

# Finite-Time Identification Algorithm based on Time-Varying Homogeneity and Lyapunov Approach

Hector Ríos, Denis Efimov, Jaime Moreno, Wilfrid Perruquetti

► **To cite this version:**

Hector Ríos, Denis Efimov, Jaime Moreno, Wilfrid Perruquetti. Finite-Time Identification Algorithm based on Time-Varying Homogeneity and Lyapunov Approach. Proc. 10th IFAC Symposium on Nonlinear Control Systems (NOLCOS), Aug 2016, Monterey, United States. hal-01372047

**HAL Id: hal-01372047**

**<https://hal.inria.fr/hal-01372047>**

Submitted on 26 Sep 2016

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Finite-Time Identification Algorithm based on Time-Varying Homogeneity and Lyapunov Approach <sup>\*</sup>

H. Ríos <sup>\*</sup>, D. Efimov <sup>\*\*,\*</sup>, J. A. Moreno <sup>\*\*\*</sup> and W. Perruquetti <sup>\*\*</sup>.

<sup>\*</sup> *Electrical & Computer Engineering Department, University of California, Santa Barbara, CA 93106-9560, USA. Email: hector.riosbarajas@ece.ucsb.edu*

<sup>\*\*</sup> *Non-A team @ Inria, Parc Scientifique de la Haute Borne, 40 avenue Halley, 59650 Villeneuve d'Ascq, France; and CRISTAL (UMR CNRS 9189), Ecole Centrale de Lille, BP 48, Cit Scientifique, 59651 Villeneuve-d'Ascq, France. Emails: denis.efimov@inria.fr; wilfrid.perruquetti@ec-lille.fr*

<sup>\*\*\*</sup> *Eléctrica y Computación, Instituto de Ingeniería, Universidad Nacional Autónoma de México (UNAM), C.P. 04510, México, D.F., Mexico. Email: JMorenoP@ii.unam.mx*

<sup>\*\*\*\*</sup> *Department of Control Systems and Informatics, Saint Petersburg State University of Information Technologies Mechanics and Optics (ITMO), Kronverkskiy av. 49, Saint Petersburg, 197101, Russia.*

---

**Abstract:** In this paper the problem of time-varying parameter identification is studied. To this aim, an identification algorithm is developed in order to identify time-varying parameters in a finite-time. The convergence proofs are based on a notion of finite-time stability over finite intervals of time, *i.e.* Short-finite-time stability; homogeneity for time-varying systems; and Lyapunov function approach. The algorithm asks for a condition over the regressor term which is related to the classic identifiability condition corresponding to the injectivity of such a term. Simulation results illustrate the feasibility of the proposed algorithm.

*Keywords:* Time-varying parameters, Identification, Finite-Time

---

## 1. INTRODUCTION

The parameter identification problem for different kind of systems has been extensively studied during the last decades. One of the more important reasons is the need for accurate and efficient control for systems. The challenge of providing better models of physical phenomena leads to that the parameter identification problem becomes fundamental in industrial applications. System identification techniques are also used in signal processing applications (such as communications (Huijberts et al., 2000), geophysical engineering (Zhongfang, 1994) and mechanical engineering (Sun et al., 2014)), in nontechnical fields such as biology (Hasenauer et al., 2010), environmental sciences and econometrics to improve the knowledge on the identified object, prediction and control.

The identification theory basically deals with the problem of the efficient extraction of signal and system dynamic properties based on available data measurements. In the literature there exist many methods to identify parameters, and the most popular ones belong to the group of least squares (LS) methods; *e.g.* non-recursive method of LS,

recursive methods of LS, method of weighted LS, exponential forgetting with constant forgetting factor, exponential forgetting with variable forgetting factor, etc. There exist also many modifications of the LS methods; *e.g.* method of generalized LS, method of extended LS, method of bias correction, instrumental variables Method, etc., and some others like Bayes Method, Maximum Likelihood Methods, Extended Kalman Filter, Modulating functions methods, Subspaces Methods, etc. (see, *e.g.* Ioannou and Sun (1996), Soderstrom and Stoica (1989), Ljung (1999) and Isermann and Münchhof (2011)). It is worth mentioning that most of these methods were established for identifying constant parameters. For the time-varying parameter identification problem, the methods of recursive least square can be also used to deal with this problem (Isermann and Münchhof, 2011). However, they are only able to follow slowly time-varying process parameters. Moreover, to the best of our knowledge, these methods have a common characteristic: exponential or asymptotic convergence to the real value.

In the context of finite-time (FT) convergence (Polyakov and Fridman, 2014), a recursive FT convergent algorithm has been presented in Moreno and Guzman (2011). Such an algorithm is a non-linear recursive version of the LS algorithm, where the nonlinear injection terms are designed based on the generalized Super-Twisting Algorithm (STA)

---

<sup>\*</sup> This work was supported in part by the Government of Russian Federation (Grant 074-U01) and the Ministry of Education and Science of Russian Federation (Project 14.Z50.31.0031).

(Moreno and Osorio, 2012); leading to a FT convergence. In this line of research, in Davila et al. (2006) and M'sirdi et al. (2006), the STA has also been used for parameter identification of mechanical systems. However, the linearly filtered equivalent output injection signal of the STA is used to obtain the regressor, from which a standard LS recursive algorithm identifies the parameters asymptotically. Other parameter identification methods, using first order sliding-modes, are also based on the reconstruction of the equivalent control signals leading to asymptotic reconstruction algorithms (see, *e.g.* Xu et al. (2003) where an identification scheme is developed for time-varying parameters). A FT and non-recursive LS algorithm is presented by Adetola and Guay (2008) for constant parameters. Such an algorithm is based on adaptive control, it requires to solve matrix valued ordinary differential equations and checking the invertibility of a matrix (persistence of excitation condition) online. Another non-recursive LS method is proposed by Chen et al. (2011) for time-varying parameters. In this method, a polynomial approximator, based on Taylor expansion, with a bounded regressor vector is built and used to approximate the time-varying parameters.

