

# A Parametric Propagator for Pairs of Sum Constraints with a Discrete Convexity Property

Jean-Noël Monette, Nicolas Beldiceanu, Pierre Flener, Justin Pearson

► **To cite this version:**

Jean-Noël Monette, Nicolas Beldiceanu, Pierre Flener, Justin Pearson. A Parametric Propagator for Pairs of Sum Constraints with a Discrete Convexity Property. *Artificial Intelligence*, Elsevier, 2016, 241, pp.170 - 190. 10.1016/j.artint.2016.08.006 . hal-01374714

**HAL Id: hal-01374714**

**<https://hal.inria.fr/hal-01374714>**

Submitted on 30 Sep 2016

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# A Parametric Propagator for Pairs of SUM Constraints with a Discrete Convexity Property<sup>☆</sup>

Jean-Noël Monette<sup>a,\*</sup>, Nicolas Beldiceanu<sup>b,\*\*</sup>, Pierre Flener<sup>a,\*\*</sup>,  
Justin Pearson<sup>a,\*\*</sup>

<sup>a</sup>*Uppsala University, Dept of Information Technology, 751 05 Uppsala, Sweden*

<sup>b</sup>*Mines Nantes, TASC (CNRS/INRIA), 44307 Nantes, France*

---

## Abstract

We introduce a propagator for pairs of SUM constraints, where the expressions in the sums respect a form of convexity. This propagator is parametric and can be instantiated for various concrete pairs, including DEVIATION, SPREAD, and the conjunction of LINEAR<sub>≤</sub> and AMONG. We show that despite its generality, our propagator is competitive in theory and practice with state-of-the-art propagators.

*Keywords:* constraint programming, propagator, discrete convexity

---

## 1. Introduction

Constraint programming (CP) is a set of techniques to model and solve combinatorial problems using a mix of inference and search. Each constraint of the problem is associated with a *propagator* that performs pruning, i.e., the removal from the search space of impossible values for the variables of the constraint.

Many combinatorial problems involve SUM constraints, along with other constraints. It is however well-known that a propagator for a SUM constraint taken in isolation is not able to perform a lot of pruning since the estimation of the minimum or maximum of a sum does not take other constraints into account. Several authors have studied how to include other constraints (sharing some variables) in the propagator for SUM, either in particular cases (e.g., SPREAD [2], DEVIATION [3], INCREASINGSUM [4], and SUM with cliques [5]), or in general (e.g., OBJECTIVESUM [6]).

---

<sup>☆</sup>This work was supported by grants 2011-6133 and 2012-4908 of the Swedish Research Council (VR). We thank the reviewers of CP 2013 for their constructive comments on the prior version [1] of this paper; the reviewers of this journal also greatly helped us to improve this paper.

\*Principal corresponding author

\*\*Corresponding author

*Email addresses:* Jean-Noel.Monette@tacton.com (Jean-Noël Monette),  
Nicolas.Beldiceanu@mines-nantes.fr (Nicolas Beldiceanu), Pierre.Flener@it.uu.se  
(Pierre Flener), Justin.Pearson@it.uu.se (Justin Pearson)

In the present work, we focus on a parametric constraint, called `TWOSUMS` hereafter, which can be cast as

$$\sum_{i \in [1, n]} f_i(x_i) \leq \bar{f} \tag{1}$$

$$\underline{g} \leq \sum_{i \in [1, n]} g_i(x_i) \leq \bar{g} \tag{2}$$

for any  $n \geq 1$ . The parameters  $f_i$  and  $g_i$  are functions from integers to integers and the  $f_i$  (respectively  $g_i$ ) can differ for each  $i$ . Initially, we require  $\bar{f}$ ,  $\underline{g}$ , and  $\bar{g}$  to be constants, but Section 7 shows how to use variables instead, along with other extensions of this constraint. As usual when describing a constraint and its propagator, we assume that all  $x_i$  are distinct variables.

Finding a solution to the conjunction of (1) and (2) is in general NP-complete as it includes as a special case the knapsack problem (by taking  $f_i(v) = a_i \cdot v$  and  $g_i(v) = b_i \cdot v$ , where the  $a_i$  and  $b_i$  are constants). In this paper, we will discuss a large class of functions  $f_i$  and  $g_i$  for which either domain consistency,  $\text{bounds}(\mathbb{Z})$  consistency, or  $\text{bounds}(\mathbb{R})$  consistency [7, 8] can be achieved in polynomial time. We present a *parametric propagator* for this tractable class and show how to instantiate it for various functions  $f_i$  and  $g_i$ . We show that the instantiations include among others  $\text{bounds}(\mathbb{Z})$ -consistent propagators for the constraints `DEVIATION` [3] and `SPREAD` [2] and a domain-consistent propagator for the conjunction of `LINEAR<` and `AMONG` [9].

While the worst-case time complexity of our parametric propagator is  $\mathcal{O}(n \cdot d^2 + n^2 \cdot d)$ , where  $d$  is the size of the largest domain,<sup>1</sup> our propagator, once instantiated, matches the theoretical time complexity and practical efficiency of several previously published specialised propagators, as shown in Table 1. However, our propagator is not limited to the reproduction of existing propagators. Table 1 also lists the time complexity and consistency for several instantiations that we identified as being useful general cases but for which, to the best of our knowledge, no propagator existed. This list is not exhaustive and one can add many problem-specific instantiations: see for instance Example 3. Note that while achieving domain consistency on the knapsack problem is NP-hard,  $\text{bounds}(\mathbb{R})$  consistency is achieved in polynomial time by our propagator (last line in Table 1).

After introducing some notation and background in Section 2, we present our `TWOSUMS` propagator in Sections 3 to 5. We study its complexity in Section 6 and give some implementation notes. In Section 7, we show how the applicability of the propagator can be extended. Afterwards, we present in Section 8 several instantiations of the propagator, including a detailed case study for `DEVIATION`. Finally, Section 9 presents some experimental results showing that the genericity of our approach is not detrimental to performance. We review the related work in Section 10 before concluding in Section 11.

---

<sup>1</sup>The precise expression of the time complexity also depends on several other parameters introduced later and is detailed in Section 6

Table 1: Several instantiations of the TwoSums constraint. For each constraint, we give the consistency achieved by the propagator presented in this paper along with its time complexity, as well as the complexity of previously published propagators achieving the same consistency. Here  $n$  is the number of variables,  $d$  is the size of the largest domain, and  $d_{\cup}$  is the size of the union of the domains. The constraint definitions are given in Section 2.

Name	Consistency	Complexity	Specialised Propagator
LINEAR $_{\leq}$	domain	$\mathcal{O}(n)$	$\mathcal{O}(n)$ [10]
LINEAR $_{\leq} \wedge$ AMONG	domain	$\mathcal{O}(n \cdot (\log n + d))$	$\mathcal{O}(n \cdot (\log n + d))$ [9]
LINEAR $_{\leq} \wedge$ MAXIMUM	domain	$\mathcal{O}(n \cdot (\log n + d))$	–
DEVIATION	bounds( $\mathbb{Z}$ )	$\mathcal{O}(n)$	$\mathcal{O}(n)$ [3]
SPREAD	bounds( $\mathbb{Z}$ )	$\mathcal{O}(n \cdot d_{\cup})$	$\mathcal{O}(n \log n)$ [11]
L $_p$ -NORM, $0 < p < +\infty$	bounds( $\mathbb{Z}$ )	$\mathcal{O}(n \cdot d_{\cup})$	–
LINEAR $_{=}$	bounds( $\mathbb{R}$ )	$\mathcal{O}(n)$	$\mathcal{O}(n)$ [10]
LINEAR $_{\leq} \wedge$ LINEAR $_{=}$	bounds( $\mathbb{R}$ )	$\mathcal{O}(n^2)$	–

This paper is an extended version of [1], expanded with new examples, proofs, and implementation notes.

## 2. Notation and Background

For a function  $f$  and value  $v$ , we write  $f^{-1}(v)$  for the *set* of values having  $v$  as image:  $\{u \mid f(u) = v\}$ . For a function  $f$  and set  $S$ , we write  $f(S)$  for the set of images of the elements of  $S$ :  $\{f(u) \mid u \in S\}$ . We use  $x_i, v_i, f_i$  to represent single variables, values, and functions, while  $\mathbf{x}, \mathbf{v}, \mathbf{f}$  represent vectors of variables, values, and functions (e.g.,  $\mathbf{x} = \langle x_1, x_2, \dots, x_n \rangle$ ).

Constraint programming (CP) is a set of techniques to model and solve problems defined by a set of existentially quantified variables and a set of constraints over those variables. During the solution process, each variable  $x$  is associated with a current *domain*, denoted  $D_x$ , of candidate values for this variable. Each variable is given an initial domain in the model. The smallest and largest values of  $D_x$  are denoted respectively  $\min(D_x)$  and  $\max(D_x)$ . For a constraint  $c$  and the variables  $\mathbf{x}$  involved in  $c$ , an assignment of values  $v_i \in D_{x_i}$ , for all  $x_i \in \mathbf{x}$ , is called a *solution* to  $c$  if the vector  $\mathbf{v}$  of values satisfies  $c$ .

One of the main operations in CP-style constraint solving is the *filtering* of the domain of a variable with respect to a constraint: values that can be proven not to belong to any solution of a constraint  $c$  (given the domains of the other variables) are removed from the domain of each variable. This is performed by an algorithm called a *propagator* of  $c$ .

A propagator does not need to remove all values that do not participate in any solution, as doing so can be computationally too expensive. If a propagator removes all such values, then it is said to enforce *domain consistency*. If a propagator always enforces domain consistency, then it is called *domain consistent*.

Weaker notions of consistency exist. In particular, *bounds*( $\mathbb{Z}$ ) consistency (see, e.g., [7, 8]) is achieved if, for each variable  $x$ , both  $\min(D_x)$  and  $\max(D_x)$  belong to a solution in which the other variables take supporting integer values within their domain bounds. If a propagator always enforces *bounds*( $\mathbb{Z}$ ) consistency, then it is called *bounds*( $\mathbb{Z}$ ) consistent. Similarly, *bounds*( $\mathbb{R}$ ) consistency (see also, e.g., [7, 8]) is achieved if, for each variable  $x$ , both  $\min(D_x)$  and  $\max(D_x)$  belong to a solution in which the other variables take supporting real values within their domain bounds. *Bounds*( $\mathbb{R}$ ) consistency is weaker than *bounds*( $\mathbb{Z}$ ) consistency in that it considers that supporting values need not be integers. *Bounds*( $\mathbb{R}$ ) consistency is the consistency usually considered for the implementation of propagators for LINEAR constraints. If a propagator always enforces *bounds*( $\mathbb{R}$ ) consistency, then it is called *bounds*( $\mathbb{R}$ ) consistent.

In this work, we make the distinction between the SUM constraint and the LINEAR constraints. A SUM constraint is any constraint  $\sum_{i \in [1, n]} f_i(x_i) \# y$ , for any functions  $f_i$  from integers to integers, with  $\# \in \{<, \leq, =, \neq, \geq, >\}$ , and  $y$  being an integer variable or constant. The  $f_i$  in the SUM constraint may be instantiated to give rise to many existing constraints. In particular,  $\text{LINEAR}_{\leq}(\mathbf{x}, \mathbf{a}, s)$  holds if and only if the weighted sum of variables  $x_i$  with given integer weights  $a_i$  is at most the integer variable  $s$ , i.e.,  $\sum_{i \in [1, n]} a_i \cdot x_i \leq s$ . Similarly,  $\text{LINEAR}_{=}(\mathbf{x}, \mathbf{a}, s)$  holds if and only if the weighted sum of variables  $x_i$  with given integer weights  $a_i$  is equal to the integer variable  $s$ , i.e.,  $\sum_{i \in [1, n]} a_i \cdot x_i = s$ . These constraints have been studied among others in [10] where many practical improvements of the usual  $\mathcal{O}(n)$  *bounds*( $\mathbb{R}$ )-consistent propagator are introduced. In the case of  $\text{LINEAR}_{\leq}$ , *bounds*( $\mathbb{R}$ ) consistency coincides with domain consistency.

The  $\text{SPREAD}(\mathbf{x}, \mu, s)$  constraint [2] holds if and only if the average of the integer variables  $x_i$  is the given rational number  $\mu$  and the sum of their scaled squared deviations from  $\mu$  is less than or equal to the integer variable  $s$ , i.e.,  $\sum_{i \in [1, n]} x_i = n \cdot \mu \wedge \sum_{i \in [1, n]} (n \cdot x_i - n \cdot \mu)^2 \leq s$ . While a generalised constraint with a variable average exists [2], we consider here only the case of a fixed average, which is used in most applications. Following [2], all values in the second inequality are scaled by  $n$  to work with integer values, as in general the average  $\mu$  might not be integer but  $n \cdot \mu$  surely is. An  $\mathcal{O}(n \cdot \log n)$  *bounds*( $\mathbb{Z}$ )-consistent propagator for SPREAD has been introduced in [11].

Similarly, the  $\text{DEVIATION}(\mathbf{x}, \mu, d)$  constraint [3] holds if and only if the average of the integer variables  $x_i$  is the given rational number  $\mu$  and the sum of their scaled deviations from  $\mu$  is less than or equal to the integer variable  $d$ , i.e.,  $\sum_{i \in [1, n]} x_i = n \cdot \mu \wedge \sum_{i \in [1, n]} |n \cdot x_i - n \cdot \mu| \leq d$ . Again, we consider only the case of a fixed average. An  $\mathcal{O}(n)$  *bounds*( $\mathbb{Z}$ )-consistent propagator for DEVIATION has been presented in [3].

The constraint  $\text{L}_p\text{-NORM}(\mathbf{x}, \mu, d)$ , with  $0 < p < +\infty$  holds if and only if  $\sum_{i \in [1, n]} x_i = n \cdot \mu \wedge \sum_{i \in [1, n]} |n \cdot x_i - n \cdot \mu|^p \leq d$ . It generalises SPREAD (with  $p = 2$ ) and Deviation (with  $p = 1$ ).

The  $\text{AMONG}(\mathbf{x}, \mathcal{V}, c)$  constraint holds if and only if the number of integer variables  $x_i$  taking their value in the given integer set  $\mathcal{V}$  is equal to the integer

variable  $c$ . The AMONG constraint can be represented using a SUM constraint as  $\sum_{i \in [1, n]} [x_i \in \mathcal{V}] = c$ , where the notation  $[\gamma]$  is the Iverson bracket and is defined to be 1 if  $\gamma$  is true, and 0 otherwise. A domain-consistent propagator for the conjunction of a  $\text{LINEAR}_{\leq}$  constraint and an AMONG constraint has been published in [9], with an  $\mathcal{O}(n \cdot (\log n + d))$  time complexity, where  $d$  is the size of the largest  $D_{x_i}$ .

