



Transport along stochastic flows in fluid dynamics

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Transport along stochastic flows in fluid dynamics

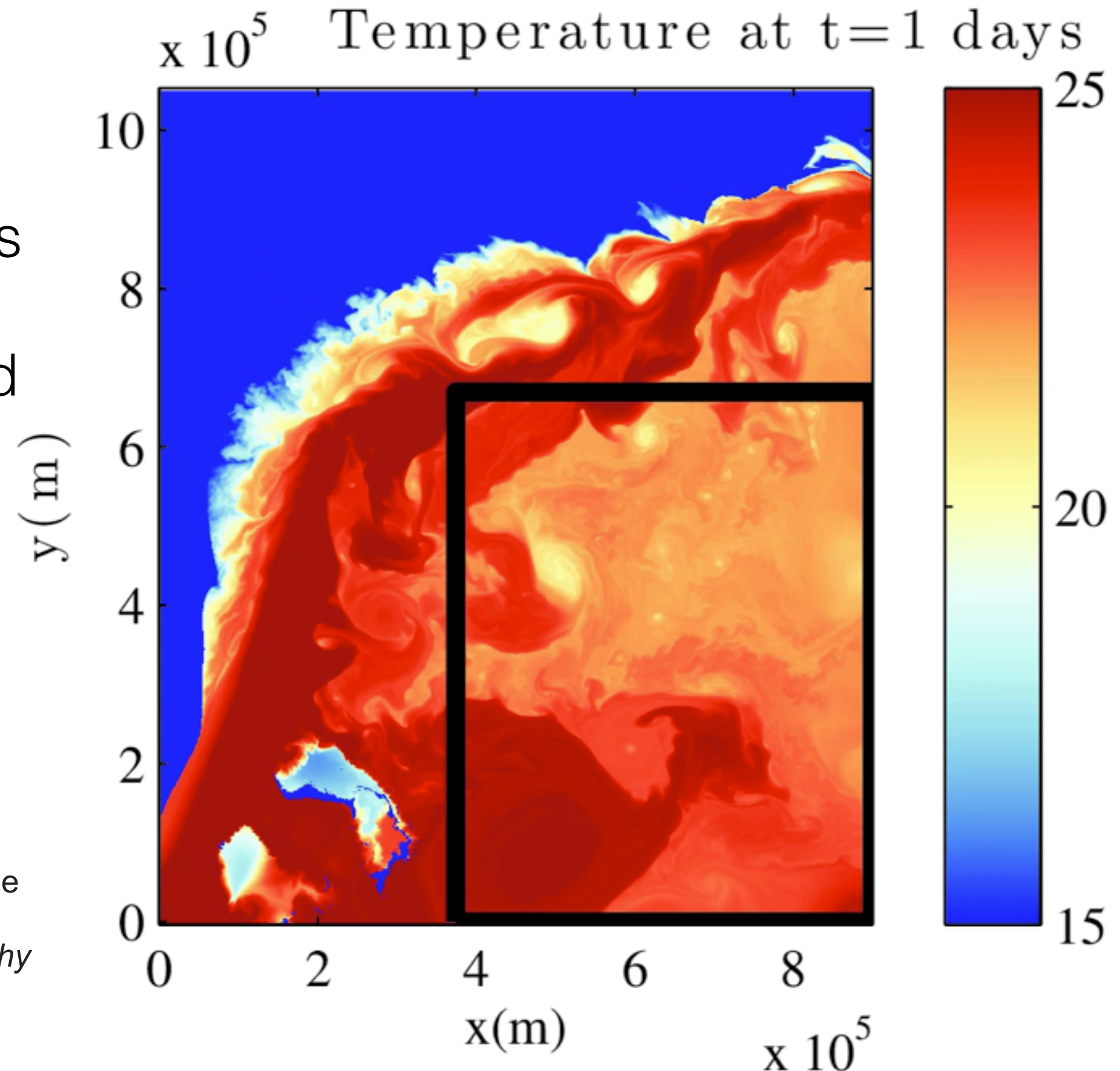
Valentin Resseguier,
Etienne Mémin,
Bertrand Chapron

Fluids are very
complex

with small vortices
interacting with
large vortices and
currents

Gula, Jonathan, M. Jeroen
Molemaker, and James C.
McWilliams

"Gulf Stream dynamics along the
southeastern US seaboard."
Journal of Physical Oceanography
45.3 (2015): 690-715.

