,t} + 1/4\,\mathrm{e}^{-2\,\frac{t(1+\epsilon)}{\epsilon}} + 1/4 \end{aligned}$$

The matrices $\tilde{Q}$ and $\hat{Q}$ are themselves generators of Markov chains. Note we that

$$\tilde{Q} = \mathrm{diag}\left(\begin{pmatrix} -\lambda_1 & \lambda_1 \\ \mu_1 & -\mu_1 \end{pmatrix}, \begin{pmatrix} -\lambda_1 & \lambda_1 \\ \mu_1 & -\mu_1 \end{pmatrix}\right)$$

with both generator weakly irreducible, with invariant distribution $\nu = \left(\frac{\mu_1}{\mu_1+\lambda_1}, \frac{\lambda_1}{\mu_1+\lambda_1}\right)$. From generator $\tilde{Q}$ we identify the classes

$$\begin{aligned} \bar{s}_0 = \{(0,0),(1,0)\} = \{i_{01}, i_{02}\} \\ \bar{s}_1 = \{(0,1),(1,1)\} = \{i_{11}, i_{12}\} \end{aligned} \tag{1.16}$$

that form the slow-scale states $\bar{S} = \{\bar{s}_0, \bar{s}_1\}$. The slow-scale generator is then given by $\bar{Q} = \mathrm{diag}\,(\nu,\nu)\,\hat{Q}\mathrm{diag}\,((1,1),(1,1))$, then

$$\bar{Q} \;=\; \begin{pmatrix} -\lambda_2 & \lambda_2 \\ \mu_2 & -\mu_2 \end{pmatrix}$$

By Corollary 1.2.6, we can approximate the law of process $(X_t)_t$ to

$$\mathbb{P}_{p_0}\left[X_t = i_{kx}\right] \approx \nu_x \mathbb{P}\left[Z_t = \bar{s}_k\right]$$

where $(Z_t)_{t\in[0,T]}$ is a Markov process on the slow-scale with states in $\bar{S}$ and generator $\bar{Q}$. We consider the probability $\theta(k,t) = \mathbb{P}\left[Z_t = \bar{s}_k\right]$, and it is solution of the forward equation

$$\begin{cases} \dfrac{d}{dt}\theta(\cdot,t) = \theta(\cdot,t)\bar{Q} \\ \theta(\cdot,0) = (p_1(0)+p_2(0), p_3(0)+p_4(0)) \end{cases}$$

which can be solved directly; in particular for $\lambda_2 = \mu_2 = 1$ the solution is

$$\begin{aligned} \theta(\bar{s}_1,t) &= 1/2 + (1/2)\mathrm{e}^{-2t} \\ \theta(\bar{s}_2,t) &= 1/2 - (1/2)\mathrm{e}^{-2t} \end{aligned}$$

Finally, the approximation, with $\nu = (1/2, 1/2)$, is equivalent to

$$\begin{aligned} p(1,t),\; p(2,t) &\approx 1/4 + 1/4\,\mathrm{e}^{-2\,t} \\ p(3,t),\; p(4,t) &\approx 1/4 - 1/4\,\mathrm{e}^{-2\,t} \end{aligned}$$

In all four cases, the reminder is given by

$$\begin{aligned} |e^{\epsilon}(t)| &= 1/4\,\mathrm{e}^{-2\,\frac{t}{\epsilon}} + 1/4\,\mathrm{e}^{-2\,\frac{t(1+\epsilon)}{\epsilon}} \\ &\leq 1/4\,\mathrm{e}^{-2t/\epsilon}\left(1 + \frac{1}{1+2t}\right) \\ &\leq \epsilon/4\;(2t\epsilon)\,\mathrm{e}^{-2\frac{t}{\epsilon}} + 1/2\,\mathrm{e}^{-2\frac{t}{\epsilon}} \\ &\leq \epsilon/4 + 1/2\,\mathrm{e}^{-2\frac{t}{\epsilon}} \end{aligned}$$

Example 2

Let define the Markov process $(X_t)_{t\in[0,T]}$ with state space $I = \{1,\ldots,9\}$, initial distribution p_0 and generated by Q^ϵ as in (1.3) for a small $\epsilon > 0$, with $\tilde{Q}$ and $\hat{Q}$ given by

$$\tilde{Q} = \mathrm{diag}\left(\begin{pmatrix} -\lambda_1 & \lambda_1 & 0 \\ \lambda_1 & -(\lambda_1+\mu_1) & \mu_1 \\ 0 & \mu_1 & -\mu_1 \end{pmatrix}, \begin{pmatrix} -\lambda_1 & \lambda_1 & 0 \\ \lambda_1 & -(\lambda_1+\mu_1) & \mu_1 \\ 0 & \mu_1 & -\mu_1 \end{pmatrix}, \begin{pmatrix} -\lambda_1 & \lambda_1 & 0 \\ \lambda_1 & -(\lambda_1+\mu_1) & \mu_1 \\ 0 & \mu_1 & -\mu_1 \end{pmatrix}\right)$$

with each sub-matrix weakly irreducible that solve for $\nu = (1/3, 1/3, 1/3)$, and

$$\hat{Q} = \begin{pmatrix} -\lambda_2 & 0 & 0 & \lambda_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\lambda_2 & 0 & 0 & \lambda_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\lambda_2 & 0 & 0 & \lambda_2 & 0 & 0 & 0 \\ \lambda_2 & 0 & 0 & -(\lambda_2+\mu_2) & 0 & 0 & \mu_2 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 & -(\lambda_2+\mu_2) & 0 & 0 & \mu_2 & 0 \\ 0 & 0 & \lambda_2 & 0 & 0 & -(\lambda_2+\mu_2) & 0 & 0 & \mu_2 \\ 0 & 0 & 0 & \mu_2 & 0 & 0 & -\mu_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu_2 & 0 & 0 & -\mu_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu_2 & 0 & 0 & -\mu_2 \end{pmatrix}$$

We consider the probability function $p^\epsilon(x,t) = \mathbb{P}_{p_0}(X_t = x)$ for $x \in I$, that denote the probability distribution of the underlying chain at time t, and it is solution of (1.2), the forward equation

$$\begin{cases} \dfrac{d}{dt}p^\epsilon(\cdot,t) = p^\epsilon(\cdot,t)\left(\dfrac{1}{\epsilon}\tilde{Q} + \hat{Q}\right) \\ p(\cdot,0) = p_0 \end{cases} \tag{1.17}$$

which can be solved directly; in particular for $\lambda_1 = \lambda_2 = \mu_1 = \mu_2 = 1$ and $p_0 = (1,0,0,0,0,0,0,0,0)$, the solution is equal to

$$\begin{aligned}
\tfrac{d}{dt}p^\epsilon(1,t) &= 1/18\,\mathrm{e}^{-3\,t} + 1/18\,\mathrm{e}^{-3\,\frac{t}{\epsilon}} + 1/36\,\mathrm{e}^{-3\,\frac{t(1+\epsilon)}{\epsilon}} + 1/6\,\mathrm{e}^{-t} + 1/12\,\mathrm{e}^{-\frac{t(\epsilon+3)}{\epsilon}} \\
&\quad + 1/6\,\mathrm{e}^{-\frac{t}{\epsilon}} + 1/12\,\mathrm{e}^{-\frac{t(3\,\epsilon+1)}{\epsilon}} + 1/4\,\mathrm{e}^{-\frac{t(1+\epsilon)}{\epsilon}} + 1/9, \\
\tfrac{d}{dt}p^\epsilon(2,t) &= 1/18\,\mathrm{e}^{-3\,t} - 1/9\,\mathrm{e}^{-3\,\frac{t}{\epsilon}} - 1/18\,\mathrm{e}^{-3\,\frac{t(1+\epsilon)}{\epsilon}} + 1/6\,\mathrm{e}^{-t} - 1/6\,\mathrm{e}^{-\frac{t(\epsilon+3)}{\epsilon}} + 1/9, \\
\tfrac{d}{dt}p^\epsilon(3,t) &= 1/18\,\mathrm{e}^{-3\,t} + 1/18\,\mathrm{e}^{-3\,\frac{t}{\epsilon}} + 1/36\,\mathrm{e}^{-3\,\frac{t(1+\epsilon)}{\epsilon}} + 1/6\,\mathrm{e}^{-t} + 1/12\,\mathrm{e}^{-\frac{t(\epsilon+3)}{\epsilon}} \\
&\quad - 1/6\,\mathrm{e}^{-\frac{t}{\epsilon}} - 1/12\,\mathrm{e}^{-\frac{t(3\,\epsilon+1)}{\epsilon}} - 1/4\,\mathrm{e}^{-\frac{t(1+\epsilon)}{\epsilon}} + 1/9, \\
\tfrac{d}{dt}p^\epsilon(4,t) &= -1/9\,\mathrm{e}^{-3\,t} - 1/18\,\mathrm{e}^{-3\,\frac{t(1+\epsilon)}{\epsilon}} + 1/18\,\mathrm{e}^{-3\,\frac{t}{\epsilon}} + 1/6\,\mathrm{e}^{-\frac{t}{\epsilon}} - 1/6\,\mathrm{e}^{-\frac{t(3\,\epsilon+1)}{\epsilon}} + 1/9, \\
\tfrac{d}{dt}p^\epsilon(5,t) &= -1/9\,\mathrm{e}^{-3\,t} + 1/9\,\mathrm{e}^{-3\,\frac{t(1+\epsilon)}{\epsilon}} - 1/9\,\mathrm{e}^{-3\,\frac{t}{\epsilon}} + 1/9, \\
\tfrac{d}{dt}p^\epsilon(6,t) &= -1/9\,\mathrm{e}^{-3\,t} - 1/18\,\mathrm{e}^{-3\,\frac{t(1+\epsilon)}{\epsilon}} + 1/18\,\mathrm{e}^{-3\,\frac{t}{\epsilon}} - 1/6\,\mathrm{e}^{-\frac{t}{\epsilon}} + 1/6\,\mathrm{e}^{-\frac{t(3\,\epsilon+1)}{\epsilon}} + 1/9, \\
\tfrac{d}{dt}p^\epsilon(7,t) &= 1/18\,\mathrm{e}^{-3\,t} + 1/36\,\mathrm{e}^{-3\,\frac{t(1+\epsilon)}{\epsilon}} + 1/18\,\mathrm{e}^{-3\,\frac{t}{\epsilon}} - 1/6\,\mathrm{e}^{-t} - 1/4\,\mathrm{e}^{-\frac{t(1+\epsilon)}{\epsilon}} \\
&\quad + 1/6\,\mathrm{e}^{-\frac{t}{\epsilon}} - 1/12\,\mathrm{e}^{-\frac{t(\epsilon+3)}{\epsilon}} + 1/12\,\mathrm{e}^{-\frac{t(3\,\epsilon+1)}{\epsilon}} + 1/9, \\
\tfrac{d}{dt}p^\epsilon(8,t) &= 1/18\,\mathrm{e}^{-3\,t} - 1/18\,\mathrm{e}^{-3\,\frac{t(1+\epsilon)}{\epsilon}} - 1/9\,\mathrm{e}^{-3\,\frac{t}{\epsilon}} - 1/6\,\mathrm{e}^{-t} + 1/6\,\mathrm{e}^{-\frac{t(\epsilon+3)}{\epsilon}} + 1/9, \\
\tfrac{d}{dt}p^\epsilon(9,t) &= 1/18\,\mathrm{e}^{-3\,t} + 1/36\,\mathrm{e}^{-3\,\frac{t(1+\epsilon)}{\epsilon}} + 1/18\,\mathrm{e}^{-3\,\frac{t}{\epsilon}} - 1/6\,\mathrm{e}^{-t} + 1/4\,\mathrm{e}^{-\frac{t(1+\epsilon)}{\epsilon}} \\
&\quad - 1/6\,\mathrm{e}^{-\frac{t}{\epsilon}} - 1/12\,\mathrm{e}^{-\frac{t(\epsilon+3)}{\epsilon}} - 1/12\,\mathrm{e}^{-\frac{t(3\,\epsilon+1)}{\epsilon}} + 1/9
\end{aligned}$$

From generator $\tilde{Q}$, we identify the classes

$$\begin{aligned} \bar{s}_1 &= \{1,2,3\} = \{i_{11}, i_{12}, i_{12}\} \\ \bar{s}_2 &= \{4,5,6\} = \{i_{21}, i_{22}, i_{23}\} \\ \bar{s}_3 &= \{7,8,9\} = \{i_{31}, i_{32}, i_{33}\} \end{aligned} \tag{1.18}$$

that form the slow-scale states $\bar{S} = \{\bar{s}_1, \bar{s}_2, \bar{s}_3\}$. By Corollary 1.2.6, we can approximate the law of process $(X_t)_t$ to

$$\mathbb{P}(X_t = i_{kx}) \approx \nu_x^k \mathbb{P}(Z_t = \bar{s}_k) = \frac{1}{3}\mathbb{P}(Z_t = \bar{s}_k) \tag{1.19}$$

for all $i_{kx} \in I$, where $(Z_t)_{t\in[0,T]}$ is a Markov process on the slow-scale with generator $\bar{Q} = \mathrm{diag}(\nu,\nu)\,\hat{Q}\mathrm{diag}((1,1),(1,1))$, then

$$\bar{Q} = \begin{pmatrix} -\lambda_2 & \lambda_2 & 0 \\ \lambda_2 & -(\lambda_2+\mu_2) & \mu_2 \\ 0 & \mu_2 & -\mu_2 \end{pmatrix}$$

We consider the probability $\theta(k,t) = \mathbb{P}(Z_t = \bar{s}_k)$, which is solution the forward equation

$$\begin{cases} \dfrac{d}{dt}\theta(\cdot,t) = \theta(\cdot,t)\bar{Q} \\ \theta(\cdot,0) = (p^\epsilon(1,0)+p^\epsilon(2,0)+p^\epsilon(3,0), p^\epsilon(4,0)+p^\epsilon(5,0)+p^\epsilon(6,0), p^\epsilon(7,0)+p^\epsilon(8,0)+p^\epsilon(9,0)) \end{cases}$$

which can be solved directly; in particular for $\lambda_2 = \mu_2 = 1$ the solution is

$$\begin{aligned} \theta(\bar{s}_1,t) &= 1/2\,\mathrm{e}^{-t} + 1/6\,\mathrm{e}^{-3t} + 1/3 \\ \theta(\bar{s}_2,t) &= -1/3\,\mathrm{e}^{-3t} + 1/3 \\ \theta(\bar{s}_3,t) &= 1/6\,\mathrm{e}^{-3t} - 1/2\,\mathrm{e}^{-t} + 1/3 \end{aligned}$$

Finally, the approximation is equivalent to

$$\begin{aligned} p^\epsilon(1,t), p^\epsilon(2,t), p^\epsilon(3,t) &\approx 1/6\,\mathrm{e}^{-t} + 1/18\,\mathrm{e}^{-3t} + 1/9 \\ p^\epsilon(4,t), p^\epsilon(5,t), p^\epsilon(6,t) &\approx -1/9\,\mathrm{e}^{-3t} + 1/9 \\ p^\epsilon(7,t), p^\epsilon(8,t), p^\epsilon(9,t) &\approx 1/18\,\mathrm{e}^{-3t} - 1/6\,\mathrm{e}^{-t} + 1/9 \end{aligned}$$

Example 3 - Monte Carlo Method

In this section, we will compare the exact distribution function of the process $(X_t)_{t\in[0,T]}$, defined in Example 2, to a Monte Carlo approximation of its law; and also, to a Monte Carlo approximation of the law of the two-scales process derived on equation (1.19).
For the simulation of the process $(X_t)_{t\in[0,T]}$ in $I = \{1,\ldots,9\}$, we follow the steps on Section 1.1.1. We set a simulation time $T \geq 0$ and a time step Δt and we define the number of steps $N = \lceil T/\Delta t \rceil$. We consider the jump rate $\lambda(x)$ and the Markov kernel $\Pi \in \mathbb{R}^{9\times 9}$ as defined in Theorem 1.1.2; in particular for $\lambda_1 = \lambda_2 = \mu_1 = \mu_2 = 1$ and $\epsilon = 0.2$, we have

$$\lambda = \begin{pmatrix} 6 \\ 11 \\ 6 \\ 7 \\ 12 \\ 7 \\ 6 \\ 11 \\ 6 \end{pmatrix}, \quad \Pi = \begin{pmatrix} 0 & 5/6 & 0 & 1/6 & 0 & 0 & 0 & 0 & 0 \\ 5/11 & 0 & 5/11 & 0 & 1/11 & 0 & 0 & 0 & 0 \\ 0 & 5/6 & 0 & 0 & 0 & 1/6 & 0 & 0 & 0 \\ 1/7 & 0 & 0 & 0 & 5/7 & 0 & 1/7 & 0 & 0 \\ 0 & 1/12 & 0 & 5/12 & 0 & 5/12 & 0 & 1/12 & 0 \\ 0 & 0 & 1/7 & 0 & 5/7 & 0 & 0 & 0 & 1/7 \\ 0 & 0 & 0 & 1/6 & 0 & 0 & 0 & 5/6 & 0 \\ 0 & 0 & 0 & 0 & 1/11 & 0 & 5/11 & 0 & 5/11 \\ 0 & 0 & 0 & 0 & 0 & 1/6 & 0 & 5/6 & 0 \end{pmatrix}$$

We set an array $(X_k)_{k=1}^{N+1}$ for the process and a variable S for the time of jumps. The pseudo-code goes as follow:

1. For the initial condition we set $X_1 = 1$, $k = 2$ and $S = 0$.
2. While $k \leq N+1$ and $S \leq T$ do:
   (a) With $U_1 \sim U(0,1)$ set $S = S - \frac{1}{\lambda(X_{k-1})} \log(U_1)$
   (b) While $k \leq N$ and $k\Delta t \leq S$ do:
      i. $X_k = X_{k-1}$
      ii. $k = k+1$
   (c) With $U_2 \sim U(0,1)$
      i. if $U_2 \leq \Pi(X_{k-1}, 1)$ then $X_k = 1$
      ii. if $\sum_{w=1}^{j-1} \Pi(X_{k-1}, w) < U_2 \leq \sum_{w=1}^{j} \Pi(X_{k-1}, w)$ for $j = 2, \ldots, 9$, then $X_k = j$
   (d) $k = k+1$

For the process $(Z_t)_{t\in[0,T]}$ over $\bar{S} = \{\bar{s}_1, \bar{s}_2, \bar{s}_3\}$, we consider its jump rate $\tilde{\lambda}(k)$ and the Markov kernel $\tilde{\Pi} \in \mathbb{R}^{3\times 3}$; in particular for $\lambda_2 = \mu_2 = 1$, we have

$$\tilde{\lambda} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \quad \tilde{\Pi} = \begin{pmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \end{pmatrix}$$

And for the simulation, we consider the array $(Z_k)_{k=1}^{N+1}$t and the same pseudo-code as before.