### 3. Overview of the Approach

Our approach for propagating the TWOSUMS constraint contains two parts. First (as discussed in Section 4), we compute a sharp lower bound  $b$  on the left-hand side  $\sum_{i \in [1, n]} f_i(x_i)$  of constraint (1) under constraint (2), together with a *support*  $\mathbf{s}^b$ , i.e., an assignment of the  $x_i$  yielding value  $b$  and satisfying constraint (2). The conjunction of (1) and (2) is feasible if and only if this lower bound, which we call the *feasibility bound*, is at most  $\bar{f}$ , the right-hand side of constraint (1). To compute this feasibility bound, we introduce new functions derived from the  $f_i$  and  $g_i$ . We show that the feasibility bound can be greedily computed if the newly introduced functions are *discretely convex* [12].

**Example 1.** For the conjunction of  $\text{LINEAR}_{\leq}(\mathbf{x}, \mathbf{a}, s)$  and  $\text{AMONG}(\mathbf{x}, \mathcal{V}, c)$  [9], the feasibility bound is computed by first taking for each variable an extreme value in its domain, and then iteratively modifying the values taken by some variables until the AMONG constraint is satisfied. The initial value is the largest in the domain of  $x_i$  if  $a_i < 0$  and is the smallest otherwise, so that the value of the weighted sum is as small as possible. The iterative modification picks a variable such that changing its value causes the smallest increase to the bound. As the increase caused by a variable is independent of the values taken by the other variables, the iteration order can be determined beforehand by sorting. Full details are provided in [9].  $\square$

In the second part (discussed in Section 5) of our propagator for TWOSUMS, the domain of each variable  $x_j$  is filtered by computing for each value  $u$  in its domain a sharp lower bound on the left hand side  $\sum_{i \in [1, n]} f_i(x_i)$  of constraint (1) under constraint (2) when  $x_j$  is assigned  $u$ . If this lower bound is larger than  $\bar{f}$ , then  $u$  is removed from the domain of  $x_j$ . The lower bound for each pair  $(x_j, u)$  is computed *incrementally* from the support  $\mathbf{s}^b$  for the feasibility bound, thanks to the discrete convexity property, and does not need to be computed explicitly for each value. We also present an improved propagator when an additional property holds on  $f_j$  and  $g_j$ , namely a form of monotonicity.

**Example 2.** For the conjunction of  $\text{LINEAR}_{\leq}(\mathbf{x}, \mathbf{a}, s)$  and  $\text{AMONG}(\mathbf{x}, \mathcal{V}, c)$ , the filtering for each variable  $x_j$  amounts to computing two values: the lower bound  $b_j^{\in}$  for the weighted sum of all the other variables when variable  $x_j$  is assigned a value in  $\mathcal{V}$ , and the lower bound  $b_j^{\notin}$  when  $x_j$  is assigned a value not in  $\mathcal{V}$ . One of these two bounds ( $b_j^{\in}$  if  $s_j^b \in \mathcal{V}$  and  $b_j^{\notin}$  otherwise) is directly computed from

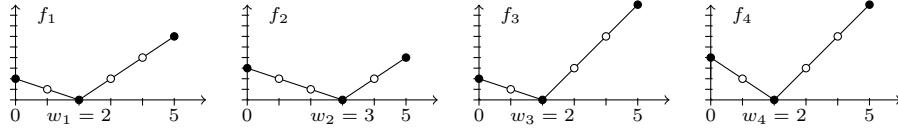


Figure 1: The  $f_i$  functions of Example 3

the feasibility bound  $b$  by subtracting the contribution of  $x_j$ . The other bound can be computed incrementally from the first bound by modifying the value of another variable than  $x_j$  so that the modification causes the smallest increase of the feasibility bound. The two new bounds are then used to remove values from the domain of  $x_j$ : the bound  $b_j^\leq$  is used to remove possibly values that are in  $\mathcal{V}$ , and  $b_j^\neq$  to remove possibly values that are not in  $\mathcal{V}$ . Again, details are provided in [9].  $\square$

Our propagator for TWOSUMS is parametric, depending on the  $f_i$  and  $g_i$ . The time complexity and the level of consistency achieved depend on the shape of the  $f_i$  and  $g_i$  and on the values given to the parameters (see Sections 6 and 7).

**Example 3.** Throughout the rest of the paper, we will use the following running example. Consider a workshop with  $n$  workers. The unknown daily workload in number of hours of each worker is encoded by variable  $x_i$  for each worker  $i \in [1, n]$ . The daily total workload must be equal to a given integer  $t$ . In addition, each worker has a given nominal workload  $w_i$ , from which they can deviate but at some cost given by the following function:  $f_i(v) = \max\{r_i \cdot (w_i - v), q_i \cdot (v - w_i)\}$ , i.e., the cost increases linearly with a slope of  $r_i > 0$  if the actual workload is below the nominal one, and with a slope of  $q_i > 0$  if the actual workload is above the nominal one. The total cost must be under some given integer upper bound  $c$ . This part of the problem (a typical workshop would have additional constraints) is modelled as:

$$\sum_{i \in [1, n]} \max\{r_i \cdot (w_i - x_i), q_i \cdot (x_i - w_i)\} \leq c$$

$$t \leq \sum_{i \in [1, n]} x_i \leq t$$

This is an instantiation of constraints (1) and (2). In subsequent examples, we will use the following values for the parameters:  $n = 4$ ,  $c = 5$ ,  $t = 10$ ,  $\mathbf{w} = \langle 2, 3, 2, 2 \rangle$ ,  $\mathbf{r} = \langle 1, 1, 1, 2 \rangle$ , and  $\mathbf{q} = \langle 2, 2, 3, 3 \rangle$ . We also define  $D_{x_i} = [0, 5]$  for all  $i \in [1, 4]$ . Figure 1 shows the functions  $f_i$  over the given domains.  $\square$

#### 4. Feasibility Test

The TWOSUMS constraint is satisfiable if and only if the *cost* (i.e., the value of the objective function) of an optimal solution to the following problem is at

most the value  $\bar{f}$ :

$$\begin{aligned}
& \text{minimise} && \sum_{i \in [1, n]} f_i(x_i) \\
& \text{such that} && \underline{g} \leq \sum_{i \in [1, n]} g_i(x_i) \leq \bar{g} \\
& && x_i \in D_{x_i}, \quad \forall i \in [1, n]
\end{aligned} \tag{3}$$

We gradually show in the next sub-sections how to compute greedily this cost, called the *feasibility bound*, together with a support, which will be used for filtering in Section 5.

#### 4.1. Problem Reformulation

We reformulate problem (3) into a simpler problem in two steps. The first step introduces for each  $i$  a new function  $h_i$  that captures the relation between  $f_i$  and  $g_i$ . This results in having only one function per variable instead of two. The second step splits the resulting reformulated problem into two subproblems that can be solved separately.

*First Step.* After introducing new variables  $y_i$ , so that  $y_i = g_i(x_i)$  for each  $i$ , we state the following new problem:

$$\begin{aligned}
& \text{minimise} && \sum_{i \in [1, n]} h_i(y_i) \\
& \text{such that} && \underline{g} \leq \sum_{i \in [1, n]} y_i \leq \bar{g} \\
& && y_i \in g_i(D_{x_i}), \quad \forall i \in [1, n]
\end{aligned} \tag{4}$$

where we introduce a new function  $h_i: g_i(D_{x_i}) \rightarrow f_i(D_{x_i})$  for each  $i$ . This function is defined as  $h_i(v) = \min f_i(g_i^{-1}(v)) = \min\{f_i(u) \mid u \in D_{x_i} \wedge g_i(u) = v\}$ , that is  $h_i(v)$  is the smallest value of  $f_i(x_i)$  that can be attained when  $g_i(x_i)$  is equal to value  $v$ . Note that the definition of  $h_i$  depends on the *current* domain of  $x_i$ .

**Example 4.** In Example 3, we have  $g_i(u) = u$  for all  $u$ , hence  $g_i^{-1}(u) = \{u\}$ . It follows that  $h_i(v) = f_i(v)$ , but restricted to the current domain of  $x_i$ .  $\square$

We now prove that the feasibility bound can also be computed from problem (4).

**Lemma 1.** *All optimal solutions to problems (3) and (4) have the same cost.*

**PROOF.** Let  $\mathbf{v}$  denote a vector of values for the vector  $\mathbf{y}$  of variables. For each value  $v_i$ , we choose an arbitrary value  $u_i$  in  $D_{x_i}$  such that  $g_i(u_i) = v_i$  and  $f_i(u_i) = h_i(v_i)$ . Such a value  $u_i$  always exists, by the definition of  $h_i$ . Then the vector  $\mathbf{u}$  is a feasible solution to problem (3) if and only if  $\mathbf{v}$  is a feasible



solution to problem (4), and they have the same cost. In addition, any other assignment  $\mathbf{u}'$  such that  $g_i(u'_i) = v_i$  for each  $i$  has a cost larger than or equal to the cost of  $\mathbf{u}$  and  $\mathbf{v}$ , by the definition of  $h_i$ . Hence  $\mathbf{u}$  is optimal if and only if  $\mathbf{v}$  is optimal.  $\square$

*Second Step.* We define a new function, called  $H$ , from integers to integers:

$$H(b) = \min \left\{ \sum_{i \in [1, n]} h_i(y_i) \left| \sum_{i \in [1, n]} y_i = b \wedge \forall i \in [1, n] : y_i \in g_i(D_{x_i}) \right. \right\} \quad (5)$$

That is,  $H(b)$  is the minimum of the sum of the  $h_i(y_i)$  when the sum of the  $y_i$  is equal to  $b$ . For a given integer  $b$ , we define  $\mathbf{s}^b$  to be an assignment to  $\mathbf{y}$  such that  $b = \sum_{i \in [1, n]} s_i^b$  and  $H(b) = \sum_{i \in [1, n]} h_i(s_i^b)$ . We call  $\mathbf{s}^b$  a *support* for  $b$ . We propose the following new problem:

$$\begin{aligned} & \text{minimise} && H(z) \\ & \text{such that} && \underline{g} \leq z \leq \bar{g} \end{aligned} \quad (6)$$

where  $z$  is a new variable. We now prove that the feasibility bound can also be computed from problem (6), as the latter has the same optimal cost as problem (4), and as problem (3) by Lemma 1.

**Lemma 2.** *All optimal solutions to problems (4) and (6) have the same cost.*

PROOF. This is shown by replacing  $H(z)$  by its definition (5) in the formulation of problem (6). This gives

$$\min \left\{ \min \left\{ \sum_{i \in [1, n]} h_i(y_i) \left| \sum_{i \in [1, n]} y_i = z \wedge \forall i \in [1, n] : y_i \in g_i(D_{x_i}) \right. \right\} \left| \underline{g} \leq z \leq \bar{g} \right. \right\}$$

which is equal to

$$\min \left\{ \sum_{i \in [1, n]} h_i(y_i) \left| \sum_{i \in [1, n]} y_i = z \wedge \forall i \in [1, n] : y_i \in g_i(D_{x_i}) \wedge \underline{g} \leq z \leq \bar{g} \right. \right\}$$

Substituting  $z$  by  $\sum_{i \in [1, n]} y_i$  leads to the formulation of problem (4).  $\square$

Problems (4) and (6) are more interesting than problem (3) in three respects. First, it is simpler to reason with only one function per variable (namely  $h_i$ ) instead of two (namely  $f_i$  and  $g_i$ ). Second, the domain  $D_{y_i}$ , which is equal to  $g_i(D_{x_i})$ , might be much smaller than  $D_{x_i}$  (but never larger), potentially reducing a lot the number of values the algorithms must consider. Third, introducing  $H$  allows us to compute the feasibility bound in two steps: (i) construct  $H$  from the  $h_i$ , and (ii) find an optimal solution to (6). This can be done greedily, as we will show in Section 4.4, if all  $h_i$  are discretely convex, which is a concept we recall now.

Table 2: Several instantiations of  $f_i$  and  $g_i$ , and the corresponding  $h_i$ . The notation  $[\gamma]$  is the Iverson bracket and is defined to be 1 if  $\gamma$  is true, and 0 otherwise.

Name	$f_i(u)$	$g_i(u)$	$h_i(v)$
LINEAR $_{\leq}$ (alone)	$a_i \cdot u$	0	$\begin{cases} a_i \cdot \min D_{x_i} & \text{if } a_i > 0 \\ a_i \cdot \max D_{x_i} & \text{if } a_i \leq 0 \end{cases}$
LINEAR $_{\leq} \wedge$ AMONG [9]	$a_i \cdot u$	$[u \in \mathcal{V}]$	$\begin{cases} a_i \cdot \min (D_{x_i} \setminus \mathcal{V}) & \text{if } v = 0 \wedge a_i > 0 \\ a_i \cdot \max (D_{x_i} \setminus \mathcal{V}) & \text{if } v = 0 \wedge a_i \leq 0 \\ a_i \cdot \min (D_{x_i} \cap \mathcal{V}) & \text{if } v = 1 \wedge a_i > 0 \\ a_i \cdot \max (D_{x_i} \cap \mathcal{V}) & \text{if } v = 1 \wedge a_i \leq 0 \end{cases}$
LINEAR $_{\leq} \wedge$ MAXIMUM	$a_i \cdot u$	$[u \geq m]$	$\begin{cases} a_i \cdot \min D_{x_i} & \text{if } v = 0 \wedge a_i > 0 \\ a_i \cdot \max \{u \mid u \in D_{x_i} \wedge u < m\} & \text{if } v = 0 \wedge a_i \leq 0 \\ a_i \cdot \min \{u \mid u \in D_{x_i} \wedge u \geq m\} & \text{if } v = 1 \wedge a_i > 0 \\ a_i \cdot \max D_{x_i} & \text{if } v = 1 \wedge a_i \leq 0 \end{cases}$
DEVIATION [3]	$ n \cdot u - n \cdot \mu $	$u$	$ n \cdot v - n \cdot \mu $
SPREAD [2]	$(n \cdot u - n \cdot \mu)^2$	$u$	$(n \cdot v - n \cdot \mu)^2$
L $_p$ -NORM, $0 < p < +\infty$	$ n \cdot u - n \cdot \mu ^p$	$u$	$ n \cdot v - n \cdot \mu ^p$
LINEAR $_{=}$	0	$b_i \cdot u$	0
LINEAR $_{\leq} \wedge$ LINEAR $_{=}$	$a_i \cdot u$	$b_i \cdot u$	$\frac{a_i}{b_i} \cdot u$
MOD_AND_DIV ( $a_i > 0$ )	$u - a_i \cdot \lfloor u/a_i \rfloor$	$\lfloor u/a_i \rfloor$	$\max \{0, \min D_{x_i} - a_i \cdot v\}$

**Definition 1 ([12]).** A function  $f: A \rightarrow B$ , where  $A, B \subseteq \mathbb{Z}$ , is *discretely convex* if

1.  $A$  is an interval, and
2.  $\forall v \in A: (v-1) \in A \wedge (v+1) \in A \Rightarrow 2 \cdot f(v) \leq f(v-1) + f(v+1)$ .

