Almost Global Attractivity of a Synchronous Generator Connected to an Infinite Bus

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Abstract— **The problem of deriving verifiable conditions for stability of the equilibria of a realistic model of a synchronous generator with constant field current connected to an infinite bus is studied in the paper. Necessary and sufficient conditions for existence and uniqueness of equilibrium points are provided. Furthermore, sufficient conditions for almost global attractivity are given. To carry out this analysis a new Lyapunov–like function is proposed to establish convergence of bounded trajectories, while the latter is proven using the powerful theoretical framework of cell structures pioneered by Leonov and Noldus.**

I. INTRODUCTION

Today's electrical power systems are very large, complex and highly nonlinear [1], [2]. They possess a huge variety of actuators and operational constraints, while persistently being subjected to disturbances. Guaranteeing a stable, reliable and efficient operation of a power system is a daunting task, while at the same time being one of the most important problems for secure power system operation [3]. Hence, it is not surprising that there exists an abundant literature on this topic dating back, at least, to the 1920s [4], [5].

Yet, in spite of these efforts, due to the complexity of the dynamics of a power system—even at the individual component level—many basic questions remain open. Therefore, typical stability analysis (and also control design) of power systems is conducted subject to several assumptions that simplify the mathematical task. Standard assumptions comprise neglecting fast dynamics [2], [6], [7] and assuming constant voltage amplitudes and small frequency variations [1, Chapter 11]. With such assumptions, it is possible to derive reduced-order synchronous generator (SG) models [1, Chapter 11] and employ algebraic models for the transmission lines [1], [2], [6]–[8], simplifying the analysis.

Unfortunately, the employed assumptions are often not physically justifiable in generic operation scenarios. In particular, the common representation of the motion of the machine rotor, *i.e.*, the swing equation, in terms of mechanical and electrical power, instead of their corresponding torques, is an approximation which is only valid for small frequency variations around the nominal frequency [1], [2], [8]–[11].

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Due to the steadily increasing penetration of fluctuating renewable energy sources, power systems worldwide often operate closer to their stability limits [12], [13]. Hence, the necessity of a more rigorous power systems stability analysis valid in large operating regions has become more compelling in the past years. In particular, it is important to derive easily and quickly verifiable analytic conditions for transient stability. This is the topic addressed in the present work.

Stability analysis of power systems employing detailed models, which are valid in a broader range of operating conditions, is a long-standing problem in the power systems literature. In this work, we consider a classical and very well known scenario called the single generator infinite bus (SGIB) model [2], [8]. Opposed to most other available analysis [1], [2], we consider a fourth-order nonlinear SG model derived from first principles. The SGIB scenario with such a model is adopted in [14], [15] where sufficient conditions for almost global asymptotic stability (GAS) are derived. The analysis in [14] proceeds along the classical lines of constructing an integro–differential equation resembling the forced pendulum equation and, subsequently, showing that the SGIB system is almost GAS if and only if the same holds for that equation. In [15] the same authors provide slightly simpler conditions for stability resulting from verifying if a real-valued nonlinear map defined on a finite interval is a contraction. But, as stated in [15], these conditions are hard to verify analytically. Furthermore, and perhaps more importantly, the geometric tools employed to establish the results in [14], [15], don't seem to be applicable to a multi–machine power system. In [10] a scenario similar to that of the SGIB system is analyzed. However, the analysis in [10] is conducted under very stringent assumptions on the specific form of the infinite bus voltage, as well as the steady-state values of the mechanical torque.

The present paper overcomes part of the limitations in the literature by providing the following main contributions.

- Necessary and sufficient conditions for uniqueness and existence of two equilibria (modulo 2π) are established.
- Sufficient conditions for almost global attractivity of one of these equilibria are derived.
- The conservativeness of the conditions, both of them given in terms of the SGIB system parameters, is evaluated via a benchmark numerical example taken from [15].

The remainder of the paper is structured as follows. The SGIB model is introduced in Section II. The steady-state solutions of this model are investigated in Section III. To establish the attractivity result, we first construct in Section IV a new Lyapunov–like function—*i.e.*, a function that is not positive definite but whose derivative is negative semi–definite. LaSalle's invariance principle [16] then establishes some convergence properties of bounded trajectories. The powerful, but little known, theoretical framework of cell structures pioneered by Leonov and co–workers [17]–[19] as well as Noldus [20] that ensures boundedness of solutions is then recalled in Section V. Finally, in Section VI we give conditions on the SGIB system parameters under which the cell structure principle is satisfied, hence completing the almost global attractivity analysis. Section VII presents a benchmark numerical example. The paper is concluded in Section VIII with a summary and an outlook on future work.

It is convenient to clarify at this point two important technical issues. First, the equilibrium of the SGIB model considered in the paper *cannot* be rendered GAS via continuous feedback, hence the need for the qualifier "almost"[1]. Indeed, the system is naturally defined on the torus, which is not diffeomorphic to the Euclidean space, and GAS is hampered by a well known topological obstruction [21]. Second, as explained above the analysis carried out in the paper *does not* rely on the construction of a *bona fide* Lyapunov function, hence we do not prove that the equilibrium is almost GAS, but only almost globally attractive.