This paper contributes to the development of an identification algorithm that is able to identify time-varying parameters in a FT. The convergence proof of the FT identification algorithm is based on a notion of finite-time stability over finite intervals of time, *i.e.* Short-finite-time (*Short-FT*) stability (Ríos et al., 2016b); and homogeneity for time-varying systems (Ríos et al., 2016a); a Lyapunov function approach is also given for this algorithm. This algorithm asks for a condition over the regressor term which is equivalent to the classic identifiability condition corresponding to the injectivity of the regressor term. Simulation results illustrate the feasibility of the proposed algorithms.

*Structure of the paper:* The problem statement is presented in the Section 2. Some preliminary concepts and results are described in Section 3. The FT identification algorithm as well as two convergence proofs are presented in Section 4 based on time-varying homogeneity and Lyapunov approach, respectively. Some simulation results are depicted in Section 5 and some concluding remarks are given in Section 6.

## 2. PROBLEM STATEMENT

Consider the following time-varying system:

$$\frac{d\theta(t)}{dt} = \Theta(\omega t), \quad (1)$$

$$y(t) = \Gamma^T(\omega t)\theta(t), \quad (2)$$

where  $\theta \in \mathbb{R}^n$  and  $y \in \mathbb{R}^m$  are the unknown parameter vector and the measurable output, respectively. The term  $\Gamma : \mathbb{R} \rightarrow \mathbb{R}^{n \times m}$  is a continuous function of time so-called *regressor*, and  $\Theta : \mathbb{R} \rightarrow \mathbb{R}^n$  is a uniformly bounded Lebesgue measurable signal such that  $|\Theta(\omega t)| \leq k(\omega t)$ , for some known continuous function  $k : \mathbb{R} \rightarrow \mathbb{R}$ , for all  $t \in \mathbb{R}_+$  and for some  $\omega > 0$  representing the frequency of time-varying parameters, while  $|q|$  represents the Euclidean norm of a vector  $q$ . The regressor  $\Gamma$  is known, and bounded, whilst  $\Theta$  represents the unknown parameter dynamics<sup>1</sup>.

<sup>1</sup> Since the function  $k$  needs to be uniformly bounded, the parameters cannot grow faster than linearly with respect to time.

The aim of this paper is to identify the time-varying parameter vector  $\theta(t)$  in a finite time.

## 3. PRELIMINARIES

A continuous function  $\sigma : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  belongs to class  $\mathcal{K}$  if it is strictly increasing and  $\sigma(0) = 0$ ; it belongs to class  $\mathcal{K}_\infty$  if it is also unbounded. A continuous function  $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$  belongs to class  $\mathcal{KL}$  if  $\beta(\cdot, r) \in \mathcal{K}$  and  $\beta(r, \cdot)$  is a strictly decreasing to zero for any fixed  $r \in \mathbb{R}_+$ . Denote a sequence of integers  $1, \dots, m$  as  $\overline{1, m}$ .

Consider a time-dependent differential equation Khalil (2002):

$$\frac{dx(t)}{dt} = f(t, x(t)), \quad t \geq t_0, \quad t_0 \in \mathbb{R}, \quad (3)$$

where  $x(t) \in \mathbb{R}^n$  is the state vector;  $f : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  is a continuous function<sup>2</sup> with respect to  $x$  and piecewise continuous with respect to  $t$ ,  $f(t, 0) = 0$  for all  $t \in \mathbb{R}$ . It is assumed that solution of the system (3) for an initial condition  $x_0 \in \mathbb{R}^n$  at time instant  $t_0 \in \mathbb{R}$  is denoted as  $x(t, t_0, x_0)$  and it is defined on some finite time interval  $[t_0, t_0 + T)$  (the notation  $x(t)$  is used to reference  $x(t, t_0, x_0)$  if the origin of  $x_0$  and  $t_0$  is clear from the context).

### 3.1 Stability definitions

Let  $\Omega, \Xi$  be open neighborhoods of the origin in  $\mathbb{R}^n$ ,  $0 \in \Omega \subset \Xi$ .

**Definition 1.** (Khalil, 2002; Haddad et al., 2008) *At the steady state  $x = 0$  the system (3) is said to be*

a) *Uniformly stable (US) if for any  $\epsilon > 0$  there is  $\delta(\epsilon)$  such that for any  $x_0 \in \Omega$ , if  $|x_0| \leq \delta(\epsilon)$  then  $|x(t, t_0, x_0)| \leq \epsilon$  for all  $t \geq t_0$ , for any  $t_0 \in \mathbb{R}$ ;*

b) *Uniformly asymptotically stable (UAS) if it is US and for any  $\kappa > 0$  and  $\epsilon > 0$  there exists  $T(\kappa, \epsilon) \geq 0$  such that for any  $x_0 \in \Omega$ , if  $|x_0| \leq \kappa$  then  $|x(t, t_0, x_0)| \leq \epsilon$  for all  $t \geq t_0 + T(\kappa, \epsilon)$  for any  $t_0 \in \mathbb{R}$ ;*

c) *Uniformly finite-time stable (UFTS) if it is US and finite-time converging from  $\Omega$ , *i.e.* for any  $x_0 \in \Omega$  there exists  $0 \leq T^{x_0} < +\infty$  such that  $x(t, t_0, x_0) = 0$  for all  $t \geq t_0 + T^{x_0}$ , for any  $t_0 \in \mathbb{R}$ . The function  $T_0(x_0) = \inf\{T^{x_0} \geq 0 : x(t, t_0, x_0) = 0 \forall t \geq t_0 + T^{x_0}\}$  is called the settling-time of the system (3).*

If  $\Omega = \mathbb{R}^n$ , then  $x = 0$  is said to be globally US (GUS) / UAS (GUAS) / UFTS (GUFTS), respectively.

