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Abstract The problem of interval state estimation is studied for systems described by parabolic Partial Differential Equations (PDEs). The proposed solution is based on a finite-element approximation of PDE, with posterior design of an interval observer for the obtained ordinary differential equation. The interval inclusion of the state function of PDE is obtained using the estimates on the error of discretization. The results are illustrated by numerical experiments with an academic example.

Keywords: Interval observers, PDE, Finite-element approximation

1. INTRODUCTION

The problem of state vector estimation is very challenging and can be encountered in many applications Meurer et al. (2005); Fossen and Nijmeijer (1999); Besanon (2007). For linear time-invariant models the theory is well developed, the most popular solutions include Luenberger observer and Kalman filter for deterministic and stochastic settings, respectively. For state estimation in nonlinear systems, a partial similarity to linear ones or representation in various canonical forms are widely used. That is why the class of Linear Parameter-Varying (LPV) systems became very popular in applications: a wide class of nonlinear systems can be presented in the LPV form (in this case the system equations are extended). A partial linearity of LPV models allows a rich spectrum of methods developed for linear systems to be applied Shamma (2012); Tan (1997).

Apart of model complexity, another difficulty for an estimator design consists in the model uncertainty (unknown parameters or/and external disturbances). In the presence of uncertainty, design of a conventional estimator, converging to the ideal value of the state, is difficult to realize. In this case an interval estimation becomes more feasible: an observer can be constructed that, using input-output information, evaluates the set of admissible values (interval) for the state at each instant of time. The interval width is proportional to the size of the model uncertainty (it has to be minimized by tuning the observer parameters). There are several approaches to design interval/set-membership estimators Jaulin (2002); Kieffer and Walter (2004); Olivier and Gouzé (2004). This work is devoted to interval observers, which form a subclass of set-membership estimators and whose design is based on the monotone systems theory Olivier and Gouzé (2004); Moisan et al. (2009); Raïssi et al. (2010, 2012); Efimov et al. (2012). This idea has been proposed rather recently in Gouzé et al. (2000), but received already numerous extensions, and in the present paper a development of this approach for estimation of systems described by PDEs is proposed.

The models of distinct physical phenomena, such as sound, heat, electrostatics, electrodynamics, fluid flow, elasticity, or quantum mechanics, can be formalized similarly in terms of PDEs. That is why control and estimation of PDEs is a very popular topic of research nowadays Bredies et al. (2013); Smyshlyaev and Krstic (2010). Frequently, for state estimation purpose the finite-dimensional approximation approach is used Alvarez and Stephanopoulos (1982); Dochain (2000); Vande Wouver and Zeitz (2002); Hagen and I. (2003), then the observation problem is addressed with the well-known tools available for finite-dimensional systems, while the convergence assessment has to be performed with respect to the actual distributed system solutions.

In this work an interval observer is proposed for systems described by PDEs using the finite-dimensional approximation approach. The discretization error estimates are used to calculate the enveloping interval for solutions of PDE. An interesting feature of the proposed approach is that being applied to a nonlinear PDE, assuming that all nonlinearities are bounded and considered as perturbations, then the obtained interval observer is linear and can be easily solved providing bounds on solutions of the originally nonlinear PDE (under the hypothesis that these solutions exist).

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The outline of this paper is as follows. After preliminaries in Section 2, and distributed system properties introduction in Section 3, the interval observer design is given in Section 4. The results of numerical experiments with a simple parabolic equation are presented in Section 5.

2. PRELIMINARIES

The real numbers are denoted by $\mathbb{R}$, $\mathbb{R}_+ = \{\tau \in \mathbb{R} : \tau \geq 0\}$. Euclidean norm for a vector $x \in \mathbb{R}^n$ will be denoted as $|x|$. The symbols I_n, $E_{n\times m}$ and E_p denote the identity matrix with dimension $n \times n$, the matrix with all elements equal 1 with dimensions $n \times m$ and $p \times 1$, respectively.

2.1 Interval relations

For two vectors $x_1, x_2 \in \mathbb{R}^n$ or matrices $A_1, A_2 \in \mathbb{R}^{n\times n}$, the relations $x_1 \leq x_2$ and $A_1 \leq A_2$ are understood elementwise. The relation $P \prec 0$ ($P \succ 0$) means that the matrix $P \in \mathbb{R}^{n\times n}$ is negative (positive) definite. Given a matrix $A \in \mathbb{R}^{m\times n}$, define $A^+ = \max\{0, A\}$, $A^- = A^+ - A$ (similarly for vectors) and denote the matrix of absolute values of all elements by $|A| = A^+ + A^-$.

Lemma 1. Efimov et al. (2012) Let $x \in \mathbb{R}^n$ be a vector variable, $\underline{x} \leq x \leq \overline{x}$ for some $\underline{x}, \overline{x} \in \mathbb{R}^n$.