II. Model of a synchronous generator connected to an infinite bus

In this section the main equations and assumptions are given for the considered SGIB system. We make some standard assumptions on the SG [22], [23], which are also used in [10], [14], [15]. First, the rotor is round, the machine has one pole pair per phase, there are no damper windings and no saturation effects as well as no Eddy currents. Second, we assume that the rotor current i_f is a real constant. This can be achieved by choosing the excitation voltage such that i_f is kept constant, see [10]. Third, we assume balanced three-phase signals throughout the paper [24]. For the SG this is equivalent to assuming a "perfectly build" SG connected in star with no neutral connection, as in [14], [15].

We follow the notation and modeling in [22]. Hence, we use a generator reference direction, *i.e*, current flowing out of the SG terminals is counted positively. We denote the electrical rotor angle of the SG by[2] $\delta : \mathbb{R}_{\geq 0} \to \mathbb{R}$ and the electrical frequency by $\omega = \dot{\delta}$. Here, δ is the angle between the axis of coil a of the SG and the d-axis, see [22, Figure 3.4]. For a constant rotor current i_f, the three-phase electromotive force (EMF) $e_{abc} : \mathbb{R}_{\geq 0} \to \mathbb{R}^3$ is given by [22], [23]

$$e_{abc} = M_f i_f \omega \begin{bmatrix}\sin(\delta) & \sin(\delta - \frac{2\pi}{3}) & \sin(\delta + \frac{2\pi}{3})\end{bmatrix}^\top, \quad (1)$$

where $M_f \in \mathbb{R}_{>0}$ is the peak mutual inductance. Likewise, we denote the three-phase voltage at the infinite bus by

$$v_{abc} := \sqrt{2}V \begin{bmatrix}\sin(\delta_g) & \sin(\delta_g - \frac{2\pi}{3}) & \sin(\delta_g + \frac{2\pi}{3})\end{bmatrix}^\top, \quad (2)$$

[1]Almost GAS means that for all initial conditions, except a set of Lebesgue measure zero, the trajectories will converge to the equilibrium.

[2]To establish an important result of this paper, namely convergence of bounded solutions, it is more convenient to work with angles defined on the real line rather than on the circle.

where $V \in \mathbb{R}_{>0}$ is the root-mean-square (RMS) value of the constant voltage amplitude (line-to-neutral) and

$$\delta_g = \delta_g(0) + \omega^s t \in \mathbb{R}, \quad (3)$$

with the grid frequency ω^s being a positive real constant. We denote the stator resistance by $R \in \mathbb{R}_{>0}$ and by $L = L_s + M_s$ the stator inductance composed of the self-inductance $L_s \in \mathbb{R}_{>0}$ and the mutual inductance $M_s \in \mathbb{R}_{<0}$. In practice, $L_s > -M_s$ and, hence, $L > 0$. Then the *electrical* equations describing the dynamics of the three-phase stator current $i_{abc} : \mathbb{R}_{\geq 0} \to \mathbb{R}^3$ are given by

$$L\frac{di_{abc}}{dt} = -Ri_{abc} + e_{abc} - v_{abc}. \quad (4)$$

The SGIB model is completed with the *mechanical* equations describing the rotor dynamics, *i.e.*,

$$\begin{aligned} \dot{\delta} &= \omega, \\ J\dot{\omega} &= -D\omega + T_m - T_e, \end{aligned} \quad (5)$$

where $J \in \mathbb{R}_{>0}$ is the total moment of inertia of the rotor masses, $D \in \mathbb{R}_{>0}$ is the damping coefficient and $T_m \in \mathbb{R}_{\geq 0}$ is the mechanical torque provided by the prime mover. Note that we assume T_m constant throughout the paper. Also, the electrical torque T_e can be written as [23]

$$T_e = \omega^{-1} i_{abc}^\top e_{abc}. \quad (6)$$

For our analysis, we represent all three-phase electrical variables in dq-coordinates with respect to the angle

$$\varphi := \omega^s t \quad (7)$$

and employ the dq-transformation matrix $T_{dq}(\cdot)$ given in [1], [2], [24]. The angle difference between the rotor angle δ and the dq-transformation angle φ is denoted by $\theta := \delta - \varphi$. In dq-coordinates, the grid voltage (2) is thus given by the *constant* vector (see [24]),

$$v_{dq} = \begin{bmatrix} v_d \\ v_q \end{bmatrix} = \sqrt{3}V \begin{bmatrix} \sin(\delta_g - \varphi) \\ \cos(\delta_g - \varphi) \end{bmatrix} = \sqrt{3}V \begin{bmatrix} \sin(\delta_g(0)) \\ \cos(\delta_g(0)) \end{bmatrix},$$

where the second equality follows from (3). By defining $b := \sqrt{3/2} M_f i_f$, the EMF in dq-coordinates is given by

$$e_{dq} = \begin{bmatrix} e_d & e_q \end{bmatrix}^\top = \begin{bmatrix} b\omega\sin(\theta) & b\omega\cos(\theta) \end{bmatrix}^\top. \quad (8)$$