In this work a special stability notion will be also used for a compact interval of initial times  $t_0$ , and only on a fixed interval of time (Kamenkov, 1953; Lebedev, 1954; Dorato, 1961; Weiss and Infante, 1965).

**Definition 2.** (Ríos et al., 2016b) *At the steady state  $x = 0$  the system (3) is said to be*

a) *Short-time stable (Short-TS) with respect to  $(\Omega, \Xi, T^0, T_f)$  if for any  $x_0 \in \Omega$ ,  $|x(t, t_0, x_0)| \in \Xi$  for all  $t \in [t_0, T_f]$  for any  $t_0 \in [-T^0, T^0]$ ;*

<sup>2</sup> System (3) may have non-unique solutions for some initial states. Selecting only solutions that stay at the origin, “weak” stability results can be presented (Haddad et al., 2008).

b) Short-finite-time stable (Short-FTS) with respect to  $(\Omega, \Xi, T^0, T_f)$  if it is Short-TS with respect to  $(\Omega, \Xi, T^0, T_f)$  and finite-time converging from  $\Omega$  with the convergence time  $T^{t_0, x_0} \leq T_f$  for all  $x_0 \in \Omega$  and  $t_0 \in [-T^0, T^0]$ ;

c) Globally short-finite-time stable (GShort-FTS) if for any bounded set  $\Omega \subset \mathbb{R}^n$  containing the origin there exist a bounded set  $\Xi \subset \mathbb{R}^n$ ,  $\Omega \subset \Xi$  and  $T_f > 0$  such that the system is Short-FTS with respect to  $(\Omega, \Xi, T^0, T_f)$  for any  $T^0$ .

In Kamenkov (1953); Lebedev (1954); Dorato (1961) and Weiss and Infante (1965) the short-time stability is considered only for a fixed initial time instant  $t_0$ . Note that this notion is used here to avoid a confusion with finite-time stability from Roxin (1966) and Bhat and Bernstein (2005); since both concepts of stability are used in this work.

### 3.2 Homogeneity for time-invariant systems

For any  $r_i > 0$ ,  $i = \overline{1, n}$  and  $\lambda > 0$ , define the dilation matrix  $\Lambda_r(\lambda) = \text{diag}\{\lambda^{r_i}\}_{i=1}^n$  and the vector of weights  $r = (r_1, \dots, r_n)^T$ . Then, for any  $x \in \mathbb{R}^n$  the homogeneous norm can be defined as follows

$$|x|_r = \left( \sum_{i=1}^n |x_i|^{\frac{\rho}{r_i}} \right)^{\frac{1}{\rho}}, \quad \rho \geq \max_{1 \leq i \leq n} (r_i).$$

For all  $x \in \mathbb{R}^n$ , its Euclidean norm is related with the homogeneous one:

$$\underline{\sigma}_r(|x|_r) \leq |x| \leq \bar{\sigma}_r(|x|_r), \quad (4)$$

for some  $\underline{\sigma}_r, \bar{\sigma}_r \in \mathcal{K}_\infty$ . In the following, due to this ‘‘equivalence’’, stability analysis with respect to the norm  $|x|$  will be substituted with analysis for the norm  $|x|_r$ . The homogeneous norm has an important property that is  $|\Lambda_r(\lambda)x|_r = \lambda|x|_r$ , for all  $x \in \mathbb{R}^n$ . Define  $\mathbb{S}_r = \{x \in \mathbb{R}^n : |x|_r = 1\}$  and  $B_\rho = \{x \in \mathbb{R}^n : |x|_r \leq \rho\}$ , for some  $0 < \rho < +\infty$ .

**Definition 3.** (Zubov, 1958) *The function  $g : \mathbb{R}^n \rightarrow \mathbb{R}$  is called  $r$ -homogeneous ( $r_i > 0$ ,  $i = \overline{1, n}$ ), if for any  $x \in \mathbb{R}^n$  the relation*

$$g(\Lambda_r(\lambda)x) = \lambda^\nu g(x),$$

holds for some  $\nu \in \mathbb{R}$  and all  $\lambda > 0$ .

*The function  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is called  $r$ -homogeneous ( $r_i > 0$ ,  $i = \overline{1, n}$ ), if for any  $x \in \mathbb{R}^n$  the relation*

$$f(\Lambda_r(\lambda)x) = \lambda^\nu \Lambda_r(\lambda)f(x),$$

holds for some  $\nu \geq -\min_{1 \leq i \leq n} r_i$  and all  $\lambda > 0$ . In both cases, the constant  $\nu$  is called the degree of homogeneity.

### 3.3 Homogeneity for time-varying systems

The weighted homogeneity property, introduced for time-invariant systems in Definition 3, is understood for the time-varying systems (3) in the following sense.

**Definition 4.** (Peuteman and Aeyels, 1999) *The function  $g : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$  is called  $r$ -homogeneous ( $r_i > 0$ ,  $i = \overline{1, n}$ ), if for any  $x \in \mathbb{R}^n$  and  $t \in \mathbb{R}$  the relation*

$$g(t, \Lambda_r(\lambda)x) = \lambda^\nu g(t, x),$$

holds for some  $\nu \in \mathbb{R}$  and all  $\lambda > 0$ .

*The function  $f : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^n$  is called  $r$ -homogeneous ( $r_i > 0$ ,  $i = \overline{1, n}$ ), if for any  $x \in \mathbb{R}^n$  and  $t \in \mathbb{R}$  the relation*

$$f(t, \Lambda_r(\lambda)x) = \lambda^\nu \Lambda_r(\lambda)f(t, x),$$

holds for some  $\nu \geq -\min_{1 \leq i \leq n} r_i$  and all  $\lambda > 0$ .