(1) If $A \in \mathbb{R}^{m\times n}$ is a constant matrix, then

$$A^+\underline{x} - A^-\overline{x} \leq Ax \leq A^+\overline{x} - A^-\underline{x}. \tag{1}$$

(2) If $A \in \mathbb{R}^{m\times n}$ is a matrix variable and $\underline{A} \leq A \leq \overline{A}$ for some $\underline{A}, \overline{A} \in \mathbb{R}^{m\times n}$, then

$$\begin{aligned} &\underline{A}^+\underline{x}^+ - \overline{A}^+\underline{x}^- - \underline{A}^-\overline{x}^+ + \overline{A}^-\overline{x}^- \leq Ax \\ &\leq \overline{A}^+\overline{x}^+ - \underline{A}^+\overline{x}^- - \overline{A}^-\underline{x}^+ + \underline{A}^-\underline{x}^-. \end{aligned} \tag{2}$$

Furthermore, if $-\overline{A} = \underline{A} \leq 0 \leq \overline{A}$, then the inequality (2) can be simplified: $-\overline{A}(\overline{x}^+ + \underline{x}^-) \leq Ax \leq \overline{A}(\overline{x}^+ + \underline{x}^-)$.

2.2 Nonnegative continuous-time linear systems

A matrix $A \in \mathbb{R}^{n\times n}$ is called Hurwitz if all its eigenvalues have negative real parts, it is called Metzler if all its elements outside the main diagonal are nonnegative. Any solution of the linear system

$$\begin{aligned} \dot{x} &= Ax + B\omega(t),\ \omega : \mathbb{R}_+ \to \mathbb{R}^q_+, \\ y &= Cx + D\omega(t), \end{aligned} \tag{3}$$

with $x \in \mathbb{R}^n$, $y \in \mathbb{R}^p$ and a Metzler matrix $A \in \mathbb{R}^{n\times n}$, is elementwise nonnegative for all $t \geq 0$ provided that $x(0) \geq 0$ and $B \in \mathbb{R}^{n\times q}_+$ Farina and Rinaldi (2000); Smith (1995). The output solution $y(t)$ is nonnegative if $C \in \mathbb{R}^{p\times n}_+$ and $D \in \mathbb{R}^{p\times q}_+$. Such dynamical systems are called cooperative (monotone) or nonnegative if only initial conditions in $\mathbb{R}^n_+$ are considered Farina and Rinaldi (2000); Smith (1995).

For a Metzler matrix $A \in \mathbb{R}^{n\times n}$ its stability can be checked verifying a Linear Programming (LP) problem

$$A^{\mathrm{T}}\lambda < 0$$

for some $\lambda \in \mathbb{R}^n_+ \setminus \{0\}$, or Lyapunov matrix equation

$$A^{\mathrm{T}}P + PA \prec 0$$

for a diagonal matrix $P \in \mathbb{R}^{n\times n}$, $P > 0$ (in general case the matrix P should not be diagonal). The L_1 and L_∞ gains for nonnegative systems (3) have been studied in Briat (2011); Ebihara et al. (2011), for this kind of systems these gains are interrelated. The conventional results and definitions on L_2/L_∞ stability for linear systems can be found in Khalil (2002).

3. DISTRIBUTED SYSTEMS

In this section the basic facts on finite-dimensional approximations of PDE and some auxiliary results are given.

3.1 Preliminaries

If X is a normed space with norm $||\cdot||_X$, $\Omega \subset \mathbb{R}^n$ for some $n \geq 1$ and $\phi : \Omega \to X$, define

$$||\phi||^2_{L^2(\Omega,X)} = \int_\Omega ||\phi(s)||^2_X ds,$$
$$||\phi||_{L^\infty(\Omega,X)} = ess\sup_{s\in\Omega} ||\phi(s)||_X.$$

By $L^\infty(\Omega, X)$ and $L^2(\Omega, X)$ denote the set of functions $\Omega \to X$ with the properties $||\cdot||_{L^\infty(\Omega,X)} < +\infty$ and $||\cdot||_{L^2(\Omega,X)} < +\infty$, respectively. Denote $I = [0, 1]$, let $C^k(I, \mathbb{R})$ be the set of functions having continuous derivatives through order $k \geq 0$ on I. For any $q > 0$ and an interval $I' \subseteq I$ define $W^{q,\infty}(I', \mathbb{R})$ as a subset of functions $y \in C^{q-1}(I', \mathbb{R})$ with an absolutely continuous $y^{(q-1)}$ and bounded $y^{(q)}$ on I', $||y||_{W^{q,\infty}} = \sum_{i=0}^q ||y^{(r)}||_{L^\infty(I',\mathbb{R})}$. Denote by $H^q(I, \mathbb{R})$ with $q \geq 0$ the Sobolev space of functions with derivatives through order q in $L^2(I, \mathbb{R})$.