By replacing the rotor angle dynamics, *i.e.*, $\dot{\delta}$, with the relative rotor angle dynamics, *i.e.*, $\dot{\theta}$, the SGIB model given by (4), (5) and (6) becomes in dq-coordinates

$$\begin{aligned} \dot{\theta} &= \omega - \omega^s, \\ J\dot{\omega} &= -D\omega + T_m - b\left(i_q\cos(\theta) + i_d\sin(\theta)\right), \\ L\dot{i}_d &= -Ri_d - L\omega^s i_q + b\omega\sin(\theta) - v_d, \\ L\dot{i}_q &= -Ri_q + L\omega^s i_d + b\omega\cos(\theta) - v_q. \end{aligned} \quad (9)$$

Here, we have used the facts that the electrical torque T_e in (6) is given in dq-coordinates by

$$T_e = \omega^{-1} i_{abc}^\top e_{abc} = \omega^{-1} i_{dq}^\top e_{dq} = b(i_q\cos(\theta) + i_d\sin(\theta)) \quad (10)$$

and that, with φ given in (7),

$$\frac{dT_{dq}(\varphi)}{dt} i_{abc} = \omega^s \begin{bmatrix} -i_q & i_d \end{bmatrix}^\top,$$

see [24, equation (4.8)]. The model (9) is used for the analysis in this paper.

Remark 1: The analysis reported in the paper can be conducted in any coordinate frame. However, we favor the one used here since it seems to be more suitable to extend the results to the multi-machine case.

III. Existence and uniqueness of equilibria

In this section, we investigate existence and uniqueness of equilibria of the system (9), which are denoted by $(\theta^s, \omega^s, i_d^s, i_q^s)$.To simplify the notation it is convenient to introduce two important constants

$$\begin{aligned} c &:= b\sqrt{(v_d^2+v_q^2)((L\omega^s)^2+R^2)}, \\ \mathcal{P} &:= \frac{1}{c}\left[-b^2\omega^s R + (T_m - D\omega^s)((L\omega^s)^2+R^2)\right]. \end{aligned} \tag{11}$$

Clearly, c is nonzero if the rotor current i_f is nonzero, which is satisfied in any practical scenario.

Proposition 1: The system in (9) possesses two unique steady-state solutions (modulo 2π) if and only if

$$|\mathcal{P}| < 1. \tag{12}$$

If and only if (12) is satisfied with equality, the system (9) has exactly one steady-state solution (modulo 2π).

Proof: Obviously, the equilibria of the system (9) are 2π-periodic in θ. Furthermore, we have to solve the equations

$$\begin{aligned} \omega - \omega^s &= 0, \\ -D\omega - bi_d\sin(\theta) - bi_q\cos(\theta) + T_m &= 0, \\ -Ri_d - L\omega^s i_q + b\omega\sin(\theta) - v_d &= 0, \\ L\omega^s i_d - Ri_q + b\omega\cos(\theta) - v_q &= 0. \end{aligned} \tag{13}$$

Thus, equilibria $(\theta^s, \omega^s, i_d^s, i_q^s)$ are given by

$$\begin{aligned} &\omega^s = \omega^s, \\ &i_d^s = \frac{bR\omega^s\sin(\theta^s) - bL(\omega^s)^2\cos(\theta^s) - v_dR + v_qL\omega^s}{(L\omega^s)^2+R^2}, \\ &i_q^s = \frac{(\omega^s)^2Lb\sin(\theta^s) + \omega^s bR\cos(\theta^s) - v_dL\omega^s - v_qR}{(L\omega^s)^2+R^2}, \\ &\quad b(L\omega^s v_q - Rv_d)\sin(\theta^s) - b(L\omega^s v_d + Rv_q)\cos(\theta^s) \\ &\quad = -b^2\omega^s R + (T_m - D\omega^s)((L\omega^s)^2+R^2). \end{aligned} \tag{14}$$

The last equation implies that such θ^s does exist if and only if

$$\left| \frac{-b^2\omega^s R + (T_m - D\omega^s)((L\omega^s)^2+R^2)}{b\sqrt{(v_d^2+v_q^2)((L\omega^s)^2+R^2)}} \right| \leq 1. \tag{15}$$

Thus, condition (15) is necessary and sufficient for system (9) to have either one (equality) or exactly two (strict inequality) equilibria (modulo 2π), completing the proof. ■

IV. Convergence of *Bounded* Solutions

In this section, we derive a sufficient condition under which all *bounded* solutions of the system (18) converge to an equilibrium. The claim is established constructing a Lyapunov–like function and invoking LaSalle's invariance principle [16]. Throughout the rest of the paper we make the following natural assumption[3].

Assumption 1: The parameters of the system (9) are such that condition (12) of Proposition 1 is satisfied and $i_f > 0$.

Proposition 2: Consider the system (9) verifying Assumption 1 and the inequality

$$4RD[(L\omega^s)^2+R^2] > (Lb\omega^s)^2. \tag{16}$$

Every bounded solution tends to an equilibrium point.