The notion of discrete convexity is an adaptation of the usual convexity from the reals to the integers. The intuition is that a function on integers is discretely convex if its natural extension to the reals is convex. This notion has been studied in depth, for instance in [12]. It is also related to the notion of submodular functions on sets [13].

The two conditions in Definition 1 on the  $h_i$  theoretically restrict the applicability of our approach. Those restrictions and their lifting in practice are discussed further in Section 7.1 but the next example already shows the broad applicability of our approach.

**Example 5.** Table 2 presents the functions  $f_i$ ,  $g_i$ , and  $h_i$  for several pairs of constraints.

- The first line shows the example of a constant  $g_i$  function, i.e.,  $g_i(u)$  does not depend on the value of  $u$ , in which case the domain of  $h_i$  is a singleton and  $h_i$  is trivially discretely convex.

- If  $g_i$  is a characteristic function, i.e., a function taking only values 0 and 1 as when defined using the Iverson bracket (e.g.,  $\text{LINEAR}_{\leq}$  and  $\text{AMONG}$ , as well as  $\text{LINEAR}_{\leq}$  and  $\text{MAXIMUM}$ ), then the domain of  $h_i$  is composed of only two values and  $h_i$  is typically defined by giving a formula for each value. In those cases,  $h_i$  is always discretely convex.
- If  $g_i$  is the identity function, as for  $\text{DEVIATION}$ ,  $\text{SPREAD}$ , and  $L_p\text{-NORM}$ , then  $h_i$  is equal to  $f_i$ . In such a case,  $h_i$  is discretely convex as long as  $D_{x_i}$  is relaxed to its smallest enclosing interval. As discussed in Section 7.1, this relaxation maintains the correctness of our approach but only  $\text{bounds}(\mathbb{Z})$  consistency can be achieved (see Table 1).
- If  $g_i$  is a linear function, as for  $\text{LINEAR}_{=}$  alone and the conjunction of  $\text{LINEAR}_{\leq}$  and  $\text{LINEAR}_{=}$ , then  $h_i$  is not discretely convex because  $g_i(D_{x_i})$  might not be an interval even if  $D_{x_i}$  is an interval. However, as discussed in Section 7.1, relaxing the domain of  $h_i$  to its smallest enclosing interval maintains the correctness of our approach: as shown in Table 1, one achieves only  $\text{bounds}(\mathbb{R})$  consistency.
- The last line shows the example of a pair of constraints  $f_i$  and  $g_i$  that are very different from the previous pairs but still give rise to a discretely convex  $h_i$  when  $D_{x_i}$  is relaxed to its smallest enclosing interval.  $\square$

Before providing algorithms in Section 4.4 to compute the feasibility bound, we need to introduce some notions in Section 4.2 and characterise  $H$  in Section 4.3.

#### 4.2. Deltas, Segments, Slopes, and Breakpoints

Let  $f: A \rightarrow B$  be an arbitrary function with  $A, B \subseteq \mathbb{Z}$ . Given some value  $v$  in  $A$ , we call *right delta* (respectively *left delta*) the increase of  $f$  when  $v$  increases (respectively decreases) by 1. Formally:  $\Delta^+(f, v) = f(v+1) - f(v)$  and  $\Delta^-(f, v) = f(v-1) - f(v)$ ; the value of  $\Delta^+(f, v)$  (respectively  $\Delta^-(f, v)$ ) is  $+\infty$  when  $v+1$  (respectively  $v-1$ ) is not in  $A$ .

A *segment* of  $f$  is a maximal interval  $[\ell, u]$  of its domain where the right delta is constant. Formally:  $\Delta^+(f, v) = \Delta^+(f, v+1)$  for all  $v \in [\ell, u-1]$ , with  $\ell \leq u$ ,  $\Delta^+(f, \ell-1) \neq \Delta^+(f, \ell)$ , and  $\Delta^+(f, u-1) \neq \Delta^+(f, u)$ . The endpoints  $\ell$  and  $u$  of a segment  $[\ell, u]$  of  $f$  are called *breakpoints* of  $f$ . The *length* of a segment  $[\ell, u]$  is  $u - \ell$ , that is the length of a line from  $\ell$  to  $u$ , and not the number  $u - \ell + 1$  of its integer elements. The *slope* of a segment  $[\ell, u]$  is  $\Delta^+(f, \ell)$ . Hence the slope of a function is constant inside any of its segments and changes at its breakpoints.

The domain of  $f$  can be uniquely partitioned into its segments, and each value of the domain belongs to one or two segments. For a value  $v$ , the *breakpoint on the right* of  $v$ , denoted by  $\text{bp}^+(f, v)$ , is  $u$  if  $v$  belongs to some segment  $[\ell, u]$  with  $u \neq v$ , and otherwise undefined, denoted by  $+\infty$ . Similarly,  $\text{bp}^-(f, v)$  denotes the *breakpoint on the left* of  $v$ , if any, otherwise  $-\infty$ .

Let  $f$  be a discretely convex function. For any two consecutive segments, the slope of the former is smaller than the slope of the latter, hence no two

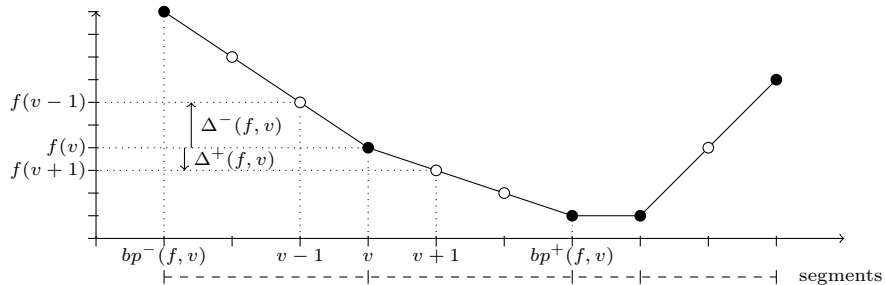


Figure 2: Illustration of the notions of Section 4.2. Filled points appear at breakpoints.

segments have the same slope. Also,  $\Delta^+(f, v) = +\infty$  only for the largest value  $v$  in  $A$ , because  $A$  is an interval, and  $\Delta^-(f, v) = +\infty$  only for the smallest value  $v$  in  $A$ .

Figure 2 illustrates these notions on a discretely convex function. The points are part of the actual function, while the segments joining them are used as a visual guide to identify the segments and their slopes.

**Example 6.** Each function  $h_i$  of Example 3 is composed of two segments (see Figure 1). For  $i = 1$ , the function  $h_1$ , defined as  $h_1(u) = \max\{1 \cdot (2 - u), 2 \cdot (u - 2)\}$  over the interval  $[0, 5]$ , is composed of the following two segments: one spans the interval  $[0, 2]$  with slope  $-1$ , the other spans the interval  $[2, 5]$  with slope  $2$ . For  $u \in [0, 1]$ , we have  $\Delta^+(h_1, u) = -1$  and  $\text{bp}^+(h_1, u) = 2$ . For  $u \in [2, 4]$ , we have  $\Delta^+(h_1, u) = 2$  and  $\text{bp}^+(h_1, u) = 5$ . Finally,  $\Delta^+(h_1, 5) = +\infty$  and  $\text{bp}^+(h_1, 5) = +\infty$ .  $\square$

The basic properties of the special values  $+\infty$  and  $-\infty$  used in our algorithms are, for any  $v \in \mathbb{Z}$ :  $-\infty < v < +\infty$ ,  $v + (+\infty) = +\infty$ ,  $v + (-\infty) = -\infty$ ,  $v - (-\infty) = +\infty$ ,  $v - (+\infty) = -\infty$ ,  $\min(v, +\infty) = v$ , and  $v / +\infty = 0$ .

### 4.3. Characterisation of the $H$ Function

We will show in Section 4.4 that when the  $h_i$  are discretely convex, problem (6) is easy to solve by greedy search, because  $H$  is then also discretely convex and can be calculated efficiently. In order to prove those claims, we first need to study closely the functions  $H$  and  $h_i$ , and the relationship between them and between supports. We first show how one can incrementally get a support for a value  $b + 1$  from a support for a value  $b$ .

**Lemma 3.** *If each  $h_i$  is discretely convex, then given a support  $\mathbf{s}^b$  for some value  $b$ , there exists a support  $\mathbf{s}^{b+1}$  for  $b + 1$  and some  $j \in [1, n]$  such that  $s_i^{b+1} = s_i^b$  for all  $i \neq j$ , and  $s_j^{b+1} = s_j^b + 1$  (assuming  $b$  and  $b + 1$  are in the domain of  $H$ ).*

PROOF. By definition, a support  $\mathbf{s}^b$  for any  $b$  is such that  $\sum_{i \in [1, n]} s_i^b = b$  and  $\sum_{i \in [1, n]} h_i(s_i^b) = H(b)$ . For any  $j$  and  $k$  with  $k \neq j$ , the sum

$$s_1^b + \cdots + (s_j^b + 1) + \cdots + (s_k^b - 1) + \cdots + s_n^b$$

also equals  $b$  as we added 1 to one value and removed 1 from another one.<sup>2</sup> Hence by definition of  $H$  (since the  $s_i^b$  are the values that minimise  $H(b)$ ), we have:

$$H(b) \leq h_1(s_1^b) + \cdots + h_j(s_j^b + 1) + \cdots + h_k(s_k^b - 1) + \cdots + h_n(s_n^b)$$

Rearranging and cancelling out common terms gives:

$$h_k(s_k^b) - h_k(s_k^b - 1) \leq h_j(s_j^b + 1) - h_j(s_j^b) \quad (7)$$

If  $h_j$  is discretely convex, then we have that:

$$h_k(s_k^b) - h_k(s_k^b - 1) \leq h_j(s_j^b + 1) - h_j(s_j^b) \leq h_j(s_j^b + 2) - h_j(s_j^b + 1)$$

And:

$$\begin{aligned} & h_1(s_1^b) + \cdots + h_j(s_j^b + 1) + \cdots + h_k(s_k^b) + \cdots + h_n(s_n^b) \\ & \leq \\ & h_1(s_1^b) + \cdots + h_j(s_j^b + 2) + \cdots + h_k(s_k^b - 1) + \cdots + h_n(s_n^b) \end{aligned}$$

This means that adding two to any single  $s_j^b$  and reducing another  $s_k^b$  by one to arrive at the sum  $b + 1$  will have a higher cost than simply adding one to a single  $s_j^b$ . Because each  $h_i$  is discretely convex, this is true for any increment larger than one. Hence it is possible to find a support  $\mathbf{s}^{b+1}$  for  $b + 1$  from a support  $\mathbf{s}^b$  for  $b$  by increasing any suitable  $s_i^b$  by one.  $\square$

Lemma 3 also shows that it is possible to find a support  $\mathbf{s}^{b-1}$  by subtracting one from any suitable  $s_i^b$ . We can now prove the following important result:

**Theorem 4.** *If each  $h_i$  is discretely convex, then  $H$  is discretely convex.*

PROOF. The domain of each  $h_i$  is an interval  $[s_i, u_i]$ , so that the domain of  $H$  is the interval  $[\sum_{i \in [1, n]} s_i, \sum_{i \in [1, n]} u_i]$ . We need to show that  $H(b) - H(b - 1) \leq H(b + 1) - H(b)$ . If  $s_i^b$  is a support for some  $b$  then by Lemma 3 there are some  $k$  and  $j$  such that  $H(b - 1) = h_1(s_1^b) + \cdots + h_k(s_k^b - 1) + \cdots + h_n(s_n^b)$  and  $H(b + 1) = h_1(s_1^b) + \cdots + h_j(s_j^b + 1) + \cdots + h_n(s_n^b)$ . Therefore  $H(b) - H(b - 1) = h_k(s_k^b) - h_k(s_k^b - 1)$  and  $H(b + 1) - H(b) = h_j(s_j^b + 1) - h_j(s_j^b)$  and, by (7),  $H(b) - H(b - 1) \leq H(b + 1) - H(b)$ . Hence  $H$  is discretely convex.  $\square$

<sup>2</sup>Visually, we here take  $j < k$  but the reasoning does not depend on their order.

We now show how to calculate  $H$  efficiently by giving a characterisation of its minimum and its segments. Here, for any non-empty set  $S$  and function  $f$ , the expression  $\operatorname{argmin}_{i \in S} f(i)$  returns *one* arbitrary value  $i \in S$  that minimises  $f(i)$ . In the following,  $b^*$  represents a value minimising the value of  $H(b^*)$ .

**Lemma 5.** *A support  $\mathbf{s}^{b^*}$  for a value  $b^*$  that minimises  $H$  is such that  $s_i^{b^*} = \operatorname{argmin}_{v_i \in g_i(D_{x_i})} h_i(v_i)$  for all  $i \in [1, n]$ .*

PROOF. If  $\mathbf{s}^{b^*}$  is a support for  $b^*$ , then  $b^*$  is equal to  $\sum_{i \in [1, n]} s_i^{b^*}$  and  $H(b^*) = \sum_{i \in [1, n]} h_i(s_i^{b^*})$ . Since each  $s_i^{b^*} = \operatorname{argmin}_{v_i \in g_i(D_{x_i})} h_i(v_i)$  corresponds to the minimum value obtainable by  $h_i$ , it is not possible to reduce the value  $\sum_{i \in [1, n]} h_i(s_i^{b^*})$  by picking a different value for any  $s_i^{b^*}$ .  $\square$

There are potentially several  $\mathbf{s}^{b^*}$  that minimise  $H$ . The correctness of our approach does not depend on a particular choice of support.