For two functions $z_1, z_2 : I \to \mathbb{R}$ their relation $z_1 \leq z_2$ has to be understood as $z_1(x) \leq z_2(x)$ for all $x \in I$, the inner product is defined in a standard way:

$$(z_1, z_2) = \int_0^1 z_1(x)z_2(x)dx.$$

3.2 Approximation

Following Wheeler (1973), consider the following PDE with associated boundary conditions:

$$\begin{aligned} \rho(x)\frac{\partial z(x,t)}{\partial t} &= L[x, z(x,t)] + r(x,t) \quad \forall (x,t) \in I \times (0,T), \\ z(x,0) &= z_0(x) \quad \forall x \in I, \\ 0 = z(0,t) &= z(1,t) \quad \forall t \in (0,T), \end{aligned} \tag{4}$$

where $I = [0, 1]$ and $T > 0$,

$$L(x,z) = \frac{\partial}{\partial x}\left(a(x)\frac{\partial z}{\partial x}\right) - b(x)\frac{\partial z}{\partial x} - q(x)z,$$

$a, b, q, \rho \in L^\infty(I, \mathbb{R})$ and there exist $a_0, a_1, \rho_0, \rho_1 \in \mathbb{R}_+$ such that

$$0 < a_0 \leq a(x) \leq a_1,\ 0 < \rho_0 \leq \rho(x) \leq \rho_1 \quad \forall x \in I,$$

and $a', b' \in L^2(I, \mathbb{R})$, where $a' = \partial a(x)/dx$.

Let $\Delta = \{x_j\}_{j=0}^{N'}$ for some $N' > 0$, where $0 = x_0 < x_1 < \cdots < x_{N'} = 1$, and $I_j = (x_{j-1}, x_j)$, $h_j = x_j - x_{j-1}$, $h = \max_{1\leq j\leq N'} h_j$. Let $P_s(I')$ be the set of polynomials of degree less than $s + 1$, $s > 0$ on an interval $I' \subseteq I$, then adopt the notation:

$$\begin{aligned} M^{s,\Delta} = \{&v \in C^0(I, \mathbb{R}) : v(x) = v_j(x)\ \forall x \in I_j, \\ &v_j \in P_s(I_j)\ \forall 1 \leq j \leq N'\} \end{aligned}$$

and $M = M_0^{s,\Delta} = \{v \in M^{s,\Delta} : v(0) = v(1) = 0\}$.

Introduce a bilinear form:

$$\mathcal{L}(y, v) = -(ay', v') - (by', v) - (qy, v) \quad y, v \in H^1(I, \mathbb{R}),$$

and define

$$\lambda \geq \frac{1}{2a_0}(\sup_{x\in I} b^2(x) - \inf_{x\in I} q(x)).$$

The continuous-time Galerkin approximation $Z(\cdot, t) \in M$ to the solution $z(x, t)$ of the parabolic system (4) is defined by

$$\left(\rho\frac{\partial Z}{\partial t}, V\right) = \mathcal{L}(Z, V) + (r, V) \quad \forall V \in M,\ \forall t \in (0, T); \tag{5}$$

$$\mathcal{L}(Z - z_0, V) - \lambda(Z - z_0, V) = 0 \quad \forall V \in M,\ t = 0.$$

Assumption 1. There exist $s > 0$, $l_1 > 0$ and $l_2 > 0$ such that the solution z of (4) belongs to $L^\infty([0, T), W^{s+1,\infty}(I, \mathbb{R}))$ and $\partial z/\partial t \in L^2([0, T), H^{s+1}(I, \mathbb{R}))$,

$$||z||_{L^\infty([0,T),W^{s+1,\infty}(I,\mathbb{R}))} \leq l_1,$$
$$||\partial z/\partial t||_{L^2([0,T),H^{s+1}(I,\mathbb{R}))} \leq l_2.$$

Proposition 2. Wheeler (1973) Let Assumption 1 be satisfied, then there is $\varrho > 0$ such that

$$||Z - z||_{L^\infty(I\times(0,T),\mathbb{R})} \leq \varrho h^{s+1}(l_1 + l_2),$$

where z and Z are the solutions of (4) and (5), respectively.

In order to calculate Z, let $\Phi_j \in M$, $1 \leq j \leq N$ with $N \geq N'$ be a basis in M, then following the Galerkin method Thomée (2006) the solution $Z(x, t)$ of (5) can be presented as

$$Z(x, t) = \sum_{j=1}^{N} \xi_j(t)\Phi_j(x),$$

where $\xi = [\xi_1 \dots \xi_N]^T \in \mathbb{R}^N$ is the vector of coefficients satisfying the ordinary differential equations for all $1 \leq j \leq N$:

$$(\rho\sum_{i=1}^{N}\dot{\xi}_i\Phi_i, \Phi_j) = \mathcal{L}(\sum_{i=1}^{N}\xi_i\Phi_i, \Phi_j) + (r, \Phi_j) \quad \forall t \in (0, T);$$
$$\mathcal{L}(\sum_{j=1}^{N}\xi_i(0)\Phi_i - z_0, \Phi_j) - \lambda(\sum_{j=1}^{N}\xi_i(0)\Phi_i - z_0, \Phi_j) = 0,$$

which finally can be presented in the form (*a.a.* means "for almost all"):

$$\Upsilon\dot{\xi}(t) = \Lambda\xi(t) + \bar{r}(t) \quad a.a.\ t \in (0, T);$$
$$\Psi\xi(0) = \varpi,$$

where for all $1 \leq i, j \leq N$

$$\Upsilon_{j,i} = (\rho\Phi_i, \Phi_j),\ \Lambda_{j,i} = \mathcal{L}(\Phi_i, \Phi_j),\ \bar{r}_j = (r, \Phi_j),$$
$$\Psi_{j,i} = \mathcal{L}(\Phi_i, \Phi_j) - \lambda(\Phi_i, \Phi_j),\ \varpi_j = \lambda(z_0, \Phi_j) - \mathcal{L}(z_0, \Phi_j).$$

Under introduced restrictions on (4) and by construction of the basis functions Φ_j, we assume that the matrices Υ and Ψ are nonsingular, therefore

$$\dot{\xi}(t) = A\xi(t) + G\bar{r}(t) \quad a.a.\ t \in (0, T),\ \xi(0) = \xi_0, \tag{6}$$

where $A = \Upsilon^{-1}\Lambda \in \mathbb{R}^{N\times N}$, $G = \Upsilon^{-1}$ and $\xi_0 = \Psi^{-1}\varpi \in \mathbb{R}^N$. The solution to Cauchy problem (6) can be easily (numerically) calculated.