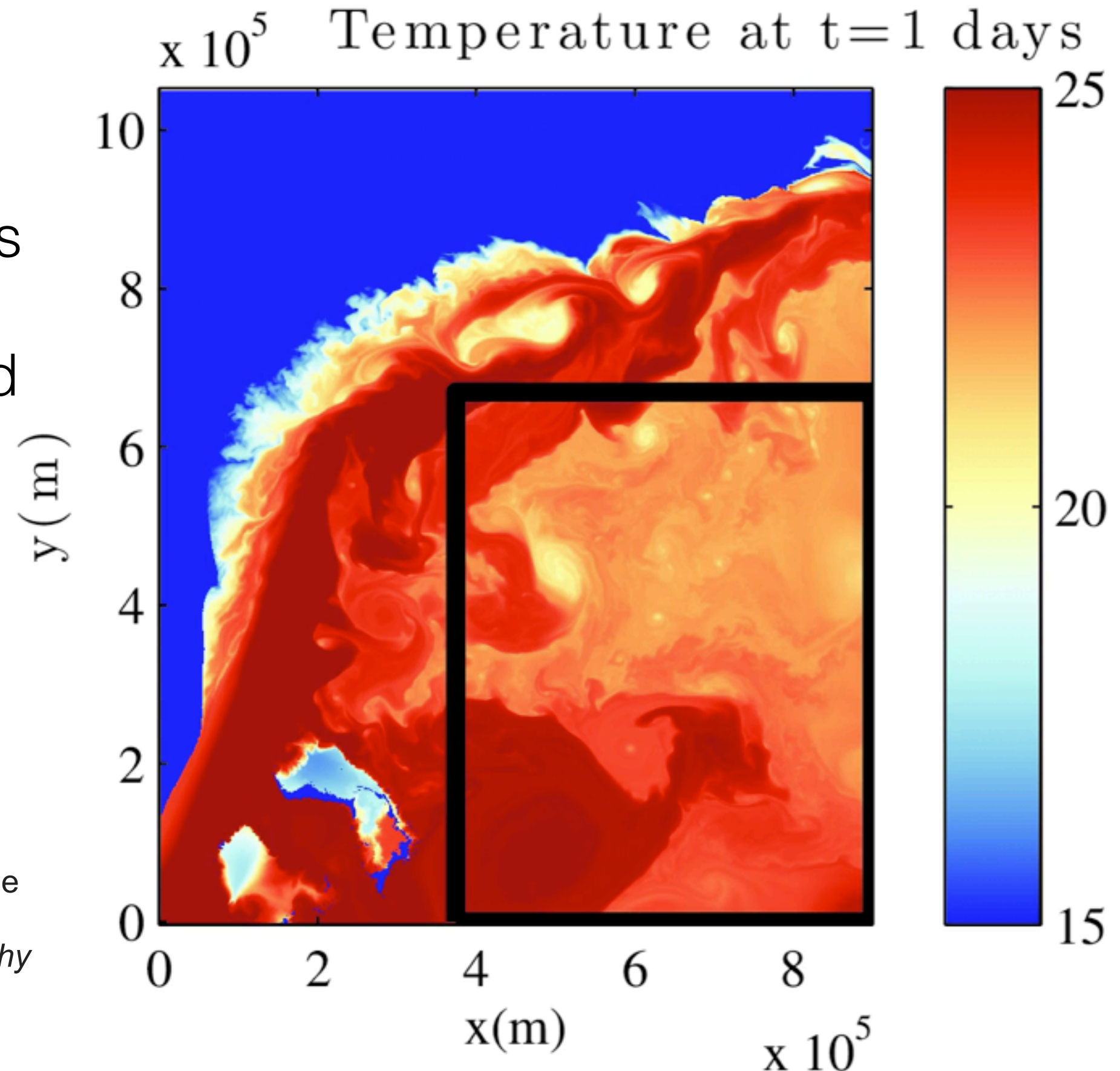


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Why a random fluid dynamics?

- Take into account unresolved processes (small scales)
- Physical justification of empirical models
- Predicting possible distinct scenarios, extreme-events, ...
- Quantification of modeling errors for **data assimilation:**
ensemble forecasts

Contents

- Randomized dynamics
- Simulation of the SQG under Moderate Uncertainty
- Stochastic reduced order model

Randomized dynamics

Advection of tracer Θ

- Stochastic flow:

$$d\mathbf{X}_t = \mathbf{w}(\mathbf{X}_t, t)dt + \boldsymbol{\sigma}(\mathbf{X}_t, t)d\mathbf{B}_t$$

Advection of tracer Θ

- Stochastic flow:

$$d\mathbf{X}_t = \mathbf{w}(\mathbf{X}_t, t)dt + \boldsymbol{\sigma}(\mathbf{X}_t, t)d\mathbf{B}_t$$

with infinite-dimensional Brownian motion

$$\boldsymbol{\sigma}(\bullet, t)d\mathbf{B}_t \triangleq \int_{\Omega} dz \, \check{\boldsymbol{\sigma}}(\bullet, z, t)d\mathbf{B}_t(z)$$

Advection of tracer Θ

- Stochastic flow:

$$d\mathbf{X}_t = \mathbf{w}(\mathbf{X}_t, t)dt + \boldsymbol{\sigma}(\mathbf{X}_t, t)d\mathbf{B}_t$$

with infinite-dimensional Brownian motion

$$\boldsymbol{\sigma}(\bullet, t)d\mathbf{B}_t \triangleq \int_{\Omega} dz \check{\boldsymbol{\sigma}}(\bullet, z, t)d\mathbf{B}_t(z)$$