Proof: Assumption 1 ensures the existence of equilibria. It is convenient to shift one of the equilibrium points to the origin via the the change of coordinates

$$\theta = \tilde{\theta} + \theta^s, \quad \omega = \tilde{\omega} + \omega^s, \quad i_d = \tilde{i}_d + i_d^s, \quad i_q = \tilde{i}_q + i_q^s.$$

In the variables $(\tilde{\theta}, \tilde{\omega}, \tilde{i}_d, \tilde{i}_q)$ the system (9) has the form

$$\begin{aligned} \dot{\tilde{\theta}} =& \tilde{\omega}, \\ J\dot{\tilde{\omega}} =& -D(\tilde{\omega}+\omega^s) - b(i_d^s + \tilde{i}_d)\sin(\theta^s+\tilde{\theta}) \\ & - b(i_q^s+\tilde{i}_q)\cos(\theta^s+\tilde{\theta}) + T_m, \\ L\dot{\tilde{i}}_d =& -R(i_d^s+\tilde{i}_d) - L\omega^s(i_q^s+\tilde{i}_q) \\ & + b(\tilde{\omega}+\omega^s)\sin(\theta^s+\tilde{\theta}) - v_d, \\ L\dot{\tilde{i}}_q =& -R(i_q^s+\tilde{i}_q) + L\omega^s(i_d^s+\tilde{i}_d) \\ & + b(\tilde{\omega}+\omega^s)\cos(\theta^s+\tilde{\theta}) - v_q. \end{aligned} \tag{17}$$

Furthermore, taking into account equations (14), we get

$$\begin{aligned} \dot{\tilde{\theta}} =& \tilde{\omega}, \\ J\dot{\tilde{\omega}} =& -D\tilde{\omega} - b\tilde{i}_d\sin(\theta^s+\tilde{\theta}) - bi_d^s(\sin(\theta^s+\tilde{\theta}) - \sin(\theta^s)) \\ & - b\tilde{i}_q\cos(\theta^s+\tilde{\theta}) - bi_q^s(\cos(\theta^s+\tilde{\theta}) - \cos(\theta^s)), \\ L\dot{\tilde{i}}_d =& -R\tilde{i}_d - L\omega^s\tilde{i}_q + b\omega^s(\sin(\theta^s+\tilde{\theta}) - \sin(\theta^s)) \\ & + b\tilde{\omega}\sin(\theta^s+\tilde{\theta}), \\ L\dot{\tilde{i}}_q =& -R\tilde{i}_q + L\omega^s\tilde{i}_d + b\omega^s(\cos(\theta^s+\tilde{\theta}) - \cos(\theta^s)) \\ & + b\tilde{\omega}\cos(\theta^s+\tilde{\theta}). \end{aligned} \tag{18}$$

The second step is to construct the Lyapunov–like function. Note that the electrical dynamics takes the form

$$L\dot{\tilde{i}}_{dq} = \begin{bmatrix} u \\ w \end{bmatrix} + \begin{bmatrix} b\tilde{\omega}\sin(\theta^s+\tilde{\theta}) \\ b\tilde{\omega}\cos(\theta^s+\tilde{\theta}) \end{bmatrix}, \tag{19}$$

where we defined

$$\begin{bmatrix} u \\ w \end{bmatrix} := \begin{bmatrix} -R & -L\omega^s \\ L\omega^s & -R \end{bmatrix}\tilde{i}_{dq} + \begin{bmatrix} b\omega^s(\sin(\tilde{\theta}+\theta^s) - \sin(\theta^s)) \\ b\omega^s(\cos(\tilde{\theta}+\theta^s) - \cos(\theta^s)) \end{bmatrix}.$$

[3]The signs of c and, hence, $\mathcal{P}$ defined in (11) depend on that of the constant rotor current i_f. For the subsequent analysis, it is important to know the sign of c, as it determines which of the two equilibria of the system (9) is stable. Yet, we remark that for $i_f < 0$ the analysis can be conducted in an analogous manner, see the numerical example in Section VII.

The expression above suggests the following function

$$\begin{aligned}V(\chi) =&\frac{L}{2}(u^2+w^2)+\frac{J\tilde{\omega}^2}{2}\left[(L\omega^s)^2+R^2\right]\\&+b^2R\omega^s(\tilde{\theta}-\sin\tilde{\theta})+L(b\omega^s)^2(1-\cos\tilde{\theta})\\&+b\left[(L\omega^s)^2+R^2\right]\Big[i_d^s\left(\cos\theta^s-\cos(\tilde{\theta}+\theta^s)-\tilde{\theta}\sin\theta^s\right)\\&+i_q^s\left(\sin(\tilde{\theta}+\theta^s)-\sin\theta^s-\tilde{\theta}\cos\theta^s\right)\Big],\end{aligned}\tag{20}$$

where we defined the four–dimensional state vector $\chi := (\tilde{\theta}, \tilde{\omega}, \tilde{i}_d, \tilde{i}_q)$. Some simple calculations show that