Thus, in the time-varying case (3), the homogeneity can be verified interpreting  $t$  as a constant parameter. Consider also the following modification of the system (3):

$$\frac{dx(t)}{dt} = f(\omega t, x(t)), \quad t \geq t_0, \quad t_0 \in \mathbb{R}, \quad (5)$$

for some  $\omega > 0$ . The parameter  $\omega$  represents dependence on the convergence rate of time processes in the system or the frequency of time-varying part. For an initial condition  $x_0 \in \mathbb{R}^n$  at initial time  $t_0$  denote the corresponding solution of (5) as  $x_\omega(t, t_0, x_0)$ , thus  $x(t, t_0, x_0) = x_{\omega=1}(t, t_0, x_0)$ . In this case the following proposition is provided.

**Proposition 1.** (Ríos et al., 2016a; Peuteman and Aeyels, 1999) *Let  $x(t, t_0, x_0)$  be a solution of the  $r$ -homogeneous system (3) with the degree  $\nu$  for an initial condition  $x_0 \in \mathbb{R}^n$  and  $t_0 \in \mathbb{R}$ . For any  $\lambda > 0$  the system (5) with  $\omega = \lambda^\nu$  has a solution  $y(t, t_0, y_0) = \Lambda_r(\lambda)x(\lambda^\nu t, \lambda^\nu t_0, x_0)$  for all  $t \geq t_0$  with the initial condition  $y_0 = \Lambda_r(\lambda)x_0$ .*

It is a well known fact that for linear time-varying systems (i.e. homogeneous systems of degree  $\nu = 0$ ) its stability for some  $\omega$  does not imply stability for all  $\omega \in (0, +\infty)$ . However, for nonlinear homogeneous time-varying systems with degree  $\nu \neq 0$  it does:

**Lemma 2.** (Ríos et al., 2016a; Peuteman and Aeyels, 1999) *Let the system (3) be  $r$ -homogeneous with degree  $\nu \neq 0$  and GUAS, i.e. there is  $\beta \in \mathcal{KL}$  such that*

$$|x(t, t_0, x_0)|_r \leq \beta(|x_0|_r, t - t_0), \quad \forall t \geq t_0,$$

for any  $x_0 \in \mathbb{R}^n$  and any  $t_0 \in \mathbb{R}$ . Then, (5) is GUAS for any  $\omega > 0$  and

$$|x_\omega(t, t_0, x_0)|_r \leq \beta_\omega(|x_0|_r, t - t_0), \quad \forall t \geq t_0,$$

for any  $x_0 \in \mathbb{R}^n$  and any  $t_0 \in \mathbb{R}$ , where  $\beta_\omega(s, t) = \omega^{1/\nu} \beta(\omega^{-1/\nu} s, \omega t)$ .

Thus, according to Lemma 2, the rate of convergence will be scaled by  $\omega$ , then the time of transients in these systems is predefined by the time-varying part, which is not the case for the degree  $\nu = 0$ , where the rate of convergence cannot be modified by  $\omega$ .

### 3.4 Short-Finite-Time Stability

The following time-continuity restriction is imposed on  $f$  in (5).

**Assumption 1.** *For (5), there exists a function  $\sigma \in \mathcal{K}_\infty$  such that*

$$\sup_{\xi \in \mathbb{S}_r} |f(\tau, \xi) - f(0, \xi)| \leq \sigma(|\tau|), \quad \forall \tau \in \mathbb{R}.$$

Let us introduce the following class of functions for  $\varrho \in \mathcal{K}$  and  $\delta > 0$ :

$$\begin{aligned} \mathcal{L}_{\varrho, \delta}^m = \{ & d : \mathbb{R} \rightarrow \mathbb{R}^m : |d(s)| \leq \varrho(s) \quad \forall s \geq 0; \\ & \exists \tau > 0 : d(s) = 0, \quad \forall |s| \geq \tau; \\ & \max\{|d|_1, |d|_\infty\} \leq \delta \}, \end{aligned}$$

where

$$|d|_1 = \int_{-\infty}^{+\infty} |d(t)|dt, \quad |d|_\infty = \sup_{t \in \mathbb{R}} |d(t)|.$$

In the following, the stability robustness is considered with respect to time-varying part. Assuming that for  $\omega = 0$  the system (5) is stable and homogeneous, a certain stability is preserved for a frequency spectrum sufficiently close to zero.

**Lemma 3.** (Ríos et al., 2016b) *Let the system (5) be  $r$ -homogeneous with degree  $\nu < 0$ , asymptotically stable for  $\omega = 0$ , and Assumption 1 be satisfied. Then, for any  $\rho > 0$  and  $T^0 > 0$  there are  $\omega_0 > 0$ ,  $\vartheta \geq 1$  and  $T_f > T^0$  such that (5) with  $\omega \in [-\omega_0, \omega_0]$  is Short-FTS at the origin with respect to  $(B_\rho, B_{\vartheta\rho}, T^0, T_f)$ .*

**Corollary 4.** (Ríos et al., 2016b) *Let the system in (5) possess a Lyapunov function  $V : \mathbb{R} \times \Omega \rightarrow \mathbb{R}_+$ , where  $0 \in \Omega \subset \mathbb{R}^n$  is an open neighborhood of the origin, such that for all  $x \in \Omega$  and  $t \in \mathbb{R}$*

$$\alpha_1(|x|) \leq V(t, x) \leq \alpha_2(|x|), \quad \alpha_1, \alpha_2 \in \mathcal{K}_\infty;$$

$$\dot{V}(t, x) \leq -\alpha V^\eta + k(\omega t)V^\eta, \quad \alpha > 0, \eta \in (0, 1),$$

for a continuous  $k : \mathbb{R} \rightarrow \mathbb{R}$ ,  $k(0) = 0$ . Then for any  $T^0 > 0$  there exist  $\omega_0 > 0$  such that for  $|\omega| \leq \omega_0$  the system (5) is Short-FTS with respect to  $(\Omega, \Xi, T^0, T_f)$  for some  $\Omega \subset \Xi \subset \mathbb{R}^n$  and  $T_f \geq T^0$ .