- A tracer is a function conserved along the flow:

$$D_t\Theta(t, \mathbf{X}_t) \triangleq d[\Theta(t, \mathbf{X}_t)] = 0$$

Ito-Wentzel formula

(Kunita 1997)

If both $\mathbf{X}$ and Θ are semimartingales (w.r.t. time)
and Θ is twice differentiable w.r.t. space

Then

$$\begin{aligned} d [\Theta(t, \mathbf{X}(t, y))] &= d_t \Theta + (\nabla \Theta)^T d\mathbf{X} + \frac{1}{2} tr(H_\Theta d \langle \mathbf{X}, \mathbf{X}^T \rangle) \\ &\quad + d_t \langle (\nabla \Theta)^T, \mathbf{X} \rangle \end{aligned}$$

Advection of tracer Θ

$$D_t \Theta = 0$$

Advection of tracer Θ

Advection of tracer Θ

$$d_t \Theta + \boldsymbol{w}^* \cdot \nabla \Theta dt + \boldsymbol{\sigma} d\boldsymbol{B}_t \cdot \nabla \Theta = \nabla \cdot \left(\frac{1}{2} \boldsymbol{a} \nabla \Theta \right) dt$$

Advection of tracer Θ

Advection

$$d_t \Theta + \boldsymbol{w}^* \cdot \nabla \Theta dt + \sigma dB_t \cdot \nabla \Theta = \nabla \cdot \left(\frac{1}{2} \boldsymbol{a} \nabla \Theta \right) dt$$

Advection of tracer Θ

$$d_t \Theta + \underbrace{w^* \cdot \nabla \Theta dt + \sigma dB_t \cdot \nabla \Theta}_{\text{Advection}} = \underbrace{\nabla \cdot \left(\frac{1}{2} a \nabla \Theta \right)}_{\text{Diffusion}} dt$$

Advection of tracer Θ

$$d_t \Theta + \underbrace{w^* \cdot \nabla \Theta dt + \sigma dB_t \cdot \nabla \Theta}_{\text{Advection}} = \underbrace{\nabla \cdot \left(\frac{1}{2} a \nabla \Theta \right)}_{\text{Diffusion}} dt$$

$$a = \sigma \sigma^T$$

$$w^* = w - \frac{1}{2} (\nabla \cdot a)^T$$

Advection of tracer Θ

$$d_t \Theta + \underbrace{w^* \cdot \nabla \Theta dt + \sigma dB_t \cdot \nabla \Theta}_{\text{Advection}} = \underbrace{\nabla \cdot \left(\frac{1}{2} a \nabla \Theta \right)}_{\text{Diffusion}} dt$$

Drift
correction

$$a = \sigma \sigma^T$$

$$w^* = w - \frac{1}{2} (\nabla \cdot a)^T$$

Advection of tracer Θ

$$d_t \Theta + \underbrace{w^* \cdot \nabla \Theta dt + \sigma dB_t \cdot \nabla \Theta}_{\text{Advection}} = \underbrace{\nabla \cdot \left(\frac{1}{2} a \nabla \Theta \right)}_{\text{Diffusion}} dt$$

Drift correction

Multiplicative random forcing

$$a = \sigma \sigma^T$$

$$w^* = w - \frac{1}{2} (\nabla \cdot a)^T$$

Advection of tracer Θ

The diagram illustrates the decomposition of the stochastic advection-diffusion equation. The left side, labeled "Advection", is enclosed in a light blue rounded rectangle and contains the terms $d_t \Theta + w^* \cdot \nabla \Theta dt + \sigma dB_t \cdot \nabla \Theta$. The term w^* is circled in orange, and the term $\sigma dB_t \cdot \nabla \Theta$ is circled in purple. An orange arrow points from the text "Drift correction" to the w^* term. A purple arrow points from the text "Multiplicative random forcing" to the $\sigma dB_t \cdot \nabla \Theta$ term. The right side, labeled "Diffusion", is enclosed in a light green rounded rectangle and contains the term $\nabla \cdot \left(\frac{1}{2} a \nabla \Theta \right) dt$. A green arrow points from the diffusion term back to the $\sigma dB_t \cdot \nabla \Theta$ term, indicating its contribution to the drift correction.

$$d_t \Theta + w^* \cdot \nabla \Theta dt + \sigma dB_t \cdot \nabla \Theta = \nabla \cdot \left(\frac{1}{2} a \nabla \Theta \right) dt$$

Drift correction

Multiplicative random forcing

$$a = \sigma \sigma^T$$

$$w^* = w - \frac{1}{2} (\nabla \cdot a)^T$$

Advection of tracer Θ

$$d_t \Theta + \underbrace{w^* \cdot \nabla \Theta dt}_{\text{Drift correction}} + \underbrace{\sigma dB_t \cdot \nabla \Theta}_{\text{Multiplicative random forcing}} = \underbrace{\nabla \cdot \left(\frac{1}{2} a \nabla \Theta \right) dt}_{\text{Diffusion}}$$