**Example 7.** For the constraints of Example 3, there is only one value minimising  $h_i$  for each  $i$ , namely the nominal workload  $w_i$ . Hence, the unique  $\mathbf{s}^{b^*}$  is equal to the nominal workloads  $\mathbf{w} = \langle 2, 3, 2, 2 \rangle$ . Then  $b^* = 9 = 2 + 3 + 2 + 2$  and  $H(b^*) = 0$  as  $h_i(w_i) = 0$  for all  $i$ .  $\square$

We now characterise the segments of  $H$ . We first establish the relation between the slope of  $H$  at some value  $b$  and the slope of each  $h_i$  at  $s_i^b$ .

**Lemma 6.** *If  $\mathbf{s}^b$  is a support for some value  $b$ , then  $\Delta^+(h_i, s_i^b) \geq \Delta^+(H, b)$  and  $\Delta^-(h_i, s_i^b) \geq \Delta^-(H, b)$  for all  $i \in [1, n]$ .*

PROOF. If  $b$  is increased by one, then one of the  $s_i^b$  must be increased by one (by Lemma 3). To reach the minimum value for  $b + 1$ , one needs to increase the value of a variable  $y_k$  for which  $\Delta^+(h_k, s_k^b)$  is the smallest. So the increase of  $H$ , namely  $\Delta^+(H, b)$ , is equal to  $\Delta^+(h_k, s_k^b)$ , which is smaller than or equal to  $\Delta^+(h_i, s_i^b)$  for any other  $i$ . A similar argument is used for a decrease of  $b$ .  $\square$

The proof of Lemma 6 also shows that there is at least one  $i$  such that  $\Delta^+(h_i, s_i^b)$  is equal to  $\Delta^+(H, b)$ . This gives us a way to define the segments of  $H$ :

**Lemma 7.** *The length of each segment of  $H$  is equal to the sum of the lengths of the segments in the functions  $h_i$  with the same slope.*

PROOF. In the proof of Lemma 6, we saw that  $\Delta^+(H, b)$  is equal to a minimal  $\Delta^+(h_k, s_k^b)$ . If one wants to increase  $b$  by more than one, then the increase per unit stays constant as long as there is at least one variable with slope equal to  $\Delta^+(H, b)$ . This defines a segment of slope  $\Delta^+(H, b)$ , whose length is equal to the sum of the lengths of the segments of all functions  $h_i$  with the same slope.  $\square$

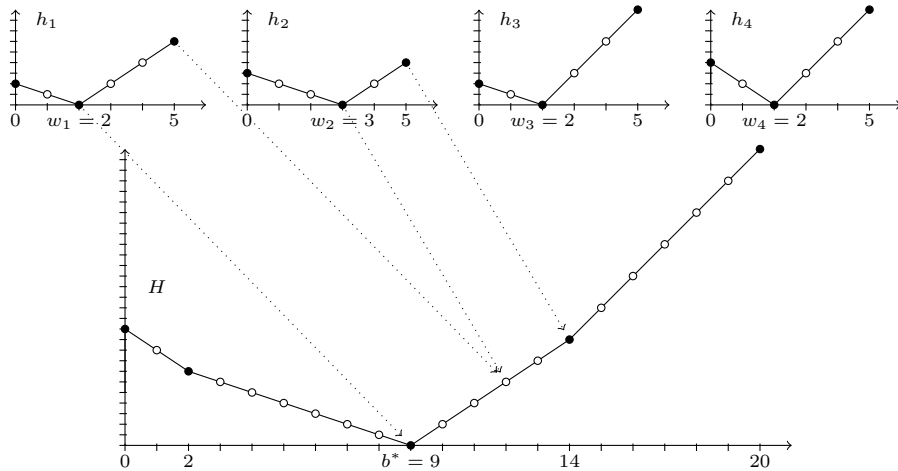


Figure 3: The  $h_i$  and  $H$  functions of Example 8

We can use Lemmas 5 and 7 to construct  $H$  efficiently, as shown in the following example. Section 6.1 presents several ways to implement this construction in practice.

**Example 8.** Given the domain  $[0, 5]$  for all  $x_i$ , the function  $H$  for the constraints of Example 3 can be constructed as follows (see also Figure 3). Each  $h_i$  has two segments joining at  $w_i$ : the first spans  $[0, w_i]$  and has slope  $-r_i$ , while the second spans  $[w_i, 5]$  and has slope  $q_i$ . Starting from  $b^* = 9$ , we can define the segment of  $H$  for which  $b^*$  is the left breakpoint using the second segment of each  $h_i$  with minimal  $q_i$ . There are two of them (namely for  $i = 1$  and  $i = 2$ ) with respective lengths 3 and 2, both with slope 2. This defines a segment of length 5 and slope 2 spanning the interval  $[9, 14]$ . The next segment has slope 3 and is constructed from the second segments of  $h_i$  for  $i = 3$  and  $i = 4$ . With a length of 6, it spans the interval  $[14, 20]$ . The same reasoning for values smaller than 9 leads to two more segments: one spans  $[0, 2]$  with a slope of  $-2$ , and the other one spans  $[2, 9]$  with a slope of  $-1$ . Figure 3 shows how  $H$  is formed of the segments of the  $h_i$ .  $\square$

As can be seen in the previous example, we do not compute an analytical definition of  $H$  but only its minimum and its segments. As will be made clear in Section 6, we are never interested in the value of  $H(b)$  for an arbitrary value  $b$  but only for  $b^*$  and for incremental modifications of  $b$  that can be computed using the slopes of the segments.

#### 4.4. Computing the Feasibility Bound and a Support

We now show that problem (6) can be solved in a greedy way.

**Theorem 8.** *Problem (6) can be solved greedily if each  $h_i$  is discretely convex.*

PROOF. If each function  $h_i$  is discretely convex, then the function  $H$  is also discretely convex (by Theorem 4) and can be constructed from the  $h_i$  (by Lemmas 5 and 7). Finding the minimum of a discretely convex function under some bound constraints can be done greedily, as a local minimum of a discretely convex function is also a global minimum (see, e.g., Theorem 2.2 in [12]).  $\square$

Given the function  $H$ , problem (6) can be solved by first finding a value  $b$  minimising  $H$  (i.e.,  $b^*$ ), and then greedily increasing or decreasing  $b$  if  $b^*$  is not in  $[g, \bar{g}]$ . In addition, computing a support  $\mathbf{s}^b$  is useful for the filtering (to be discussed in Section 5).

Thanks to Lemma 6, this can be achieved by Algorithm 1.<sup>3</sup> From now on, we write  $\mathbf{s}$  to refer to  $\mathbf{s}^b$ . An assignment  $\mathbf{s}$  that minimises the value of  $H$  without considering the bounds for  $b$  is initially constructed (lines 2–4). If  $b$  is in  $[g, \bar{g}]$ , then the initial assignment is the final one. Otherwise the assignment is iteratively modified in order to satisfy the bounds of  $b$ . We assume  $b < \underline{g}$  happens in line 5. Then some  $s_i$  must be increased until  $b$  is equal to  $\underline{g}$ . This is done in two steps. In lines 6–10, the segment of  $H$  where  $\underline{g}$  lies is found. Its slope is stored in  $\Delta^{\max}$ , and the distance between  $\text{bp}^-(H, \underline{g})$  and  $\underline{g}$  is stored in *slack*. Those two values allow us to modify each  $s_i$  separately (lines 11–17). For each  $i$ , first  $s_i$  is moved from breakpoint to breakpoint of  $h_i$  while the slope of the segment is smaller than  $\Delta^{\max}$ . Next, if the slope of the segment on the right of  $s_i$  is equal to  $\Delta^{\max}$ , then  $s_i$  is moved further on this segment, without exceeding the remaining slack (line 15). Lines 18–30 show the symmetrical case when  $b > \bar{g}$ : the left deltas and left breakpoints are used, and *slack* takes on negative values.

The algorithm returns the support  $\mathbf{s}$  (line 31), or “null” if the constraint is unsatisfiable (lines 8 and 21), which triggers propagator failure and happens if there exists no value in the domains of the  $h_i$  such that  $b \in [g, \bar{g}]$ .

We now prove that Algorithm 1 is correct:

**Theorem 9.** *If the  $h_i$  are discretely convex and  $H$  is defined as in equation (5), then Algorithm 1 returns an optimal solution to problem (4), and hence to problem (6), if one exists, and “null” otherwise.*

PROOF. If the  $h_i$  are discretely convex, then  $H$  is also discretely convex (by Theorem 4) and lines 2–4 store its minimum in variable  $b$  (using Lemma 5). If  $\underline{g} \leq b \leq \bar{g}$ , then  $b$  is feasible and we are done. Otherwise, assume  $b < \underline{g}$  (the other case is symmetrical), then by Theorem 2.2 in [12], the optimal solution is at  $b = \underline{g}$ . Now, we need to find  $\mathbf{s}^{\underline{g}}$ . We do this in two steps. Lines 6–7 locate the segment of  $H$  where  $\underline{g}$  lies. If such a segment does not exist, then  $\underline{g}$  is not in the domain of  $H$  and “null” is returned in line 8. Otherwise, lines 9 and 10 compute the slope  $\Delta^{\max}$  of that segment, and the distance *slack* between  $\text{bp}^-(H, \underline{g})$  and  $\underline{g}$ . Lines 11–17 modify the value of each  $w_i$  to reach  $\underline{g}$ : lines 12–13 position each

---

<sup>3</sup>Line 16 is corrected here with respect to our [1] where a sign error was present. Our implementation was correct.



---

**Algorithm 1** Greedy algorithm to compute a support
 

---

```

1: function GETSUPPORTLOWERBOUND
2:   for all  $i \in [1, n]$  do
3:      $s_i := \operatorname{argmin}_{v \in g_i(D_{x_i})} h_i(v)$ 
4:    $b := \sum_{i \in [1, n]} s_i$ 
5:   if  $b < \underline{g}$  then
6:     while  $\Delta^+(H, b) < +\infty$  and  $\operatorname{bp}^+(H, b) < \underline{g}$  do
7:        $b := \operatorname{bp}^+(H, b)$ 
8:     if  $\Delta^+(H, b) = +\infty \wedge b < \underline{g}$  then return null
9:      $\Delta^{\max} := \Delta^+(H, b)$ 
10:     $\operatorname{slack} := \underline{g} - b$ 
11:    for all  $i \in [1, n]$  do
12:      while  $\Delta^+(h_i, s_i) < \Delta^{\max}$  do
13:         $s_i := \operatorname{bp}^+(h_i, s_i)$ 
14:      if  $\Delta^+(h_i, s_i) = \Delta^{\max}$  and  $\operatorname{slack} > 0$  then
15:         $s' := \min(\operatorname{bp}^+(h_i, s_i), s_i + \operatorname{slack})$ 
16:         $\operatorname{slack} := \operatorname{slack} - s' + s_i$ 
17:         $s_i := s'$ 
18:    else if  $b > \bar{g}$  then
19:      while  $\Delta^-(H, b) < +\infty$  and  $\operatorname{bp}^-(H, b) > \bar{g}$  do
20:         $b := \operatorname{bp}^-(H, b)$ 
21:      if  $\Delta^-(H, b) = +\infty \wedge b > \bar{g}$  then return null
22:       $\Delta^{\max} := \Delta^-(H, b)$ 
23:       $\operatorname{slack} := \bar{g} - b$ 
24:      for all  $i \in [1, n]$  do
25:        while  $\Delta^-(h_i, s_i) < \Delta^{\max}$  do
26:           $s_i := \operatorname{bp}^-(h_i, s_i)$ 
27:        if  $\Delta^-(h_i, s_i) = \Delta^{\max}$  and  $\operatorname{slack} < 0$  then
28:           $s' := \min(\operatorname{bp}^-(h_i, s_i), s_i + \operatorname{slack})$ 
29:           $\operatorname{slack} := \operatorname{slack} - s' + s_i$ 
30:           $s_i := s'$ 
31:    return s

```

$s_i$  at the right breakpoint of the correct segment as defined by Lemma 6, and lines 14–17 ensure that  $\sum_{i \in [1, n]} s_i$  is equal to  $\underline{g}$ .  $\square$

**Example 9.** We now show an execution of Algorithm 1 on the problem of our running example. Important values at some steps of Algorithm 1 are given in Table 3. We have already shown in Example 7 that  $b^* = 9$ , so we have  $b = 9$  at line 4. As  $b < \underline{g} = 10$  on line 5, the algorithm enters the conditional branch starting at line 6 with  $b = 9$  and  $\mathbf{s} = \langle 2, 3, 2, 2 \rangle$ . We have that  $\Delta^+(H, 9) = 2$  and  $\operatorname{bp}^+(H, 9) = 14$ . As  $14 \not< \underline{g} = 10$ , line 7 is never executed. Lines 9 and 10 set

Table 3: Values at some steps of Algorithm 1.

Step	$b$	$slack$	$\mathbf{s}$
after initial bound	9	-	$\langle 2, 3, 2, 2 \rangle$
after sharp bound	10	1	$\langle 2, 3, 2, 2 \rangle$
after modifying $s_1$	10	0	$\langle 3, 3, 2, 2 \rangle$

$\Delta^{\max} = 2$  and  $slack = 1$ . The loop of lines 11–17 is executed for each  $i \in [1, 4]$ . Here, line 13 is never executed. For  $i = 1$ , the condition of line 14 is true and lines 15–17 are executed: they set  $s' = \min\{5, 2 + 1\} = 3$ ,  $slack = 1 - 3 + 2 = 0$ , and  $s^i = 3$ . As the slack is now equal to zero, no other value will be modified for  $i \in [2, 4]$ . The final support is  $\mathbf{s} = \langle 3, 3, 2, 2 \rangle$ .