3.3 Interval estimates

For $\phi \in \mathbb{R}$ define two operators $\cdot^+$ and $\cdot^-$ as follows:

$$\phi^+ = \max\{0, \phi\},\ \phi^- = \phi^+ - \phi.$$

Lemma 3. Let $s, \underline{s}, \overline{s} : I \to \mathbb{R}$ admit the relations $\underline{s} \leq s \leq \overline{s}$, then for any $\phi : I \to \mathbb{R}$ we have

$$(\underline{s}, \phi^+) - (\overline{s}, \phi^-) \leq (s, \phi) \leq (\overline{s}, \phi^+) - (\underline{s}, \phi^-).$$

All proofs are excluded due to the page limitation.

Lemma 4. Let there exist $\underline{\xi}, \overline{\xi} \in L^\infty([0, T), \mathbb{R}^N)$ such that for solution ξ of (6) we have

$$\underline{\xi}(t) \leq \xi(t) \leq \overline{\xi}(t) \quad \forall t \in [0, T),$$

then for solution Z of (5),

$$\underline{Z}(x, t) \leq Z(x, t) \leq \overline{Z}(x, t) \quad \forall (x, t) \in I \times [0, T) \tag{7}$$

and $\underline{Z}, \overline{Z} \in L^\infty(I \times [0, T), \mathbb{R})$, where

$$\underline{Z}(x, t) = \sum_{j=1}^{N}(\underline{\xi}_j(t)\Phi_j^+(x) - \overline{\xi}_j(t)\Phi_j^-(x)), \tag{8}$$
$$\overline{Z}(x, t) = \sum_{j=1}^{N}(\overline{\xi}_j(t)\Phi_j^+(x) - \underline{\xi}_j(t)\Phi_j^-(x)).$$

The result of this lemma connects the interval estimates obtained for a real vector of coefficients ξ and the approximated solution Z, and can be extended to z using Proposition 2.

Lemma 5. Let Assumption 1 be satisfied and there exist $\underline{Z}, \overline{Z} \in L^\infty(I \times [0, T), \mathbb{R})$ such that (7) be true for solution Z of (5), then there is $\varrho > 0$ such that for solution z of (4),

$$\underline{z}(x, t) \leq z(x, t) \leq \overline{z}(x, t) \quad \forall (x, t) \in I \times [0, T) \tag{9}$$

and $\underline{z}, \overline{z} \in L^\infty(I \times [0, T), \mathbb{R})$, where

$$\underline{z}(x, t) = \underline{Z}(x, t) - \varrho h^{s+1}(l_1 + l_2), \tag{10}$$
$$\overline{z}(x, t) = \overline{Z}(x, t) + \varrho h^{s+1}(l_1 + l_2).$$

Therefore, according to lemmas 4 and 5, in order to calculate interval estimates for (4) it is enough to design an interval observer for (6).

4. INTERVAL OBSERVER DESIGN

Assume that the state $z(x, t)$ is available for measurements in certain points $x_i^m \in I$ for $1 \leq i \leq p$:

$$y_i(t) = z(x_i^m, t) + \nu_i(t), \tag{11}$$

where $y(t), \nu(t) \in \mathbb{R}^p$, $\nu \in L^\infty(\mathbb{R}_+, \mathbb{R}^p)$ is the measurement noise. Under Assumption 1 from Proposition 2, for a finite elements approximation we can assign

$$y_i(t) = Z(x_i^m, t) + \nu_i(t) + e_i(t),$$

where $||e||_{L^\infty([0,T),\mathbb{R}^p)} \leq \varrho h^{s+1}(l_1 + l_2)$ for some $\varrho > 0$, $e = [e_1 \dots e_p]^T$. Next,

$$y_i(t) = \sum_{j=1}^{N}\xi_j(t)\Phi_j(x_i^m) + \nu_i(t) + e_i(t)$$

and

$$y(t) = C\xi(t) + v(t), \tag{12}$$

where $v(t) = \nu(t) + e(t) \in \mathbb{R}^p$ is the new measurement noise and $C \in \mathbb{R}^{p\times N}$ is the appropriate matrix:

$$C = \begin{bmatrix} \Phi_1(x_1^m) & \dots & \Phi_N(x_1^m) \\ \vdots & \ddots & \vdots \\ \Phi_1(x_p^m) & \dots & \Phi_N(x_p^m) \end{bmatrix}.$$