Diffusion

Drift correction

Multiplicative random forcing

Balanced energy exchanges

$$a = \sigma \sigma^T$$

$$w^* = w - \frac{1}{2} (\nabla \cdot a)^T$$

$$\sigma = ?$$

$$\sigma(\bullet, t) d\mathbf{B}_t \triangleq \int_{\Omega} dz \check{\sigma}(\bullet, z, t) d\mathbf{B}_t(z)$$

- Parametric or non-parametric estimation

- Observations can be:

- Small-scale eulerian velocity $(\sigma(\mathbf{x}_i, t_j) d\mathbf{B}_{t_j})_{ij}$
- Small-scale Lagrangian velocity $(\sigma(\mathbf{X}_{t_j}(\mathbf{x}_i), t_j) d\mathbf{B}_{t_j})_{ij}$
- **Tracer (solution of the SPDE)** $(\Theta(\mathbf{x}_i, t_j))_{ij}$

- In the following simulations, very simple model (without estimations)

$$\sigma d\mathbf{B}_t = \int_{\Omega} dz \nabla^{\perp} \check{\psi}(\bullet - z) d\mathbf{B}_t(z) = \nabla^{\perp} \check{\psi} * d\mathbf{B}_t$$

$$\hat{\check{\psi}}(\mathbf{k}) = A \mathbf{1}_{\{\kappa_1 < \|\mathbf{k}\| < \kappa_2\}} \|\mathbf{k}\|^{-\alpha}$$

Simulation of the SQG under Moderate Uncertainty

SQG MU

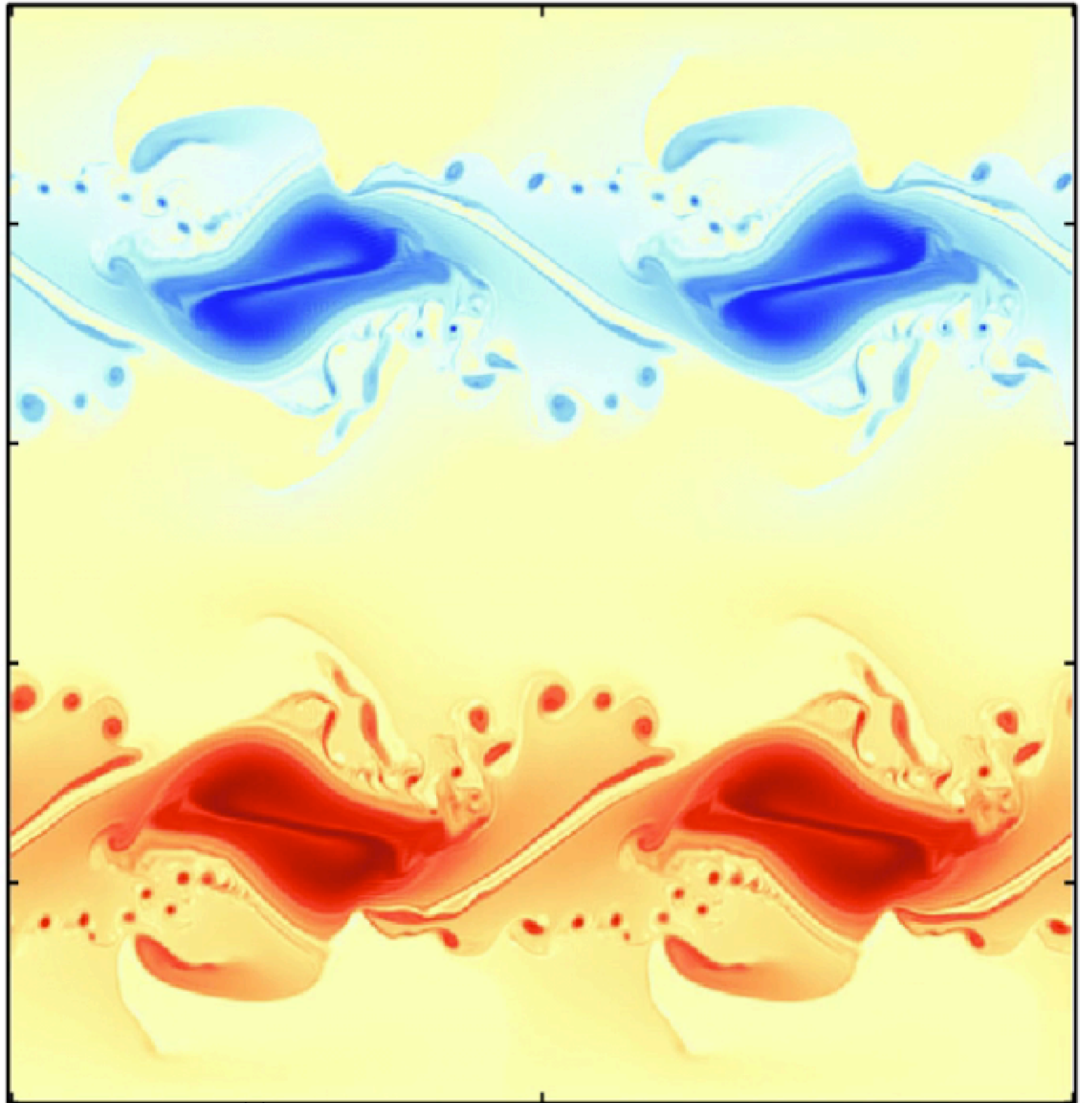
Code available online

$t = 17$ days

SQG model:

$$D_t \Theta = \nu \Delta^4 \Theta dt$$

$$\mathbf{w} = \alpha \nabla^\perp (-\Delta)^{-\frac{1}{2}} \Theta$$



Reference flow:

deterministic SQG

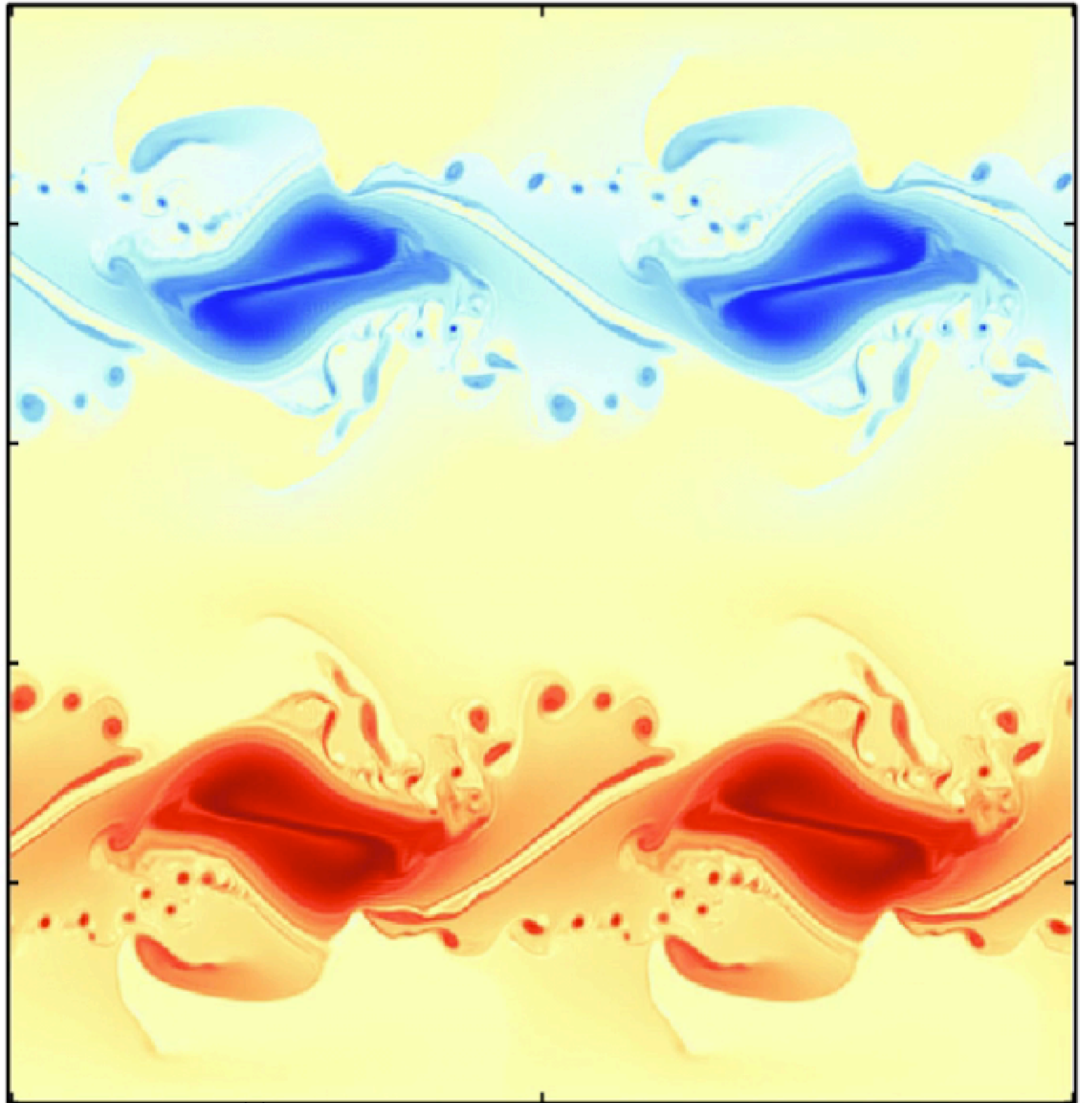
512 x 512

$t = 17$ days

SQG model:

$$D_t \Theta = \nu \Delta^4 \Theta dt$$

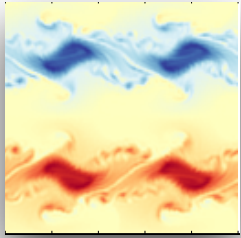
$$\mathbf{w} = \alpha \nabla^\perp (-\Delta)^{-\frac{1}{2}} \Theta$$