Note that we iterate in line 11 over the indices in increasing order. If we iterated in decreasing order, then we would obtain  $\mathbf{s} = \langle 2, 4, 2, 2 \rangle$ . The two assignments are both correct supports for  $b = 10$  and  $H(b) = 2$ . The conjunction of constraints in this example is feasible as the optimal value, namely 2, is at most the upper bound  $\bar{f} = 5$ .  $\square$

## 5. Domain Filtering

To filter the domain of a variable, we extend the reasoning presented in Section 4.1. Indeed, variable  $x_j$  can take a value  $u \in D_{x_j}$  if and only if the cost of an optimal solution to the following problem is smaller than or equal to  $\bar{f}$ :

$$\begin{aligned}
 & \text{minimise} && f_j(u) + \sum_{i \neq j \in [1, n]} f_i(x_i) \\
 & \text{such that} && \underline{g} \leq g_j(u) + \sum_{i \neq j \in [1, n]} g_i(x_i) \leq \bar{g} \\
 & && x_i \in D_{x_i}, \quad \forall i \neq j \in [1, n]
 \end{aligned} \tag{8}$$

Problem (8) resembles problem (3) but  $x_j$  is fixed to  $u$ . Hence we can use the same reformulation as in Section 4.1. We introduce the following new function:

$$H_j(b) = \min \left\{ \sum_{i \neq j \in [1, n]} h_i(y_i) \mid \sum_{i \neq j \in [1, n]} y_i = b \wedge \forall i \neq j \in [1, n] : y_i \in g_i(D_{x_i}) \right\}$$

That is,  $H_j(b)$  is similar to  $H(b)$  in (5) but it only uses the functions  $h_i$  for  $i$  different from  $j$ . The optimal cost of problem (8) is the optimal cost of the following new problem:

$$\begin{aligned}
 & \text{minimise} && f_j(u) + H_j(z) \\
 & \text{such that} && \underline{g} \leq g_j(u) + z \leq \bar{g}
 \end{aligned} \tag{9}$$

where value  $u$  is given and  $z$  is the only variable. The result of the following lemma can be used to compute  $H_j$ .

---

**Algorithm 2** Main filtering algorithm
 

---

```

1: function FILTERTWO SUMS(f, g, h,  $\bar{f}$ ,  $\underline{g}$ ,  $\bar{g}$ )
2:   Construct  $H$  as discussed in Section 6.1
3:   s := GETSUPPORTLOWERBOUND
4:   for all  $j \in [1, n]$  do
5:     FORWARDFILTER( $j$ )
6:     BACKWARDFILTER( $j$ )

```

---

**Lemma 10.** *The function  $H_j$  is discretely convex if all  $h_i$  are discretely convex. The value  $b_j^*$  that minimises  $H_j$  is equal to the value  $b^*$  that minimises  $H$  minus the value  $v^*$  that minimises  $h_j$ . The length of each segment of  $H_j$  is equal to the length of the linear segment of  $H$  of the same slope minus the length of the linear segment of  $h_j$  of the same slope, if any.*

PROOF. As  $H$  and  $H_j$  are defined identically except for the set of indices they consider, one can apply all results concerning  $H$  to  $H_j$ . So the first statement is a consequence of Theorem 4, replacing  $H$  by  $H_j$ . In the same way, applying Lemma 5 to  $H_j$ , we get  $s_i^{b_j^*} = s_i^{b^*}$  for all  $i \neq j$ . Hence  $b_j^* = b^* - s_i^{b^*}$ , which proves the second statement. Applying Lemma 7 to  $H_j$  gives that the length of each segment of  $H_j$  is equal to the sum of the lengths of the segments of the same slope of the  $h_i$  with  $i \neq j$ . Hence the only difference between the length of a segment of  $H_j$  and the length of the segment of  $H$  of the same slope is the length of the segment of  $h_j$  with the same slope, if it exists.  $\square$

**Example 10.** Given the functions of Example 3,  $H_1$  is characterised as follows. Its minimum is at  $9 - 2 = 7$  and it has 4 segments spanning respectively the interval  $[0, 2]$  with slope  $-2$ , the interval  $[2, 7]$  with slope  $-1$ , the interval  $[7, 9]$  with slope  $2$ , and the interval  $[9, 15]$  with slope  $3$ . Figure 4 on page 21 illustrates the function and its use in the forthcoming Example 11.  $\square$

Our filtering algorithm is given in Algorithm 2. It iterates over all variables and, for each variable  $y_j$ , filters in turn for the values larger than  $s_j$  and for the values smaller than  $s_j$ . To avoid cluttering the algorithm descriptions with numerous parameters, we consider  $\mathbf{f}, \mathbf{g}, \mathbf{h}, \bar{f}, \underline{g}, \bar{g}, H$ , and  $\mathbf{s}$  to be globally available. We show hereafter two ways to use  $H_j$  to implement FORWARDFILTER and BACKWARDFILTER. The first way is applicable in general, provided  $H_j$  is discretely convex. The second way makes use of an additional property that  $f_j$  and  $g_j$  might have.

### 5.1. Filtering in the General Case

As several values  $u$  of  $x_j$  can have the same image  $v$  through  $g_j$ , the set of values in  $D_{x_j}$  that are consistent with constraints (1) and (2) can be partitioned as:

$$\bigcup_{v \in g_j(D_{x_j})} \left\{ u \mid g_j(u) = v \wedge f_j(u) \leq \bar{f} - \min_{\underline{g} \leq z + v \leq \bar{g}} H_j(z) \right\}$$

That is, for each  $v$ , we have the set of values  $u$  in  $g_j^{-1}(v)$  such that the optimal cost of problem (9) is at most  $\bar{f}$ , hence the set of values that are feasible. The domain of  $x_j$  can be made domain consistent by filtering the following unary constraint for each value  $v \in g_j(D_{x_j})$ :

$$g_j(x_j) = v \Rightarrow f_j(x_j) \leq \bar{f} - \min_{g \leq z+v \leq \bar{g}} H_j(z) \quad (10)$$

The function  $H_j$  being discretely convex, one can compute  $\min_{g \leq z+v \leq \bar{g}} H_j(z)$ , which is independent of any particular  $u$ , incrementally from a value  $v$  to  $v+1$ . In addition, if  $v$  is equal to  $s_j$ , which is the value of  $y_j$  in the support  $\mathbf{s}$  computed in Section 4.4, then

$$H_j \left( \sum_{i \neq j \in [1, n]} s_i \right) + h_j(s_j) = H \left( \sum_{i \in [1, n]} s_i \right) \quad (11)$$

This leads to Algorithm 3,<sup>4</sup> which is used to filter the domain of  $x_j$  for values larger than  $s_j$ . This algorithm traverses the functions  $h_j$  and  $H_j$ . The only complication is that in some cases (captured by the Boolean variable  $dec_b$  defined in lines 6 and 12) reaching an optimal solution to  $\min_{g \leq z+v \leq \bar{g}} H_j(z)$  involves decrementing  $b$ , which is the current value of  $z$  (line 10). Domain filtering according to constraint (10) takes place in lines 5 and 11. We assume that the FILTER procedure to filter the domain of a variable for a unary constraint is provided by the user. The semantics of  $\text{FILTER}(\phi(x))$  for some unary predicate  $\phi$  is that it removes from the domain of  $x$  all values  $u$  such that  $\phi(u)$  does not hold. The algorithm ends when the optimal cost to problem (9) for  $v+1$  is larger than  $\bar{f}$  (line 8), at which point values of  $x_j$  for which  $g_j(x_j) > v$  are filtered (line 14). The description of the procedure COMPUTEHJ on line 2 will be given in Section 6.

Algorithm 4 presents the complementary algorithm for values smaller than  $s_j$ . In that case,  $v$  is iteratively decreased while  $b$  is increased. Hence the Boolean  $dec_b$  is replaced by  $inc_b$  defined as  $b+v \leq \underline{g} \vee \Delta^+(H_j, b) < 0$ . Note that line 5 is redundant with line 5 in Algorithm 3 but has been left for symmetry.

**Theorem 11.** *If each  $h_i$  is discretely convex,  $H$  is defined as in equation (5),  $\mathbf{s}$  is the result of Algorithm 1, and each FILTER call achieves domain consistency on the unary constraint given as argument, then Algorithms 3 and 4 achieve domain consistency on  $x_j$ .*

PROOF. First, we show that the value  $b$  used in each FILTER call of lines 5 and 11 is

$$\operatorname{argmin}_{g \leq b+v \leq \bar{g}} H_j(b) \quad (12)$$

---

<sup>4</sup>Line 8 is a clarification of our previously published version [1].

---

**Algorithm 3** Filtering algorithm for values larger than  $s_j$  (general case)

---

```

1: function FORWARDFILTER( $j$ )
2:    $H_j := \text{COMPUTE}HJ(H, h_j)$ 
3:    $b := \sum_{i \in [1, n]} s_i - s_j$ 
4:    $v := s_j$ 
5:   FILTER( $g_j(x_j) = v \Rightarrow f_j(x_j) \leq \bar{f} - H_j(b)$ )
6:    $dec_b := b + v \geq \bar{g} \vee \Delta^-(H_j, b) < 0$ 
7:    $valH_j := \text{if } dec_b \text{ then } H_j(b-1) \text{ else } H_j(b)$ 
8:   while  $valH_j + h_j(v+1) \leq \bar{f}$  do
9:      $v := v + 1$ 
10:    if  $dec_b$  then  $b := b - 1$ 
11:    FILTER( $g_j(x_j) = v \Rightarrow f_j(x_j) \leq \bar{f} - H_j(b)$ )
12:     $dec_b := b + v \geq \bar{g} \vee \Delta^-(H_j, b) < 0$ 
13:     $valH_j := \text{if } dec_b \text{ then } H_j(b-1) \text{ else } H_j(b)$ 
14:  FILTER( $g_j(x_j) \leq v$ )

```

---

Table 4: Values at some steps of Algorithms 3 and 4, as per Example 11.

Step	$b$	$v$	$dec_b/inc_b$	$valH_j$	$h_j(v+1)$
Forward, after the initialisation	8	2	true	0	2
Forward, after one iteration	7	3	true	1	4
Forward, after two iterations	6	4	true	2	6
Backward, after the initialisation	8	2	true	4	1
Backward, after one iteration	7	3	true	7	2

This is verified for the call in line 5 by equation (11) and the computed values of  $b$  and  $v$ . From now on, we only consider the case of the values  $v$  larger than  $w_j$ , i.e., Algorithm 3, the other case being symmetrical. Each iteration of the loop of lines 8–13 increments  $v$  by 1. If  $b + v = \bar{g}$ , then  $b$  must be decreased by 1 in order to satisfy the condition in expression (12) when  $v$  is incremented. If  $\Delta^-(H_j, b) < 0$ , then  $H_j(b-1) < H_j(b)$ . The value  $b-1$  is not feasible for  $v$  (as  $b$  is optimal, meaning that  $b + v = \underline{g}$ ) but is feasible for  $v+1$ , so  $b$  must be decremented in that case. In all other cases,  $b$  stays constant. Hence, in each case, the value of  $b$  at line 11 is equal to the value of expression (12).

Second, we show that the value  $v$  in line 14 is the largest feasible value in the domain of  $y_j$ . By the argument above,  $b$  is the optimal value at each iteration of the loop. Hence the test of line 8 evaluates to false if and only if  $v+1$  is not feasible. As  $H_j$  is discretely convex, there is no feasible value larger than  $v$ .  $\square$

We note that domain consistency is usually easy to achieve for any unary constraint appearing in a FILTER call.

In Section 7.1, we will characterise the consistency level achieved by Algorithms 3 and 4 when the  $h_i$  are not discretely convex.

---

**Algorithm 4** Filtering algorithm for values smaller than  $s_j$  (general case)

---

```

1: function BACKWARDFILTER( $j$ )
2:    $H_j := \text{COMPUTE}H_j(H, h_j)$ 
3:    $b := \sum_{i \in [1, n]} s_i - s_j$ 
4:    $v := s_j$ 
5:   FILTER( $g_j(x_j) = v \Rightarrow f_j(x_j) \leq \bar{f} - H_j(b)$ )
6:    $inc_b := b + v \leq \underline{g} \vee \Delta^+(H_j, b) < 0$ 
7:    $valH_j := \text{if } inc_b \text{ then } H_j(b+1) \text{ else } H_j(b)$ 
8:   while  $valH_j + h_j(v-1) \leq \bar{f}$  do
9:      $v := v - 1$ 
10:    if  $inc_b$  then  $b := b + 1$ 
11:    FILTER( $g_j(x_j) = v \Rightarrow f_j(x_j) \leq \bar{f} - H_j(b)$ )
12:     $inc_b := b + v \leq \underline{g} \vee \Delta^+(H_j, b) < 0$ 
13:     $valH_j := \text{if } inc_b \text{ then } H_j(b+1) \text{ else } H_j(b)$ 
14:  FILTER( $g_j(x_j) \geq v$ )

```

---

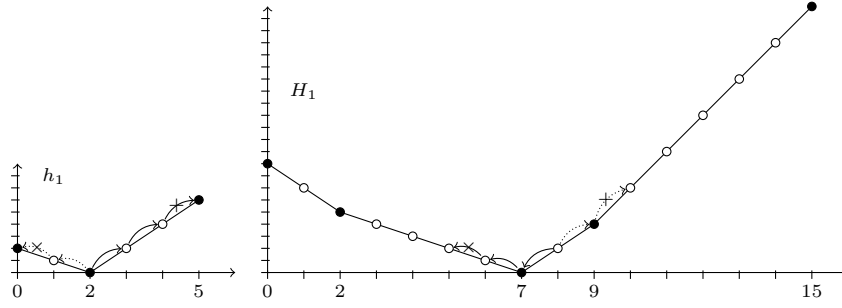


Figure 4: The  $h_1$  and  $H_1$  functions used in Examples 10 and 11. The solid arrows correspond to the successive iterations of Algorithm 3. The loop ends after two iterations as  $H_1(5) + h_1(5) = 2 + 6 = 8 > 5$ . The dotted arrows correspond to the successive iterations of Algorithm 4. The loop ends after one iteration as  $H_1(10) + h_1(0) = 7 + 2 = 9 > 5$ .