$$\begin{aligned}\dot{V} =& -R[u^2+w^2]-D((L\omega^s)^2+R^2)\tilde{\omega}^2+\\&+\tilde{\omega}u\,Lb\omega^s\cos(\tilde{\theta}+\theta^s)-\tilde{\omega}w\,Lb\omega^s\sin(\tilde{\theta}+\theta^s)\\=&\begin{bmatrix}u & w & \tilde{\omega}\end{bmatrix}M\begin{bmatrix}u & w & \tilde{\omega}\end{bmatrix}^\top,\end{aligned}\tag{21}$$

where we defined

$$M := \begin{bmatrix}-R & 0 & \frac{Lb\omega^s\cos(\tilde{\theta}+\theta^s)}{2}\\ 0 & -R & -\frac{Lb\omega^s\sin(\tilde{\theta}+\theta^s)}{2}\\ \frac{Lb\omega^s\cos(\tilde{\theta}+\theta^s)}{2} & -\frac{Lb\omega^s\sin(\tilde{\theta}+\theta^s)}{2} & -D((L\omega^s)^2+R^2)\end{bmatrix}.\tag{22}$$

A Schur complement analysis yields that $M < 0$ if and only if (16) holds. Now, by LaSalle's invariance principle [16] all bounded solutions of the system (18) converge to the largest invariant set contained in the set of $\{\chi \in \mathbb{R}^4 : \dot{V} = 0\}$. Clearly, $\dot{V} = 0 \;\Leftrightarrow\; w = u = \tilde{\omega} = 0$. Hence, $\tilde{\theta}$ is constant and, from (19), we have that $\tilde{i}_{dq}$ is also constant. Consequently, the set $\{\chi \in \mathbb{R}^4 : \dot{V} = 0\}$ is an equilibrium set, completing the proof. ■

V. Boundedness of Solutions

We recall here that, to ensure continuity of the function V, we are viewing the system evolving in $\mathbb{R}^4$, therefore $\tilde{\theta}$ is not a–priori bounded. To prove this fact we use the cell structure principle of Leonov and co–workers [17]–[19] as well as Noldus [20]. Although the proof of the proposition is an immediate corollary of Theorem 16 in [25, Chapter 8], (see also [19]), it is given here for the sake of completeness.

Proposition 3: Consider the function V defined in (20). Assume there exist positive real numbers ϵ and λ such that along the solutions of the system (18) the function

$$\bar{V} := V - \frac{\epsilon}{2}\tilde{\theta}^2\tag{23}$$

verifies

$$\dot{\bar{V}} \le -\lambda\bar{V}.\tag{24}$$

Then, all solutions $\chi = (\tilde{\theta}, \tilde{\omega}, \tilde{i}_d, \tilde{i}_q)$ of the system (18) are bounded.

Proof: Note that from (20) and (21) we have that the evolution of u, w and $\tilde{\omega}$—along solutions of the system (18)—is bounded. This implies that $\tilde{i}_{dq}$ is also bounded and it only remains to show that $\tilde{\theta}$ is bounded. To show the latter, we begin by simplifying the function V defined in (20). Denote by ϕ the unique number in $[0, \pi)$ such that

$$\cos\phi = \frac{b}{c}(L\omega^s v_q - Rv_d),\; \sin\phi = \frac{b}{c}(L\omega^s v_d + Rv_q),\tag{25}$$

where the constant c is defined in (11). Together with (14), we have that

$$\begin{aligned}&b^2R\omega^s(\tilde{\theta}-\sin\tilde{\theta})+L(b\omega^s)^2(1-\cos\tilde{\theta})\\&+b\left[(L\omega^s)^2+R^2\right]\Big[i_d^s\left(\cos\theta^s-\cos(\tilde{\theta}+\theta^s)-\tilde{\theta}\sin\theta^s\right)\\&+i_q^s\left(\sin(\tilde{\theta}+\theta^s)-\sin\theta^s-\tilde{\theta}\cos\theta^s\right)\Big]=\\&c\int_0^{\tilde{\theta}}[\sin(\theta^s-\phi+s)-\sin(\theta^s-\phi)]ds.\end{aligned}$$

Hence, V can be written compactly as

$$\begin{aligned}V(\chi) =&\frac{L}{2}(u^2+w^2)+\frac{J\tilde{\omega}^2}{2}((L\omega^s)^2+R^2)+\\&+c\int_0^{\tilde{\theta}}[\sin(\theta^s-\phi+s)-\sin(\theta^s-\phi)]ds.\end{aligned}\tag{26}$$

From the definition of V in (26) it follows that the function $\bar{V}$ is positive definite on the hyperplane $\tilde{\theta} = 0$. For every integer $k = 0, \pm1, \pm2, \ldots$ consider the function

$$\begin{aligned}\bar{V}_k(\chi) =&\frac{L}{2}(u^2+w^2)+\frac{J\tilde{\omega}^2}{2}((L\omega^s)^2+R^2)-\frac{\epsilon}{2}(\tilde{\theta}-2\pi k)^2\\&+c\int_{2\pi k}^{\tilde{\theta}}[\sin(\theta^s-\phi+s)-\sin(\theta^s-\phi)]ds.\end{aligned}\tag{27}$$

It follows immediately that $\bar{V}_k$ is positive definite on the hyperplane $\tilde{\theta} = 2\pi k$. Observe that the function $\bar{V}_k$ defined in (27) differs from $\bar{V}$ defined in (23), in that the integer k appears as a scalar in the quadratic term $-0.5\epsilon(\tilde{\theta} - 2\pi k)^2$ and in the lower limit of the integral expression, which is 2π-periodic. Hence, the fact that by assumption $\dot{\bar{V}} \le -\lambda\bar{V}$, *i.e.*, condition (23), implies that also for every integer k, $\dot{\bar{V}}_k \le -\lambda\bar{V}_k$. This and the 2π-periodicity of the system (18) with respect to $\tilde{\theta}$, imply that for every integer k the set

$$Z_k = \{\chi \in \mathbb{R}^4 : \bar{V}_k(\chi) \le 0\}\tag{28}$$

is invariant with respect to solutions of the system (18).