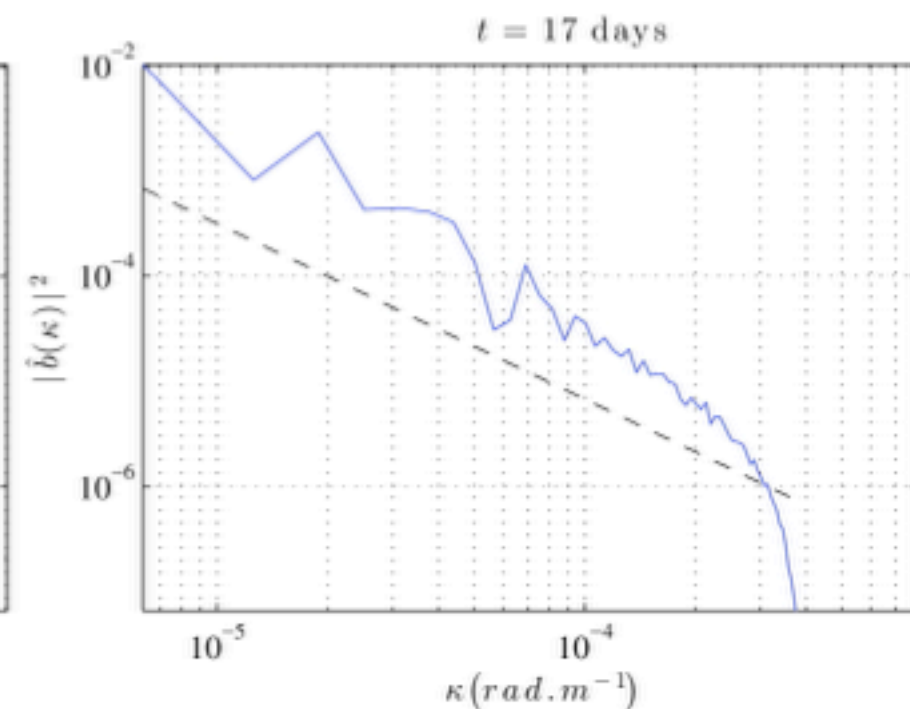
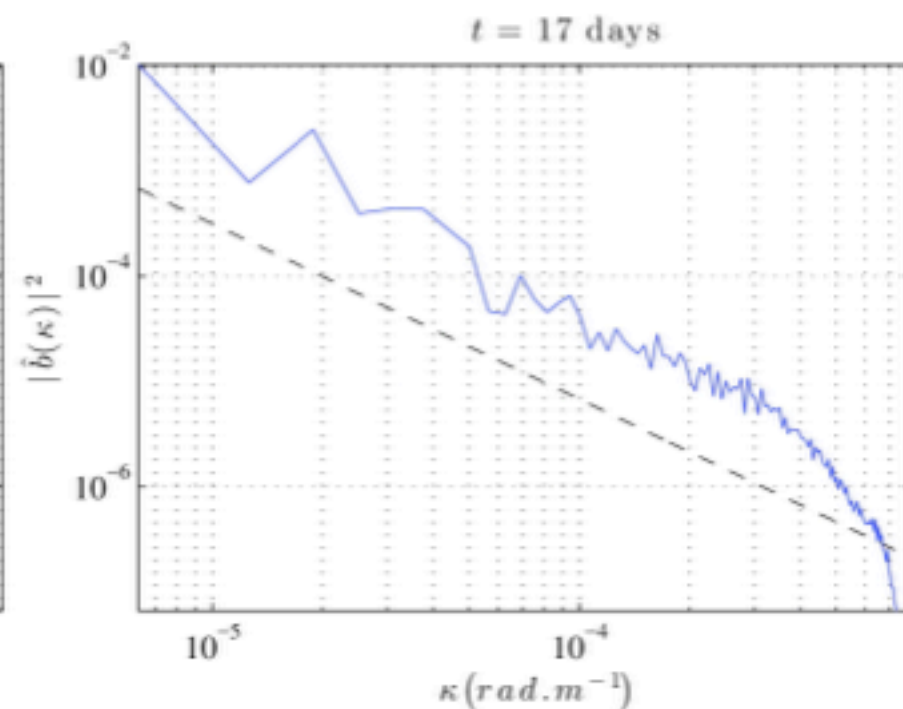
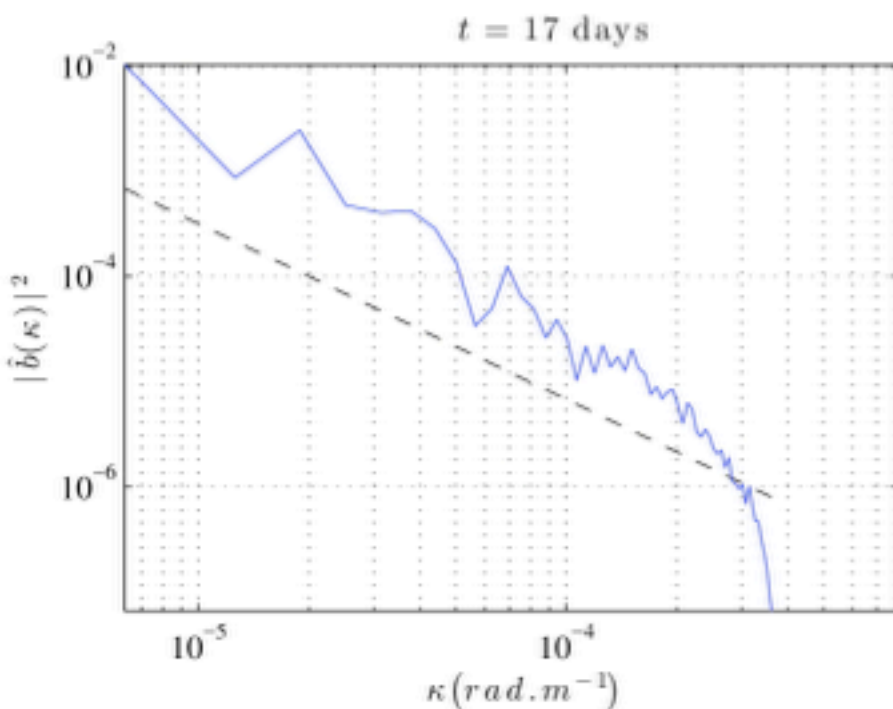