**Example 11.** The execution of Algorithm 3 on variable  $x_1$  in the problem of Example 3 is as follows. Table 4 gives the values of the variables appearing in the algorithm at different moments and Figure 4 shows how the functions  $h_j$  and  $H_j$  are traversed. We start from the support  $\mathbf{s} = \langle 2, 4, 2, 2 \rangle$  and have  $\bar{f} = 5$ . The result of line 2, namely  $H_j$ , was given in Example 10. Lines 3 and 4 set  $b = 8$  and  $v = 2$ . The FILTER calls at lines 5 and 11 amount, after instantiating and simplifying the formula, to “if  $\max\{2 - v, 2v - 4\} + H_j(b) > 5$ , then remove  $v$  from  $D_{x_1}$ ”. In line 5, we have  $H_j(b) = 2$ , hence  $v = 2$  stays in the domain. Line 6 sets  $dec_b$  to true. As  $\underline{g} = \bar{g}$  in this example,  $dec_b$  is always true and  $b$  will be decremented at each iteration. The tested value in line 8 is equal to  $0 + 2 = 2$ , which is smaller than 5, hence the loop is entered. Now, we have  $v = 3$ ,  $b = 7$ , and  $H_j(b) = 0$ , and nothing is filtered by line 11. To start the next iteration, we have that the tested value in line 8 is  $1 + 4 = 5$  and the body

of the loop is executed again:  $v = 4$ ,  $b = 6$ ,  $H_j(b) = 1$ , and nothing is filtered. Finally, the tested value in line 8 becomes  $2 + 6 = 8$ , which is larger than 5, and the loop ends. The execution of line 14 enforces  $x_1 \leq v = 4$ , which removes 5 from  $D_{x_1}$ . The domain of  $x_1$  after the execution of Algorithm 3 is thus  $[0, 4]$ .

The execution of Algorithm 4 is similar and illustrated with the dotted arrows in Figure 4. The loop ends when  $v = 1$  and the execution of line 14 enforces  $x_1 \geq v = 1$ , which removes 0 from  $D_{x_1}$ . The domain of  $x_1$  after the execution of Algorithm 4 is thus  $[1, 4]$ .  $\square$

### 5.2. Filtering in a Special Case

We now present a special case that allows us to avoid useless computation, namely a form of monotonicity. Let us define  $k_j(v) = \max f_j(g_j^{-1}(v))$ , that is  $k_j(v)$  is the largest value  $f_j(u)$  for  $u$  such that  $g_j(u) = v$ . The function  $k_j$  is similar to  $h_j$  but the ‘max’ operator replaces the ‘min’ one.

If  $h_j(v) \geq k_j(v - 1)$  for any value  $v$  larger than  $v^* = \operatorname{argmin}_{u \in g_j(D_{x_j})} h_j(u)$  and  $h_j(v) \geq k_j(v + 1)$  for any  $v$  smaller than  $v^*$ , then there exists a value  $v^{\max}$  such that for all values  $v \in g_j(D_{x_j})$  smaller than  $v^{\max}$  (but larger than or equal to  $s_j$ ) we have that all values  $u \in g_j^{-1}(v)$  are feasible, and for all  $v$  larger than  $v^{\max}$ , there is no feasible  $u$ . We then need not consider all values but only find  $v^{\max}$  and filter according to the two constraints  $g_j(x_j) \leq v^{\max}$  and  $g_j(x_j) = v^{\max} \Rightarrow f_j(x_j) \leq \bar{f} - \min_{g \leq z + v^{\max} \leq \bar{g}} H_j(z)$ . A similar argument holds for a similarly defined  $v^{\min}$ .

Finding  $v^{\max}$  amounts to computing the largest value  $v$  such that  $h_j(v) + \min_{g \leq z + v \leq \bar{g}} H_j(z) \leq \bar{f}$ . As  $h_j$  and  $H_j$  are both convex, this problem can be solved by incrementally increasing  $v$  until the bound is reached. Algorithm 5 presents the steps to find  $v^{\max}$ .<sup>5</sup> This algorithm is very similar to Algorithm 3, but it does not need to iterate over all the values  $v$ , but only over the ones that are at a breakpoint of  $h_j$  or  $H_j$ . The increment is stored in  $\ell$  (lines 6, 12, and 14). For conciseness, we use **if-then-else** expressions inside the expressions giving the value of  $\ell$  in lines 6, 12, and 14. The correctness of Algorithm 5 stems from the correctness of Algorithm 3 and the fact that  $\ell$  is the largest increment such that the values of  $\Delta^+(h_j, v)$ ,  $\Delta^-(H_j, b)$ , and  $dec_b$  are constant.

An example of the special case is when  $g_j$  is the identity function. Then  $g_j$  is injective. Hence  $h_j = k_j$  and, by convexity,  $h_j$  is non-decreasing to the right of  $v^*$  and non-increasing to the left of  $v^*$ .

**Example 12.** The filtering of Example 11 can be rerun with Algorithm 5 since the functions  $g_j$  are the identity function. Table 5 gives the values of the variables appearing in the algorithm at different moments. Initially, we have  $v = 2$ ,  $b = 8$ , and  $\ell = 8 - 7 = 1$ , as  $H_j$  has a breakpoint at  $b = 7$ . After one iteration, we have  $v = 3$ ,  $b = 7$ , and  $\ell = 5 - 3 = 2$ , as  $h_j$  has a breakpoint at 5. At this point, the tested value in line 8 is  $2 + 6 = 8$ , and the loop ends. Lines 14 and 15

<sup>5</sup>Lines 6 and 12 have been corrected and line 8 is a clarification of our previously published version [1]. Our implementation was correct.

---

**Algorithm 5** Filtering algorithm for values larger than  $s_j$  (special case)

---

```

1: function FORWARDFILTER( $j$ )
2:    $H_j := \text{COMPUTE}HJ(H, h_j)$ 
3:    $b := \sum_{i \in [1, n]} s_i - s_j$ 
4:    $v := s_j$ 
5:    $dec_b := b + v \geq \bar{g} \vee \Delta^-(H_j, b) < 0$ 
6:    $\ell := \min(\text{bp}^+(h_j, v) - v, \text{if } dec_b \text{ then } b - \text{bp}^-(H_j, b) \text{ else } \bar{g} - b - v)$ 
7:    $valH_j := \text{if } dec_b \text{ then } H_j(b - \ell) \text{ else } H_j(b)$ 
8:   while  $valH_j + h_j(v + \ell) \leq \bar{f}$  do
9:      $v := v + \ell$ 
10:    if  $dec_b$  then  $b := b - \ell$ 
11:     $dec_b := b + v \geq \bar{g} \vee \Delta^-(H_j, b) < 0$ 
12:     $\ell := \min(\text{bp}^+(h_j, v) - v, \text{if } dec_b \text{ then } b - \text{bp}^-(H_j, b) \text{ else } \bar{g} - b - v)$ 
13:     $valH_j := \text{if } dec_b \text{ then } H_j(b - \ell) \text{ else } H_j(b)$ 
14:     $\ell := (\bar{f} - H_j(b) - h_j(v)) / (\Delta^+(h_j, v) + (\text{if } dec_b \text{ then } \Delta^-(H_j, b) \text{ else } 0))$ 
15:     $v := v + \ell$ 
16:  FILTER( $g_j(x_j) \leq v$ )
17:  FILTER( $g_j(x_j) = v \Rightarrow f_j(x_j) \leq \bar{f} - H_j(b)$ )

```

---

Table 5: Values at some steps of Algorithm 5, as per Example 12.

Step	$b$	$v$	$dec_b$	$\ell$	$valH_j$	$h_j(v - 1)$
after the initialisation	8	2	true	1	0	2
after one iteration	7	3	true	2	2	6

set  $\ell = (5 - 0 - 2) / (2 + 1) = 1$  and  $v = 4$ . As in Example 11, line 16 removes all values larger than 4 (i.e., 5) from  $D_{x_1}$ . Line 17 does not remove anything here. Similarly, the execution of Algorithm 6 stops after one iteration and removes value 0 from the domain of  $x_1$ .  $\square$

Algorithm 6 presents the complementary algorithm to find  $v^{\min}$ . Again, it is very similar to Algorithm 4 but, like Algorithm 5, it decreases the value of  $v$  in each step by  $\ell$ , which is potentially larger than 1.

## 6. A Parametric Propagator and its Complexity

Our propagator is generic in the sense that it works correctly for any functions  $f_i$  and  $g_i$  that respect the condition of Theorem 8. However, we call it a *parametric* propagator, because rather than resorting to a fully generic implementation, we use hook functions and procedures that need to be provided. This often allows us to get a lower time complexity. The parameters for each instantiation are shown in Table 6: they are used in Algorithms 1 to 6. We now study the time and space complexity of our propagator, after describing some implementation strategies.



---

**Algorithm 6** Filtering algorithm for values smaller than  $s_j$  (special case)

---

```

1: function BACKWARDFILTER( $j$ )
2:    $H_j := \text{COMPUTE}HJ(H, h_j)$ 
3:    $b := \sum_{i \in [1, n]} s_i - s_j$ 
4:    $v := s_j$ 
5:    $inc_b := b + v \leq \underline{g} \vee \Delta^+(H_j, b) < 0$ 
6:    $\ell := \min(v - \text{bp}^-(h_j, v), \text{if } inc_b \text{ then } \text{bp}^+(H_j, b) - b \text{ else } b + v - \underline{g})$ 
7:    $valH_j := \text{if } inc_b \text{ then } H_j(b + \ell) \text{ else } H_j(b)$ 
8:   while  $valH_j + h_j(v - \ell) \leq \bar{f}$  do
9:      $v := v - \ell$ 
10:    if  $inc_b$  then  $b := b + \ell$ 
11:     $inc_b := b + v \leq \underline{g} \vee \Delta^+(H_j, b) < 0$ 
12:     $\ell := \min(v - \text{bp}^-(h_j, v), \text{if } inc_b \text{ then } \text{bp}^+(H_j, b) - b \text{ else } b + v - \underline{g})$ 
13:     $valH_j := \text{if } inc_b \text{ then } H_j(b + \ell) \text{ else } H_j(b)$ 
14:   $\ell := (\bar{f} - H_j(b) - h_j(v)) / (\Delta^-(h_j, v) + (\text{if } inc_b \text{ then } \Delta^+(H_j, b) \text{ else } 0))$ 
15:   $v := v - \ell$ 
16:  FILTER( $g_j(x_j) \geq v$ )
17:  FILTER( $g_j(x_j) = v \Rightarrow f_j(x_j) \leq \bar{f} - H_j(b)$ )

```

---

Table 6: Parameters to instantiate

Functions	Procedures
$\text{argmin}_{v \in g_i(D_{x_i})} h_i(v)$	FILTER( $g_i(x_i) \leq v$ )
$\min_{v \in g_i(D_{x_i})} h_i(v)$	FILTER( $g_i(x_i) \geq v$ )
$\Delta^+(h_i, v)$	FILTER( $g_i(x_i) = v \Rightarrow f_i(x_i) \leq u$ )
$\Delta^-(h_i, v)$	
$\text{bp}^+(h_i, v)$	
$\text{bp}^-(h_i, v)$	

### 6.1. Constructing $H$

We represent the  $H$  function as two linked lists of segments, plus two integers for the value  $b^*$  minimising  $H(b^*)$  and for  $H(b^*)$  itself. For each segment, its slope and length are stored. One linked list chains all the segments to the right of  $b^*$  by increasing order of the slope value and is terminated by a dummy segment with slope  $+\infty$ . The other linked list chains the segments to the left of  $b^*$  by decreasing order of the slope value and is terminated by a dummy segment with slope  $-\infty$ .

Constructing the linked lists of  $H$ , i.e., line 2 in Algorithm 2, can be implemented in various ways.

A first way is to traverse each function  $h_i$  in turn and to build  $H$  incrementally by traversing the linked lists in parallel. This takes  $\mathcal{O}(n \cdot (s(h) \cdot p + s(H)))$  time, where  $s(h)$  is the maximum number of segments among the  $h_i$  functions,  $s(H)$  is the number of segments of  $H$ , and  $p$  is the highest complexity of the parametric functions in Table 6.

---

**Algorithm 7** Maintaining  $H(b)$ ,  $s$ , and  $\ell$  when incrementing  $b$  by  $k$

---

**In/Out:**  $b$  ▷ The identifier  $b$  appearing in Algorithm 1  
**In/Out:**  $H.b$  ▷ The current value of  $H(b)$   
**In/Out:**  $s$  ▷ The segment of  $H$  on which  $b$  and  $b + 1$  lie  
**In/Out:**  $\ell$  ▷ The distance between  $b$  and the right breakpoint of  $s$   
**Require:**  $k \leq \ell$

- 1: **procedure** INCREMENT $_{H,b}(k)$
- 2:    $b := b + k$
- 3:    $H.b := H.b + k \cdot \Delta(s)$  ▷  $\Delta$  returns the slope of the segment
- 4:    $\ell := \ell - k$
- 5:   **if**  $\ell = 0$  **then**
- 6:      $s := \text{SUCC}(s)$  ▷ SUCC returns the next segment in the linked list
- 7:      $\ell := \text{LENGTH}(s)$  ▷ LENGTH returns the length of the segment

---

A second way is to collect all the segments from all the functions in a list, to sort this list, and to construct  $H$  by traversing the list. This takes  $\mathcal{O}(n \cdot s(h) \cdot (p + \log(n \cdot s(h))))$  time and is asymptotically better than the first way when  $s(H) > s(h) \cdot \log(n \cdot s(h))$ .

Although it might be interesting to construct  $H$  lazily because some parts of  $H$  might never be used, preliminary experiments have shown that the construction of  $H$  takes only a very small portion of the running time so that we did not explore this direction further.

### 6.2. Computing $H(b)$ , $h_j(v)$ , and $H_j(b)$

The value of  $H(b)$  is never queried for arbitrary values of  $b$ , but only for a value  $b^*$  minimising  $H(b^*)$  and for incrementally modified values of  $b$ , so that  $H(b)$  can also be computed incrementally.

The computation of  $H(b)$ ,  $\Delta^+(H, b)$ ,  $\Delta^-(H, b)$ ,  $\text{bp}^+(H, b)$ , and  $\text{bp}^-(H, b)$  can be performed as follows in constant time for all values of  $b$  used in the algorithms. Note first that in any algorithm the value of  $b$  is either only increased or only decreased, starting from  $b^*$ . We only discuss here the case of increasing  $b$ , the decreasing case being symmetrical. At any point in the algorithms, instead of maintaining only the value  $b$ , we also maintain a pointer to the segment  $s$  of  $H$  in which  $b$  and  $b + 1$  lie (there is always a unique such segment), the distance  $\ell$  between  $b$  and the right breakpoint of  $s$ , and the value of  $H(b)$ .

Algorithm 7 shows how those quantities can be maintained upon incrementing  $b$ . If  $b$  is incremented by  $k$  with  $k \leq \ell$  (which is ensured in all algorithms), then  $H(b)$  is incremented by  $k \cdot \Delta(s)$  and  $\ell$  is decreased by  $k$ . If  $\ell$  becomes equal to 0, then  $s$  is replaced by its successor in the linked list and  $\ell$  is set to the length of the new segment. The expression  $\Delta^+(H, b)$  appearing in Algorithm 1 is equal to  $\Delta(s)$  and  $\text{bp}^+(H, b)$  can be computed as  $b + \ell$ .