We will also assume that in (4),

$$r(x,t) = \sum_{k=1}^{m} r_{1k}(x)u_k(t) + r_0(x,t)$$

where $u(t) \in \mathbb{R}^m$ is a known input or control, $r_{1k} \in L^\infty(I,\mathbb{R})$ and $r_0 \in L^\infty(I \times [0,T),\mathbb{R})$. Then in (6):

$$G\bar{r}(t) = G \begin{bmatrix} (\sum_{k=1}^{m} r_{1k}(x)u_k(t) + r_0(x,t), \Phi_1) \\ \vdots \\ (\sum_{k=1}^{m} r_{1k}(x)u_k(t) + r_0(x,t), \Phi_N) \end{bmatrix}$$
$$= Bu(t) + Gd(t),$$

where

$$B = G \begin{bmatrix} (r_{11}, \Phi_1) & \dots & (r_{1m}, \Phi_1) \\ \vdots & \ddots & \vdots \\ (r_{11}, \Phi_N) & \dots & (r_{1m}, \Phi_N) \end{bmatrix} \in \mathbb{R}^{N\times m},$$

$$d(t) = \begin{bmatrix} (r_0, \Phi_1) \\ \vdots \\ (r_0, \Phi_N) \end{bmatrix} \in \mathbb{R}^N$$

is an external unknown disturbance.

The idea of the work consists in design of interval observer for the approximation (5), (12) with the aim to calculate an interval estimate for the state of (4), (11) taking into account the approximation error evaluated in Proposition 2 and the results of lemmas 4 and 5. For this purpose we need the following hypothesis.

Assumption 2. Let $\underline{z}_0 \le z_0 \le \overline{z}_0$ for some known $\underline{z}_0, \overline{z}_0 \in L^\infty(I,\mathbb{R})$, let also two functions $\underline{r}_0, \overline{r}_0 \in L^\infty(I \times [0,T),\mathbb{R})$ and a constant $\nu_0 > 0$ be given such that

$$\underline{r}_0(x,t) \le r_0(x,t) \le \overline{r}_0(x,t),\ |\nu(t)| \le \nu_0\ \forall (x,t) \in I\times(0,T).$$

Assumption 3. There are a matrix $L \in \mathbb{R}^{N\times p}$ and a Metzler matrix $D \in \mathbb{R}^{N\times N}$ such that the matrices $A-LC$ and D have the same eigenvalues and the pairs $(A-LC, \chi_1)$ and (D, χ_2) are observable for some $\chi_1 \in \mathbb{R}^{1\times N}$, $\chi_2 \in \mathbb{R}^{1\times N}$.

Thus, by Assumption 2 three intervals $[\underline{z}_0, \overline{z}_0]$, $[\underline{r}_0(x,t), \overline{r}_0(x,t)]$ and $[-\nu_0, \nu_0]$ determine for all $(x,t) \in I\times[0,T)$ in (4), (11) uncertainty of values of z_0, $r_0(x,t)$ and $\nu(t)$, respectively. Using Lemma 3 we obtain:

$$\underline{d}(t) \le d(t) \le \overline{d}(t) \quad \forall t \in [0,T),$$

$$\underline{d}(t) = \begin{bmatrix} (\underline{r}_0, \Phi_1^+) - (\overline{r}_0, \Phi_1^-) \\ \vdots \\ (\underline{r}_0, \Phi_N^+) - (\overline{r}_0, \Phi_N^-) \end{bmatrix},$$

$$\overline{d}(t) = \begin{bmatrix} (\overline{r}_0, \Phi_1^+) - (\underline{r}_0, \Phi_1^-) \\ \vdots \\ (\overline{r}_0, \Phi_N^+) - (\underline{r}_0, \Phi_N^-) \end{bmatrix}$$

and under Assumption 1

$$-V \le v(t) \le V = \nu_0 + \varrho h^{s+1}(l_1 + l_2).$$

Finally,

$$\underline{\xi}_0 \le \xi(0) \le \overline{\xi}_0,$$

where

$$\underline{\xi}_0 = (\Psi^{-1})^+ \underline{\varpi} - (\Psi^{-1})^- \overline{\varpi},$$
$$\overline{\xi}_0 = (\Psi^{-1})^+ \overline{\varpi} - (\Psi^{-1})^- \underline{\varpi}$$

and $\underline{\varpi}_j \le \varpi_j \le \overline{\varpi}_j$ for all $1 \le j \le N$ with

$$\begin{aligned} \overline{\varpi}_j = {} & \lambda[(\overline{z}_0, \Phi_j^-) - (\underline{z}_0, \Phi_j^+)] - (a\underline{z}_0', \Phi_j'^+) + (a\overline{z}_0', \Phi_j'^-) \\ & -(b^+\underline{z}_0' - b^-\overline{z}_0', \Phi_j^+) + (b^+\overline{z}_0' - b^-\underline{z}_0', \Phi_j^-) \\ & -(q^+\underline{z}_0 - q^-\overline{z}_0, \Phi_j^+) + (q^+\overline{z}_0 - q^-\underline{z}_0, \Phi_j^-), \\ \underline{\varpi}_j = {} & \lambda[(\underline{z}_0, \Phi_j^-) - (\overline{z}_0, \Phi_j^+)] - (a\overline{z}_0', \Phi_j'^+) + (a\underline{z}_0', \Phi_j'^-) \\ & -(b^+\overline{z}_0' - b^-\underline{z}_0', \Phi_j^+) + (b^+\underline{z}_0' - b^-\overline{z}_0', \Phi_j^-) \\ & -(q^+\overline{z}_0 - q^-\underline{z}_0, \Phi_j^+) + (q^+\underline{z}_0 - q^-\overline{z}_0, \Phi_j^-), \end{aligned}$$