If  $\Omega = \mathbb{R}^n$  and  $k(t)$  is periodic, then there exist  $\omega_0 > 0$  and  $\delta > 0$  such that for  $k \in \mathcal{L}_{\rho, \delta}^1$ ,  $\varrho(s) = \sup_{|t| \leq s} k(\omega_0 t)$  the system (5) is GShort-FTS.

#### 4. SHORT-FT IDENTIFICATION ALGORITHM

In this section the FT identification algorithm is presented. The convergence to zero of the parameter identification error will be proved based on the statements previously described in previous sections 3.3 and 3.4, *i.e.* homogeneity for time-varying systems and *Short-FT* stability. For simplicity and brevity it is assumed that  $t_0 = 0$ .

In order to estimate the parameter vector  $\theta$ , the following nonlinear algorithm can be introduced<sup>3</sup>

$$\dot{\hat{\theta}}(t) = -K\Gamma(\omega t) \left[ \Gamma^T(\omega t)\hat{\theta}(t) - y(t) \right]^\gamma, \quad (6)$$

where  $[\cdot]^\gamma \doteq |\cdot|^\gamma \text{sign}(\cdot)$ , with  $|\cdot|$  and  $\text{sign}(\cdot)$  understood in the component-wise sense, and  $\gamma \in [0, 1]$ ; the matrix  $K \in \mathbb{R}^{n \times n}$  is symmetric and positive definite, *i.e.*  $K = K^T > 0$ . Define  $\sigma_{\Gamma_{\min}}$  as the minimum singular value of  $\Gamma(\omega t)$  for all  $t \geq 0$ . Then, let us introduce the following assumption.

**Assumption 2.** *The regressor term  $\Gamma(t)$  is such that  $\sigma_{\Gamma_{\min}} > 0$ , for all  $t \geq 0$ .*

The Assumption 2 implies that  $m \geq n$  and it is equivalent to the classic identifiability condition corresponding to the injectivity of the regressor term, *i.e.*  $\text{rank}(\Gamma(\omega t)) = n$ , for each instant of time  $t$ .

Let us define the error  $\tilde{\theta}(t) = \hat{\theta}(t) - \theta(t)$ . Hence, the error dynamics is given by

$$\dot{\tilde{\theta}}(t) = -K\Gamma(\omega t) \left[ \Gamma^T(\omega t)\tilde{\theta}(t) \right]^\gamma - \Theta(\omega t). \quad (7)$$

<sup>3</sup> A similar algorithm was previously presented in Ríos et al. (2016a) for the adaptive estimation problem, *i.e.* state estimation and parameter identification; but for constant parameters.

In the following, the *Short-FT* stability statements given by Lemma 3 and Corollary 4 will be applied, separately, to prove that error dynamics (7) is *GShort-FTS*.

##### 4.1 Case 1: Homogeneity based approach

Let us apply the statements given in Lemma 3 in order to prove that the system (7) is *Short-FTS*.

- (1) System (7) is  $r$ -homogeneous with degree  $\nu = -1$  for  $(r_1, r_2, \dots, r_n) = (1, 1, \dots, 1)$  and  $\gamma = 0$ .
- (2) Let us prove that system (7) is *GAS* for  $\omega = 0$ . Assume that  $\Theta(0) = 0$  and define  $\Gamma_0 = \Gamma(0)$ . Then, let us consider the following candidate Lyapunov function

$$V(t) = \frac{1}{2} \tilde{\theta}^T K^{-1} \tilde{\theta}. \quad (8)$$

The time derivative along the trajectories of system (7) is given as follows

$$\dot{V}(t) = \tilde{\theta}^T K^{-1} \left( -K\Gamma_0 \left[ \Gamma_0^T \tilde{\theta} \right]^\gamma \right) = -\tilde{\theta}^T \Gamma_0 \left[ \Gamma_0^T \tilde{\theta} \right]^\gamma.$$

Note that

$$\tilde{\theta}^T \Gamma_0 \left[ \Gamma_0^T \tilde{\theta} \right]^\gamma = \sum_{i=1}^m |(\tilde{\theta}^T \Gamma_0)_i|^{\gamma+1} = |\tilde{\theta}^T \Gamma_0|^{\gamma+1},$$

and since  $|\tilde{\theta}^T \Gamma_0| \leq |\tilde{\theta}^T \Gamma_0|_{\gamma+1}$  holds for all  $2 > \gamma + 1 > 0$ ,  $\dot{V}$  may be bounded as follows

$$\begin{aligned} \dot{V}(t) &\leq -|\tilde{\theta}^T \Gamma_0|_{\gamma+1}^{\gamma+1} \leq -|\tilde{\theta}^T \Gamma_0|^{\gamma+1}, \\ &\leq -\left( \tilde{\theta}^T \Gamma_0 \Gamma_0^T \tilde{\theta} \right)^{\frac{\gamma+1}{2}} \leq -\sigma_{\Gamma_{\min}}^{\gamma+1} |\tilde{\theta}|^{\gamma+1}. \end{aligned}$$

Hence,  $\dot{V}$  is negative definite and thus, *GAS* is concluded for  $\omega = 0$ .