Similarly, the value of  $h_j(v)$  is only queried for  $s_i^{b^*}$  and incrementally modified values of  $v$ . This is reflected by the absence of  $h_j(v)$  from the parameters in Table 6. As shown in Algorithm 8, the bookkeeping associated with  $h_j$  is

---

**Algorithm 8** Maintaining  $h_j(v)$  when incrementing  $v$  by  $k$

---

**In/Out:**  $v$  ▷ The identifier  $v$  appearing in Algorithms 3 to 6  
**In/Out:**  $h_{j\_v}$  ▷ The current value of  $h_j(v)$   
**Require:**  $k \leq \text{bp}^+(h_j, v) - v$   
1: **procedure** INCREMENT $_{h_j, v}(k)$   
2:    $v := v + k$   
3:    $h_{j\_v} := h_{j\_v} + k \cdot \Delta^+(h_j, v)$  ▷  $\Delta^+(h_j, v)$  is a parameter to instantiate

---

**Algorithm 9** Maintaining  $H_j(b)$  and related quantities when incrementing  $b$  by  $k$

---

**In/Out:**  $b$  ▷ The identifier  $b$  appearing in Algorithms 3 to 6  
**In/Out:**  $H_{j\_b}$  ▷ The current value of  $H_j(b)$   
**In/Out:**  $d$  ▷ The slope of the segment of  $H_j$  on which  $b$  and  $b + 1$  lie  
**In/Out:**  $\ell$  ▷ The distance between  $b$  and its right breakpoint in  $H_j$   
**In/Out:**  $v'$  ▷ A value such that  $-\Delta^-(h_j, v') \leq d \wedge \Delta^+(h_j, v') > d$   
**In/Out:**  $s'$  ▷ The segment of  $H$  on which  $b + v'$  and  $b + v' + 1$  lie  
**Require:**  $k \leq \ell$   
1: **procedure** INCREMENT $_{H_j, b}(k)$   
2:    $b := b + k$   
3:    $H_{j\_b} := H_{j\_b} + k \cdot d$   
4:    $\ell := \ell - k$   
5:   **while**  $\ell = 0$  **do**  
6:      $s' := \text{SUCC}(s')$   
7:      $d := \Delta(s')$   
8:      $\ell := \text{LENGTH}(s')$   
9:     **if**  $d = \Delta^+(h_j, v')$  **then**  
10:        $\ell := \ell - (\text{bp}^+(h_j, v') - v')$   
11:        $v' := \text{bp}^+(h_j, v')$

---

simpler than the one of  $H$ : it suffices to maintain  $h_j(v)$  using the parameter function  $\Delta^+(h_j, v)$  whenever  $v$  is modified.

The approach for querying the value of  $H_j(b)$  is the same as for  $H(b)$ : the value of  $b$  is initialised to a value for which  $H_j(b)$  is known, and then either only increased or only decreased. Hence  $H_j(b)$  and the other related quantities can be maintained incrementally as is done for  $H$ . Using this fact and Lemma 10, we actually do not need to compute  $H_j$  in line 2 of Algorithms 3 to 6. Instead, we only maintain the segment  $s$  of  $H_j$  on which  $b$  and  $b + 1$  lie (in case of increasing  $b$ ). As shown in Algorithm 9, to compute the length  $\ell$  and slope  $d$  of  $s$ , we also maintain a pointer to the segment  $s'$  in  $H$  from which the segment of  $H_j$  is built and a value  $v'$  such that  $-\Delta^-(h_j, v') \leq \Delta(s) < \Delta^+(h_j, v')$ . When we need to access the next segment of  $H_j$  (because the value  $\ell$  reached 0), we can construct it from the successor of  $s'$  in  $H$  and the segment on the right of  $v'$  in  $h_j$ . As a segment in  $H$  might be built from a single segment of only  $h_j$ , Algorithm 9 must loop while the remaining length  $\ell$  is equal to zero.

Table 7: Time complexity of the different versions of the propagator

Case	Construct $H$	Time complexity
General	Traversing	$\mathcal{O}(n \cdot (s(h) \cdot p + s(H) + r(h) \cdot c))$
	Sorting	$\mathcal{O}(n \cdot (s(h) \cdot p + s(h) \cdot \log(n \cdot s(h)) + r(h) \cdot c))$
Special	Traversing	$\mathcal{O}(n \cdot (s(h) \cdot p + s(H) + c))$
	Sorting	$\mathcal{O}(n \cdot (s(h) \cdot p + s(h) \cdot \log(n \cdot s(h)) + s(H) + c))$

### 6.3. Complexity Analysis

*Feasibility Test.* Algorithm 1 computes a support in  $\mathcal{O}(s(H) + n \cdot s(h))$  time, that is  $\mathcal{O}(n \cdot s(h))$  time, as  $s(H) \leq n \cdot s(h)$ . This is dominated by the prior construction of  $H$  discussed in Section 6.1.

*Filtering.* We implement Algorithms 3 and 4 to run in  $\mathcal{O}(r(h) \cdot (p + c))$  time, where  $r(h) = |g_j(D_{x_j})|$ ,  $p$  is the highest complexity of the parametric functions in Table 6, and  $c$  is the highest complexity of the FILTER procedures in Table 6. The segments of  $H_j$  are computed on the fly from  $h_j$  and  $H$ , as explained in Section 6.2. The sum in line 3 of Algorithms 3 and 4 is actually provided by our representation of  $H$ , so it need not be recomputed each time. Algorithms 5 and 6 take  $\mathcal{O}(s(h) \cdot p + s(H) + c)$  time.

*The Whole Propagator.* The time complexity of our propagator, given in Algorithm 2, is obtained by multiplying the filtering complexity by  $n$  (the number of variables) and adding the complexity of computing  $H$ . Table 7 summarises this for the different versions of the propagator. Note that  $s(h) \leq r(h) \leq |D_x|$  and  $s(H) \leq n \cdot s(h)$ . In the worst case, one can assume that  $p = c = \mathcal{O}(d)$ , where  $d$  is the size of the largest domain, leading to the complexity announced in Section 1 for the general case and constructing  $H$  by traversal:  $\mathcal{O}(n \cdot d^2 + n^2 \cdot d)$ . However, we show in Table 1 on page 3 and in Section 8 that the time complexity for specific instantiations of the propagator is much lower.

The space complexity of our propagator is  $\mathcal{O}(n + s(H))$ , as we need to store a constant amount of information for each variable (namely the value  $s_i$ ), as well as the whole function  $H$  (which amounts to a constant amount for each of its  $s(H)$  segments). The functions  $h_i$  and  $H_j$  are not stored explicitly.

## 7. Instantiating the Parametric Propagator

We now present a number of relaxations and extensions of the problem as covered in the previous sections. At the same time, we discuss the consistency levels that can be achieved by our propagator. Those results are used in Section 8 when instantiating the parametric propagator for particular pairs of constraints.

### 7.1. Relaxations

The required discrete convexity of the  $h_i$  functions puts a strong restriction on the shape of the  $g_i$ . Recall that  $g_i(D_{x_i})$  must be an interval by the first condition in Definition 1. As the discrete convexity must be respected for *all*  $D_{x_i}$  that arise during the search, the only instantiations of  $g_i$  satisfying the first condition of Definition 1 are those whose image contains only two values, which must be one unit apart. We call these *characteristic functions*. In such a case, the second condition of Definition 1 is always respected and the  $f_i$  can be any (integer) functions. This is for instance the case of the constraint pairs LINEAR $_{\leq}$  and AMONG, and LINEAR $_{\leq}$  and MAXIMUM shown in Table 2 on page 9.

If  $D_{x_i}$  is restricted to be an interval, then the class of  $g_i$  functions satisfying the first condition of Definition 1 is more general, namely all functions where

$$|g_i(u) - g_i(u + 1)| \leq 1 \quad \forall u, u + 1 \in D_{x_i} \quad (13)$$

If there are holes in a domain  $D_{x_i}$ , then  $D_{x_i}$  can be relaxed to the smallest enclosing interval, but some propagation may be lost: this compromise is often acceptable for global constraints. In this case, we do not achieve domain consistency, but bounds( $\mathbb{Z}$ ) consistency. Among others, the identity function respects equation (13). If some  $g_i$  is the identity function, then  $f_i$  must be discretely convex, because  $h_i = f_i$ .

In general, if  $h_i$  is not discretely convex, then one can replace it by a discretely convex function  $h'_i: S \rightarrow T$  that *underapproximates*  $h_i$ , i.e., such that  $g_i(D_{x_i}) \subseteq S$  and  $h'_i(v) \leq h_i(v)$  for all  $v \in g_i(D_{x_i})$ . Using  $h'_i$ , the propagator remains correct but might miss propagation.

Even when  $h_i$  is discretely convex, it can be beneficial to replace it by an underapproximation in order to reduce the time complexity of the algorithms, at the sacrifice of a potentially weaker filtering. For example, a generally applicable relaxation is to replace all the segments of a function  $h_i$  on the right of some  $s_i^b$  by a single segment with slope  $\Delta^+(h_i, s_i^b)$ , and all segments on the left of  $s_i^b$  by a single segment with slope  $-\Delta^-(h_i, s_i^b)$ . This reduces the term  $s(h)$  in the complexity analysis to the constant 2 but may lead to some missed filtering.

### 7.2. Extensions

If some  $h_i$  is a linear function, then  $-h_i$  is also discretely convex. Hence, one can put a lower bound  $\underline{f}$  on  $\sum_{i \in [1, n]} f_i(x_i)$  and run the propagator twice, first with constraint (1) being  $\sum_{i \in [1, n]} f_i(x_i) \leq \bar{f}$ , and then with constraint (1) being  $-\sum_{i \in [1, n]} f_i(x_i) \leq -\underline{f}$ .

Our propagator can be extended to handle variables as the upper and lower bounds of the constraints. In such a case, the largest values in the domains of  $\bar{f}$  and  $\bar{g}$ , as well as the smallest values in the domains of  $\underline{f}$  and  $\underline{g}$  are used in the propagator. In addition, the other bound of each variable can be constrained using the  $H$  function without changing the time complexity. Only bounds( $\mathbb{Z}$ ) consistency can be achieved on those variables.

Our propagator can also be adapted to work with the  $f_i$  being functions from integers to *reals*. As long as the  $g_i$  are defined from integers to integers,

the domains of the intermediate variables  $y_i$  and  $z$  stay subsets of the integers. However, care must be taken when implementing operations on reals using floating point numbers.

## 8. Example Instantiations

We now show that many existing (pairs of) constraints fit our parametric constraint, optionally relaxed or extended as in Section 7. Table 2 on page 9 presents several instantiations of the  $f_i$  and  $g_i$ , together with the derived  $h_i$ . We discuss below some of these constraints and the time complexity of the parametric propagator in those cases, also summarised in Table 1 on page 3. The instantiated complexities are derived from the parametric complexities in Table 7 by replacing  $s(h)$ ,  $s(H)$ ,  $r(h)$ ,  $p$ , and  $c$  by suitable values derived from the  $h_i$ .

If  $g_i(u) = 0$  for all  $i$ , then the second constraint vanishes and we can use our propagator for a single  $\text{SUM}_{\leq}$  constraint. In such a case, our parametric propagator achieves domain consistency in  $\mathcal{O}(n \cdot (p + c))$  time. For the particular case of a linear inequality ( $\text{LINEAR}_{\leq}$ ), our parametric propagator runs in  $\mathcal{O}(n)$  time. Although this complexity matches the theoretical complexity of a dedicated propagator, our propagator is too general for this simple case and does not use any practical improvement such as the ones presented in [10].

Similarly, if  $f_i(u) = 0$  for all  $i$ , then the first constraint vanishes and we can use our propagator for a single  $\text{SUM}_{=}$  constraint, with  $\underline{g} = \bar{g}$ , achieving  $\text{bounds}(\mathbb{R})$  consistency in  $\mathcal{O}(n \cdot d)$  time. Again, for  $\text{LINEAR}_{=}$ , we match the theoretical  $\mathcal{O}(n)$  complexity of dedicated propagators but without any practical improvement such as the ones presented in [10].

The case  $g_i(u) = u$  for all  $i$  covers many interesting constraints already presented in the literature. In particular, it covers the  $\text{bounds}(\mathbb{Z})$ -consistent propagators for the statistical constraints  $\text{DEVIATION}$  and  $\text{SPREAD}$  with a fixed rational average. Interestingly, it can be generalised to any  $L_p$ -norm, with  $p > 0$ , except  $L_{+\infty}$ . One can also give a different penalty for deviations over and under the average. This may be very useful as in many practical situations it is less problematic to deviate from the average in one direction than in the other. The time complexity of our propagator is  $\mathcal{O}(n)$  for  $\text{DEVIATION}$ , which matches the best published propagator [3]. For  $\text{SPREAD}$  and higher norms, the time complexity of our propagator is  $\mathcal{O}(n \cdot d_{\cup})$ , with  $d_{\cup} = |\cup_{i \in [1, n]} D_{x_i}|$ . This is incomparable to the  $\mathcal{O}(n \cdot \log n)$  complexity of the best published propagator [2]. Note that our propagator achieves  $\text{bounds}(\mathbb{Z})$  consistency, which has only been achieved recently and independently in the case of  $\text{SPREAD}$  [11].

As an example, we show in Table 8 the instantiations of the parameters for  $\text{DEVIATION}$  (symmetric parameters are omitted): note that each  $h_i$  has (up to) three segments, joining at the breakpoints  $\lfloor \mu \rfloor$  and  $\lceil \mu \rceil$ .

If  $g_i$  is a characteristic function, then  $f_i$  can be any function. A characteristic function may be used to count, as in the case of the  $\text{COUNT}$  family of constraints (e.g.,  $\text{AMONG}$  [14, 15]). But characteristic functions can also be used to represent

Table 8: Expressions for instantiating a propagator for DEVIATION. The conditions are not always mutually exclusive and are to be evaluated in top-down order.