According to Assumption 3 and Raïssi et al. (2012) there is a nonsingular matrix $S \in \mathbb{R}^{N\times N}$ such that $D = S(A-LC)S^{-1}$. Now, applying the results of Gouzé et al. (2000); Chebotarev et al. (2015) two estimates $\underline{\xi}, \overline{\xi} \in L^\infty([0,T),\mathbb{R}^N)$ can be calculated, based on the available information on these intervals and $y(t)$, such that

$$\underline{\xi}(t) \le \xi(t) \le \overline{\xi}(t) \quad \forall t \in [0,T). \tag{13}$$

In other words, an interval observer can be designed for the approximating dynamics. For this purpose, following Gouzé et al. (2000); Chebotarev et al. (2015), rewrite (5):

$$\begin{aligned} \dot{\xi}(t) = {} & (A-LC)\xi(t) + Bu(t) \\ & +Ly(t) - Lv(t) + Gd(t). \end{aligned}$$

In the new coordinates $\zeta = Sz$ the system (5) takes the form:

$$\begin{aligned} \dot{\zeta}(t) = {} & D\zeta(t) + SBu(t) + SLy(t) \\ & +\delta(t),\ \delta(t) = S[Gd(t) - Lv(t)]. \end{aligned} \tag{14}$$

And using Lemma 1 we obtain that

$$\underline{\delta}(t) \le \delta(t) \le \overline{\delta}(t),$$

where $\underline{\delta}(t) = (SG)^+\underline{d}(t) - (SG)^-\overline{d}(t) - |SL|E_pV$ and $\overline{\delta}(t) = (SG)^+\overline{d}(t) - (SG)^-\underline{d}(t) + |SL|E_pV$. Next, for the system (14) an interval observer can be proposed:

$$\begin{aligned} \dot{\underline{\zeta}}(t) &= D\underline{\zeta}(t) + SBu(t) + SLy(t) + \underline{\delta}(t), \\ \dot{\overline{\zeta}}(t) &= D\overline{\zeta}(t) + SBu(t) + SLy(t) + \overline{\delta}(t), \\ \underline{\zeta}(0) &= S^+\underline{\xi}_0 - S^-\overline{\xi}_0,\ \overline{\zeta}(0) = S^+\overline{\xi}_0 - S^-\underline{\xi}_0, \\ \underline{\xi}(t) &= (S^{-1})^+\underline{\zeta}(t) - (S^{-1})^-\overline{\zeta}(t), \\ \overline{\xi}(t) &= (S^{-1})^+\overline{\zeta}(t) - (S^{-1})^-\underline{\zeta}(t), \end{aligned} \tag{15}$$

where the relations (1) are used to calculate the initial conditions for $\underline{\zeta}, \overline{\zeta}$ and the estimates $\underline{\xi}, \overline{\xi}$.

Proposition 6. Let assumptions 1, 2 and 3 be satisfied. Then for (5), (12) with the interval observer (15) the relations (13) are satisfied. In addition, $\underline{\xi}, \overline{\xi} \in L^\infty([0,T),\mathbb{R}^N)$ if $A-LC$ is Hurwitz.

Remark 7. In order to regulate the estimation accuracy it is worth to strengthen the conditions of stability for $\underline{\xi}, \overline{\xi}$ (Hurwitz property of the matrix $A-LC$) to a requirement that L_∞ gain of transfer $\begin{bmatrix} \delta - \underline{\delta} \\ \overline{\delta} - \delta \end{bmatrix} \to \begin{bmatrix} \overline{e} \\ \underline{e} \end{bmatrix}$ is less than γ for some $\gamma > 0$. To this end, coupling this restriction with the conditions of Assumption 3 the following nonlinear matrix inequalities can be obtained:

$$\begin{bmatrix} \gamma^{-1} I_N & P \\ P & -W^T - W - \gamma I_N \end{bmatrix} \succeq 0, \tag{16}$$

$$W + Z \geq 0,\ P > 0,\ Z > 0, \tag{17}$$

$$SA - FC = P^{-1} W S, \tag{18}$$

which have to be solved with respect to diagonal matrices $P \in \mathbb{R}^{N\times N}$ and $Z \in \mathbb{R}^{N\times N}$, nonsingular matrices $S \in \mathbb{R}^{N\times N}$ and $W \in \mathbb{R}^{N\times N}$, and some $F \in \mathbb{R}^{N\times p}$. Then $D = P^{-1}W$ and $L = S^{-1}F$. It is easy to see that this system can be easily solved iteratively: first, a solution $P^{-1}W$ of the LMIs (16), (17) can be found for given $N > 0$ and $\gamma > 0$, second, existence of a solution S and F of the LMI (18) can be checked. If such a solution does not exist, then another iteration can be performed for some other values of N and/or γ.