In order to approximate the law p^ϵ of $(X_t)_{t\in[0,T]}$ via the Monte Carlo method, we take into account M realizations of $(X_k^m)_{k=1}^{N+1}$ for $m = 1, \ldots, M$, then we have that

$$p_{MC}(i,T) = \frac{1}{M} \sum_{m=1}^{M} \mathbb{1}_{\{X_N^m = i\}}, \quad i = 1, \ldots, 9$$

with standard deviation $\sqrt{\frac{p_{MC}(x,T)(1-p_{MC}(x,T))}{M}}$. And similarly, for the two-scales approximation (1.19), we set M realizations of $(Z_k^m)_{k=1}^{N+1}$ for $m = 1, \ldots, M$, and then it follows

$$p_{MC}^*(i,T) = \frac{1}{3}\theta_{MC}(k,T) = \frac{1}{3M} \sum_{m=1}^{M} \mathbb{1}_{\{Z_N^m = \bar{s}_k\}}, \quad i = 1, \ldots, 9$$

for k such that $x \in \bar{s}_k$, and standard deviation $\frac{1}{3}\sqrt{\frac{p_{MC}^*(x,T)(1-p_{MC}^*(x,T))}{M}}$.

Solving for $\lambda_1 = \lambda_2 = \mu_1 = \mu_2 = 1$, $p_0 = (1,0,0,0,0,0,0,0,0)$, $\epsilon = 0.2$ and $M = 1000$, the results are the followings:

$T=1$	p^ϵ	p_{MC}	std	p^*_{MC}	std
1	0.1769	0.177	0.0120694	0.172	0.0039779
2	0.1751	0.179	0.0121227	0.172	0.0039779
3	0.1734	0.167	0.0117945	0.172	0.0039779
4	0.1066	0.097	0.0093590	0.105	0.0032314
5	0.1056	0.115	0.0100884	0.105	0.0032314
6	0.1046	0.095	0.0092723	0.105	0.0032314
7	0.05307	0.054	0.0071473	0.0563	0.0024304
8	0.05256	0.057	0.0073315	0.0563	0.0024304
9	0.05203	0.059	0.0074511	0.0563	0.0024304

Table 1.1: Comparatives result for the distribution function of X_t at $t=1$, between the exact solution p^ϵ, the Monte Carlo approx. p_{MC} and the Monte Carlo approx. of the two-scales process p^*_{MC}.

$T=2$	p^ϵ	p_{MC}	std	p^*_{MC}	std
1	0.1338	0.132	0.0107040	0.135	0.0036021
2	0.1338	0.145	0.0111344	0.135	0.0036021
3	0.1337	0.144	0.0111024	0.135	0.0036021
4	0.1108	0.113	0.0100115	0.1113	0.0033156
5	0.1108	0.103	0.0096120	0.1113	0.0033156
6	0.1108	0.103	0.0096120	0.1113	0.0033156
7	0.08869	0.094	0.0092284	0.087	0.0029708
8	0.08869	0.086	0.0088659	0.087	0.0029708
9	0.08868	0.08	0.0085790	0.087	0.0029708

Table 1.2: Comparatives result for the distribution function of X_t at $t=2$, between the exact solution p^ϵ, the Monte Carlo approx. p_{MC} and the Monte Carlo approx. of the two-scales process p^*_{MC}.

$T=5$	p^ϵ	p_{MC}	std	p^*_{MC}	std
1	0.1122	0.107	0.0097750	0.1126	0.0033329
2	0.1122	0.101	0.0095289	0.1126	0.0033329
3	0.1122	0.109	0.0097750	0.1126	0.0033329
4	0.1111	0.109	0.0098549	0.1086	0.0032806
5	0.1111	0.107	0.0097750	0.1086	0.0032806
6	0.1111	0.117	0.0101642	0.1086	0.0032806
7	0.1099	0.123	0.0104940	0.112	0.0033243
8	0.1099	0.108	0.0098151	0.112	0.0033243
9	0.1099	0.118	0.0102018	0.112	0.0033243

Table 1.3: Comparatives result for the distribution function of X_t at $t=5$, between the exact solution p^ϵ, the Monte Carlo approx. p_{MC} and the Monte Carlo approx. of the two-scales process p^*_{MC}.

Chapter 2

Piecewise Deterministic Markov Process

Piecewise deterministic Markov processes, or PDMP, are a family of càdlàg Markov processes involving a deterministic motion punctuated by random jumps. The motion of the PDMP $(Z_t)_{t\geq 0}$ depends on three local characteristics, namely the jump rate λ, the flow ϕ solution of an ordinary differential equation, and the transition kernel Π according to which the location of the process at the jump time is chosen. The process starts from z and follows the flow $\phi(z,t)$ until the first jump time T_1, then the location of the process at T_1 is selected by the transition measure $\Pi(\phi(x,T_1),\cdot)$ and the motion restarts from this new point as before.
In order to perform the state reduction approximation on PDMP, we consider a subclass called Markov switching model where the jumps occurs only on the discrete part of the process. In the case when there are multiple weakly irreducible classes and the generator can be written as a double scale generator for a small parameter ϵ, we present a method to approximate the process to a two-scale process: a slow-process on a reduced state space and fast-process inside each new class.

2.1 Ordinary differential equations and vector fields

First we study the deterministic part of the process. Let D be an open set of $\mathbb{R}^d$ and $\psi : D \to D$ be a Lipschitz continuous function, i.e. there exists a constant C_D such that $|\psi(x)-\psi(y)| \leq C_D|x-y|$ for all x, y in D. Then the differential equation

$$\begin{cases} \dfrac{d}{dt}x(t) &= \psi(x(t)) \\ x(0) &= x \in D \end{cases} \tag{2.1}$$

has a unique solution $\phi(x,t)$ determined for all $t \leq t_D$, where t_D is the time at which the solution exit from D, i.e.

$$t_D := \inf\{t \geq 0 | \phi(x,t) \in \partial D\}$$

and ∂D is the boundary of D. The unique solution ϕ has the following properties

1. The map $\phi(\cdot,t) : x \to \phi(x,t)$ is one-to-one and onto.
2. The family $(\phi(\cdot,t))_{t\geq 0}$ is a semi-group, i.e. for any $t,s \geq 0$ it holds that

$$\phi(x,t+s) = \phi(\phi(x,s),t) \text{ for all } x \in D$$

Remark It is possible that t_D fails to converge to ∞ as $D \uparrow \mathbb{R}^d$, in which case there is said to be an explosion.

Let $f : D \to \mathbb{R}$ be a C^1 function. Then with $x(t) = \phi(x,t)$,

$$\frac{d}{dt} f(x(t)) = \sum_{l=1}^{d} \frac{\partial f(x(t))}{\partial x_l} \psi_l(x(t))$$

where ψ_l is the lth component of ψ. Let denote F the first order differential operator

$$Ff(x) = \sum_{l=1}^{d} \frac{\partial}{\partial x_l} f(x) \psi_l(x)$$

Then $x(t)$ satisfies (2.1) if and only if it satisfies

$$\frac{d}{dt} f(x(t)) = Ff(x(t)), \text{ for all } f \in C^1(\mathbb{R}^d)$$

the operator F is the *vector field*.

Remark It is interesting to think ODEs as Markov Process. Assuming that F is locally Lipschitz and that there are no explosion, the trajectory of the process is $X_t = \phi(x,t)$ and the semi-group is $P_t f(x) = f(\phi(x,t))$ for $f \in C^1$. Then

$$P_t f(x) - f(x) = \int_0^t Ff(X_s) ds$$

and $P_t f(x) - f(x) - \int_0^t Ff(X_s)ds = 0$ and certainly a martingale. Thus F is the infinitesimal generator of the deterministic process.

2.2 Definition PDMP

We present a formal definition of a PDMP as shown by Davis (see [4], [5]). We start with the state space E defined as follows. Let consider the finite set $I = \{1, \dots, N\}$ for a $N \in \mathbb{N}$, and for every $i \in I$ let D_i be an open subset of $\mathbb{R}^d$, then

$$E = \bigcup_{i \in I} (\{i\} \times D_i) = \{(i,x) : i \in I, x \in D_i\} \tag{2.2}$$

Let $\mathcal{E}$ denotes the following class of measurable sets in E

$$\mathcal{E} = \left\{ \bigcup_{i \in I} A_i : A_i \in \mathcal{M}_i \right\}$$

where $\mathcal{M}_i$ denotes the Borel sets of D_i; then $(E, \mathcal{E})$ is a Borel space. Let consider a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F})_t, \mathbb{P})$ and we define

$$(Z_t)_{t\geq 0} = (I_t, X_t)_{t\geq 0}$$

a Piecewiese deterministic Markov process with states on E and initial condition $Z_0 = z = (i,x) \in E$. The process I_t corresponds to the discrete part and has states in I and X_t is the continuous part and has states in $\mathbb{R}^d$. The process $(Z_t)_t$ is completely defined by $((F_i)_{i\in I}, \lambda, \Pi)$ called the *local characteristic* of the PDMP, where

- F_i is a locally Lipschitz continuous vector field characterized by the Lipschitz function $\psi(i,x)$ for all $i \in I$, determining a flow $\phi(i,x,t)$ that set the motion of the PDMP between the jumps as shown in section 2.1. On a point of notation, any function $f : E \to \mathbb{R}$ is identified with its component functions $f_i : D_i \to \mathbb{R}$ and we write $Ff(i,x)$ instead of $F_i f(x)$ for all $(i,x) \in E$.
  We will suppose that if $t_\infty(x)$ denote the explosion time of the flow $\phi(i,x,\cdot)$ then we assume that $t_\infty(x) = \infty$ when $t_{D_i}(x) = \infty$, thus excluding explosions.

- $\lambda : E \to \mathbb{R}_+$ is the jump rate, a measurable function where for each $(i,x) \in E$ and all $t > 0$ it holds that
  $$\int_0^t \lambda(i, \phi(i,x,s))ds < \infty$$
  and we suppose that $\bar{\lambda} = \sup_{z\in E} \lambda(z)$ is bounded.

- $\Pi : E \times \mathcal{E} \to [0,1]$ is a Markov transition kernel that maps E into the set $\mathcal{P}(E)$ of probability measures on $(E, \mathcal{E})$, with properties that:
  - for each $A \in \mathcal{E}$ the map $z \to \Pi(z, A)$ is measurable and
  - $\forall z \in E, \Pi(z, \{z\}) = 0$

The dynamic of the process $(Z_t)_t$ with initial condition $(i,x) \in E$ is constructed as follows. Consider a random variable T_1 such that

$$\mathbb{P}_{(i,x)}[T_1 > t] = \begin{cases} \exp\left(-\int_0^t \lambda(i, \phi(i,x,s))\, ds\right) & \text{for } t < t_{D_i}(x) \\ 0 & \text{for } t \geq t_{D_i}(x) \end{cases}$$

If T_1 is equal to infinity then

$$Z_t = (i, \phi(i,x,t)) \text{ for all } t \geq 0$$

Otherwise if $T_1 < \infty$, we select independently a E-valued random variable θ_1 having distribution $\Pi((i, \phi(i,x,T_1)), \cdot)$, namely $\mathbb{P}_{(i,x)}[\theta_1 \in A] = \Pi((i, \phi(i,x,T_1)), A)$ for any $A \in \mathcal{E}$. The trajectory of $(Z_t)_t$ for $t \in [0, T_1]$, is given by

$$Z_t = \begin{cases} (i, \phi(i,x,t)) & \text{for } t < T_1 \\ \theta_1 & \text{for } t = T_1 \end{cases}$$

Starting from $X_{T_1} = \theta_1$, we now select the next inter-jump time $T_2 - T_1$ and post-jump location $X_{T_2} = \theta_2$ in a similar way.

Lemma 2.2.1 *The times between jumps of process $(Z_t)_t$, satisfy the lack of memory property; i.e. the random variable $T_{n+1} - T_n$ is such that*

$$\mathbb{P}_{(I_{T_n}, X_{T_n})}\left[T_{n+1} - T_n > t + u \,|\, T_n,\, T_{n+1} - T_n > u\right] = \mathbb{P}_{(I_{T_n}, \phi(I_{T_n}, X_{T_n}, u))}\left[T_{n+1} - T_n > t\right]$$

for all n.

Proof Let T_1 be the first time-jump of the process $(Z_t)_t$, then for $u > 0$ and $t + u < t_{D_i}(x)$ we have

$$\begin{aligned}
\mathbb{P}_{(i,x)}\left[T_1 > t + u | T_1 > u\right] &= \frac{\mathbb{P}_{(i,x)}\left[T_1 > t + u,\ T_1 > u\right]}{\mathbb{P}_{(i,x)}\left[T_1 > u\right]} = \frac{\mathbb{P}_{(i,x)}\left[T_1 > t + u\right]}{\mathbb{P}_{(i,x)}\left[T_1 > u\right]} \\
&= \exp\left(-\int_0^{t+u} \lambda\left(i, \phi(i, x, s)\right) ds\right) / \exp\left(-\int_0^{u} \lambda\left(i, \phi(i, x, s)\right) ds\right) \\
&= \exp\left(-\int_u^{t+u} \lambda\left(i, \phi(i, x, s)\right) ds\right) \\
&= \exp\left(-\int_0^{t} \lambda\left(i, \phi(i, x, s + u)\right) ds\right) \\
&= \exp\left(-\int_0^{t} \lambda\left(\phi(i, \phi(i, x, u), s)\right) ds\right) \\
&= \mathbb{P}_{(i, \phi(i,x,u))}\left[T_1 > t\right]
\end{aligned}$$

where we have used the semigroup property of ϕ. ∎

We now present that PDMP satisfies the strong and normal Markov property.

Theorem 2.2.2 (Markov property) *The process $(Z_t)_{t \geq 0}$ is a homogeneous strong Markov process; i.e. for any $z \in E$, a stopping time T and function f, it holds that*

$$\mathbb{E}_z\left[f(Z_{T+s}) \mathbb{1}_{\{T < \infty\}} | \mathcal{F}_T\right] = \mathbb{E}_{Z_T}\left[f(Z_s)\right] \mathbb{1}_{\{T < \infty\}}$$

where $\mathbb{E}_z\left[f(Z_t)\right]$ corresponds to the expectation value of the process $f(Z_t)$ with starting condition $Z_0 = z$.

The proof can be found in [5] (theorem 25.5).

Remark In particular, the process also satisfies the normal Markov property; i.e. for all $n \geq 0$ and $0 \leq s_1 < \ldots < s_n < t$ and $s \geq 0$ and $z_1, \ldots, z_n, z, w$ in I,

$$\mathbb{P}\left[Z_{t+s} = w |\, Z_{s_1} = z_1, \ldots, Z_{s_n} = z_n, Z_t = z\right] = \mathbb{P}\left[Z_s = w |\, Z_0 = z\right]$$

This comes form the fact that any $t > 0$ is a stopping time.

In order to better understand the process, we define the following operators.

Definition For all $t \geq 0$, the semigroup P_t of the process $(Z_t)_t$ is defined for all measurable function f such that

$$P_t f(z) = \mathbb{E}_z\left[f(Z_t)\right]$$

We continue with the definition of another operator closely related to the semigropup.

Definition The infinitesimal generator L of the process $(Z_t)_t$ is defined such that

$$Lf(z) = \lim_{h\to 0} \frac{P_h f(z) - f(z)}{h}$$

for all $f \in D(L)$, where $D(L)$ is the set of all measurable function $f : E \to \mathbb{R}^d$ such that $t \to f(i, \phi(i, x, t))$ is absolutely continuous on $\mathbb{R}_+$ for all $(i, x) \in E$.

Lemma 2.2.3 *The infinitesimal generator of the PDMP with* local characteristics $((F_i)_{i\in I}, \lambda, \Pi)$ *corresponds to the operator L such that*

$$Lf(z) = Ff(z) + \lambda(z) \int_E (f(v) - f(z))\, \Pi(z, dv) \tag{2.3}$$

defined for all $f \in D(L)$ and $z \in E$.