Assume $\chi(\cdot)$ is a solution of system (18) with initial condition $\chi(0) = \chi_0$. From the definition of the function $\bar{V}_k$ in (27) we see that $\bar{V}_k(\chi_0)$ is decreasing with respect to $|k|$ quadratically. Hence, for any χ_0 there exist integers k_1 and k_2, with $k_1 < k_2$, such that $\bar{V}_{k_1}(\chi_0) \le 0$, $\tilde{\theta}(0) \ge 2\pi k_1$, and $\bar{V}_{k_2}(\chi_0) \le 0$, $\tilde{\theta}(0) \le 2\pi k_2$. The function $\bar{V}_{k_1}$ is positive on the plane $\tilde{\theta} = 2\pi k_1$, and the function $\bar{V}_{k_2}$ is positive on the plane $\tilde{\theta} = 2\pi k_2$. Furthermore, the sets Z_{k_1} and Z_{k_2} are invariant. Consequently, we have that $2\pi k_1 \le \tilde{\theta}(t) \le 2\pi k_2$ for all $t \ge 0$. This completes the proof. ■

VI. Main Result: Almost Global Attractivity

To streamline the presentation of our main result, we need the following. Given the equilibrium values θ^s and ω^s, define the functions

$$q(\tilde{\theta}) := c\int_0^{\tilde{\theta}}[\sin(\theta^s-\phi+s)-\sin(\theta^s-\phi)]ds,$$

$$g(\lambda) := 4(R - \frac{L\lambda}{2})[((L\omega^s)^2 + R^2)(D - \frac{J\lambda}{2}) - \frac{2\epsilon_{min}}{\lambda}],$$

with c and ϕ defined in (11) and (25), respectively, and the constant ϵ_{min} given by

$$\epsilon_{min} := \inf_{\epsilon \in \mathbb{R}_{>0}} \{q(\tilde{\theta}) \leq \frac{\epsilon}{2}\tilde{\theta}^2,\ \forall \tilde{\theta} \in \mathbb{R}\}. \tag{29}$$

Assumption 2: There exists $\lambda_{max} > 0$—a point of local maximum of the function $g(\lambda)$—such that

$$2R > \quad \lambda_{max} L \quad \text{and} \quad g(\lambda_{max}) > (Lb\omega^s)^2. \tag{30}$$

The lemma below provides an explicit, though conservative, estimate for ϵ_{min} defined in (29).

Lemma 1: Consider the function

$$h(\tilde{\theta}) = q(\tilde{\theta}) - \frac{\bar{\epsilon}}{2}\tilde{\theta}^2. \tag{31}$$

Select $\bar{\epsilon} > c$. Then, $h(\tilde{\theta}) \leq 0$ for all $\tilde{\theta} \in \mathbb{R}$.

Proof: Straight-forward calculations yield

$$q(\tilde{\theta}) = c[-\cos(\theta^s - \phi + \tilde{\theta}) - \sin(\theta^s - \phi)\tilde{\theta} + \cos(\theta^s - \phi)].$$

The critical points of h are attained at values of $\tilde{\theta}^*$ satisfying

$$\frac{\partial h}{\partial \tilde{\theta}}\Big|_{\tilde{\theta} = \tilde{\theta}^*} = c[\sin(\theta^s - \phi + \tilde{\theta}^*) - \sin(\theta^s - \phi)] - \bar{\epsilon}\tilde{\theta}^* = 0. \tag{32}$$

It follows from the mean value theorem that

$$\sin(\theta^s - \phi + \tilde{\theta}^*) - \sin(\theta^s - \phi) \leq |\tilde{\theta}^*|.$$

Thus, for $\bar{\epsilon} > c$, the only solution of (32) is $\tilde{\theta}^* = 0$ and

$$\frac{\partial^2 h}{\partial \tilde{\theta}^2}\Big|_{\tilde{\theta} = \tilde{\theta}^*} = c\cos(\theta^s - \phi + \tilde{\theta}^*) - \bar{\epsilon},$$

which shows that for $\bar{\epsilon} > c \geq c\cos(\theta^s - \phi)$, $\tilde{\theta}^* = 0$ is a maximum of h, completing the proof. ■

We are now ready to state our main result.