Parameter	Instantiation
$\operatorname{argmin}_{v \in g_i(D_{x_i})} h_i(v)$	$\begin{cases} \lceil \mu \rceil & \text{if } \min D_{x_i} \leq \mu \leq \max D_{x_i} \wedge \lceil \mu \rceil - \mu < \mu - \lfloor \mu \rfloor \\ \lfloor \mu \rfloor & \text{if } \min D_{x_i} \leq \mu \leq \max D_{x_i} \wedge \lceil \mu \rceil - \mu \geq \mu - \lfloor \mu \rfloor \\ \min D_{x_i} & \text{if } \mu < \min D_{x_i} \\ \max D_{x_i} & \text{if } \mu > \max D_{x_i} \end{cases}$
$\Delta^+(h_i, v)$	$\begin{cases} +\infty & \text{if } v = \max D_{x_i} \\ -n & \text{if } v < \lfloor \mu \rfloor \\ n \cdot (\lceil \mu \rceil + \lfloor \mu \rfloor) - 2 \cdot n \cdot \mu & \text{if } v = \lfloor \mu \rfloor \wedge \lceil \mu \rceil \neq \lceil \mu \rceil \\ n & \text{if } v \geq \lceil \mu \rceil \end{cases}$
$\operatorname{bp}^+(h_i, v)$	$\begin{cases} +\infty & \text{if } v = \max D_{x_i} \\ \min(\max D_{x_i}, \lfloor \mu \rfloor) & \text{if } v < \lfloor \mu \rfloor \\ \lceil \mu \rceil & \text{if } v = \lfloor \mu \rfloor \wedge \lceil \mu \rceil \neq \lceil \mu \rceil \\ \max D_{x_i} & \text{if } v \geq \lceil \mu \rceil \end{cases}$
$\operatorname{FILTER}(g_i(x_i) \leq v)$	$\operatorname{FILTER}(x_i \leq v)$
$\operatorname{FILTER}(g_i(x_i) = v \Rightarrow f_i(x_i) \leq u)$	$\operatorname{FILTER}( n \cdot v - n \cdot \mu  > u \Rightarrow x_i \neq v)$

the MAXIMUM constraint with a fixed maximum  $m$ . Indeed, the constraint  $m = \max_{i \in [1, n]} x_i$  can be decomposed into  $\forall i \in [1, n] : m \geq x_i \wedge \sum_{1 \in [1, n]} [x_i = m] \geq 1$ . Table 2 gives a definition of the functions  $h_i$  for  $\operatorname{LINEAR}_{\leq}$  and  $\operatorname{EXACTLY}$ , in which case our propagator achieves domain consistency and runs in  $\mathcal{O}(n \cdot (\log n + p + c))$  time, as does the dedicated propagator presented in [9].

Many other pairs of SUM constraints can be instantiated. Note that the functions  $f_i$  and  $g_i$  can differ for each  $i$ , i.e., one can mix in the same sum terms of different forms (e.g., some linear and some quadratic), as long as each corresponding function  $h_i$  is discretely convex.

**Example 13.** We now complete our running example. Applying Algorithm 2 achieves bounds( $\mathbb{Z}$ ) consistency and yields the following reduced domains:  $x_1 \in [1, 4]$ ,  $x_2 \in [2, 5]$ ,  $x_3 \in [1, 3]$ , and  $x_4 \in [1, 3]$ . In total, 9 values were removed from the domains. In contrast, if the two SUM constraints were filtered separately, then the domains would be  $x_1 \in [0, 4]$ ,  $x_2 \in [0, 5]$ ,  $x_3 \in [0, 3]$ , and  $x_4 \in [0, 3]$ , for a total of only 5 removed values.

Generalising to any number  $n$  of workers, our parametric propagator takes  $\mathcal{O}(n^2)$  time. This is obtained by instantiating the time complexity given in Table 7 for the special case with  $p = c = \mathcal{O}(1)$  as all parametric functions and procedures can be implemented in constant time,  $s(h) = 2$  as each  $h_i$  function has 2 segments, and  $s(H) = 2 \cdot n$  as, in the worst case, the parameters  $\mathbf{r}$  and  $\mathbf{s}$  are all different.  $\square$

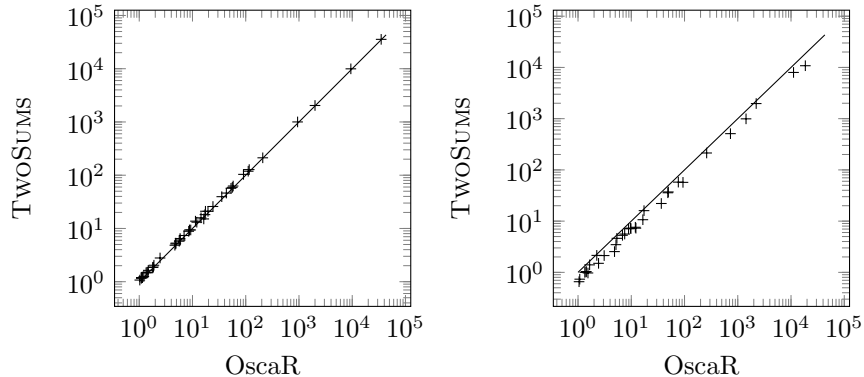


Figure 5: Results for DEVIATION (left) and SPREAD (right) comparing the time in seconds for solving an instance using either the specialised propagator or our propagator. We only report instances that take more than 1 second but less than 12 hours.

## 9. Experimental Evaluation

To show that the parametricity of our propagator is not detrimental not only to asymptotic complexity (as seen in Section 8) but also to efficiency, we make an experiment to compare custom propagators with instantiations of our parametric propagator. We selected the DEVIATION [3] and SPREAD [11] constraints as their bounds( $\mathbb{Z}$ )-consistent propagators are freely available in the distribution of OsaR [16]. We implemented our propagator and its instantiations on top of OsaR. We performed the comparison on the 100 instances of the balanced academic curriculum problem (BACP) that were introduced in [17],<sup>6</sup> modelled as in the OsaR distribution (we only slightly modified the search heuristic in order to make it deterministic, so that the search trees are the same for comparison purposes).

For DEVIATION, we used the 44 instances that are solved to optimality in more than 1 second (to avoid measurement errors) but less than 12 hours (3 instances timed out). When using our parametric propagator, the time to solve an instance is on average only 7% longer than when using the custom propagator, with a standard deviation of 5%: see also Figure 5, left. The numbers of nodes in the search tree and calls to the propagator are *exactly* the same for both propagators due to their common level of consistency and the deterministic search procedure.

For SPREAD, we used the 33 instances that are solved to optimality in more than 1 second but less than 12 hours (2 instances timed out). When using our parametric propagator, the time to solve an instance is on average 28% *shorter* than when using the custom propagator, with a standard deviation of 10%: see also Figure 5, right. This improvement is explained by a different algorithmic

<sup>6</sup>They are available at <http://becool.info.ucl.ac.be/resources/bacp>



approach, which is in our favour when the domains of the variables are small, as is the case for the BACP instances. The numbers of nodes in the search tree and calls to the propagator are *exactly* the same for both propagators.

Our Java implementation and the raw data for the results reported here are available at <http://www.it.uu.se/research/group/astra/software#convexpairs>. A package for replication is at <http://recomputation.org> [18].

## 10. Related Work

Our approach of first computing a feasibility bound and then incrementally adapting it is not new and has been used in the design of several propagators. Among others, this is the case for the cited propagators covered by our own propagator. However, the novelty of our work is that for the first time we abstract from the details of each constraint to focus on their common properties. This is close in spirit to what has been done with SEQBIN [19, 20] for another class of constraints.

When the  $g_i$  are characteristic functions, our conjunction of SUM constraints can also be represented using COSTGCC [21]. However, this would require the explicit representation of all variable-value pairs and induce a higher time complexity than our propagator. On the other hand, COSTGCC can handle more than one counting constraint in one propagator.

To the best of our knowledge, the notion of discrete convexity has not been used before for the design of propagators. However, several researchers have exploited other forms of convexity in constraint programming, namely row convexity [22] and tree convexity [23]. Those forms of convexity are unrelated to the one considered here and the focus of the cited works is on global properties of the constraint network rather than a propagator.

Finally, it should be noted that there are domain-consistent propagators for several constraints for which our propagator achieves  $\text{bounds}(\mathbb{Z})$  consistency or  $\text{bounds}(\mathbb{R})$  consistency only. In particular, Trick [24] presented a domain-consistent propagator for `Linear=` and Pesant [25] presented a domain-consistent propagator for `Lp-NORM` (including `SPREAD` and `DEVIATION`). Those propagators are based on dynamic programming ideas and have a higher time complexity than our approach.

## 11. Conclusion and Future Work

We have studied how to propagate pairs of SUM constraints that respect a discrete convexity condition. From this condition, we have derived a parametric propagator, which can be instantiated to be competitive with previously published propagators, often matching their time complexity, despite its generality.

There are a number of open questions we plan to address in the future. Can we *automatically* generate the instantiation of the parameters from the definitions of the  $f_i$  and  $g_i$ ? Can we make an *incremental* propagator that has a better time complexity along a branch of the search tree? Can we extend

the approach to functions that take more than one argument, say  $f_i(x_i, y_i)$  for variables  $y_i$  distinct from each other, or  $f_i(x_i, y)$  for a shared variable  $y$ , covering for instance the variable-average version of SPREAD [2, 26]? Can we deal with more than two sum constraints in one propagator?

## References

- [1] J.-N. Monette, N. Beldiceanu, P. Flener, J. Pearson, A parametric propagator for discretely convex pairs of Sum constraints, in: C. Schulte (Ed.), CP 2013, Vol. 8124 of LNCS, Springer, 2013, pp. 529–544.
- [2] G. Pesant, J.-C. Régin, SPREAD: A balancing constraint based on statistics, in: P. van Beek (Ed.), CP 2005, Vol. 3709 of LNCS, Springer, 2005, pp. 460–474.
- [3] P. Schaus, Y. Deville, P. Dupont, Bound-consistent Deviation constraint, in: C. Bessière (Ed.), CP 2007, Vol. 4741 of LNCS, Springer, 2007, pp. 620–634.
- [4] T. Petit, J.-C. Régin, N. Beldiceanu, A  $\Theta(n)$  bound-consistency algorithm for the Increasing Sum constraint, in: J. Lee (Ed.), CP 2011, Vol. 6876 of LNCS, Springer, 2011, pp. 721–728.
- [5] J.-F. Puget, Improved bound computation in presence of several Clique constraints, in: M. Wallace (Ed.), CP 2004, Vol. 3258 of LNCS, Springer, 2004, pp. 527–541.
- [6] J.-C. Régin, T. Petit, The Objective Sum constraint, in: T. Achterberg, J. C. Beck (Eds.), CPAIOR 2011, Vol. 6697 of LNCS, Springer, 2011, pp. 190–195.
- [7] C. Schulte, P. J. Stuckey, When do bounds and domain propagation lead to the same search space?, ACM Transactions on Programming Languages and Systems 27 (3) (2005) 388–425.
- [8] C. Choi, W. Harvey, J. Lee, P. Stuckey, Finite domain bounds consistency revisited, in: A. Sattar, B.-h. Kang (Eds.), AI 2006: Advances in Artificial Intelligence, Vol. 4304 of LNCS, Springer, 2006, pp. 49–58.
- [9] N. Razakarison, N. Beldiceanu, M. Carlsson, H. Simonis, GAC for a linear inequality and an Atleast constraint with an application to learning simple polynomials, in: SoCS 2013, AAAI Press, 2013, pp. 149–157.
- [10] W. Harvey, J. Schimpf, Bounds consistency techniques for long Linear constraints, in: Proceedings of TRICS 2002, the workshop on Techniques for Implementing Constraint programming Systems, 2002, pp. 39–46.
- [11] P. Schaus, J.-C. Régin, Bound-consistent Spread constraint, application to load balancing in nurse-to-patient assignments, EURO Journal on Computational Optimization (2013) 1–24.

- [12] K. Murota, Recent developments in discrete convex analysis, in: W. Cook, L. Lovász, J. Vygen (Eds.), *Research Trends in Combinatorial Optimization*, Springer, 2009, pp. 219–260.
- [13] S. Fujishige, *Submodular Functions and Optimization*, 2nd Edition, *Annals of Discrete Mathematics*, Elsevier, 2005.
- [14] N. Beldiceanu, E. Contejean, Introducing global constraints in CHIP, *Mathematical and Computer Modelling* 20 (12) (1994) 97–123.
- [15] C. Bessière, E. Hebrard, B. Hnich, Z. Kızıltan, T. Walsh, Among, Common and Disjoint constraints, in: B. Hnich, M. Carlsson, F. Fages, F. Rossi (Eds.), *CSCLP 2005; Revised Selected and Invited Papers*, Vol. 3978 of *LNAI*, Springer-Verlag, 2006, pp. 28–43.
- [16] Oscar Team, *Oscar: Scala in OR*, available from <http://www.oscarlib.org> (2012).
- [17] P. Schaus, Solving balancing and bin-packing problems with constraint programming, PhD Thesis, Université catholique de Louvain, Belgium (2009).
- [18] I. P. Gent, The recomputation manifesto, CoRR abs/1304.3674, available at <http://arxiv.org/abs/1304.3674>.
- [19] T. Petit, N. Beldiceanu, X. Lorca, A generalized arc-consistency algorithm for a class of counting constraints, in: *IJCAI 2011*, AAAI Press, 2011, pp. 643–648, revised edition available at <http://arxiv.org/abs/1110.4719>.
- [20] G. Katsirelos, N. Narodytska, T. Walsh, The SeqBin constraint revisited, in: M. Milano (Ed.), *CP 2012*, Vol. 7514 of *Lecture Notes in Computer Science*, Springer, 2012, pp. 332–347.
- [21] J.-C. Régin, Cost-based arc consistency for global cardinality constraints, *Constraints* 7 (3–4) (2002) 387–405.
- [22] Y. Deville, O. Barette, P. Van Hentenryck, Constraint satisfaction over connected row-convex constraints, *Artificial Intelligence* 109 (1–2) (1999) 243–271.
- [23] Y. Zhang, E. C. Freuder, Properties of tree convex constraints, *Artificial Intelligence* 172 (12–13) (2008) 1605–1612.
- [24] M. A. Trick, A dynamic programming approach for consistency and propagation for knapsack constraints, *Annals of Operations Research* 118 (1–4) (2003) 73–84.
- [25] G. Pesant, Achieving domain consistency and counting solutions for dispersion constraints, *INFORMS Journal on Computing* 27 (4) (2015) 690–703.
- [26] S. C. Loong, W.-Y. Ku, J. C. Beck,  $\mathbb{Q}$ -bounds consistency for the spread constraint with variable mean, *Constraints* (2016) 1–7.