Proof To prove this result, we observe that when $h \downarrow 0$

$$\begin{aligned}
\mathbb{P}_{(i,x)}\left[Z_h = (i, \phi(i, x, h))\right] &\geq \mathbb{P}_{(i,x)}\left[T_1 \geq h\right] \\
&= \exp\left(-\int_0^h \lambda\left(i, \phi(i, x, s)\right) ds\right) \\
&= 1 - \int_0^h \lambda\left(i, \phi(i, x, s)\right) ds + o(h)
\end{aligned}$$

and that for $(i, x) \neq (j, y)$

$$\begin{aligned}
&\mathbb{P}_{(i,x)}\left[Z_h = (j, \phi(j, y, h))\right] \geq \mathbb{P}\left[T_1 \leq h, Z_{T_1} = (j, y), T_2 - T_1 \geq h\right] \\
&= \left(1 - \exp\left(-\int_0^h \lambda\left(i, \phi(i, x, s)\right) ds\right)\right) \Pi((i, x), \{(j, y)\}) \exp\left(-\int_0^h \lambda\left(j, \phi(j, y, s)\right) ds\right) \\
&= \left(\int_0^h \lambda\left(i, \phi(i, x, s)\right) ds\right) \Pi((i, x), \{(j, y)\}) + o(h)
\end{aligned}$$

We observe that as the sum over all $(j, y) \in E$ on the right side is 1, last inequalities have to be equalities. To prove this, let $(T_k)_{n\in\mathbb{N}}$ denotes the sequence of time-jumps of the process $(Z_t)_t$, then we have that

$$N_t = \sum_{k\in\mathbb{N}} \mathbb{1}_{\{t \geq T_k\}}$$

is a Poisson process; and if we consider $\bar{\lambda} = \sup_{z\in E} \lambda(z)$ and $h \downarrow 0$, we have

$$\begin{aligned}
\mathbb{P}_{(i,x)}\left[N_h = 0\right] &= 1 - h\bar{\lambda} + o(h) \\
\mathbb{P}_{(i,x)}\left[N_h = 1\right] &= h\bar{\lambda} + o(h) \\
\mathbb{P}_{(i,x)}\left[N_h \geq 2\right] &= o(h)
\end{aligned}$$

The inequality $\leq$ is due the bound of the Poisson law; and the equality comes form the fact that the sum at both side of the inequality has to be 1 at any h small. This allow us to control the number of jumps in a small interval, and we can conclude that

$$\begin{aligned}
\mathbb{P}_{(i,x)}\left[Z_h = (i, \phi(i,x,h))\right] &= 1 - \int_0^h \lambda\left(i, \phi(i,x,s)\right) ds + o(h) \\
\mathbb{P}_{(i,x)}\left[Z_h = (j, \phi(j,y,h))\right] &= \left(\int_0^h \lambda\left(i, \phi(i,x,s)\right) ds\right) \Pi((i,x), \{(j,y)\}) + o(h), \quad \text{for } (i,x) \neq (j,y)
\end{aligned}$$

Now, for all $f \in D(L)$ and $h \downarrow 0$, we have

$$\begin{aligned}
&\mathbb{E}_{(i,x)}\left[f(Z_h)\right] = f(i,\phi(i,x,h))\mathbb{P}_{(i,x)}\left[Z_h = (i,\phi(i,x,h)]\right) \\
&\qquad\qquad + \sum_{j\in I\setminus\{i\}} \int_{D_j\setminus\{x\}} f(j,\phi(j,y,h))\mathbb{P}_{(i,x)}\left[Z_h = (j,\phi(j,dy,h))\right] \\
&= f(i,\phi(i,x,h)) - \left(\int_0^h \lambda\left(i,\phi(i,x,s)\right) ds\right) f(i,\phi(i,x,h)) \\
&\quad + \left(\int_0^h \lambda\left(i,\phi(i,x,s)\right) ds\right) \sum_{j\in I\setminus\{i\}} \int_{D_j\setminus\{x\}} f(j,\phi(j,y,h))\Pi((i,x),\{(j,dy)\}) + o(h) \\
&= f(i,\phi(i,x,h)) \\
&\quad + \left(\int_0^h \lambda\left(i,\phi(i,x,s)\right) ds\right) \sum_{j\in I} \int_{D_j} \left(f(j,\phi(j,y,h)) - f(j,\phi(i,x,h))\right) \Pi((i,x),\{(j,dy)\}) + o(h)
\end{aligned}$$

as $\Pi((i,\phi(i,x,h)),\cdot)$ is a probability measure where $\Pi(z,\{z\}) = 0$. In order to compute the infinitesimal generator we observe that

$$\begin{aligned}
&\frac{\mathbb{E}_{(i,x)}\left[f(Z_h)\right] - f(i,x)}{h} = \frac{f(i,\phi(i,x,h)) - f(i,x)}{h} \\
&+ \left(\frac{1}{h}\int_0^h \lambda\left(i,\phi(i,x,s)\right) ds\right) \sum_{j\in I} \int_{D_j} \left(f(j,\phi(j,y,h)) - f(j,\phi(i,x,h))\right) \Pi((i,x),\{(j,dy)\}) + o(1)
\end{aligned}$$

taking the limit and taking $o(1) \downarrow 0$ when $h \downarrow 0$, it follows that

$$Lf(i,x) = \sum_{l=1}^{d} \frac{\partial f(i,x)}{\partial x_l}\psi_l(i,x) + \lambda(i,x)\sum_{j\in I}\int_{D_j} \left(f(j,y) - f(i,x)\right) \Pi((i,x),\{(j,dy)\})$$

and considering $v = (j,y) \in E$ and $dv = (j,dy)$, we have (2.3). ∎

An important property of the generator is that it satisfies the *Dynkin formula*:

$$P_t f(z) = f(z) + \int_0^t P_s Lf(z) ds \tag{2.4}$$

that it is equivalent to the statement that the process

$$C_t^f := f(Z_t) - f(z) - \int_0^t Lf(Z_s) ds$$

is an $\mathcal{F}_t$-martingale. To prove this, we use the Markov property and the time-homogeneity; we see that for all $t > s$

$$\begin{aligned}
\mathbb{E}\left[C_t^f - C_s^f|\mathcal{F}_s\right] &= \mathbb{E}\left[f(Z_t) - f(Z_s) - \int_s^t Lf(Z_u)du|\mathcal{F}_s\right] \\
&= \mathbb{E}\left[f(Z_t) - f(Z_s) - \int_s^t Lf(Z_u)du|Z_s\right] \\
&= P_{t-s}f(Z_s) - f(Z_s) - \int_s^t P_{u-s}Lf(Z_s)du \\
&= 0
\end{aligned}$$

With the objective to calculate the probability measure for the process $(Z_t)_t$, we define for all $t \geq 0$ the distribution $p(\cdot, t) \in \mathcal{P}(E, \mathcal{E})$ such that

$$p(A,t) = \mathbb{E}\left[\mathbb{1}_{\{Z_t \in A\}}\right]$$

for $A \in \mathcal{E}$ and some initial distribution $p(\cdot, 0) = p_0$. If we rewrite the expectation as the integral over the probability measure in equation (2.4), we have

$$\frac{\partial}{\partial t}\int_E f(z)p(dz,t) = \int_E \left(Ff(z) + \lambda(z)\int_E \left(f(v) - f(z)\right)\Pi(z,dv)\right)p(dz,t)$$

Now, with $z = (i,x) \in E$, replacing $Ff(i,x)$ for $\sum_{l=1}^d \frac{\partial f(i,x)}{\partial x_l}\psi_l(i,x)$, integrating by parts and by Fubini's lemma, it leads to

$$\begin{aligned}
\sum_{i\in I}\int_{D_i} f(i,x)\frac{d}{dt}p(i,dx,t) = \; & -\sum_{i\in I}\int_{D_i} f(i,x)\sum_{l=1}^d \frac{\partial}{\partial x_l}\left[\psi_l(i,x)p(i,dx,t)\right] - \sum_{i\in I}\int_{D_i} f(i,x)\lambda(i,x)p(i,dx,t) \\
& + \sum_{i\in I}\int_{D_i}\lambda(i,x)\left(\sum_{j\in I}\int_{D_j} f(j,y)\Pi((i,x),\{(j,dy)\})\right)p(i,dx,t)
\end{aligned}$$

we conclude that in a weak sense, the distribution of the process solves the Fokker-Planck equation,

$$\begin{aligned}
\frac{\partial}{\partial t}p(i,x,t) &= -\sum_{l=1}^d \frac{\partial}{\partial x_l}\left[\psi_l(i,x)p(i,x,t)\right] \\
&\quad -\lambda(i,x)p(i,x,t) + \sum_{j\in I}\int_{D_j}\lambda(j,y)\Pi((j,y),\{(i,x)\})p(j,dy,t)
\end{aligned} \tag{2.5}$$

for all $(i,x) \in E$, $t \geq 0$ and for some initial condition distribution $p(\cdot, 0) = p_0$.

Remark If we define the generator matrix $Q = \left(Q_{ij}\right)_{i,j\in I}$ such that for each $i,j \in I$, $Q_{ij} : D_i \times D_j \to \mathbb{R}$ where

$$\begin{aligned}
Q_{ii}(x,x) &= -\lambda(i,x) && \text{for all } (i,x) \in E \\
Q_{ij}(x,y) &= \lambda(i,x)\Pi((i,x),\{(j,y)\}) && \text{for all } (i,x),(j,y) \in E, (i,x) \neq (j,y).
\end{aligned}$$

the equation (2.5) for the probability measure can be rewritten as

$$\frac{\partial}{\partial t}p(i,x,t) = -\sum_{l=1}^{d} \frac{\partial}{\partial x_l}\left[\psi_l(i,x)p(i,x,t)\right] + \sum_{j\in I}\int_{D_j} p(j,dy,t)Q_{ji}(y,x)$$

for all $(i,x) \in E$, $t \geq 0$ and initial condition $p(\cdot,0) = p_0$.

Example 1

Let consider a PDMP $(Z_t)_t = (I_t, X_t)_t$ with states in $E = \{0,1\} \times \mathbb{R}$ and *local characteristics* given by $((F_i)_{i=1,2}, \lambda, \Pi)$, where

- $F_i = 1$ is the vector field for $i = 1,2$, determining a flow $\phi(i,x,t) = x + t$,
- $\lambda : E \to \mathbb{R}_+$ is the jump rate, such that $\lambda(i,x) = \begin{cases} \alpha(x) & \text{if } i = 0 \\ \beta(x) & \text{if } i = 1 \end{cases}$
- Π is a Markov transition kernel such that $\begin{cases} \Pi((0,x),\{(1,0)\}) &= 1 \quad \text{for all } x \in \mathbb{R} \\ \Pi((1,x),\{(0,x)\}) &= 1 \quad \text{for all } x \in \mathbb{R} \end{cases}$

Then, the infinitesimal generator is given by

$$\begin{aligned} Lf(i,x) &= \frac{\partial}{\partial x}f(i,x) + \lambda(i,x)\sum_{j=0,1}\int_{\mathbb{R}}(f(j,y) - f(i,x))\Pi((i,x),\{j,dy\}) \\ &= \frac{\partial}{\partial x}f(i,x) + \alpha(x)(f(1,0) - f(0,x))\mathbb{1}_{\{i=0\}} + \beta(x)(f(0,x) - f(1,x))\mathbb{1}_{\{i=1\}} \end{aligned}$$

for $f \in D(L)$. And the Fokker-Plank equation is given by

$$\frac{\partial}{\partial t}p(i,x,t) = -\frac{\partial}{\partial x}p(i,x,t) - \lambda(i,x)p(i,x,t) + \sum_{j=0,1}\int_{\mathbb{R}}\lambda(j,y)\Pi((j,y),\{(i,x)\})p(j,dy,t)$$

$$= -\frac{\partial}{\partial x}p(i,x,t) + \left[\beta(x)p(1,x,t) - \alpha(x)p(0,x,t)\right]\mathbb{1}_{\{i=0\}} + \left[\alpha(x)p(0,x,t)\mathbb{1}_{\{x=0\}} - \beta(x)p(1,x,t)\right]\mathbb{1}_{\{i=1\}}$$

Remark In this model, X_t could represent the time since the last spike of a neuron denote by I_t, such that $I_t = 0$ when the neuron is resting and $I_t = 1$ during the spike.

2.2.1 Simulation

In order to simulate a PDMP $(Z_t)_{t\in[0,T]}$ with values in E, initial distribution p_0 and *local characteristics* $((F_i)_{i\in I}, \lambda, \Pi,)$, the idea is:

- draw a sample z_0 from p_0 for the initial value,
- then iteratively, if the process is in state (i,x),
  A. determine the flow ϕ from vector field F_i,
  B. draw a sample from an $\mathcal{E}(\lambda(i,\phi(i,x,t)))$ exponential law for the duration time,

C. between jumps set $Z_t = (i, \phi(i,x,t))$,

D. draw the next state $w \neq (i,x)$ from the law $\Pi((i,x),\cdot)$.

To achieve this, we consider the following lemmas that allows to draw samples of the distribution needed from the uniform distribution.

Lemma 2.2.4 (Rejection Method) *Let $\lambda : [0,T] \to \mathbb{R}_+$ be a bound rate function and a random variable $\tau \sim \exp(\lambda(t))$. Let consider $\bar{\lambda} = \sup_{t\in[0,T]} \lambda(t)$ and the auxiliary random variables $\xi \sim \exp(\bar{\lambda})$ and $U \sim U(0,1)$, then*

$$\mathbb{P}\left[\xi \in dt \middle| U \leq \frac{\lambda(\xi)}{\bar{\lambda}}\right] = \lambda(t)dt$$

Proof Let define the functions

$$f^\delta(t) = \mathbb{P}\left[\tau \in [t,t+\delta)\right] = \int_t^{t+\delta} \lambda(s)\exp\left(-\int_0^s \lambda(r)dr\right)ds = \lambda(t)\delta + o(\delta^2)$$
$$\bar{f}^\delta(t) = \mathbb{P}\left[\xi \in [t,t+\delta)\right] = \int_t^{t+\delta} \bar{\lambda}\exp\left(-\bar{\lambda}s\right)ds = \bar{\lambda}\delta + o(\delta^2)$$

Now, noting that $\mathbb{P}[A|B] = \mathbb{P}[B|A]\,\mathbb{P}[A]/\mathbb{P}[B]$ we have

$$\mathbb{P}\left[\xi \in [t,t+\delta) \middle| U \leq \frac{f^\delta(\xi)}{\bar{f}^\delta(\xi)}\right] = \mathbb{P}\left[U \leq \frac{f^\delta(\xi)}{\bar{f}^\delta(\xi)} \middle| \xi \in [t,t+\delta)\right] \frac{\bar{f}^\delta(t)}{\mathbb{P}\left[U \leq \frac{f^\delta(\xi)}{\bar{f}^\delta(\xi)}\right]}$$

where

$$\begin{aligned}
\mathbb{P}\left[U \leq \frac{f^\delta(\xi)}{\bar{f}^\delta(\xi)}\right] &= \sum_{k=0}^{\infty} \mathbb{P}\left[U \leq \frac{f^\delta(k\delta)}{\bar{f}^\delta(k\delta)}\right]\bar{f}^\delta(k\delta) \\
&= \sum_{k=0}^{\infty} \frac{f^\delta(k\delta)}{\bar{f}^\delta(k\delta)}\bar{f}^\delta(k\delta) = \sum_{k=0}^{\infty} f^\delta(k\delta) = 1
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{P}\left[U \leq \frac{f^\delta(\xi)}{\bar{f}^\delta(\xi)} \middle| \xi \in [t,t+\delta)\right] &= \frac{1}{\bar{f}^\delta(t)}\mathbb{P}\left[U \leq \frac{f^\delta(\xi)}{\bar{f}^\delta(\xi)},\ \xi \in [t,t+\delta)\right] \\
&= \frac{1}{\bar{f}^\delta(t)}\sum_{k=0}^{\infty}\mathbb{P}\left[U \leq \frac{f^\delta(k\delta)}{\bar{f}^\delta(k\delta)},\ k\delta \in [t,t+\delta)\right]\bar{f}^\delta(k\delta) \\
&= \frac{1}{\bar{f}^\delta(t)}\mathbb{P}\left[U \leq \frac{f^\delta(\tilde{t})}{\bar{f}^\delta(\tilde{t})}\right]\bar{f}^\delta(\tilde{t}) \\
&= \frac{f^\delta(\tilde{t})}{\bar{f}^\delta(t)}
\end{aligned}$$

for some $\tilde{t} \in [t,t+\delta)$. Finally, we obtain

$$\mathbb{P}\left[\xi \in [t,t+\delta) \middle| U \leq \frac{f^\delta(\xi)}{\bar{f}^\delta(\xi)}\right] = f^\delta(\tilde{t})$$

and when $\delta \to 0$, we have the result wanted. ∎

Where we use Lemma 1.1.3 to simulate $\xi \sim \mathcal{E}(\bar{\lambda})$. For draw a sample of $\Pi((i,x),\cdot)$, it will depend on the form of the distribution. In the case it has a finite support, we can use Lemma 1.1.4.

2.3 Markov switching model with multiple weakly irreducible classes

In order to perform the state reduction approximation, we will consider a subclass of PDMP known as the Markov switching model (see [6], [7]). Let consider the state space $E = I \times \mathbb{R}^d$ for a finite space $I = \{1, 2, \ldots, N\}$, the Borel space $(E, \mathcal{E})$ and a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_t, \mathbb{P})$; we define for a $T < \infty$

$$(Z_t)_{t\in[0,T]} = (I_t, X_t)_{t\in[0,T]}$$

a Markov switching process with states on E, initial condition $Z_0 = z = (i, x) \in E$ and *local characteristics* given by $((F_i)_{i\in I}, \lambda, \Pi)$, where

- F_i is a continuous vector field characterized by the Lipschitz functions $\psi(i, x)$, for all $(i, x) \in E$; that determines a flow $f(\phi(i, x, t))$ via the ODE

$$\frac{d}{dt} f(x(t)) = F f(i, x(t)) = \sum_{l=1}^{d} \frac{\partial f(x(t))}{\partial x_l} \psi_l(i, x(t))$$

- $\lambda : E \to \mathbb{R}_+$ is the jump rate,
- $\Pi : E \times E \to [0, 1]$ is a Markov transition kernel that allows only jumps in I; i.e. we demand that

$$\begin{aligned} \Pi\left((i, x), \{(j, x)\}\right) &\geq 0, \quad \text{for all } (i, x) \in E \text{ and } j \in I, \\ \Pi\left((i, x), \{(j, y)\}\right) &= 0, \quad \text{otherwise} \end{aligned}$$

The Markov switching model is a process $Z_t = (I_t, X_t) \in I \times \mathbb{R}^d$ can be described such that

A. X_t is driven by the vector field F_{I_t},

B. If $I_t = i$ and $X_t = x$, then I_t jumps to j with rate $\lambda(i, x)\Pi((i, x), \{(j, x)\})$ for $j \neq i$,

C. Its generator infinitesimal is given by

$$Lf(i, x) = F f(i, x) + \lambda(i, x) \sum_{j\in I} \left(f(j, x) - f(i, x)\right) \Pi((i, x), \{(j, x)\})$$

with a generator matrix $Q(x) \in \mathbb{R}^{N\times N}$ defined by

$$Q_{ij}(x) = -\lambda(i, x)\mathbb{1}_{\{i=j\}} + \lambda(i, x)\Pi((i, x), \{(j, x)\})\mathbb{1}_{\{i\neq j\}} \tag{2.6}$$

for $i, j = 1, \ldots, N$.

In this model, the trajectory of the continuous part X_t does not jump and it evolves continuously; the jumps only occur on the discrete part I_t. Let us also note that there are no boundary in this model for the continuous part, so we don't have to consider border condition and there is no risk of explosion times.

Remark In general, $(I_t)_t$ is not a Markov process on its own since its jump rates depend on $(X_t)_t$. If the jump rate λ does not depend on $(X_t)_t$, then $(I_t)_t$ is a Markov process on the finite space I and X_t is a function of $(I_s)_{s\leq t}$.