Theorem 1: Consider the system (9) verifying Assumptions 1 and 2. The equilibrium point $(\theta^s, \omega^s, i_d^s, i_q^s)$ satisfying $|\theta^s - \phi| < \frac{\pi}{2}$ (modulo 2π) with ϕ defined in (25) is locally asymptotically stable and almost globally attractive, *i.e.*, for all initial conditions, except a set of measure zero, the solutions of the system (9) tend to that equilibrium point.

Proof: Assumption 1 ensures, via Proposition 1, that an equilibrium exists. To establish the local stability claim we note that from (14) and the definition of $q(\tilde{\theta})$ above, it follows that

$$q(0) = 0, \quad q'(0) = 0, \quad q''(0) = c\cos(\theta^s - \phi)$$

and

$$\sin(\theta^s - \phi) = \mathcal{P}, \tag{33}$$

with $\mathcal{P}$ defined in (11). From Assumption 1 we have $|\mathcal{P}| < 1$. Therefore, the equation (33) has two roots θ^s in the interval $[\phi, \phi + 2\pi)$. If $|\theta^s - \phi| < \frac{\pi}{2}$, then $q''(0) > 0$. This implies that the function V has a local minimum at the origin. Furthermore, the parameters λ and $\epsilon_{\min}$ only enter with negative sign in $g(\lambda)$. Hence, Assumption 2 implies that (30) is also satisfied for $\lambda = \epsilon_{\min} = 0$ (with $(\epsilon_{\min}/\lambda)|_{(0,0)} := 0$) which is exactly condition (16). Thus, $\dot{V} \leq 0$ and the zero solution of the system (18) is Lyapunov asymptotically stable (see Proposition 2). If $|\theta^s - \phi| > \frac{\pi}{2}$, then $q''(0) < 0$, and the zero solution of the system (18) is Lyapunov unstable.

To show almost global attractivity of the stable equilibrium, we assume in the following that the zero solution of the system (18) is Lyapunov unstable (and therefore $|\theta^s - \phi| > \frac{\pi}{2}$). Recall the sets Z_k defined in (28) and note that every intersection of sets Z_k is also invariant. The set Z_k is equal to Z_0 shifted in the coordinate $\tilde{\theta}$ by $2\pi k$ to the right since $\theta_k^s = \theta^s + 2\pi k$. Now,

$$\begin{aligned} Z_0 = \{\chi \in \mathbb{R}^4 \ :\ & \frac{L}{2}(u^2 + w^2) + \frac{J\tilde{\omega}^2}{2}((L\omega^s)^2 + R^2) \\ & + q(\tilde{\theta}) - \frac{\epsilon}{2}\tilde{\theta}^2 \leq 0\}. \end{aligned} \tag{34}$$

Recall that by Lemma 1, the number ϵ_{min} defined in (29) indeed exists. Now we check the condition of Proposition 3. To this end, we evaluate $\frac{d\bar{V}}{dt} + \lambda\bar{V}$, which yields

$$\begin{aligned} \frac{d\bar{V}}{dt} + \lambda\bar{V} =& - R[u^2 + w^2] - D((L\omega^s)^2 + R^2)\tilde{\omega}^2 \\ & + \tilde{\omega}u\ Lb\omega^s\cos(\tilde{\theta} + \theta^s) - \tilde{\omega}w\ Lb\omega^s\sin(\tilde{\theta} + \theta^s) \\ & - \epsilon\tilde{\theta}\tilde{\omega} + \lambda[\frac{L}{2}(u^2 + w^2) + \frac{J\tilde{\omega}^2}{2}((L\omega^s)^2 + R^2) \\ & + c\int_0^{\tilde{\theta}}[\sin(\theta^s - \phi + s) - \sin(\theta^s - \phi)]ds - \frac{\epsilon}{2}\tilde{\theta}^2] \\ \leq& (-R + \frac{L\lambda}{2})[u^2 + w^2] - \epsilon\tilde{\theta}\tilde{\omega} - \frac{\lambda(\epsilon - \epsilon_{min})}{2}\tilde{\theta}^2 \\ & - ((L\omega^s)^2 + R^2)(D - \frac{J\lambda}{2})\tilde{\omega}^2 \\ & + \tilde{\omega}u\ Lb\omega^s\cos(\tilde{\theta} + \theta^s) - \tilde{\omega}w\ Lb\omega^s\sin(\tilde{\theta} + \theta^s) \\ =& \begin{bmatrix} u & w & \tilde{\omega} & \tilde{\theta} \end{bmatrix} M_1 \begin{bmatrix} u & w & \tilde{\omega} & \tilde{\theta} \end{bmatrix}^\top, \end{aligned}$$

with M_1 given in (37). The matrix M_1 is negative definite if and only if $\epsilon > \epsilon_{min}$ and the matrix M_2 defined in (38) is negative definite. Similarly to the matrix M defined in (22), M_2 is negative definite if and only if $2R > L\lambda$ and

$$4(R - \frac{L\lambda}{2})[(L\omega^s)^2 + R^2)(D - \frac{J\lambda}{2}) - \frac{\epsilon^2}{2\lambda(\epsilon - \epsilon_{min})}] > (Lb\omega^s)^2. \tag{35}$$