Combining the results of Proposition 6 and lemmas 4 and 5, an interval estimate can be obtained for $z(t)$.

Theorem 8. Let assumptions 1, 2 and 3 be satisfied and the matrix $A - LC$ be Hurwitz. Then for (4), (11) with the interval observer (8), (10), (15) the relations (9) are satisfied and $\underline{z}, \overline{z} \in L^\infty(I \times [0,T), \mathbb{R})$.

Remark 9. The obtained interval observer can also be applied to a nonlinear PDE. If in Assumption 2,

$$\underline{r}_0(x,t) \leq r_0(x,t,z,\frac{\partial z}{\partial x},\frac{\partial z}{\partial t}) \leq \overline{r}_0(x,t)$$

for some known $\underline{r}_0, \overline{r}_0 \in L^\infty(I \times [0,T), \mathbb{R})$ for all $x \in I$, $t \in [0,T]$ and the corresponding solutions $z(x,t)$ (provided that they exist for such a nonlinear PDE), then the interval observer (8), (10), (15) saves its form and the result of Theorem 8 stays correct. In such a case the proposed interval observer can be used for a fast and reliable calculation of envelops for solutions of nonlinear PDEs.

5. EXAMPLE

Consider an academic example of (4) for

$$\rho(x) = 1,\ a(x) = 1 + \frac{1}{2}\sin(\frac{\pi}{4}x),\ b(x) = \frac{1}{4}\cos(\pi x),$$
$$q(x) = \frac{1}{10}\cos(\frac{\pi}{2}x),\ r_1(x) = x - 0.7,$$
$$u(t) = \sin(t),\ r_0(x,t) = r_{01}(x) r_{02}(t),$$
$$r_{01}(x) = 0.2\cos(\pi x),\ |r_{02}(t)| \leq 1,$$

with $T = 20$, then $\lambda = 1$ is an admissible choice and r_{02} is an uncertain part of the input r_0 (for simulation $r_{02}(t) = \cos(t)$), then

$$\underline{r}_0(x,t) = -|r_{01}(x)|,\ \overline{r}_0(x,t) = |r_{01}(x)|.$$

The uncertainty of initial conditions is given by the interval

$$\underline{z}_0(x) = z_0(x) - 1,\ \overline{z}_0(x) = z_0(x) + 1,$$

where $z_0(x) = \sin(\pi x)$ is the function used as initial condition for simulation of this PDE. Take $\Delta = \{0, h, 2h, \ldots, 1-h, 1\}$ with $h = 1/N'$, and a pyramidal basis

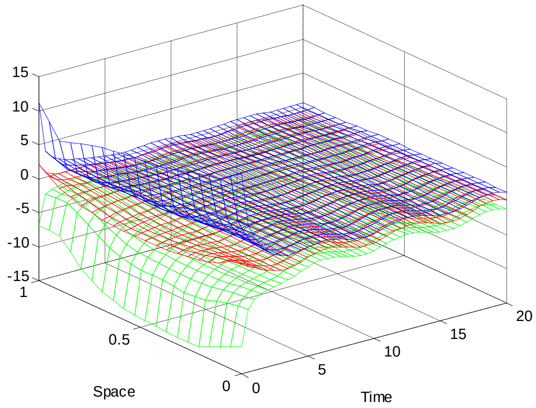

Figure 1. The results of interval estimation for $N = 10$

$$\Phi_i(x) = \begin{cases} 0 & x \leq x_{i-1}, \\ \dfrac{x - x_{i-1}}{x_i - x_{i-1}} & x_{i-1} < x \leq x_i, \\ \dfrac{x_{i+1} - x}{x_{i+1} - x_i} & x_i < x \leq x_{i+1}, \\ 0 & x \geq x_{i+1} \end{cases}$$

for $i = 1, \ldots, N = N'$ (it is assumed $x_{-1} = -1$ and $x_{N+1} = 2$). For simulation we took $N = 10$, then the approximated dynamics (6), (12) is an observable system, and assume that $\varrho h^{s+1}(l_1 + l_2) = 0.1$. Let $p = 2$ with $x_1^m = 2$, $x_2^m = 8$, and

$$\nu(t) = 0.1[\sin(25t)\ \cos(20t)]^{\mathrm{T}},$$

then $\nu_0 = 0.14$. For calculation of scalar product in space or for simulation of the approximated PDE in time, the explicit Euler method has been used with the step 0.01. The matrix L has been selected to ensure distinct eigenvalues of the matrix $A - LC$ in the interval $[-12.94, -0.55]$, then S^{-1} has been composed by eigenvectors of the matrix $A - LC$ and the matrix D has been selected diagonal. The results of interval estimation are shown in Fig. 1, where the red surface corresponds to $Z(x,t)$, while green and blue ones represent $\underline{z}(x,t)$ and $\overline{z}(x,t)$, respectively (20 and 40 points are used for plotting in space and in time).

6. CONCLUSION

Taking a parabolic PDE with Dirichlet boundary conditions, a method of design of interval observers is proposed, which is based on the Galerkin approximation. The errors of discretization are taken into account by the interval estimates. The efficiency of the proposed interval observer is demonstrated through numerical experiments.