We consider that the process $(Z_t^\epsilon)_t$ has multiple irreducible classes, which means that its generator $Q^\epsilon(x)$ for any fix $x \in \mathbb{R}^d$ depends on a small parameter $\epsilon > 0$ and two generators $\tilde{Q}(x)$ and $\hat{Q}(x)$, such that

$$Q^\epsilon(x) = \frac{1}{\epsilon}\tilde{Q}(x) + \hat{Q}(x) \tag{2.7}$$

where $\hat{Q}(x)$ governs the dynamic on the slow-scale and $\tilde{Q}(x)$ on the fast-scale. The generator $\tilde{Q}(x)$ can be put into block-diagonal form

$$\tilde{Q}(x) \quad = \quad \begin{pmatrix} \tilde{Q}^1(x) & 0 & & 0 \\ 0 & \tilde{Q}^2(x) & & \\ & & \ddots & 0 \\ 0 & & 0 & \tilde{Q}^{\bar{N}}(x) \end{pmatrix}$$

where each $\tilde{Q}^k(x)$ is a generator of dimension $m_k \times m_k$ for some integer m_k, such that $\sum_{k=1}^{\bar{N}} m_k = N$, and it defines the class

$$\bar{s}_k = \{i_{k1}, \ldots, i_{km_k}\} \subset I$$

that corresponds to the states associated with the generator $\tilde{Q}^k(x)$. We define the new set

$$\bar{S} = \{\bar{s}_1, \ldots, \bar{s}_{\bar{N}}\}$$

that will be the space of the slow process, and that satisfies $\bigcup_{k=1}^{\bar{N}} \bar{s}_k = I$.

Assumption 2.3.1 *For all $k = 1, \ldots, \bar{N}$ and $x \in \mathbb{R}^d$, $\tilde{Q}^k(x)$ is a weakly irreducible generator; that is for any fix $x \in \mathbb{R}^d$ the system of equations*

$$\begin{cases} \displaystyle\sum_{j=1}^{m_k} \nu_j^k(x)\tilde{Q}_{ji}^k(x) &=& 0, \quad \textit{for } i = 1, \ldots, m_k \\ \displaystyle\sum_{i=1}^{m_k} \nu_i^k(x) &=& 1 \end{cases}$$

has a unique non-negative solution.

We consider for all $t \geq 0$ the distribution $p^\epsilon(\cdot, t) \in \mathcal{P}(E, \mathcal{E})$ such that

$$p^\epsilon(A, t) = \mathbb{E}_{p_0}\left[\mathbb{1}_{\{Z\in A\}}\right]$$

for $A \in \mathcal{E}$, then the Fokker-Planck equation (2.5) corresponds to

$$\begin{cases} \dfrac{\partial}{\partial t}p^\epsilon(i,x,t) &=& -\displaystyle\sum_{l=1}^{d} \frac{\partial}{\partial x_l}\left[\psi_l(i,x)p^\epsilon(i,x,t)\right] + \sum_{j\in I} p^\epsilon(j,x,t)\left(\frac{1}{\epsilon}\tilde{Q}_{ji}(x) + \hat{Q}_{ji}(x)\right) \\ p^\epsilon(i,x,0) &=& p_0(i,x) \end{cases} \tag{2.8}$$

for all $(i, x) \in E$, $t \in [0, T]$ and initial distribution p_0.

2.3.1 Asymptotic expansions

Now we present the most important theorem of the section, that presents a characterization for the probably measure of the process $(Z_t^\epsilon)_t$, based on results given in [2] and [9].

Theorem 2.3.2 (Asymptotic expansion) *The probability measure $p^\epsilon(\cdot,t)$ (2.8) of the process $(Z_t^\epsilon)_t$ can be expanded in the form:*

$$p^\epsilon(i,x,t) = \varphi(i,x,t) + \gamma\left(i,x,\frac{t}{\epsilon}\right) + e^\epsilon(i,x,t) \tag{2.9}$$

In this approach φ is set to be an approximation on the slow-scale t away from 0, γ approximate the fast-scale $\tau = t/\epsilon$ and e^ϵ corresponds to the error of the expansion. The functions φ, γ and e^ϵ are such that

- *$\varphi(i,x,t)$ is differentiable for all $t \in [0,T]$.*
- *there exist a constant $\kappa_0 > 0$ such that*

  $$|\gamma(i,x,\tau)| \leq K\exp(-\kappa_0\tau)$$

  uniformly for all $(i,x) \in E$.
- *and the following estimate holds*

  $$\sup_{t\in[0,T]} |e^\epsilon(i,x,t)| \leq K\epsilon$$

  uniformly for all $(i,x) \in E$.

To proceed, we define an operator $\mathcal{L}^\epsilon$ by

$$\mathcal{L}^\epsilon f(i,x,t) = \epsilon\frac{d}{dt}f(i,x,t) + \epsilon\sum_{l=1}^{d}\frac{\partial}{\partial x_l}\left[\psi_l(i,x)f(i,x,t)\right] - \sum_{j\in I} f(i,x,t)\left(\tilde{Q}_{ji}(x) + \epsilon\hat{Q}_{ji}(x)\right) \tag{2.10}$$

for any smooth function f, then $\mathcal{L}^\epsilon f = 0$ iff f is a solution to the Fokker-Planck equation (2.8). We set that both φ and γ are solution to the forward equation, then they satisfy

$$\mathcal{L}^\epsilon\varphi(i,x,t) = 0 \text{ and } \mathcal{L}^\epsilon\gamma\left(i,x,\frac{t}{\epsilon}\right) = 0$$

that is, we have

$$\begin{aligned}
\epsilon\frac{\partial}{\partial t}\varphi(i,x,t) &= -\epsilon\sum_{l=1}^{d}\frac{\partial}{\partial x_l}\left[\psi_l(i,x)\varphi(i,x,t)\right] + \sum_{j\in I}\varphi(j,x,t)\left(\tilde{Q}_{ji}(x) + \epsilon\hat{Q}_{ji}(x)\right) \\
\epsilon\frac{\partial}{\partial t}\gamma\left(i,x,\frac{t}{\epsilon}\right) &= -\epsilon\sum_{l=1}^{d}\frac{\partial}{\partial x_l}\left[\psi_l(i,x)\varphi\left(i,x,\frac{t}{\epsilon}\right)\right] + \sum_{j\in I}\gamma\left(j,x,\frac{t}{\epsilon}\right)\left(\tilde{Q}_{ji}(x) + \epsilon\hat{Q}_{ji}(x)\right)
\end{aligned} \tag{2.11}$$

and we write a new time scale $\tau = t/\epsilon$ for the second equation

$$\frac{\partial}{\partial \tau}\gamma(i,x,\tau) = -\epsilon\sum_{l=1}^{d}\frac{\partial}{\partial x_l}\left[\psi_l(i,x)\varphi\left(i,x,\tau\right)\right] + \sum_{j\in I}\gamma(j,x,\tau)\left(\tilde{Q}_{ji}(x) + \epsilon\hat{Q}_{ji}(x)\right)$$

And if we match the terms over ϵ^0 and ϵ^1 in the time scale t and setting that $\psi(\tau)$ must not depends on ϵ, we have the following set of equations

$$\begin{cases} \sum_{j\in I}\varphi(j,x,t)\tilde{Q}_{ji}(x) = 0 \\ \frac{\partial}{\partial t}\varphi(i,x,t) = -\sum_{l=1}^{d}\frac{\partial}{\partial x_l}\left[\psi_l(i,x)\varphi(i,x,t)\right] + \sum_{j\in I}\varphi(j,x,t)\hat{Q}_{ji}(x) \end{cases}$$

and

$$\left\{ \frac{\partial}{\partial \tau}\gamma(i,x,\tau) = \sum_{j\in I}\gamma(j,x,\tau)\tilde{Q}_{ji}(x) \right.$$

Remark From the last term comes the approximation error of this expansion, and due the fact that $\hat{Q}$ is not weakly irreducible.

In order to match the asymptotic expansion, we have necessarily at $t = 0$ that

$$p^\epsilon(i,x,0) = \varphi(i,x,0) + \gamma(i,x,0)$$

Sending $\epsilon \to 0$ in the expansion (2.11), the fast-scale and error disappear and only remains the slow-scale, then as $\sum_{i\in I}\int_D p^\epsilon(i,x,t)dx = 1$ for all $\epsilon > 0$, we conclude

$$\lim_{\epsilon\to 0}\sum_{i\in I}\int_D p^\epsilon(i,x,t)dx = \sum_{i\in I}\int_D \varphi(i,x,t)dx = 1$$

Determining φ

We need to determine $\varphi(i,x,t)$ for $(i,x)\in E$ and $t\in[0,T]$ such that

$$\begin{cases} \sum_{j\in I}\varphi(j,x,t)\tilde{Q}_{ji}(x) = 0 \\ \frac{\partial}{\partial t}\varphi(i,x,t) = -\sum_{l=1}^{d}\frac{\partial}{\partial x_l}\left[\psi_l(i,x)\varphi(i,x,t)\right] + \sum_{j\in I}\varphi(j,x,t)\hat{Q}_{ji}(x) \\ \sum_{i\in I}\int_{\mathbb{R}^d}\varphi(i,x,t)dx = 1 \end{cases} \tag{2.12}$$

Since $\tilde{Q}(x) = \text{diag}\left(\tilde{Q}^1(x),\dots,\tilde{Q}^{\bar{N}}(x)\right)$ where each $\tilde{Q}^k(x)$ is weakly irreducible, therefore if we consider $\varphi^k(i,x,t)$, the function $\varphi(i,x,t)$ restricted on $i\in\bar{s}_k$, we have that it satisfies

$$\sum_{j\in\bar{s}_k}\varphi^k(j,x,t)\tilde{Q}^k_{ji}(x) = 0, \text{ for all } i\in\bar{s}_k$$

then its solution is $\varphi^k(i,x,t) = \theta(k,x,t)\nu_i^k(x)$, the product of the invariant measure $\nu^k(x)$ of $\tilde{Q}^k(x)$ and a function multiplier $\theta(k,x,t)$ with values in $k \in \{1,\dots,\bar{N}\}$, $x \in \mathbb{R}^d$ and $t \in [0,T]$. And as $\nu^k(x)$ is a distribution in $\bar{s}_k$, we have that

$$\begin{aligned}\sum_{i\in\bar{s}_k}\int_{\mathbb{R}^d}\varphi^k(i,x,t)dx &= \sum_{i\in\bar{s}_k}\int_{\mathbb{R}^d}\theta(k,x,t)\nu_i^k(x)dx \\ &= \int_{\mathbb{R}^d}\theta(k,x,t)dx\end{aligned}$$

and by consequence of the third equation on (2.12), it also holds that

$$\sum_{k=1}^{\bar{N}}\int_{\mathbb{R}^d}\theta(k,x,t)dx = 1$$

Setting all this with the second equation in (2.12), for each $k = 1,\dots,\bar{N}$ it follows that

$$\begin{aligned}&\frac{\partial}{\partial t}\theta(k,x,t) = \sum_{i\in\bar{s}_k}\frac{d}{dt}\varphi(i,x,t) \\ &= -\sum_{i\in\bar{s}_k}\sum_{l=1}^{d}\frac{\partial}{\partial x_l}\left[\psi_l(i,x)\varphi(i,x,t)\right] + \sum_{i\in\bar{s}_k}\sum_{j\in I}\varphi(j,x,t)\hat{Q}_{ji}(x) \\ &= -\sum_{i\in\bar{s}_k}\sum_{l=1}^{d}\frac{\partial}{\partial x_l}\left[\psi_l(i,x)\nu_i^k(x)\theta(k,x,t)\right] + \sum_{i\in\bar{s}_k}\sum_{p=1}^{\bar{N}}\sum_{j\in\bar{s}_p}\theta(p,x,t)\nu_j^p(x)\hat{Q}_{ji}(x)\end{aligned}$$

reorganizing the terms, we have that for $k = 1,\dots,\bar{N}$, it holds

$$\frac{\partial}{\partial t}\theta(k,x,t) = -\sum_{l=1}^{d}\frac{\partial}{\partial x_l}\left[\left(\sum_{i\in\bar{s}_k}\nu_i^k\psi_l(i,x)\right)\theta(k,x,t)\right] + \sum_{p=1}^{\bar{N}}\theta(p,x,t)\left(\sum_{j\in\bar{s}_p}\sum_{i\in\bar{s}_k}\nu_j^p(x)\hat{Q}_{ji}(x)\right) \tag{2.13}$$

where we observe the emergence of a new generator $\bar{Q}(x) \in \mathbb{R}^{\bar{N}\times\bar{N}}$ such that

$$\bar{Q}_{pk}(x) = \sum_{j\in\bar{s}_p}\sum_{i\in\bar{s}_k}\nu_j^p(x)\hat{Q}_{ji}(x) \tag{2.14}$$

for $k,p = 1,\dots,\bar{N}$, and a new function $\bar{\psi}$ define by

$$\bar{\psi}(k,x) = \sum_{i\in\bar{s}_k}\nu_i^k(x)\psi(i,x) \tag{2.15}$$

for $k = 1,\dots,\bar{N}$. In order to determine the initial conditions $\theta(k,x,0)$, we fist observe that in the asymptotic expansion it has to hold that

$$\sum_{i\in\bar{s}_k}\varphi(i,x,0) = \lim_{\delta\to 0}\lim_{\epsilon\to 0}\sum_{i\in\bar{s}_k}p^\epsilon(i,x,\delta) \tag{2.16}$$

moreover, in view of the forward equation (2.8) and that $\sum_{i\in\bar{s}_k}\tilde{Q}^k_{ji}(x)=0$ for all $j\in\bar{s}_k$, we have

$$\sum_{i\in\bar{s}_k}p^{\epsilon}(i,x,\delta)=\sum_{i\in\bar{s}_k}p^{\epsilon}(i,x,0)+\int_0^{\delta}\left(-\sum_{i\in\bar{s}_k}\sum_{l=1}^{d}\frac{\partial}{\partial x_l}\left[\varphi_l(i,x)p^{\epsilon}(i,x,s)\right]+\sum_{i\in\bar{s}_k}\sum_{j\in\bar{s}_k}p^{\epsilon}(j,x,s)\hat{Q}_{ji}(x)\right)ds$$

and since $p^{\epsilon}(i,x,t)$ is bounded it follows that

$$\lim_{\delta\to 0}\limsup_{\epsilon\to 0}\int_0^{\delta}\left(-\sum_{i\in\bar{s}_k}\sum_{l=1}^{d}\frac{\partial}{\partial x_l}\left[\varphi_l(i,x)p^{\epsilon}(i,x,s)\right]+\sum_{i\in\bar{s}_k}\sum_{j\in\bar{s}_k}p^{\epsilon}(j,x,s)\hat{Q}_{ji}(x)\right)ds=0$$

therefore by (2.16) it yields

$$\sum_{i\in\bar{s}_k}\int_{\mathbb{R}^d}\varphi(i,x,0)dx=\lim_{\delta\to 0}\left(\lim_{\epsilon\to 0}\sum_{i\in\bar{s}_k}\int_{\mathbb{R}^d}\varphi(i,x,\delta)dx\right)=\sum_{i\in\bar{s}_k}\int_{\mathbb{R}^d}p^{\epsilon}(i,x,0)dx$$

and we finally have

$$\theta(k,x,0)=\sum_{i\in\bar{s}_k}p(i,x,0)$$

To conclude, we present the following Corollary that synthesized the results of this section.

Corollary 2.3.3 *The system* (2.12) *for* $\varphi(i,x,t)$*, is equivalent to the system*

$$\begin{cases}\displaystyle\sum_{j\in\bar{s}_k}\varphi(j,x,t)\tilde{Q}_{ji}(x)=0,\ \textit{for } i\in\bar{s}_k\\ \displaystyle\sum_{i\in\bar{s}_k}\varphi(i,x,t)=\theta(k,x,t)\\ \displaystyle\frac{\partial}{\partial t}\theta(k,x,t)=-\sum_{l=1}^{d}\frac{\partial}{\partial x_l}\left[\bar{\psi}_l(k,x)\theta(k,x,t)\right]+\sum_{p=1}^{\bar{N}}\theta(p,x,t)\bar{Q}_{pk}(x)\\ \displaystyle\theta(k,x,0)=\sum_{i\in\bar{s}_k}p_0(i,x)\end{cases}$$

for $k=1,\dots,\bar{N}$*; where the function* $\bar{\psi}$ *is defined in (2.15) and the generator* $\bar{Q}$ *in (2.14).*

Remark These $\theta(i,x,t)$ can be interpreted as the probability measure of an PDMP $(M_t)_{t\geq 0}$ defined over the aggregate states space $\bar{S}\times\mathbb{R}^d$, where $\bar{S}$ is the finite space $\{\bar{s}_1,\dots,\bar{s}_{\bar{N}}\}$, and defined by its *local characteristics* given by the generator $\bar{Q}$ and the vector field $\left(\bar{F}_{\bar{s}}\right)_{\bar{s}\in\bar{S}}$ given by

$$Ff(\bar{s}_k,x)=\sum_{l=1}^{d}\frac{\partial f(x)}{\partial x_l}\bar{\psi}(\bar{s}_k,x)\tag{2.17}$$

Determining γ

We consider $\gamma(i,x,\tau)$ for $(i,x)\in E$ and $t\in[0,T]$ solution to

$$\left\{ \frac{\partial}{\partial\tau}\gamma(i,x,\tau)=\sum_{j\in I}\gamma(j,x,\tau)\tilde{Q}_{ji}(x) \right. \tag{2.18}$$

To matched asymptotic expansion, we have necessarily at $t=0$ that

$$\varphi(i,x,0)+\gamma(i,0)=p^{\epsilon}(i,x,0)$$

In order to solve equation (2.18), we observe that we can solve it directly and together with the above initial condition, we obtain

$$\psi(\cdot,x,\tau)=(p_0(\cdot,x)-\varphi(\cdot,x,0))\exp\left(\tilde{Q}(x)\tau\right) \tag{2.19}$$