- (3) Since  $\Gamma$  is a continuous function of time and  $\Theta$  is a uniformly bounded Lebesgue measurable signal, Assumption 1 is satisfied for all  $\gamma \in [0, 1]$ .

Then, according to Lemma 3, the following result has been proved.

**Theorem 5.** *Let Assumption 2 be satisfied. If  $\Theta(0) = 0$ ; then, for any  $\rho > 0$  and  $T^0 > 0$  there exist  $\omega_0 > 0$ ,  $\vartheta \geq 1$  and  $T_f > T^0$ ; such that system (7) with  $\omega \in [-\omega_0, \omega_0]$ , for  $\gamma = 0$ , is Short-FTS at the origin with respect to  $(B_\rho, B_{\vartheta\rho}, T^0, T_f)$ .*

**Remark 1.** *According to Theorem 5, the Short-FT stability is preserved for a frequency spectrum sufficiently close to zero.*

##### 4.2 Case 2: Lyapunov function based approach

Let us consider the Lyapunov function (8) which satisfies the following inequalities

$$c_1^{-1} |\tilde{\theta}|^2 \leq V \leq c_2^{-1} |\tilde{\theta}|^2, \quad (9)$$

$$c_1^{-\frac{\gamma+1}{2}} |\tilde{\theta}|^{\gamma+1} \leq V^{\frac{\gamma+1}{2}} \leq c_2^{-\frac{\gamma+1}{2}} |\tilde{\theta}|^{\gamma+1}, \quad (10)$$

where  $c_1 = 2\lambda_{\max}(K)$  and  $c_2 = 2\lambda_{\min}(K)$ . The function  $V$  is positive definite, radially unbounded, and continuously differentiable with its time derivative satisfying

$$\begin{aligned} \dot{V}(t) &= \tilde{\theta}^T K^{-1} \left( -K\Gamma \left[ \Gamma^T \tilde{\theta} \right]^\gamma - \Theta(t) \right), \\ &\leq -\tilde{\theta}^T \Gamma \left[ \Gamma^T \tilde{\theta} \right]^\gamma + \frac{|\tilde{\theta}| |\Theta(t)|}{\lambda_{\min}(K)}. \end{aligned}$$

## 5. SIMULATION RESULTS

Then, recalling that  $\tilde{\theta}^T \Gamma \left[ \Gamma^T \tilde{\theta} \right]^\gamma = |\tilde{\theta}^T \Gamma|_{\gamma+1}^{\gamma+1}$ , and since  $|\tilde{\theta}^T \Gamma| \leq |\tilde{\theta}^T \Gamma|_{\gamma+1}$  holds for all  $2 > \gamma + 1 > 0$ , and  $|\Theta(t)| \leq k(\omega t)$ ,  $\dot{V}$  may be bounded as follows

$$\begin{aligned} \dot{V}(t) &\leq -|\tilde{\theta}^T \Gamma|_{\gamma+1}^{\gamma+1} + \frac{k(\omega t)}{\lambda_{\min}(K)} |\tilde{\theta}|, \\ &\leq -|\tilde{\theta}^T \Gamma|_{\gamma+1}^{\gamma+1} + \frac{k(\omega t)}{\lambda_{\min}(K)} |\tilde{\theta}|, \\ &\leq -\left(\tilde{\theta}^T \Gamma \Gamma^T \tilde{\theta}\right)^{\frac{\gamma+1}{2}} + \frac{k(\omega t)}{\lambda_{\min}(K)} |\tilde{\theta}|, \\ &\leq -\sigma_{\Gamma_{\min}}^{\gamma+1} c_2^{\frac{\gamma+1}{2}} V^{\frac{\gamma+1}{2}}(t) + \frac{2\sqrt{c_1}}{c_2} k(\omega t) V^{\frac{1}{2}}(t). \end{aligned} \quad (11)$$

Let us assume that  $\gamma = 0$ . Therefore, from (11) and (9), it follows that

$$\dot{V}(t) \leq -\sigma_{\Gamma_{\min}} \sqrt{c_2} V^{\frac{1}{2}}(t) + \frac{2\sqrt{c_1}}{c_2} k(\omega t) V^{\frac{1}{2}}(t).$$

Applying Corollary 4, with  $\alpha = \sigma_{\Gamma_{\min}} \sqrt{c_2}$ ,  $\eta = 0.5$ , and assuming that  $k(t)$  is periodic, and such that  $k \in \mathcal{L}_{\varrho, \delta}^1$ , with  $\varrho(s) = \sup_{|t| \leq s} k(\omega_0 t)$ , one can conclude that the error (7) is *Short-FTS*. Thus, based on the statements given by Corollary 4, the following result has been proven.

**Theorem 6.** *Let Assumption 2 be satisfied. If there exist  $\omega_0 > 0$  and  $\delta > 0$  such that  $k(t)$  is a periodic continuous function such that  $k \in \mathcal{L}_{\varrho, \delta}^1$ , with  $\varrho(s) = \sup_{|t| \leq s} k(\omega_0 t)$ ; then, the system (7), for the case  $\gamma = 0$ , is GShort-FTS.*

**Remark 2.** *From (11), if the matrix  $K$  is such that  $\lambda_{\min}(K) > (\sqrt{\lambda_{\max}(K)} k(\omega t) / \sigma_{\Gamma_{\min}})^{2/3}$ , then the system (7) is UFTS.*

Let us consider the case in which  $\gamma \in (0, 1)$ . Then, based on the statements given by Theorem 6, the following result is established.