For future research, the proposed interval observer can be used for control design of an uncertain PDE system in the spirit of Efimov et al. (2013). More complex uncertainty of PDE equation can also be incorporated in the design procedure.

REFERENCES

Alvarez, J. and Stephanopoulos, G. (1982). An estimator for a class of non-linear distributed systems. *International Journal of Control*, 5(36), 787–802.

Besanon, G. (ed.) (2007). *Nonlinear Observers and Applications*, volume 363 of *Lecture Notes in Control and Information Sciences*. Springer.

Bredies, K., Clason, C., Kunisch, K., and von Winckel, G. (2013). *Control and optimization with PDE constraints*, volume 164 of *International Series of Numerical Mathematics*. Birkhäuser, Basel.

Briat, C. (2011). Robust stability analysis of uncertain linear positive systems via integral linear constraints: l_1- and l_∞-gain characterizations. In *Proc. 50th IEEE CDC and ECC*, 6337–6342. Orlando.

Chebotarev, S., Efimov, D., Raïssi, T., and Zolghadri, A. (2015). Interval observers for continuous-time LPV systems with l_1/l_2 performance. *Automatica*, 58(8), 82–89.

Dochain, D. (2000). State observers for tubular reactors with unknown kinetics. *Journal of Process Control*, 10, 259–268.

Ebihara, Y., Peaucelle, D., and Arzelier, D. (2011). L1 gain analysis of linear positive systems and its application. In *Proc. 50th IEEE CDC and ECC*, 4029–4035. Orlando.

Efimov, D., Fridman, L., Raïssi, T., Zolghadri, A., and Seydou, R. (2012). Interval estimation for LPV systems applying high order sliding mode techniques. *Automatica*, 48, 2365–2371. doi:10.1016/j.automatica.2012.06.073.

Efimov, D., Raïssi, T., and Zolghadri, A. (2013). Control of nonlinear and lpv systems: interval observer-based framework. *IEEE Trans. Automatic Control*, 58(3), 773–782.

Farina, L. and Rinaldi, S. (2000). *Positive Linear Systems: Theory and Applications*. Wiley, New York.

Fossen, T. and Nijmeijer, H. (1999). *New Directions in Nonlinear Observer Design*. Springer.

Gouzé, J., Rapaport, A., and Hadj-Sadok, M. (2000). Interval observers for uncertain biological systems. *Ecological Modelling*, 133, 46–56.

Hagen, G. and I., M. (2003). Spillover stabilization in finite-dimensional control and observer design for dissipative evolution equations. *SIAM Journal on Control and Optimization*, 2(42), 746–768.

Jaulin, L. (2002). Nonlinear bounded-error state estimation of continuous time systems. *Automatica*, 38(2), 1079–1082.

Khalil, H.K. (2002). *Nonlinear Systems*. Prentice Hall PTR, 3rd edition.

Kieffer, M. and Walter, E. (2004). Guaranteed nonlinear state estimator for cooperative systems. *Numerical Algorithms*, 37, 187–198.

Meurer, T., Graichen, K., and Gilles, E.D. (eds.) (2005). *Control and Observer Design for Nonlinear Finite and Infinite Dimensional Systems*, volume 322 of *Lecture Notes in Control and Information Sciences*. Springer.

Moisan, M., Bernard, O., and Gouzé, J. (2009). Near optimal interval observers bundle for uncertain bioreactors. *Automatica*, 45(1), 291–295.

Olivier, B. and Gouzé, J. (2004). Closed loop observers bundle for uncertain biotechnological models. *Journal of Process Control*, 14(7), 765–774.

Raïssi, T., Efimov, D., and Zolghadri, A. (2012). Interval state estimation for a class of nonlinear systems. *IEEE Trans. Automatic Control*, 57(1), 260–265.

Raïssi, T., Videau, G., and Zolghadri, A. (2010). Interval observers design for consistency checks of nonlinear continuous-time systems. *Automatica*, 46(3), 518–527.

Shamma, J. (2012). *Control of Linear Parameter Varying Systems with Applications*, chapter An overview of LPV systems, 1–22. Springer.

Smith, H. (1995). *Monotone Dynamical Systems: An Introduction to the Theory of Competitive and Cooperative Systems*, volume 41 of *Surveys and Monographs*. AMS, Providence.

Smyshlyaev, A. and Krstic, M. (2010). *Adaptive Control of Parabolic PDEs*. Princeton University Press.

Tan, W. (1997). *Applications of Linear Parameter-Varying Control Theory*. Ph.D. thesis, Dept. of Mechanical Engineering, University of California at Berkeley.

Thomée, V. (2006). *Galerkin Finite Element Methods for Parabolic Problems*. Springer, Berlin.

Vande Wouver, A. and Zeitz, M. (2002). *Encyclopedia of Life Support Systems (EOLSS)*, chapter State estimation in distributed parameter systems. Eolss Publishers. Developed under the Auspices of the UNESCO.

Wheeler, M. (1973). l^∞ estimates of optimal orders for Galerkin methods for one-dimensional second order parabolic and hyperbolic equations. *SIAM J. Numer. Anal.*, 10(5), 908–913.