Considering that each $\tilde{Q}^k(x)$ is weakly irreducible, we need to prove that $\gamma(i,x,\tau)$ can be obtain by (2.19), and there is a positive number κ_0 such that

$$|\psi(i,x,\tau)|\le K\exp\left(-\kappa_0\tau\right)$$

uniformly for $(i,x)\in E$. To prove this, let $\nu^k(x)$ be the stationary distribution corresponding to the generator $\tilde{Q}^k(x)$, and we define the column vector $\mathbb{1}_m=(1,1,\ldots,1)'\in\mathbb{R}^{1\times m}$, then

$$\pi(x) \;=\; \begin{pmatrix} \mathbb{1}_{m_1}\nu^1(x) & 0 & & 0 \\ 0 & \mathbb{1}_{m_2}\nu^2(x) & & \\ & & \ddots & 0 \\ 0 & & 0 & \mathbb{1}_{m_{\bar{N}}}\nu^{\bar{N}}(x) \end{pmatrix}$$

where

$$\mathbb{1}_{m_k}\nu^k(x)=\begin{pmatrix} \nu_1^k(x) & \cdots & \nu_{m_k}^k(x) \\ & \vdots & \\ \nu_1^k(x) & \cdots & \nu_{m_k}^k(x) \end{pmatrix}$$

Noting the block-diagonal structure of $\tilde{Q}$, we have

$$\exp\left(\tilde{Q}(x)\tau\right)=\begin{pmatrix} \exp\left(\tilde{Q}^1(x)\tau\right) & 0 & & 0 \\ 0 & \exp\left(\tilde{Q}^2(x)\tau\right) & & \\ & & \ddots & 0 \\ 0 & & 0 & \exp\left(\tilde{Q}^{\bar{N}}(x)\tau\right) \end{pmatrix}$$

Furthermore, we see that for $k=1,\ldots,\bar{N}$ it holds

$$\sum_{i\in\bar{s}_k}(p_0(i,x)-\varphi(i,x,0))=\sum_{i\in\bar{s}_k}p_0(i,x)-\sum_{i\in\bar{s}_k}\varphi(i,x,0)=\sum_{i\in\bar{s}_k}p_0(i,x)-\theta(k,x,0)=0$$

and then, the initial condition $(p_0(\cdot,x)-\varphi(\cdot,x,0))$ is orthogonal to $\pi(x)$. By virtue of Lemma (1.2.2), for each $k=1,\dots,\bar{N}$ there exists $\kappa_k>0$ such that

$$\left|\exp\left(\tilde{Q}^k(x)\tau\right)-\mathbb{1}_{m_k}\nu^k(x)\right|\leq K\exp\left(-\kappa_k\tau\right)$$

then we have

$$\begin{aligned}|\gamma(\cdot,x,\tau)| &= \left|(p_0(\cdot,x)-\varphi(\cdot,x,0))\left(\exp\left(\tilde{Q}(x)\tau\right)-\pi(x)\right)\right|\\ &\leq K\sup_{k\leq\bar{N}}\left|\exp\left(\tilde{Q}^k(x)\tau\right)-\mathbb{1}_{m_k}\nu^k(x)\right|\\ &\leq K\exp\left(-\kappa_0\tau\right)\end{aligned}$$

where $\kappa_0=\min\limits_{k\leq\bar{N}}\kappa_k$.

Analysis of remainder

The remainder of the asymptotic expansion (2.9) correspond to

$$e^\epsilon(i,x,t)=\varphi(i,x,t)+\gamma\left(i,x,\frac{t}{\epsilon}\right)-p^\epsilon(i,x,t)$$

where $e^\epsilon(0)=0$, and if we consider the operator $\mathcal{L}^\epsilon$ as in (2.10), it holds that $\mathcal{L}^\epsilon p^\epsilon(t)=0$ and from equations (2.12) and (2.18), we have

$$\mathcal{L}^\epsilon e^\epsilon(i,x,t)=-\epsilon\sum_{l=1}^{d}\frac{\partial}{\partial x_l}\left[\psi_l(i,x)\gamma\left(i,x,\frac{t}{\epsilon}\right)\right]+\epsilon\sum_{j\in I}\gamma\left(j,x,\frac{t}{\epsilon}\right)\hat{Q}_{ji}(x)$$

expanding the operator $\mathcal{L}^\epsilon$ we have

$$\begin{aligned}\frac{d}{dt}e^\epsilon(i,x,t) &= -\sum_{l=1}^{d}\frac{\partial}{\partial x_l}\left[\psi_l(i,x)e^\epsilon(i,x,t)\right]+\sum_{j\in I}e^\epsilon(j,x,t)Q^\epsilon_{ji}(x)\\ &\quad -\sum_{l=1}^{d}\frac{\partial}{\partial x_l}\left[\psi_l(i,x)\gamma\left(i,x,\frac{t}{\epsilon}\right)\right]+\sum_{j\in I}\gamma\left(j,x,\frac{t}{\epsilon}\right)\hat{Q}_{ji}(x)\end{aligned}$$

Now by integrating, using Poincare inequality, taking the norm and making use of the exponential decay property of γ, it holds that

$$\begin{aligned}|e^\epsilon(i,x,t)| &\leq |e^\epsilon(i,x,0)|+C\left|\int_0^t\psi(i,x)e^\epsilon(i,x,s)ds\right|+\left|\int_0^t\sum_{j\in I}e^\epsilon(j,x,s)Q^\epsilon_{ji}(x)ds\right|\\ &\quad +K\left|\int_0^t\psi(i,x)\gamma\left(i,x,\frac{s}{\epsilon}\right)ds\right|+\left|\int_0^t\sum_{j\in I}\gamma\left(j,y,\frac{s}{\epsilon}\right)\hat{Q}_{ji}(x)ds\right|\\ &\leq C\int_0^t|e^\epsilon(i,x,s)|\,ds+K\int_0^t\exp\left(-\kappa_0\frac{s}{\epsilon}\right)ds\\ &\leq C\int_0^t|e^\epsilon(i,x,s)|\,ds+K\frac{\epsilon}{\kappa_0}\left(1-\exp\left(-\kappa_0\frac{t}{\epsilon}\right)\right)\\ &\leq C\int_0^t|e^\epsilon(i,x,s)|\,ds+K'\epsilon\end{aligned}$$

by Gronwall's lemma (1.2.3), we conclude

$$|e^\epsilon(i,x,t)| \leq K'\epsilon \exp(Ct)$$

and taking the supremum on time, the remainder satisfies

$$\sup_{t\in[0,T]} \left| p^\epsilon(i,x,t) - \varphi(i,x,t) - \gamma\left(i,x,\frac{t}{\epsilon}\right) \right| = K\epsilon$$

where K is a positive constant that depends on T.

2.3.2 Two-scales approximation

We now present a corollary that allows us, under the conditions already discussed, to represent the PDMP as a two-scale process: one on a slow-scale that goes over the state classes and a fast-scale that acknowledge the dynamic inside each class.

Corollary 2.3.4 *Let be $(Z^\epsilon)_{t\in[0,T]}$ a Markov switching process defined over $E = I \times \mathbb{R}^d$ with $I = \{1,\ldots,N\}$, initial distribution p_0, and* local characteristics *given by the functions ψ and a two-scales generator $Q^\epsilon(x) \in \mathbb{R}^{N\times N}$ that depends on $\epsilon > 0$ and two generators $\tilde{Q}(x)$ and $\hat{Q}(x)$ such that*

$$Q^\epsilon(x) = \frac{1}{\epsilon}\tilde{Q}(x) + \hat{Q}(x)$$

where $\tilde{Q}(x) = diag(\tilde{Q}^1(x),\ldots,\tilde{Q}^{\bar{N}}(x))$ with each sub-generator $\tilde{Q}^k(x) \in \mathbb{R}^{m_k\times m_k}$ weakly irreducible and determines the class $\bar{s}_k \subset I$. Then there exists positive constants K_T, K and κ_0 such that

$$\left|\mathbb{P}\left[Z_t^\epsilon = (i_{kj},x)\right] - \nu_j^k(x)\theta(k,x,t)\right| \leq K_T\epsilon + K\exp\left(-\kappa_0\frac{t}{\epsilon}\right)$$

for all $x \in \mathbb{R}^d$ and $i_{kj} \in \bar{s}_k$ for $j = 1,\ldots,m_k$ and $k = 1,\ldots,\bar{N}$. Here $\nu^k(x)$ is the stationary distribution in class $\bar{s}_k$ given by $\tilde{Q}^k(x)$, and $\theta(k,x,t)$ is a function defined for $k = 1,\ldots,\bar{N}$, $x \in \mathbb{R}^d$ and $t \in [0,T]$ that satisfies

$$\begin{cases} \dfrac{\partial}{\partial t}\theta(k,x,t) = -\displaystyle\sum_{l=1}^{d}\frac{\partial}{\partial x_l}\left[\bar{\psi}_l(k,x)\theta(k,x,t)\right] + \sum_{p=1}^{\bar{N}}\theta(p,x,t)\bar{Q}_{pk}(x) \\ \theta(k,x,0) = \displaystyle\sum_{i\in\bar{s}_k} p(i,x,0) \end{cases}$$

for the generator $\bar{Q}(x) \in \mathbb{R}^{\bar{N}\times\bar{N}}$ defined by

$$\bar{Q}_{pk}(x) = \sum_{j\in\bar{s}_p}\sum_{i\in\bar{s}_k}\nu_j^p(x)\hat{Q}_{ji}(x)$$

for $p,k = 1\ldots,\bar{N}$; and function $\bar{\psi}$ given by

$$\bar{\psi}(k,x) = \sum_{i\in\bar{s}_k}\nu_i^k(x)\psi(i,x)$$

for $k = 1\ldots,\bar{N}$.

In this corollary, we interpret $\theta(k,x,t)$ as the probability measure of an PDMP $(M_t)_{t\in[0,T]}$ defined over $\bar{S}\times\mathbb{R}^d$ with $\bar{S}=\{\bar{s}_1,\ldots,\bar{s}_{\bar{N}}\}$ and *local characteristics* given by the functions $\bar{\psi}$ and generator $\bar{Q}(x)$, which holds the dynamics on the slow-scale. We can think of this process as

$$M_t=(\bar{s}_k,x)\Longleftrightarrow Z_t=(i,x) \text{ for some } i\in\bar{s}_k$$

and that $\theta(k,x,t)=\mathbb{P}\left[M_t=(\bar{s}_k,x)\right]$ for $i=1,\ldots,\bar{N}$. On the other hand, the dynamic in the fast-scale will be defined punctually on the position of M_t by the random variable I_{fast} such that

$$\mathbb{P}\left[I_{\text{fast}}=i_{kj}\,|\,M_t=(\bar{s}_k,x)\right]=\nu_j^k(x)$$

for each $i_{kj}\in\bar{s}_k$. We observe that the value of the fast process depends instantaneously on the prosition of the slow process M_t.

Example 2

Continuation of Example 1 Chapter 1 (1.2.2). Let consider now that the proper functioning of the two machines depends on the heat in the factory. If the heat on the factory is too high, the machines are more likely to malfunction; and at the same time by working they contribute to the rise of the heat. For this we present the following model: let $(Z_t)_{t\in[0,t]}=(X_t,H_t)_{t\in[0,t]}$ be a Markov switching model, where X_t represents capacity of the workshop that correspond to a four-state chain with state space $I=\{(0,0),(1,0),(0,1),(1,1)\}$, and H_t represents the heat in the factory with values in $\mathbb{R}$. Suppose that the first machine breaks down much more often and produce more heat than the second one; to reflect this situation, consider that the process is generated by $Q^\epsilon(h)$ as (2.7) for a small $\epsilon>0$, with $\tilde{Q}(h)$ and $\hat{Q}(h)$ given by

$$Q^\epsilon(h)=\frac{1}{\epsilon}\tilde{Q}(h)+\hat{Q}(h)$$

where

$$\tilde{Q}(h)=\begin{pmatrix}-\lambda(h) & \lambda(h) & 0 & 0\\ \mu(h) & -\mu(h) & 0 & 0\\ 0 & 0 & -\lambda(h) & \lambda(h)\\ 0 & 0 & \mu(h) & -\mu(h)\end{pmatrix},\quad \hat{Q}(h)=\begin{pmatrix}-\lambda(h) & 0 & \lambda(h) & 0\\ 0 & -\lambda(h) & 0 & \lambda(h)\\ \mu(h) & 0 & -\mu(h) & 0\\ 0 & \mu(h) & 0 & -\mu(h)\end{pmatrix}$$

where $\lambda(h)$ and $\mu(h)$ are the rates of repair and breakdown respectively, and follow the logistic equations

$$\begin{aligned}\lambda(h) &= \frac{1}{1+\exp(2h)} &&\in[0,1]\\ \mu(h) &= \frac{1}{1+\exp(-2h)} &&\in[0,1]\end{aligned}$$

The heat is determined vector field functions

$$\psi(x,h)=\alpha(2x_1-1)+\beta(2x_2-1)$$

with $\alpha \geq \beta$, which determines the flow $\phi(x, h, t) = h + \alpha(2x_1 - 1)t + \beta(2x_2 - 1)t$; and we define the matrix $\Psi \in \mathbb{R}^{4\times 4}$ by

$$\Psi = \begin{pmatrix} \psi(i_1) & 0 & 0 & 0 \\ 0 & \psi(i_2) & 0 & 0 \\ 0 & 0 & \psi(i_3) & 0 \\ 0 & 0 & 0 & \psi(i_4) \end{pmatrix} = \begin{pmatrix} -\alpha-\beta & 0 & 0 & 0 \\ 0 & \alpha-\beta & 0 & 0 \\ 0 & 0 & -\alpha+\beta & 0 \\ 0 & 0 & 0 & \alpha+\beta \end{pmatrix}$$

We consider the probability $p^\epsilon(x, h, t) = \mathbb{P}_{p_0}[Z_t = (x, h)]$, that denotes the probability distribution of the underlying process at time t, and it is solution of the forward equation

$$\begin{cases} \frac{\partial}{\partial t} p^\epsilon(\cdot, h, t) = -\frac{\partial}{\partial h} p^\epsilon(\cdot, h, t)\Psi + p^\epsilon(\cdot, h, t)Q^\epsilon(h) \\ p^\epsilon(\cdot, h, 0) = p_0(\cdot, h) \end{cases} \tag{2.20}$$

where p_0 is the initial distribution.

In order to find the two-scale approximation, we notice that $\tilde{Q}$ can be rewritten as

$$\tilde{Q}(h) = \operatorname{diag}\left(\begin{pmatrix} -\lambda(h) & \lambda(h) \\ \mu(h) & -\mu(h) \end{pmatrix}, \begin{pmatrix} -\lambda(h) & \lambda(h) \\ \mu(h) & -\mu(h) \end{pmatrix}\right)$$

with both sub-generator weakly irreducible, with invariant measure

$$\nu(h) = \left(\frac{\mu(h)}{\mu(h)+\lambda(h)}, \frac{\lambda(h)}{\mu(h)+\lambda(h)}\right) = (\mu(h), \lambda(h))$$

and we identify the classes

$$\begin{aligned} \bar{s}_0 &= \{(0,0), (1,0)\} = \{i_{01}, i_{02}\} \\ \bar{s}_1 &= \{(0,1), (1,1)\} = \{i_{11}, i_{12}\} \end{aligned}$$

that form the slow-scale states $\bar{S} = \{\bar{s}_0, \bar{s}_1\}$. By Corollary 2.3.4, we approximate the law of process $(Z_t)_t = (X_t, H_t)_t$ to the law of a two-scale process $(W_t)_t = (\tilde{X}_t, H_t)_t$ with states in $\bar{S} \times \mathbb{R}$, and X_{fast} with states in $\bar{s}_{\tilde{X}_t}$, such that

$$\mathbb{P}[Z_t = (i_{kx}, h)] \approx \mathbb{P}\left[X_{\text{fast}} = i_{kx}\right] \mathbb{P}[W_t = (\bar{s}_k, h)] \tag{2.21}$$

with a magnitude error of order ϵ. On the fast-scale, $X_{\text{fast}} \in \bar{s}_{\tilde{X}_t}$ and it follows that

$$\begin{aligned} \mathbb{P}\left[X_{\text{fast}} = i_{k1}\right] &= \mu(H_t), \quad \text{for } k = 0, 1 \\ \mathbb{P}\left[X_{\text{fast}} = i_{k2}\right] &= \lambda(H_t), \quad \text{for } k = 0, 1 \end{aligned}$$

On the other hand, $(\tilde{X}_t, H_t)_t \in \bar{S} \times \mathbb{R}$ is a Markov switching process on the slow-scale with generator $\bar{Q}(h)$ given by

$$\begin{aligned} \bar{Q}(h) &= \begin{pmatrix} \nu(h) & 0 \\ 0 & \nu(h) \end{pmatrix} \hat{Q}(h) \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & \mathbb{1}_2 \end{pmatrix} \\ &= \begin{pmatrix} -\lambda(h) & \lambda(h) \\ \mu(h) & -\mu(h) \end{pmatrix} \end{aligned}$$

and vector field functions $\bar{\psi}$ that satisfies

$$\begin{aligned}
\bar{\psi}(k,h) &= \sum_{x\in\bar{s}_k} \nu_x(h)\left[\alpha\left(2x_1-1\right)+\beta\left(2x_2-1\right)\right] \\
&= \mu(h)\left[-\alpha+\beta\left(2k-1\right)\right]+\lambda(h)\left[\alpha+\beta\left(2k-1\right)\right] \\
&= \alpha\left(\lambda(h)-\mu(h)\right)+\beta\left(2k-1\right)
\end{aligned}$$

We consider the probability $\theta(k,h,t)=\mathbb{P}\left[W_t=(\bar{s}_k,h)\right]$ and it is solution of the forward equation

$$\begin{cases}
\dfrac{\partial}{\partial t}\theta(k,h,t)=-\dfrac{\partial}{\partial h}\left[\bar{\psi}(k,h)\theta(k,h,t)\right]+\displaystyle\sum_{p=1}^{2}\theta(p,h,t)\bar{Q}_{pk}(h) \\
\theta(k,h,0)=\displaystyle\sum_{i\in\bar{s}_k} p_0(i,h)
\end{cases}$$

Example 3 - Monte Carlo

In this section, we will compute the distribution function of the process $(Z_t)_{t\in[0,T]}=(X_t,H_t)_{t\in[0,T]}$, defined in Example 2, via a Monte Carlo estimate of its law; and also, via a Monte Carlo estimate of the law of the two-scale approximation process derived on equation (2.21). For the simulation of the process X_t in $I=\{1,\dots,4\}$ and H_t in $\mathbb{R}$, we follow the steps on Section 2.2.1. We set a simulation time $T\geq 0$ and a time step Δt and we define the number of steps $N=T/\Delta t$. We define jump rate $\Lambda(x,\phi(x,h,t))$ and the Markov Kernel $\Pi(h)=\left(\Pi(x,y,h)\right)_{x,y\in I}$ as

$$\Lambda(\cdot,h,t)=\begin{cases}
\left(\dfrac{1+\epsilon}{\epsilon}\right)\lambda(h-(\alpha+\beta)t) \\
\dfrac{1}{\epsilon}\mu(h+(\alpha-\beta)t)+\lambda(h+(\alpha-\beta)t) \\
\dfrac{1}{\epsilon}\lambda(h+(-\alpha+\beta)t)+\mu(h+(-\alpha+\beta)t) \\
\left(\dfrac{1+\epsilon}{\epsilon}\right)\mu(h+(\alpha+\beta)t)
\end{cases}$$

and

$$\Pi(h)=\begin{pmatrix}
0 & \left(\frac{1}{\epsilon+1}\right) & \left(\frac{\epsilon}{\epsilon+1}\right) & 0 \\
\left(\frac{\mu(h)}{\lambda(h)\epsilon+\mu(h)}\right) & 0 & 0 & \left(\frac{\lambda(h)\epsilon}{\lambda(h)\epsilon+\mu(h)}\right) \\
\left(\frac{\mu(h)\epsilon}{\lambda(h)+\mu(h)\epsilon}\right) & 0 & 0 & \left(\frac{\lambda(h)}{\lambda(h)+\mu(h)\epsilon}\right) \\
0 & \left(\frac{\epsilon}{\epsilon+1}\right) & \left(\frac{1}{\epsilon+1}\right) & 0
\end{pmatrix}$$

We also need to compute $\bar{\Lambda}(x,h)$, a rate that bound of $\Lambda(x,\phi(x,h,t))$ for all $t\geq 0$. As $\lambda(h),\mu(h)\leq 1$ for all h, we have $\bar{\Lambda}(x,h)=\frac{1+\epsilon}{\epsilon}$. We set an array $(X_k,H_k)_{k=1}^{N+1}$ for the process and a variable S for the time of jumps. The pseudo-code goes as follow:

1. For the initial condition we set $X_1=x$, $H_1=h$, $k=2$ and $S=0$.
2. While $k\leq N+1$ and $S\leq T$ do:

(a) To compute the sejour time

i. With $U_1 \sim U(0,1)$ set $\xi = -\bar{\Lambda}^{-1}\log(U_1)$

ii. With $U_2 \sim U(0,1)$, if $U_2 \leq \frac{\Lambda(X_{k-1},H_{k-1},\xi)}{\bar{\Lambda}}$ set $S = S + \xi$

iii. else return to i.