**Corollary 7.** *Let Assumption 2 be satisfied. If  $|k(t)| \leq \Lambda$ , for all  $t \in \mathbb{R}$ , with a known positive constant  $\Lambda$ ; and  $\gamma \in (0, 1)$ ; then, the system (7) is globally ultimate bounded, and its trajectories satisfy the following bound*

$$|\tilde{\theta}(t)| \leq \sqrt{\frac{\lambda_{\max}(K)}{\lambda_{\min}(K)}} \mu, \quad \forall t \geq T(\tilde{\theta}(0)), \quad (12)$$

with

$$\mu = (\Lambda / \lambda_{\min}(K) \sigma_{\Gamma_{\min}}^{\gamma+1} \delta)^{\frac{1}{\gamma}},$$

$$T(\tilde{\theta}(0)) \leq 2c_2^{\frac{\gamma-1}{2}} |\tilde{\theta}(0)|^{\frac{1-\gamma}{2}} / \sigma_{\Gamma_{\min}}^{\gamma+1} (1-\delta) c_1^{\frac{\gamma+1}{2}} (1-\gamma),$$

for any  $\delta \in (0, 1)$ , and any  $\tilde{\theta}(0) \in \mathbb{R}^n$ .

**Proof.** It is omitted due to lack of space. ■

**Remark 3.** *Note that the solutions of system (7) enter the bound (12) at most in a finite time  $T(\tilde{\theta}(0))$ . In addition, similarly to Definition 2, system (7) is GShort-FTS with respect to the set  $\{\tilde{\theta} \in \mathbb{R}^n : |\tilde{\theta}(t)| \leq \sqrt{\lambda_{\max}(K) / \lambda_{\min}(K)} \mu\}$ .*

**Remark 4.** *Corollary 7 shows that the parameter identification error may be reduced according to the choice of the gain  $K$  and the parameter  $\gamma$  since the size of  $\mu$  depends on the value of both of them.*

Consider the following parameter dynamics, *i.e.*

$$\begin{aligned} \dot{\theta}_1(t) &= \sin(4t) + \cos(2t), \quad \theta_1(0) = 0, \\ \dot{\theta}_2(t) &= \sin(t) + \cos(3t), \quad \theta_2(0) = 0, \\ y(t) &= \Gamma^T(t) \theta(t), \end{aligned}$$

where the regressor term, that satisfies Assumption 2, has the following structure

$$\Gamma(t) = \begin{bmatrix} 2 \cos(t) + \sin(t) & \sin(t+1) - \cos(t+1) \cdots \\ \cos(t) + \sin(2t) & \cos(0.5t) + \sin(0.5t) \cdots \\ 3 \cos(2t) + \sin(2t) & 2 \cos(0.3t) + \sin(0.3t) \\ 2 \cos(2.5t) + \sin(1.5t) & \sin(1.3t) - 3 \cos(1.3t) \end{bmatrix},$$

The FT algorithm (6) is implemented with  $\gamma = 0$  and  $K = 2I_2$ , and different initial conditions. The results are depicted by Figs. 1 and 2. The results illustrate the statements given by Theorem 6, *i.e.* *FT convergence*.

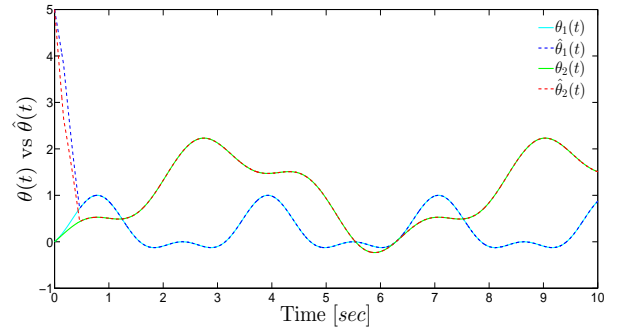


Fig. 1. Parameter Identification. *This graph shows the parameter identification for the case  $\gamma = 0.0$ , with  $\theta(0) = (0, 0)^T$  and  $\hat{\theta}(0) = (5, 5)^T$ .*

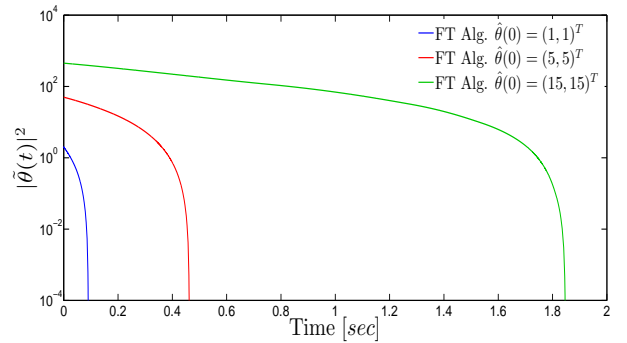


Fig. 2. Parameter Identification Error. *The graph shows that the convergence time increases when the initial conditions of the error dynamics also increases.*

Now, the algorithm (6) is implemented with  $K = 2I_2$  and different values of  $\gamma \in (0, 1)$ . The parameter identification error is depicted by Fig. 3. The results illustrate the statements given by Corollary 7, *i.e.* *global ultimate boundedness*.

## 6. CONCLUSIONS

A FT identification algorithm is proposed that is able to identify time-varying parameters in a finite-time. The

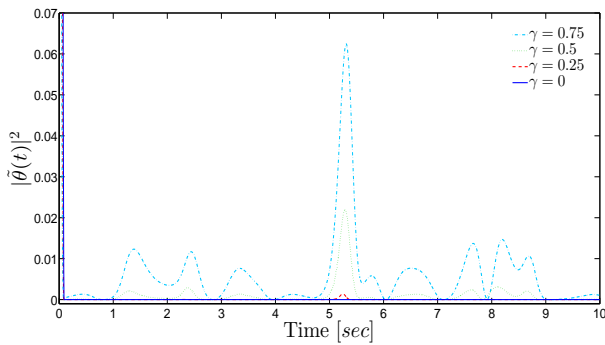


Fig. 3. Parameter Identification Error. This graph shows the parameter identification error for different values of  $\gamma \in (0, 1)$ , with  $\theta(0) = (0, 0)^T$  and  $\hat{\theta}(0) = (1, 1)^T$ .

convergence proof of the FT identification algorithm is based on *Short-FT* stability and homogeneity for time-varying systems; and also a Lyapunov function approach is given for this algorithm. This algorithm asks for a condition over the regressor term which is equivalent to the classic identifiability condition corresponding to the injectivity of the regressor term. Simulation results depict the feasibility of the proposed algorithm. The noise measurement problem is in the scope of future research.