In (35) the positive parameters ϵ and λ have to be chosen. The maximum of the left hand side with respect to ϵ is attained at $\epsilon = 2\epsilon_{min}$. For this choice, (35) takes the form

$$4(R - \frac{L\lambda}{2})[((L\omega^s)^2 + R^2)(D - \frac{J\lambda}{2}) - \frac{2\epsilon_{min}}{\lambda}] > (Lb\omega^s)^2. \tag{36}$$

Consider the following polynomial

$$f(\lambda) := ((L\omega^s)^2 + R^2)(D\lambda - \frac{J\lambda^2}{2}) - 2\epsilon_{min},$$

and denote by λ_1 its smallest root, that is,

$$\lambda_1 = \frac{D - \sqrt{D^2 - \frac{4J\epsilon_{min}}{(L\omega^s)^2 + R^2}}}{J}.$$

If $\lambda_1 < \frac{2R}{L}$, then on the interval $[\lambda_1, \frac{2R}{L}]$ there is a unique

$$M_1 = \begin{bmatrix} -R + \frac{L\lambda}{2} & 0 & \frac{Lb\omega^s \cos(\tilde{\theta}+\theta^s)}{2} & 0 \\ 0 & -R + \frac{L\lambda}{2} & \frac{-Lb\omega^s \sin(\tilde{\theta}+\theta^s)}{2} & 0 \\ \frac{Lb\omega^s \cos(\tilde{\theta}+\theta^s)}{2} & \frac{-Lb\omega^s \sin(\tilde{\theta}+\theta^s)}{2} & -((L\omega^s)^2 + R^2)(D - \frac{J\lambda}{2}) & -\frac{\epsilon}{2} \\ 0 & 0 & -\frac{\epsilon}{2} & -\frac{\lambda(\epsilon-\epsilon_{min})}{2} \end{bmatrix} \quad (37)$$

$$M_2 = \begin{bmatrix} -R + \frac{L\lambda}{2} & 0 & \frac{Lb\omega^s \cos(\tilde{\theta}+\theta^s)}{2} \\ 0 & -R + \frac{L\lambda}{2} & \frac{-Lb\omega^s \sin(\tilde{\theta}+\theta^s)}{2} \\ \frac{Lb\omega^s \cos(\tilde{\theta}+\theta^s)}{2} & \frac{-Lb\omega^s \sin(\tilde{\theta}+\theta^s)}{2} & -((L\omega^s)^2 + R^2)(D - \frac{J\lambda}{2}) + \frac{\epsilon^2}{2\lambda(\epsilon-\epsilon_{min})} \end{bmatrix} \quad (38)$$

point λ_{max} of local maximum of the left hand side in (36).

The derivations above, together with the definition of $g(\lambda)$, prove that the inequalities (30) of Assumption 2 ensure M_1 is negative definite. Hence, the conditions of Proposition 3 are satisfied. Recall that Assumption 2 also implies that condition (16) of Proposition 2 is satisfied. Consequently, all solutions $(\theta, \omega, i_d, i_q)$ of the system (9) are bounded and tend to an equilibrium point. Recall that one of the two equilibria of the system (9) (modulo 2π) is stable and the other one is unstable. Thus, for all initial conditions, except a set of measure zero, the solutions of the system (9) tend to the stable equilibrium point. This shows that the latter is almost globally attractive and completes the proof. ∎

Remark 2: Note that if $\mathcal{P} = 0$ (and therefore $|\theta^s - \phi| = \frac{\pi}{2}$) then $\epsilon_{min} = 0$, and the inequality (36) is equivalent to (16).

Remark 3: The related analysis in [10] critically relies on imposing a specific value for the mechanical torque T_m and on the knowledge of the stationary rotor currents i^s_{dq}. Such restrictions do not apply in the present case.

VII. Numerical example

We investigate the effectiveness of the inequalities (30) of Assumption 2 via a numerical benchmark example taken directly from [15]. Note that in the example of [15] the rotor current $i_f < 0$. Thus, $b < 0$ and $c < 0$, see (11). In our notation, this corresponds to the (potentially) stable equilibrium being shifted by π. Indeed, conditions (11), (12) and (16) are satisfied for this example. Hence, the system (9) has two equilibria and the proof of Theorem 1 implies that the equilibrium with $|\theta^s - \phi| > \pi/2$ is *locally* asymptotically stable. In addition, inequalities (30) of Assumption 2 are satisfied with $\epsilon_{\min} = 82.12$ and $\lambda = 23.81$. Consequently, by Theorem 1, the equilibrium with $|\theta^s - \phi| > \pi/2$ is also an *almost globally* attractive equilibrium. This result coincides with the conclusions in [15].

VIII. Conclusions

A complete stability analysis of a realistic SGIB model has been presented. First, it is shown that (12)—with $\mathcal{P}$ defined in (11)—is a necessary and sufficient condition for existence of equilibria. Then, it is proven that if the inequalities (30) of Assumption 2 hold then almost all trajectories converge to a stable equilibrium point. The conservativeness of the estimates have been assessed via a numerical benchmark problem.

The main topic of future research is the extension of these results to the multi-machine case. Given the "scalable" nature of the analysis tools employed here this seems a feasible—albeit difficult—task.

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