(b) While $k \leq N$ and $k\Delta t \leq S$ do:

i. $X_k = X_{k-1}$

ii. $H_k = H_{k-1} + \Delta t\, \psi(X_{k-1}, H_{k-1})$

iii. $k = k+1$

(c) With $U_3 \sim U(0,1)$

i. if $U_3 \leq \Pi(X_{k-1}, 1, H_{k-1})$ then $X_k = 1$ and $H_k = H_{k-1}$

ii. if $\displaystyle\sum_{w=1}^{j-1} \Pi(X_{k-1}, w, H_{k-1}) < U_3 \leq \sum_{w=1}^{j} \Pi(X_{k-1}, w, H_{k-1})$ for $j = 2,3,4$, then $X_k = j$ and $H_k = H_{k-1}$

(d) $k = k+1$

For the process $(W_t)_{t\in[0,T]} = \left(\tilde{X}_t, H_t\right)_{t\in[0,T]}$ over $\bar{S}\times\mathbb{R}$, we then define the jump rate $\tilde{\Lambda}(k, \bar{\phi}(k,h,t))$ and the Markov Kernel $\tilde{\Pi}(h) = \left(\tilde{\Pi}(k,p,h)\right)_{k,p\in\bar{S}}$ as

$$\tilde{\Lambda}(\cdot,h,t) = \begin{cases} \lambda(h + \alpha\left(\lambda(h) - \mu(h)\right)t - \beta t) \\ \mu(h + \alpha\left(\lambda(h) - \mu(h)\right)t + \beta t) \end{cases}, \ \tilde{\Pi}(h) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

We also need to compute $\bar{\tilde{\Lambda}}(x,h)$, a rate that bound of $\tilde{\Lambda}(x,\phi(x,h,t))$ for all $t \geq 0$. As $\lambda(h), \mu(h) \leq 1$ for all h, we have $\bar{\Lambda}(x,h) = 1$. We set an array $\left(\tilde{X}_k, H_k\right)_{k=1}^{N+1}$ for the process the pseudo-code is identical to the one already described.

In order to approximate the law p^ϵ of $(X_t, H_t)_{t\in[0,T]}$ via the Monte Carlo method, we take into account M realizations of $(X_k^m, H_k^m)_{k=1}^{N+1}$ for $m = 1,\ldots,M$, then we have that

$$p_{MC}(x,A,T) = \frac{1}{M}\sum_{m=1}^{M} \mathbb{1}_{\{X_N^m = x,\ H_N^m \in A\}}, \quad x = 1,\ldots,4,\ A \in \mathbb{R}$$

with standard deviation $\sqrt{\frac{p_{MC}(x,A,T)(1-p_{MC}(x,A,T))}{M}}$. And similarly, for the two-sale approximation (2.21), we set M realizations of $\left(\tilde{X}_k^m, H_k^m\right)_{k=1}^{N+1}$ for $m = 1,\ldots,M$, and then it follows

$$p^*_{MC}(\cdot,A,T) = \begin{cases} \frac{1}{M}\sum_{m=1}^{M} \mu(H_N^m)\mathbb{1}_{\{\tilde{X}_N^m = \bar{s}_0,\ H_N^m \in A\}} \\ \frac{1}{M}\sum_{m=1}^{M} \lambda(H_N^m)\mathbb{1}_{\{\tilde{X}_N^m = \bar{s}_0,\ H_N^m \in A\}} \\ \frac{1}{M}\sum_{m=1}^{M} \mu(H_N^m)\mathbb{1}_{\{\tilde{X}_N^m = \bar{s}_1,\ H_N^m \in A\}} \\ \frac{1}{M}\sum_{m=1}^{M} \lambda(H_N^m)\mathbb{1}_{\{\tilde{X}_N^m = \bar{s}_1,\ H_N^m \in A\}} \end{cases}$$

and standard deviation $\sqrt{\frac{p^*_{MC}(x,A,T)(1-p^*_{MC}(x,A,T))}{M}}$.

Solving for $p_0 = (\delta_0, 0, 0, 0)$, $\epsilon = 0.2$, $\alpha = 2$, $\beta = 1$, $\epsilon = 0.2$ and $M = 3000$, the results are the followings:

X_T at $T = 5$	p_{MC}	std	p^*_{MC}	std
1	0.3133	0.0084662	0.3366	0.0086275
2	0.1906	0.0071720	0.1780	0.0069846
3	0.1913	0.0071816	0.1952	0.0072371
4	0.305	0.0084059	0.2901	0.0082852

Table 2.1: Comparatives result for the distribution function of X_t at $t = 5$, between the Monte Carlo approx. p_{MC}, and the Monte Carlo approx. of the two-scales approximation p^*_{MC}.

X_T at $T = 10$	p_{MC}	std	p^*_{MC}	std
1	0.3133	0.0084687	0.3546	0.0087343
2	0.1876	0.0071285	0.1633	0.0067501
3	0.1936	0.0072148	0.1873	0.0071242
4	0.3053	0.0084084	0.2946	0.0083231

Table 2.2: Comparatives result for the distribution function of X_t at $t = 10$, between the Monte Carlo approx. p_{MC}, and the Monte Carlo approx. of the two-scales approximation p^*_{MC}.

X_T at $T = 15$	p_{MC}	std	p^*_{MC}	std
1	0.296	0.0083343	0.3502	0.0087097
2	0.191	0.0071863	0.1577	0.0066548
3	0.195	0.0072336	0.1881	0.0071352
4	0.3173	0.0084977	0.3038	0.0083971

Table 2.3: Comparatives result for the distribution function of X_t at $t = 15$, between the Monte Carlo approx. p_{MC}, and the Monte Carlo approx. of the two-scales approximation p^*_{MC}.

Chapter 3

Hodgkin and Huxley model

The Hodgkin and Huxley model (H&H) [8], describes the membrane potential V of a typical neuron on the mean behavior of the potassium (K^+) and calcium (Na^+) ion channels, through its voltage-gated processes. Each channel contains four separate voltage-gates that are open or closed depending on the voltage variation; and it is in an open-state (conductance) if all four gates are open, and close-state (non-conductance) if at least one gate is close. The voltage equation is given by

$$\begin{cases} \dfrac{dV(t)}{dt} & = & f\left(V(t), m(t), h(t), n(t)\right) \\ \dfrac{dm(t)}{dt} & = & \alpha_m(V(t))\left(1-m(t)\right) - \beta_m(V(t))m(t) \\ \dfrac{dh(t)}{dt} & = & \alpha_h(V(t))\left(1-h(t)\right) - \beta_h(V(t))h(t) \\ \dfrac{dn(t)}{dt} & = & \alpha_n(V(t))\left(1-n(t)\right) - \beta_n(V(t))n(t) \end{cases} \tag{3.1}$$

for all $t \geq 0$ and initial condition $V(0) = V_0, m(0) = m_0, h(0) = h_0$ and $n(0) = n_0$; where $f : \mathbb{R}^4 \to \mathbb{R}$ such that

$$f\left(V, m, h, n\right) = \frac{1}{C}(I_{\text{ext}} - \overbrace{\mathrm{g}_{\text{Na}} m^3 h\left(V - E_{\text{Na}}\right)}^{I_{\text{Na}}} - \overbrace{\mathrm{g}_{\text{K}} n^4\left(V - E_{\text{K}}\right)}^{I_{\text{K}}} - \overbrace{\mathrm{g}_{\text{L}}\left(V - E_{\text{L}}\right)}^{I_{\text{L}}})$$

The dimensionless variables m, h, and n describe the probability of open voltage-gates on each ion-channel. The rate functions that appear in the equations were also determined by Hodgkin and Huxley and are given by

$$\begin{aligned} \alpha_m(V) &= \frac{(25-V)/10}{e^{(25-V)/10}-1}, & \beta_m(V) &= 4e^{-V/18} \\ \alpha_h(V) &= 0.07e^{-V/20}, & \beta_h(V) &= \frac{1}{e^{(30-V)/10}+1} \\ \alpha_n(V) &= \frac{(10-V)/100}{e^{(10-V)/10}-1}, & \beta_n(V) &= 0.125e^{-V/80} \end{aligned} \tag{3.2}$$

The parameters provided in the original paper correspond to the membrane potential shifted by approximately 65 mV so that the resting potential is at $V \approx 0$. The equilibrium potentials and typical conductance are

$$\begin{array}{lll} E_{\mathrm{K}} = -12 \text{ mV}, & E_{\mathrm{Na}} = 120 \text{ mV}, & E_{\mathrm{L}} = 10.6 \text{ mV}, \\ g_{\mathrm{K}} = 36 \text{ mS/cm}^2, & g_{\mathrm{Na}} = 120 \text{ mS/cm}^2, & g_{\mathrm{L}} = 0.3 \text{ mS/cm}^2; \end{array}$$

and $C = 1\ \mu F/cm^2$ is the membrane capacitance.

3.1 H&H as limit of PDMP

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_t, \mathbb{P})$ be a complete filtered probability space and a finite $T > 0$. We present a neuron model (see [9],[10]) with N_{K} potassium channels and N_{Na} sodium channels ($N = N_{\mathrm{K}} + N_{\mathrm{Na}}$), as a PDMP

$$\left(Z_t^N\right)_{t\in[0,T]} = \left(V_t^N, m_t^{N_{\mathrm{Na}}}, h_t^{N_{\mathrm{Na}}}, n_t^{N_{\mathrm{K}}}\right)_{t\in[0,T]}$$

where V_t^N is the membrane potential with values in $\mathbb{R}$, and $m_t^{N_{\mathrm{Na}}}, h_t^{N_{\mathrm{Na}}}$ and $n_t^{N_{\mathrm{K}}}$ correspond to the proportion of voltage-gates of type m, h, n respectably. The voltage-gates are defined by the sequences $\{e_i^m(t)\}_{i=1}^{N_{\mathrm{Na}}}$, $\{e_i^h(t)\}_{i=1}^{N_{\mathrm{Na}}}$ and $\{e_i^n(t)\}_{i=1}^{N_{\mathrm{K}}}$, such that for all i and $u \in \{m, h, n\}$, $e_i^u(t) \in \{0, 1\}$ and

$$e_i^u(t) : \quad 0 \quad \underset{\beta_u(V)}{\overset{\alpha_u(V)}{\rightleftarrows}} \quad 1 \tag{3.3}$$

with rates α_u and β_u as defined in equations (3.2). Then, the variables $u_t^{N_u}$, for $u \in \{m, h, n\}$, are defined by

$$u_t^{N_u} = \frac{1}{N_u}\sum_{i=1}^{N_u} e_i^u(t)$$

characterized by

- Space state $E_{N_u} = \left\{0, \frac{1}{N_u}, \frac{2}{N_u}, \ldots, \frac{N_u - 1}{N_u}, 1\right\}$,
- jump rate $\lambda_{N_u} : \mathbb{R} \times E_{N_u} \to \mathbb{R}_+$, such that

$$\lambda_{N_u}(V, u) = N_u\left[u\beta_u(V) + (1 - u)\alpha_u(V)\right] \tag{3.4}$$

  which is time-dependent through V,
- Markov transition kernel Π such that:

$$\begin{aligned} \Pi((V, u), \{(V, u + 1/N_u)\}) &= \frac{(1 - u)\alpha_u(V)}{u\beta_u(V) + (1 - u)\alpha_u(V)} \\ \Pi((V, u), \{(V, u - 1/N_u)\}) &= \frac{u\beta_u(V)}{u\beta_u(V) + (1 - u)\alpha_u(V)} \end{aligned} \tag{3.5}$$

for all $V \in \mathbb{R}$ and $m \in E_{N_u} \setminus \{1, N_u\}$, and

$$\begin{aligned}\Pi((V,0), \{(V, 1/N_u)\}) = 1\\ \Pi((V,1), \{(V, (N_u-1)/N_u)\}) = 1\end{aligned}$$

for all $V \in \mathbb{R}$.

The corresponding membrane equation is

$$\begin{cases} \dfrac{d}{dt} V_t^N = f\left(V_t^N, m_t^{N_{\mathrm{Na}}}, h_t^{N_{\mathrm{Na}}}, n_t^{N_{\mathrm{K}}}\right) \\ u_t^{N_u} = \dfrac{1}{N_u} \displaystyle\sum_{i=1}^{N_u} e_i^u(t), \quad u = \{m, h, n\} \end{cases} \tag{3.6}$$

where e_i^u are defined in (3.3), initial condition $Z_0^N \in \mathbb{R} \times E_N$ and $f : \mathbb{R} \times [0,1]^3 \to \mathbb{R}$ such that

$$f(V, m, h, n) = \frac{1}{C}\left(I_{\mathrm{ext}} - g_{\mathrm{Na}} m^3 h (V - E_{\mathrm{Na}}) - g_{\mathrm{K}} n^4 (V - E_{\mathrm{K}}) - g_L (V - E_L)\right)$$

Under suitable initial conditions, the solution $Z_t^N = \left(V_t^N, m_t^{N_{\mathrm{Na}}}, h_t^{N_{\mathrm{Na}}}, n_t^{N_{\mathrm{K}}}\right)$ of (3.6) converges in probability as N grows to infinity, uniformly on bounded intervals $[0,T]$, to the solution $Z(t) = (v(t), n(t), m(t), h(t))$ of the deterministic equation:

$$\begin{cases} \dfrac{d}{dt} v(t) &= f(v(t), m(t), h(t), n(t)) \\ \dfrac{d}{dt} g(t) &= \alpha_g(v(t))(1 - g(t)) - \beta_g(v(t)) g(t), \quad g \in \{m, , h, n\} \end{cases} \tag{3.7}$$

when the following conditions are satisfied,

H1 α_u and $\beta_u \in C^1$ for $u \in \{m, h, n\}$

H2 $f \in C^1$

H3 The process v from (3.7) is bounded on $[0,T]$ with $T > 0$, and for all $N \geq 1$ the process V_N (3.6) is uniformly bounded on $[0,T]$.

Since the process of opening and closing by assumptions are asymptotically independent among species, we can consider the study of only one gate type. Thus, we will consider the equation

$$\begin{cases} \dfrac{d}{dt} V_t^N = f\left(V_t^N, u_t^N\right) \\ u_t^N = \dfrac{1}{N} \displaystyle\sum_{i=1}^{N} e_i(t) \end{cases} \tag{3.8}$$

where $\{e_i\}_{i=1\ldots N}$ are analogously defined as in (3.3), and the deterministic system is given by

$$\begin{cases} \dfrac{d}{dt} v(t) &= f(v(t), g(t)) \\ \dfrac{d}{dt} g(t) &= \alpha(v(t))(1 - g(t)) - \beta(v(t)) g(t) \end{cases} \tag{3.9}$$

We assume the conditions $H1 - H3$ are satisfied for this reduced case. Thus, we have the following theorem as it is presented in [10].

Theorem 3.1.1 Law of large number.
Let $\{e_i\}_{i=1\ldots N}$ *be a succession of Markov processes defined in* (3.3)*, and let* $Z_0 = (v_0, g_0) \in \mathbb{R} \times [0,1]$ *be a initial condition of* (3.9)*. For all* $\delta, \varepsilon > 0$*, there exists an initial condition* $Z_0^N = \left(V_0^N, u_0^N\right)$ *for* (3.8) *and* $N_0 = N_0(\delta, \varepsilon)$ *such that for all* $N \geq N_0$ *the solution* $Z_t^N = \left(V_t^N, u_t^N\right)$ *satisfies:*

$$\mathbb{P}\left(\sup_{0\leq t\leq T} |V_t^N - v(t)| > \delta\right) < \varepsilon \tag{3.10}$$

and

$$\mathbb{P}\left(\sup_{0\leq t\leq T} |u_t^N - g(t)| > \delta\right) < \varepsilon \tag{3.11}$$

for all fixed $T > 0$.