#### REFERENCES

- Adetola, V. and Guay, M. (2008). Finite-time parameter estimation in adaptive control of nonlinear systems. *IEEE Transactions on Automatic Control*, 53(3).
- Bhat, S. and Bernstein, D. (2005). Geometric homogeneity with applications to finite-time stability. *Mathematics of Control, Signals and Systems*, 17, 101–127.
- Chen, J., Zhang, G., and Li, Z. (2011). On-line parameter estimation for a class of time-varying continuous systems with bounded disturbances. *International Journal of Adaptive Control and Signal Processing*, (25), 18–32.
- Davila, J., Fridman, L., and Poznyak, A. (2006). Observation and identification of mechanical systems via second order sliding modes. *International Journal of Control*, 79(10), 1251–1262.
- Dorato, P. (1961). *Short-time stability in linear time-varying systems*. Ph.D. thesis, Polytechnic Institute of Brooklyn, New York.
- Haddad, W.M., Nersesov, S.G., and Du, L. (2008). Finite-time stability for time-varying nonlinear dynamical systems. In *Proc. ACC*, 4135–4139. Seattle.
- Hasenauer, J., Waldherr, S., Wagner, K., and Allgöwer, F. (2010). Parameter identification, experimental design and model falsification for biological network models using semidefinite programming. *IET Systems Biology*, 4(2), 119–130.
- Huijberts, H., Nijmeijer, H., and Willems, R. (2000). System identification in communication with chaotic systems. *IEEE Transactions on Circuits and Systems - I: Fundamental Theory and Applications*, 47(6), 800–808.
- Ioannou, P.A. and Sun, J. (1996). *Robust Adaptive Control*. Prentice Hall, Inc., New Jersey, USA.
- Isermann, R. and Münchhof, M. (2011). *Identification of Dynamic Systems, An Introduction with Applications*. Springer-Verlag, Berlin, 1st edition.
- Kamenkov, G. (1953). On stability of motion over a finite interval of time. *Journal of Applied Math. and Mechanics (PMM)*, 17, 529–540.
- Khalil, H. (2002). *Nonlinear Systems*. Prentice Hall, Upper Saddle River, New Jersey, 3rd edition.
- Lebedev, A. (1954). The problem of stability in a finite interval of time. *Journal of Applied Math. and Mechanics (PMM)*, 18, 75–94.
- Ljung, L. (1999). *System Identification: Theory for the User*. Prentice Hall Information and System Sciences Series, New Jersey, second edition.
- Moreno, J. and Osorio, M. (2012). Strict lyapunov functions for the super-twisting algorithm. *IEEE Transactions on Automatic Control*, 57(4), 1035–1040.
- Moreno, J.A. and Guzman, E. (2011). A new recursive finite-time convergent parameter estimation algorithm. In *In Proceedings of the 18th IFAC World Congress*, volume 18, 3439–3444. Milano, Italy.
- M’sirdi, N., Rabhi, A., Fridman, L., Davila, J., and Delanne, Y. (2006). Second order sliding mode observer for estimation of velocities, wheel sleep, radius and stiffness. In *Proceedings of the 2006 American Control Conference*, 3316–3321. Minneapolis, Minnesota USA.
- Peuteman, J. and Aeyels, D. (1999). Averaging results and the study of uniform asymptotic stability of homogeneous differential equations that are not fast time-varying. *SIAM J. Control Optim.*, 37(4), 997–1010.
- Polyakov, A. and Fridman, L. (2014). Stability notions and lyapunov functions for sliding mode control systems. *Journal of the Franklin Institute*, 351(4), 1831–1865.
- Ríos, H., Efimov, D., Fridman, L., Moreno, J., and Perruquetti, W. (2016a). Homogeneity based uniform stability analysis for time-varying systems. *IEEE Transactions on Automatic Control*, 61(3), 725–734.
- Ríos, H., Efimov, D., Polyakov, A., and Perruquetti, W. (2016b). Homogeneous time-varying systems: Robustness analysis. *IEEE Transactions on Automatic Control*. DOI:10.1109/TAC.2016.2540806.
- Roxin, E. (1966). On finite stability in control systems. *Rendiconti del Circolo Matematico di Palermo*, 15, 273–283.
- Soderstrom, T. and Stoica, P. (1989). *System Identification*. Prentice Hall International, Cambridge, Great Britain.
- Sun, W., Li, H., and Han, Q. (2014). Identification of mechanical parameters of hard-coating materials with strain-dependence. *Journal of Mechanical Science and Technology*, 28(1), 81–92.
- Weiss, L. and Infante, E. (1965). On the stability of systems defined over a finite time interval. *Proc. of the National Academy of Sciences*, 54, 440–448.
- Xu, J.X., Pan, Y.J., and Lee, T.H. (2003). A VSS identification scheme for time-varying parameters. *Automatica*, 39, 727–734.
- Zhongfang, Q. (1994). Distributed parameter system identification and its application to geophysical parameter identification. In *Proceedings of the IEEE International Conference on Industrial Technology*, 799–802. Guangzhou.
- Zubov, V. (1958). On systems of ordinary differential equations with generalized homogenous right-hand sides. *Izvestia vuzov. Matematika.*, 1, 80–88. In Russian.