First we decompose the difference between the stochastic and the deterministic processes as a sum of a martingale part M_N and a finite variation part F_N as follows

$$[u_t^N - u_0^N] - [g(t) - g(0)] = M_N(t) + \int_0^t F_N(s)ds$$

where we define:

$$\begin{aligned} F_N(t) &:= \alpha(V_t^N)(1 - u_t^N) - \beta(V_t^N)u_t^N - \frac{dg(t)}{dt} \\ M_N(t) &:= \left[u_t^N - g(t)\right] - \left[u_0^N - g(0)\right] - \int_0^t F_N(s)ds \end{aligned}$$

For the proof of Theorem 3.1.1, we need the following lemmas.

Lemma 3.1.2 $M_N(t)$ *is a* $\{\mathcal{F}_t\}$*-martingale*

Proof First we notice that for all i and $h \downarrow 0$

$$\begin{aligned} \mathbb{E}\left[e_i(t+h)|\mathcal{F}_t\right] &= \mathbb{E}\left[e_i(t+h)|e_i(t)\right] = \mathbb{P}\left[e_i(t+h) = 1|e_i(t)\right] \\ &= \alpha(V_t^N)(1 - e_i(t))h + \left[1 - \beta(V_t^N)\right] e_i(t)h + o(h) \end{aligned}$$

and

$$\begin{aligned} \frac{1}{h}\mathbb{E}\left[u_{t+h}^N - u_t^N|\mathcal{F}_t\right] &= \alpha(V_t^N)(1 - u_t^N) - \beta(V_t^N)u_t^N + o(1) \\ &= \mathbb{E}\left[\alpha(V_t^N)(1 - u_t^N) - \beta(V_t^N)u_t^N|\mathcal{F}_t\right] + o(1) \end{aligned}$$

Then we have that

$$\begin{aligned} \frac{1}{h}\mathbb{E}\left[M_N(t+h) - M_N(t)|\mathcal{F}_t\right] &= \frac{1}{h}\mathbb{E}\left[u_{t+h}^N - u_t^N|\mathcal{F}_t\right] \\ &\quad - \frac{1}{h}\mathbb{E}\left[\int_t^{t+h} \alpha(V_s^N)(1 - u_s^N) - \beta(V_s^N)u_s^N\ ds|\mathcal{F}_t\right] \\ &\quad - \frac{1}{h}\left[g(t+h) - g(t)\right] + \frac{1}{h}\int_t^{t+h} \frac{dg(s)}{dt}ds \end{aligned}$$

Solving gives $\lim_{h\to 0} \frac{1}{h}\mathbb{E}\left[M_N(t+h) - M_N(t)|\mathcal{F}_t\right] = 0$ and therefore $\frac{d}{ds}\mathbb{E}\left[M_N(t+s)|\mathcal{F}_t\right]|_{s=0} = 0$. Finally $\mathbb{E}\left[M_N(t+h)|\mathcal{F}_t\right] = cte = \mathbb{E}\left[M_N(t)|\mathcal{F}_t\right] = M_N(t)$. ∎

Lemma 3.1.3 *Let $T > 0$, $\varepsilon > 0$, $\delta > 0$. Then there exists N_0 such that $\forall N \geq N_0$*

$$\mathbb{P}\left(\sup_{0\leq t\leq T} M_N(t)^2 \geq \delta\right) \leq \varepsilon$$

Proof Using $(a+b+c)^2 \leq 4(a^2+b^2+c^2)$ and CauchySchwarz inequality we have that

$$\mathbb{E}\left[M_N(t)^2\right] \leq 4\mathbb{E}\left[u_t^N - g(t)\right]^2 + 4\left[u_0^N - g(0)\right]^2 + 4t\int_0^t \mathbb{E}\left[F_N^2(s)\right] ds$$

On another side, considering $\mathbb{E}\left[e_i(t) - g(t)\right] = 0$ for all i, and then

$$\mathbb{E}\left[u_t^N - g(t)\right]^2 = \mathbb{E}\left[\frac{1}{N}\sum_{i=1}^{N} e_i(t) - g(t)\right]^2 = \frac{1}{N^2}\sum_{i=1}^{N}\mathbb{E}\left[e_i(t) - g(t)\right]^2 \leq \frac{1}{N}$$

as $\mathbb{E}\left[e_i(t) - g(t)\right]^2 = g(t)\left(1 - g(t)\right) \leq 1$ for all i, and

$$\begin{aligned}
\mathbb{E}\left[F_N^2(t)\right] &= \mathbb{E}\left[\left(\alpha(v(t))g(t) - \alpha(V_t^N)u_t^N\right) - \left(\beta(v(t))g(t) - \beta(V_t^N)u_t^N\right)\right]^2 \\
&\leq 2\mathbb{E}\left[\|\alpha\|_\infty^2\left(g(t) - u_t^N\right)^2 + \|\beta\|_\infty^2\left(g(t) - u_t^N\right)^2\right] \\
&\leq 4\max\{\|\alpha\|_\infty^2, \|\beta\|_\infty^2\}\mathbb{E}\left[u_t^N - g(t)\right]^2 \leq \tfrac{4}{N}\max\{\|\alpha\|_\infty^2, \|\beta\|_\infty^2\}
\end{aligned}$$

Finally

$$\mathbb{E}\left[M_N(t)^2\right] \leq C_1\frac{T^2}{N}\max\{\|\alpha\|_\infty^2, \|\beta\|_\infty^2\}$$

where $\|\alpha\|_\infty$ and $\|\beta\|_\infty$ are finite because α and β are continuous by assumption $H1$. Then by Chebychev inequality and Doob inequalities for martingales:

$$\mathbb{P}\left(\sup_{0\leq t\leq T} M_N(t)^2 \geq \delta\right) \leq \frac{1}{\delta}\mathbb{E}\left[\sup_{0\leq t\leq T} M_N(t)\right]^2 \leq \frac{4}{\delta}\mathbb{E}\left[M_N(t)^2\right] \leq \bar{C}\frac{4T^2}{\delta N}$$

and $\mathbb{E}\left[M_N(t)^2\right] \leq \frac{\varepsilon\delta}{4}$ for all $N \geq N_0 = \frac{\bar{C}T^2}{\varepsilon}$. ∎

Lemma 3.1.4 *For the finite variation term $F_N(t)$, there exists $C > 0$ independent of N such that*

$$|F_N(t)| \leq C\left(|u_t^N - g(t)| + |V_t^N - v(t)|\right)$$

Proof First we notice that

$$\begin{aligned}
F_N(t) &= \alpha(V_t^N)(1 - u_t^N) - \alpha(v(t))(1 - g(t)) \\
&\quad -\beta(V_t^N)u_t^N + \beta(v(t))g(t)
\end{aligned}$$

In order to use the Lipschitz property of α and β, we separate last equation into two,

$$\begin{aligned}
&F_N^1(t) := \beta(V_t^N)u_t^N - \beta(v(t))g(t) \\
&F_N^2(t) := \alpha(V_t^N)(1 - u_t^N) - \alpha(v(t))(1 - g(t))
\end{aligned}$$

For the first term we have that

$$
\begin{aligned}
F_N^1(t) &= \beta(V_t^N)u_t^N - \beta(v(t))g(t)\\
&= \beta(V_t^N)\left(u_t^N - g(t)\right) + g(t)\left(\beta(V_t^N) - \beta(v(t))\right)\\
&\leq \|\beta\|_\infty(u_t^N - g(t)) + K_\beta(V_t^N - v(t))
\end{aligned}
$$

given that $g(t) \in [0,1]$. Identically we have that

$$F_N^2(t) \leq \|\alpha\|_\infty(u_t^N - g(t)) + K_\alpha(V_t^N - v(t))$$

Finally

$$|F_N(t)| \leq C\left(|u_t^N - g(t)| + |V_t^N - v(t)|\right) \tag{3.12}$$

with $C = \max\{\|\alpha\|_\infty, \|\alpha\|_\infty, K_\alpha, K_\beta\}$. ∎

Now we have all the necessary tools to prove Theorem 3.1.1.

Proof *Theorem* 3.1.1. We want to apply the Gronwall lemma to the function

$$f(t) = |V_t^N - v(t)|^2 + |u_t^N - g(t)|^2$$

As $\left[u_t^N - g(t)\right] = \left[u_0^N - g(0)\right] + M_N(t) + \int_0^t F_N(s)ds$ and using the last lemma, $(a+b+c)^2 \leq 4(a^2 + b^c + c^2)$ and CauchySchwarz inequality we have that

$$\left[u_t^N - g(t)\right]^2 = 4\left[u_0^N - g(0)\right]^2 + 4M_N(t)^2 + 8tC^2\int_0^t \left[(u_s^N - g(s))^2 + (V_s^N - v(s))^2\right]ds$$

Wee need now to work on $\left(V_t^N - v(t)\right)^2$, using hypothesis $H1$ between the jumps, we have

$$
\begin{aligned}
K_1 &= \sup_N \sup_{0\leq s\leq T} |\frac{\partial f}{\partial v}(V_s^N, u_s^N)|\\
K_2 &= \sup_N \sup_{0\leq s\leq T} |\frac{\partial f}{\partial u}(V_s^N, u_t^N)|
\end{aligned}
$$

thus,

$$\frac{d}{dt}\left(V_s^N - v(t)\right)^2 = 2\left[f(V_s^N, u_t^N) - f(v(t), g(t))\right]\left(V_t^N - v(t)\right)$$

Then, by CauchySchwarz inequality and $ab \leq \frac{1}{2}(a^2 + b^2)$ holds

$$
\begin{aligned}
\left(V_t^N - v(t)\right)^2 &= \left(V_0^N - v(0)\right)^2 + 2\int_0^t \left[f(V_s^N, u_s^N) - f(v(s), g(s))\right]\left((V_s^N - v(s)\right)ds\\
&= \left(V_0^N - v(0)\right)^2 + 2K_1\int_0^t \left[V_s^N - v(s)\right]^2 ds\\
&+ \ 2K_1\int_0^t \left[u_s^N - g(s)\right]\left[V_s^N - v(s)\right]ds\\
&= \left(V_0^N - v(0)\right)^2 + 2K_1\int_0^t \left[V_s^N - v(s)\right]^2 ds\\
&+ \ K_2\int_0^t \left[u_s^N - g(s)\right]^2 ds + K_2\int_0^t \left[V_s^N - v(s)\right]^2 ds
\end{aligned}
$$

Putting together both inequality we obtain:

$$f(t) \leq A + B \int_0^t f(s)ds$$

where $B = \max\left(2K_1 + K_2, 8TC^2\right)$ that does not depend on N, and $A(N) = \left[V_0^N - v(0)\right]^2 + 4\left[u_0^N - g(0)\right]^2 + 4 \sup_{0\leq t\leq T} M_N(t)^2$. We control the initial condition and we control the martingale, and A can be chosen arbitrarily small with great probability, then for $\varepsilon > 0$ there exists N_0 such that $A(N) \leq \varepsilon$ for all $N \geq N_0$. By Gronwall lemma 1.2.3 we have $f(t) \leq \varepsilon \exp(BT)$ for all $t \in [0, T]$ and $N \geq N_0$. The proof conclude with Chebychev and Doob inequalities for each term. ∎

Remark In order to include the 3 different voltage-gates of type $u \in \{m, h, n\}$, one should just write the same arguments for all the 3 processes u_t^N, and include all the $|u_t^N - g(t)|^2$ in the function $f(t)$ of the Gronwall lemma.

3.2 Langevin approximation

A second result corresponds to a central limit theorem that provides a way to build a diffusion or Langevin approximation to the solution of the stochastic system (3.8). As before let $Z_t^N = (V_t^N, u_t^N) \in \mathbb{R} \times E_N$ be solution of the system (3.8), and we define the process $(R_t^N)_{t\in[0,T]}$ such that

$$R_t^N := \sqrt{N}\left(u_t^N - u_0^N - \int_0^t b(V_s^N, u_s^N)ds\right) \tag{3.13}$$

where $b(V, u) := (1 - u)\alpha(V) - u\beta(V)$.

Theorem 3.2.1 *Under the same hypothesis of Theorem 3.1.1, the process $(R_t^N)_t$ defined in* (3.13)*, converges in law as $N \to \infty$ to the process $(R_t)_{t\in[0,T]}$ with*

$$R_t = \int_0^t \sqrt{b\left(v(s), g(s)\right)} dW_s \tag{3.14}$$

where $Z(t) = (v(t), g(t))$ is solution of the deterministic system (3.9) *with initial condition $Z(0) = Z_0^N$ for all N, and $(W_t)_{t\in[0,T]}$ is a standard Brownian motion in $\mathbb{R}$.*

Proof First, we notice that

$$\mathbb{P}\left[\sup_{s\leq T}\left|R_t^N\right| \geq \delta\right] \leq \frac{TN}{\delta^2}\left(\|\alpha\|_\infty + \|\beta\|_\infty\right)$$

We want to compute the characteristic function $\phi_N(\theta, t)$ of R_t^N, defined by

$$\phi_N(\theta, t) := \mathbb{E}\left[\exp\left(i\theta R_t^N\right)\right]$$

We define the process $(M_t^N)_t$, function of (V_t^N, u_t^N) such that

$$M_t^N = \frac{1}{\sqrt{N}} R_t^N = u_t^N - u_0^N - \int_0^t b(V_s^N, u_s^N) ds$$

with infinitesimal generator L that satisfies

$$Lh(M) = h'(M) b(V,u) + \lambda_N(V,u) \sum_{w \in E_N} \left(h(w-u+M) - h(M)\right) \Pi\left((V,u), \{(V,w)\}\right)$$

for all $h \in D(L)$, and due the definition of λ_N in (3.4) and Π in (3.5), we have that

$$b(V,u) = \lambda_N(V,u) \sum_{w \in E_N} (w-u) \Pi\left((V,u), \{(V,w)\}\right)$$

If we define functions $h(M) = \exp\left(i\theta\sqrt{N}M\right)$, $\psi(u) = \left(\exp(iu) - 1 - iu + u^2/2\right)/u^2$, and $\xi(u) = \exp(iu) - 1 - iu = u^2\psi(u) - u^2/2$; then it follows that

$$\begin{aligned}
&\phi_N(\theta,t) - 1 = \mathbb{E}\left[h(M_t^N)\right] - h(0) = \int_0^t \mathbb{E}\left[Lh(M_s^N)\right] ds \\
&= \int_0^t \mathbb{E}\left[\lambda_N(V_s^N, u_s^N) \sum_{w \in E_N} \left(h(w - u_s^N + M_s^N) - h(M_s^N) - (w - u_s^N) h'(M_s^N)\right) \Pi\left((V_s^N, u_s^N), \{(V_s^N, w)\}\right)\right] ds \\
&= \int_0^t \mathbb{E}\left[\exp\left(i\theta R_s^N\right) \lambda_N(V_s^N, u_s^N) \sum_{w \in E_N} \xi\left(\theta\sqrt{N}\left(w - u_s^N\right)\right) \Pi\left((V_s^N, u_s^N), \{(V_s^N, w)\}\right)\right] ds \\
&= -\int_0^t \mathbb{E}\left[\frac{1}{2}\exp\left(i\theta R_s^N\right) \lambda_N(V_s^N, u_s^N) \sum_{w \in E_N} N\theta^2 \left(w - u_s^N\right)^2 \Pi\left((V_s^N, u_s^N), \{(V_s^N, w)\}\right)\right] ds \\
&\quad + \int_0^t \mathbb{E}\left[\frac{1}{2}\exp\left(i\theta R_s^N\right) \lambda_N(V_s^N, u_s^N) \sum_{w \in E_N} N\theta^2 \left(w - u_s^N\right)^2 \times \right. \\
&\qquad\qquad \left. \psi\left(\theta\sqrt{N}\left(w - u_s^N\right)\right) \Pi\left((V_s^N, u_s^N), \{(V_s^N, w)\}\right)\right] ds
\end{aligned}$$

the second term in the last equality, said $K^N(\theta)$, converges to 0 as $N \to \infty$ by dominated convergence, and because $\psi\left(\theta\sqrt{N}\left(w - u_s^N\right)\right) = \psi\left(\pm\theta/\sqrt{N}\right) \to 0$ as $\lim_{u \to 0} \psi(u)$. So we have

$$\begin{aligned}
\phi_N(\theta,t) &= -\int_0^t \mathbb{E}\left[\frac{1}{2}\exp\left(i\theta R_s^N\right) b\left(V_s^N, u_s^N\right)\right] ds + K^N(\theta) \\
&= -\frac{1}{2}\int_0^t \theta^2 b\left(v(s), g(s)\right) \phi_N(\theta,s) ds \\
&\quad + \frac{1}{2}\int_0^t \theta^2 \mathbb{E}\left[\left(b\left(v(s), g(s)\right) - b\left(V_s^N, u_s^N\right)\right) \exp\left(i\theta R_s^N\right)\right] ds + K^N(\theta)
\end{aligned}$$

and again, the second term of the inequality, said $J^N(\theta)$ converges to 0 as $N \to \infty$, because of the convergence of $(V_t^N, u_t^N)_t$ to $(v(t), g(t))$ proved in Theorem (3.1.1). By Gronwall lemma

(1.2.3) we conclude that $\phi_N(\theta,t) \to \phi(\theta,t)$ with

$$\phi(\theta,t) = \exp\left(-\frac{1}{2}\theta^2 \int_0^t b\left(v(s), g(s)\right) ds\right)$$

where $(v(t), g(t))_t$ is solution of the deterministic system (3.9). $\blacksquare$

It follows from Theorem 3.2.1 that when N is large enough, u_t^N degenerates and behaves in law equal to the process

$$u_0^N + \int_0^t b\left(V_s^N, u_s^N\right) ds + \int_0^t \sqrt{\frac{1}{N} b\left(V_s^N, u_s^N\right)} dW_s$$

and thus when N is large enough $Z_t^N = \left(V_t^N, u_t^N\right)$, the solution of the system

$$\begin{cases} \dfrac{d}{dt} V_t^N = f\left(V_t^N, u_t^N\right) \\ u_t^N = \dfrac{1}{N} \displaystyle\sum_{i=1}^N e_i(t) \end{cases} \tag{3.15}$$

degenerates and tends identical in law to the diffusion approximation $\tilde{Z}_t^N = \left(\tilde{V}_t^N, \tilde{u}_t^N\right)$, solution of the system

$$\begin{cases} d\tilde{V}_t^N &= f\left(\tilde{V}_t^N, \tilde{u}_t^N\right) dt \\ d\tilde{u}_t^N &= b\left(\tilde{V}_t^N, \tilde{u}_t^N\right) dt + \sqrt{\dfrac{1}{N} b\left(\tilde{V}_t^N, \tilde{u}_t^N\right)} dW_t \end{cases} \tag{3.16}$$

with initial condition $\tilde{Z}_0^N = Z_0^N$.

3.3 State reduction in sodium channels

Let present an H&H model with and alternative characterization of the sodium channels. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_t, \mathbb{P})$ be a complete filtered probability space and a finite $T > 0$. Let consider a neuron with N sodium channels an we define the PDMP $(Z_t^N)_{t\in[0,T]} = (V_t^N, u_t^N)_{t\in[0,T]}$, where V_t^N is the membrane potential with values in $\mathbb{R}$, and u_t^N corresponds to the proportion of open sodium ion channels with values in E_N. Each channel contains four separate voltage-gates: three of type m and one of type h; and it is in an open-state if all four gates are open, and closed-state if at least one gate is closed. Let consider $\{C_i(t)\}_{i=1}^N = \left\{e_i^h(t), e_i^m(t)\right\}_{i=1}^N$ be the sequence of sodium ion channels with states in $\{0,1\} \times \{0,1,2,3\}$, and it can be characterized by the chain

$$\begin{array}{cccccccccc}
 & & m=0 & & m=1 & & m=2 & & m=3 \\
 & h=0 & \bigcirc & \underset{\beta_m}{\overset{3\alpha_m}{\rightleftarrows}} & \bigcirc & \underset{2\beta_m}{\overset{2\alpha_m}{\rightleftarrows}} & \bigcirc & \underset{3\beta_m}{\overset{\alpha_m}{\rightleftarrows}} & \bigcirc \\
C_i(t): & & \alpha_h \downarrow\uparrow \beta_h & & \alpha_h \downarrow\uparrow \beta_h & & \alpha_h \downarrow\uparrow \beta_h & & \alpha_h \downarrow\uparrow \beta_h \\
 & h=1 & \bigcirc & \underset{\beta_m}{\overset{3\alpha_m}{\rightleftarrows}} & \bigcirc & \underset{2\beta_m}{\overset{2\alpha_m}{\rightleftarrows}} & \bigcirc & \underset{3\beta_m}{\overset{\alpha_m}{\rightleftarrows}} & \bigcirc
\end{array}$$

where α, β depend on V and are defined in (3.2) for voltage-gate of type h and m. For any $V \in \mathbb{R}$, $C_i(t)$ is a stochastic process with states in

$$I = \{(0,0),(0,1),(0,2),(0,3),(1,0),(1,1),(1,2),(1,3)\}$$

and generator $Q(V) = (Q_{xy}(V))_{x,y\in I}$, such that

$$Q(V) = \begin{pmatrix} -(3\alpha_m+\alpha_h) & 3\alpha_m & 0 & 0 & \alpha_h & 0 & 0 & 0 \\ \beta_m & -(2\alpha_m+\beta_m+\alpha_h) & 2\alpha_m & 0 & 0 & \alpha_h & 0 & 0 \\ 0 & 2\beta_m & -(\alpha_m+2\beta_m+\alpha_h) & \alpha_m & 0 & 0 & \alpha_h & 0 \\ 0 & 0 & 3\beta_m & -(3\beta_m+\alpha_h) & 0 & 0 & 0 & \alpha_h \\ \beta_h & 0 & 0 & 0 & -(3\alpha_m+\beta_h) & 3\alpha_m & 0 & 0 \\ 0 & \beta_h & 0 & 0 & \beta_m & -(2\alpha_m+\beta_m+\beta_h) & 2\alpha_m & 0 \\ 0 & 0 & \beta_h & 0 & 0 & 2\beta_m & -(\alpha_m+2\beta_m+\beta_h) & \alpha_m \\ 0 & 0 & 0 & \beta_h & 0 & 0 & 3\beta_m & -(3\beta_m+\beta_h) \end{pmatrix}$$

Then, the variable u_t^N is defined by

$$u_t^N = \frac{1}{N}\sum_{i=1}^{N} \mathbb{1}_{\{(1,3)\}}(C_i(t))$$

that is, the ion channel in only open in state $(1,4)$; and the corresponding membrane equation is

$$\begin{cases} \dfrac{d}{dt}V_t^N = I_{\text{ext}} - g_{\text{Na}} u_t^N (V_t^N - E_{\text{Na}}) \\ u_t^N = \dfrac{1}{N}\displaystyle\sum_{i=1}^{N} \mathbb{1}_{\{(1,3)\}}(C_i(t)), \end{cases} \tag{3.17}$$

with initial condition $V_0^N = V_0$ and $u_0^N = u_0$.

3.3.1 Two-scales approximation

In order to perform the approximation, we first need to find the time-scale separation ϵ of the two sub processes. For that, we notice that given the rate function given by H&H, the voltage-gates of type m fluctuate much more rapidly than the voltage-gate h, so we look for some ϵ such that

$$\alpha_m(V) + \beta_m(V) \approx \frac{1}{\epsilon}\left(\alpha_h(V) + \beta_h(V)\right)$$

for all $V \approx [0, 80]$, that is around where the membrane potential fluctuate; that gives us a mean value of $\epsilon = 0.2051$. Then, we have that $Q^\epsilon(V)$ depends on ϵ and two generator $\tilde{Q}(V)$ and $\hat{Q}(V)$ that satisfies

$$Q^\epsilon(V) = \frac{1}{\epsilon}\tilde{Q}(V) + \hat{Q}(V)$$

where the generator $\hat{Q}(V)$ is given by

$$\hat{Q}(V) = \begin{pmatrix} -\alpha_h & 0 & 0 & 0 & \alpha_h & 0 & 0 & 0 \\ 0 & -\alpha_h & 0 & 0 & 0 & \alpha_h & 0 & 0 \\ 0 & 0 & -\alpha_h & 0 & 0 & 0 & \alpha_h & 0 \\ 0 & 0 & 0 & -\alpha_h & 0 & 0 & 0 & \alpha_h \\ \beta_h & 0 & 0 & 0 & -\beta_h & 0 & 0 & 0 \\ 0 & \beta_h & 0 & 0 & 0 & -\beta_h & 0 & 0 \\ 0 & 0 & \beta_h & 0 & 0 & 0 & -\beta_h & 0 \\ 0 & 0 & 0 & \beta_h & 0 & 0 & 0 & -\beta_h \end{pmatrix}$$

and $\tilde{Q}(V) = \operatorname{diag}\left(\tilde{Q}^1(V), \tilde{Q}^2(V)\right)$, such that

$$\tilde{Q}^k(V) = \begin{pmatrix} -3\tilde{\alpha}_m & 3\tilde{\alpha}_m & 0 & 0 \\ \tilde{\beta}_m & -(2\tilde{\alpha}_m + \tilde{\beta}_m) & 2\tilde{\alpha}_m & 0 \\ 0 & 2\tilde{\beta}_m & -(\tilde{\alpha}_m + 2\tilde{\beta}_m) & \tilde{\alpha}_m \\ 0 & 0 & 3\tilde{\beta}_m & -3\tilde{\beta}_m \end{pmatrix}$$

for $k = 1, 2$, and rates $\tilde{\alpha}_m(V)$ and $\tilde{\beta}_m(V)$ given by

$$\begin{aligned} \tilde{\alpha}_m(V) &= \epsilon\alpha_m(V) \\ \tilde{\beta}_m(V) &= \epsilon\beta_m(V) \end{aligned}$$

Each sub-matrix $\tilde{Q}^k(V)$ has invariant distribution given by

$$\nu(V) = \begin{bmatrix} \frac{\tilde{\beta}_m^3}{\tilde{\alpha}_m^3 + 3\,\tilde{\alpha}_m^2\tilde{\beta}_m + 3\,\tilde{\alpha}_m\,\tilde{\beta}_m^2 + \tilde{\beta}_m^3} \\ \frac{3\tilde{\alpha}_m\,\tilde{\beta}_m^2}{\tilde{\alpha}_m^3 + 3\,\tilde{\alpha}_m^2\tilde{\beta}_m + 3\,\tilde{\alpha}_m\,\tilde{\beta}_m^2 + \tilde{\beta}_m^3} \\ \frac{3\tilde{\alpha}_m^2\tilde{\beta}_m}{\tilde{\alpha}_m^3 + 3\,\tilde{\alpha}_m^2\tilde{\beta}_m + 3\,\tilde{\alpha}_m\,\tilde{\beta}_m^2 + \tilde{\beta}_m^3} \\ \frac{\tilde{\alpha}_m^3}{\tilde{\alpha}_m^3 + 3\,\tilde{\alpha}_m^2\tilde{\beta}_m + 3\,\tilde{\alpha}_m\,\tilde{\beta}_m^2 + \tilde{\beta}_m^3} \end{bmatrix}$$

and determines the classes

$$\begin{aligned} \bar{s}_1 &= \{(0,0), (0,1), (0,2), (0,3)\} \\ \bar{s}_2 &= \{(1,0), (1,1), (1,2), (1,3)\} \end{aligned}$$

We define for the slow-scale the generator matrix $\bar{Q}(V)$, that satisfies

$$\begin{aligned} \bar{Q}(V) &= \begin{pmatrix} \nu(V) & 0 \\ 0 & \nu(V) \end{pmatrix} \hat{Q}(V) \begin{pmatrix} \mathbb{1}_4 & 0 \\ 0 & \mathbb{1}_4 \end{pmatrix} \\ &= \begin{pmatrix} -\alpha_h & \alpha_h \\ \beta_h & -\beta_h \end{pmatrix} \end{aligned} \tag{3.18}$$

With this results, we approximate the law of the process $\left(Z_t^N\right)_t = (V_t^N, u_t^N)_t$ to a two-scale process $(W_t)_t = (\tilde{V}_t^N, \tilde{u}_t^N)_t$ and $\{m_{i,\text{fast}}^N\}_{i=1}^N$. The dynamic on fast-scale corresponds to the dynamic of each voltage-gates of type m, $\{e^m_{i,\text{fast}}\}_{i=1}^N$ with sates in $\{0,1,2,3\}$, which satisfies

$$\mathbb{P}\left[e^N_{i,\text{fast}} = x\right] = \nu_x(\tilde{V}^N_t)$$

for all $i = 1,\ldots,N$. And the slow-scale process $(W_t)_t = (\tilde{V}^N_t, \tilde{u}^N_t)_t$, defined over $\mathbb{R}\times[0,1]$, is a Neuron model that only depends on the sequence of voltage-gate $\{e^h_i(t)\}_{i=1}^N$, each one with values in $\{0,1\}$ and generator $\bar{Q}(V)$, that is

$$e^h_i(t): \quad \textcircled{0} \quad \underset{\beta_h(V)}{\overset{\alpha_h(V)}{\rightleftarrows}} \quad \textcircled{1}$$

Then the proportion of open sodium channels $\tilde{u}^N_t$, depends dynamically on the voltage-gates of type h and statically on the voltage-gates of type m, that affects the process from the fast-scale, such that

$$\begin{aligned}\tilde{u}^N_t &= \frac{1}{N}\sum_{i=1}^N\sum_{x=1}^4 \nu_x(\tilde{V}^N_t)\mathbb{1}_{\{(1,3)\}}(e^h_i(t),x)\\ &= \frac{1}{N}\sum_{i=1}^N \nu_4(\tilde{V}^N_t)e^h_i(t)\end{aligned}$$

and the membrane equation is equal to

$$\begin{cases}\dfrac{d}{dt}\tilde{V}^N_t = I_{\text{ext}} - g_{\text{Na}}\tilde{u}^N_t(\tilde{V}^N_t - E_{\text{Na}})\\ \tilde{u}^N = \nu_4(\tilde{V}^N_t)\,\dfrac{1}{N}\displaystyle\sum_{i=1}^N e^h_i(t)\end{cases} \tag{3.19}$$

with initial condition $\tilde{V}^N_0 = V_0$ and $\tilde{u}^N_0 = u^N_0$.

Numerical results

First we present a simulation of the diffusion approximation (3.16) for the original H&H model (3.6); and next, to the same model after the state-reduction method applied on the sodium channel as shown in secction 3.3.

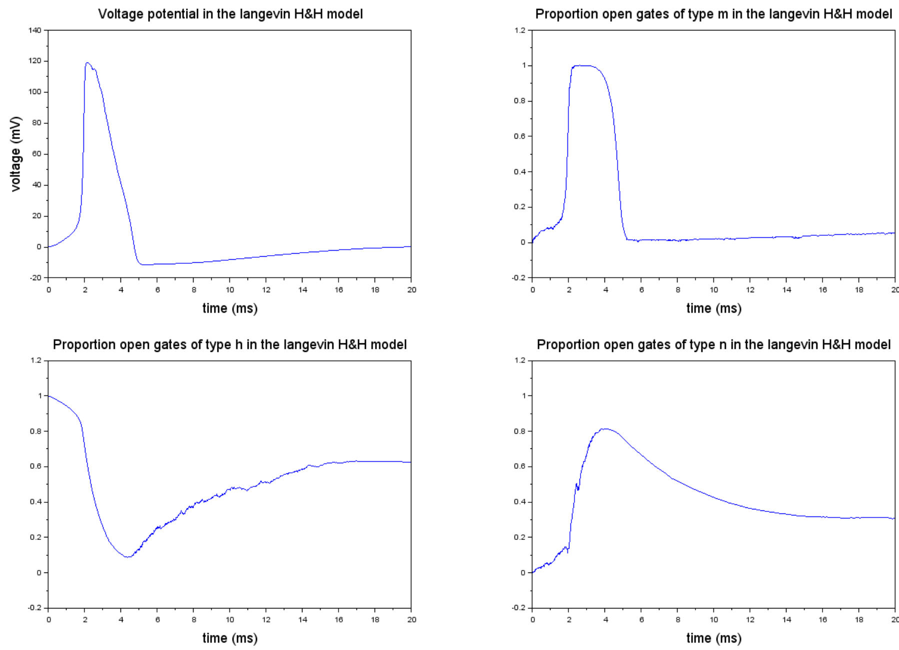

Figure 3.1: Diffusion approximation of the H&H model.

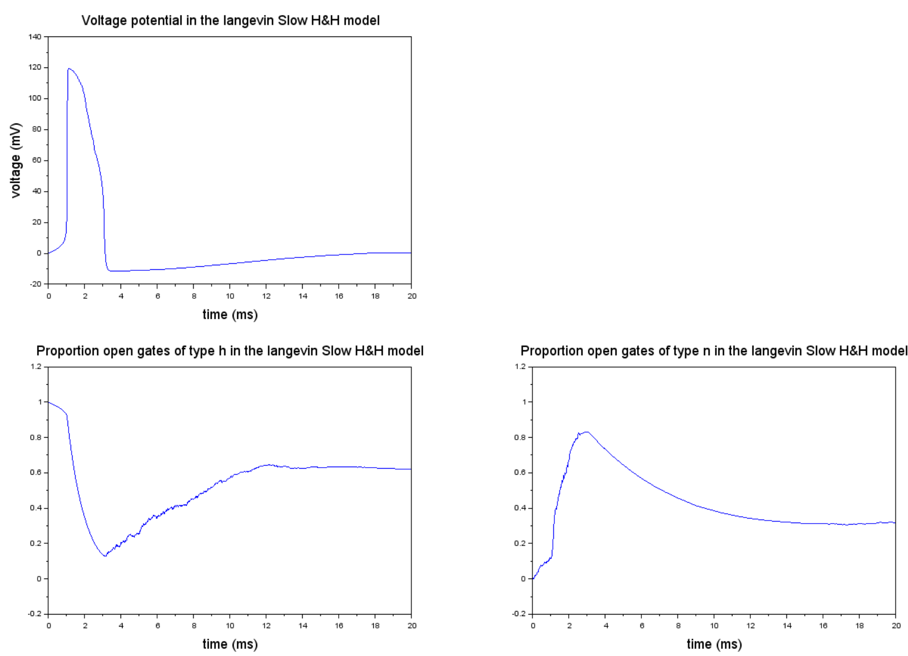

Figure 3.2: Diffusion approximation of the slow H&H model. We notice the absence the voltage-gates of type m.

Chapter 4

Conclusions

When we identify fast moving components on a Markov process and a small parameter ϵ that separates the fast component from the slow ones, we can expect that in short intervals of time the process fluctuates only on a subset of the state space, and in long intervals of time we observe the emergent of a new, slow Markov process on a reduced state space which ignores the fast fluctuation inside each class. The generator matrix Q^ϵ of this process can be rewritten as a double scales generator that depends on ϵ as in the form $Q^\epsilon = \tilde{Q}/\epsilon + \hat{Q}$, where $\tilde{Q}$ holds the information of the fast process and $\hat{Q}$ of the slow one, this characterization together with the asymptotic expansion method of the law presented in Yin [2], allows us to properly identify the components of the slow and fast processes and it gives us an approximation error of the laws of order ϵ.

This method creates an opportunity for the simulation and numerical analysis of Markov processes, because it turns a complicated problem into two simpler ones, and instead of simulating the process on the complete state space we can simulate only the slow process on a reduced space, much easier to simulate, and then use the fast process to identify the position of the Markov process at each time needed.

Markov process on a finite set are very important in their modeling capabilities but there are not sufficient for processes with more complicated dynamics, so we introduce Piecewise deterministic Markov processes or PDMP. PDMP were proposed by Davis [4] and are a family of non-diffusion processes that consists on two sub-processes, one on a continuous-space and the other on a discrete-space, and together are a Markov process involving a deterministic motion given by the solution of a differential equation, and exponentially distributed random jumps. As the rapidly moving component occurs on the finite-state process we applied the two-scales approximation method on a sub-group of PDMP called the Markov switching model, where the jumps only happen in the discrete-state process, by applying the asymptotic expansion method of the law.

By the work of Pakdamana et al [10], PDMP have a natural application in neuroscience via the PDMP interpretation of Hodgkin and Huxley model (H&H) of the membrane potential of a neuron. And as H&H model has several components moving at different rates it is well suited to perform a two-scales approximation; so we apply it to the sodium channel where we separate the voltage-gates of type h and m into two different time scales.

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