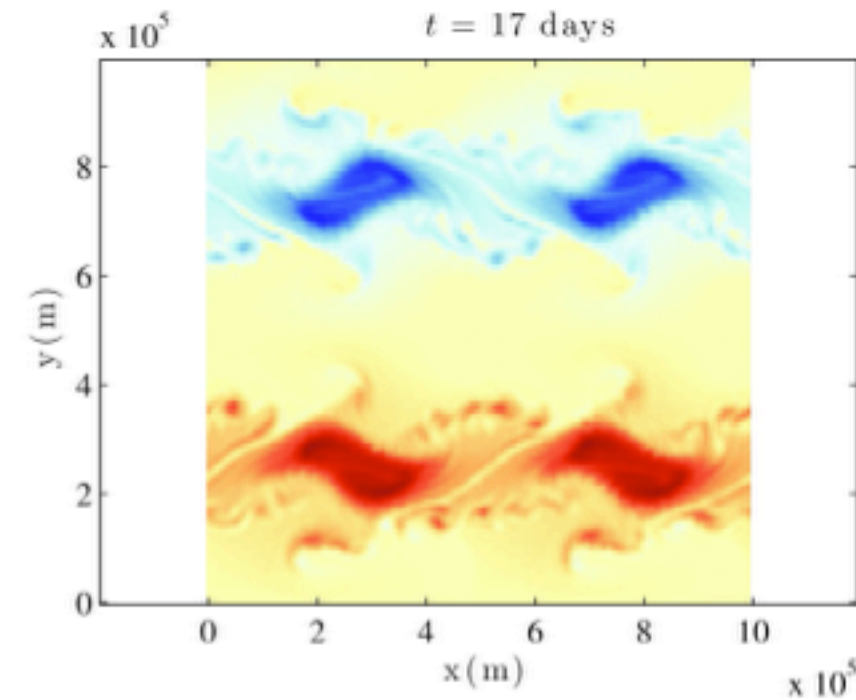
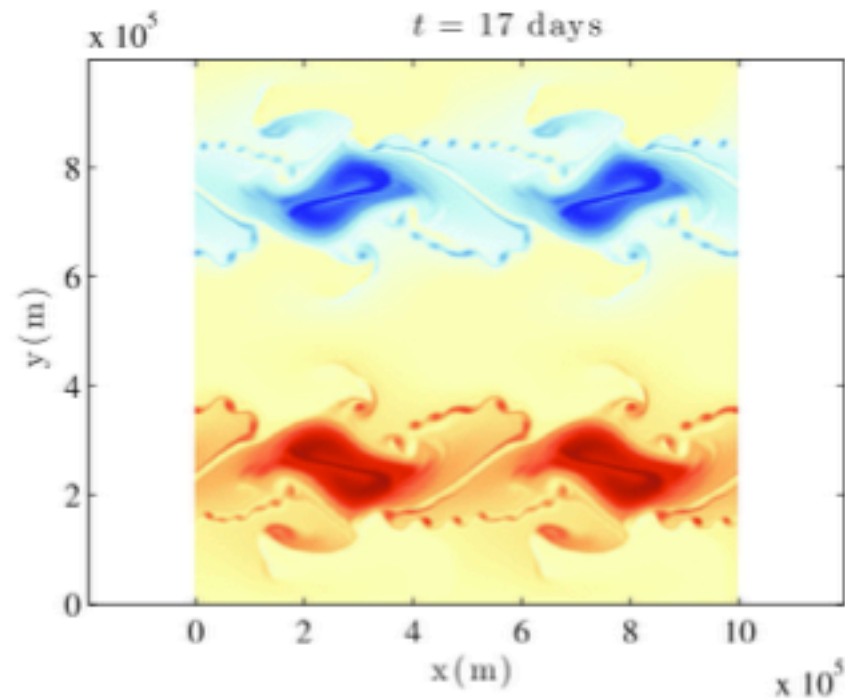
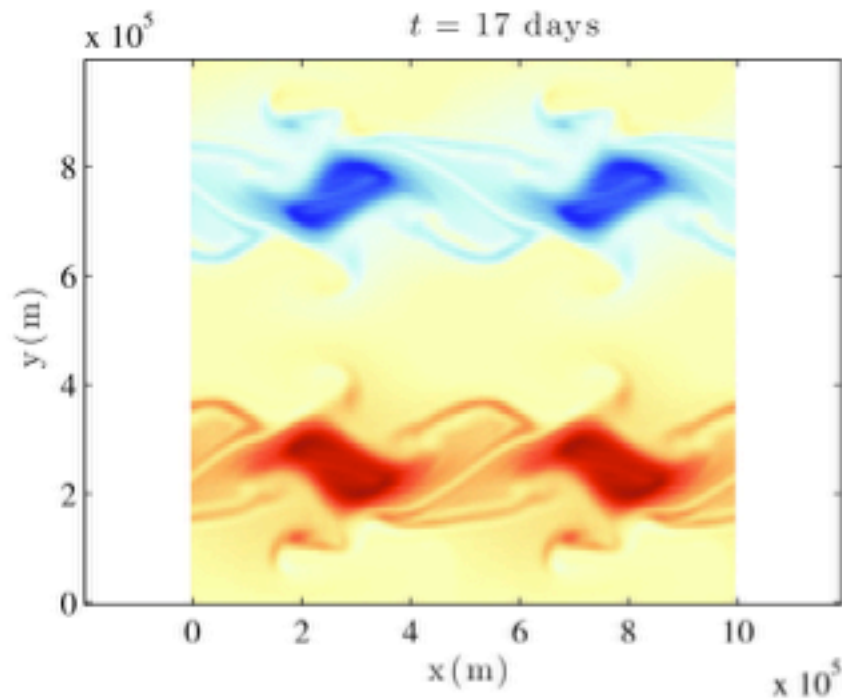
Reference flow:

deterministic SQG

512 x 512



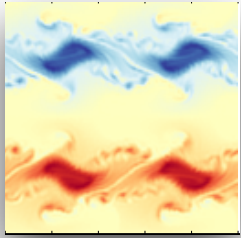
One realization



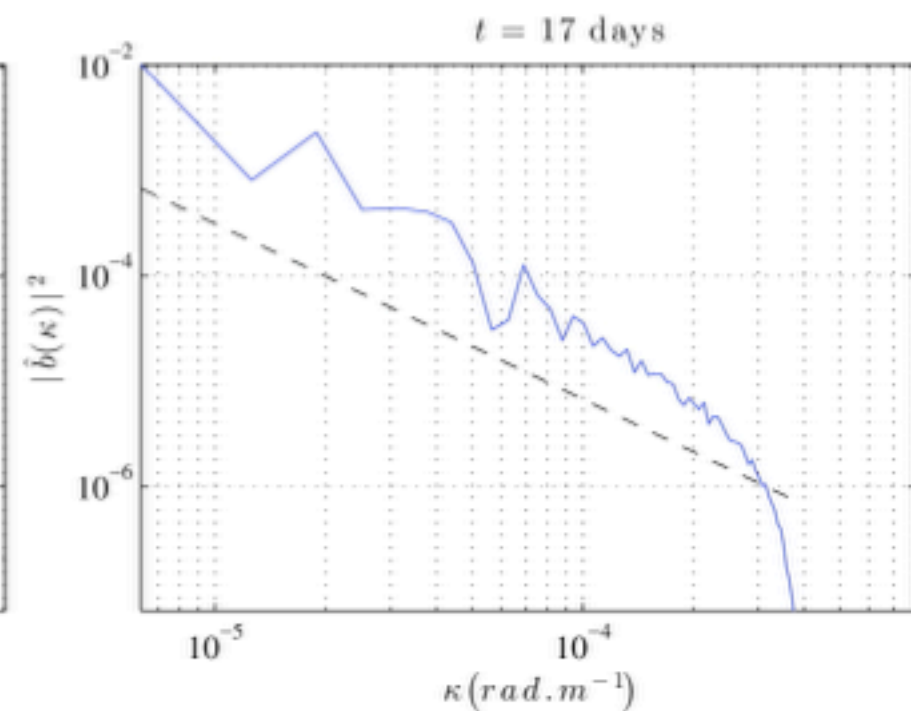
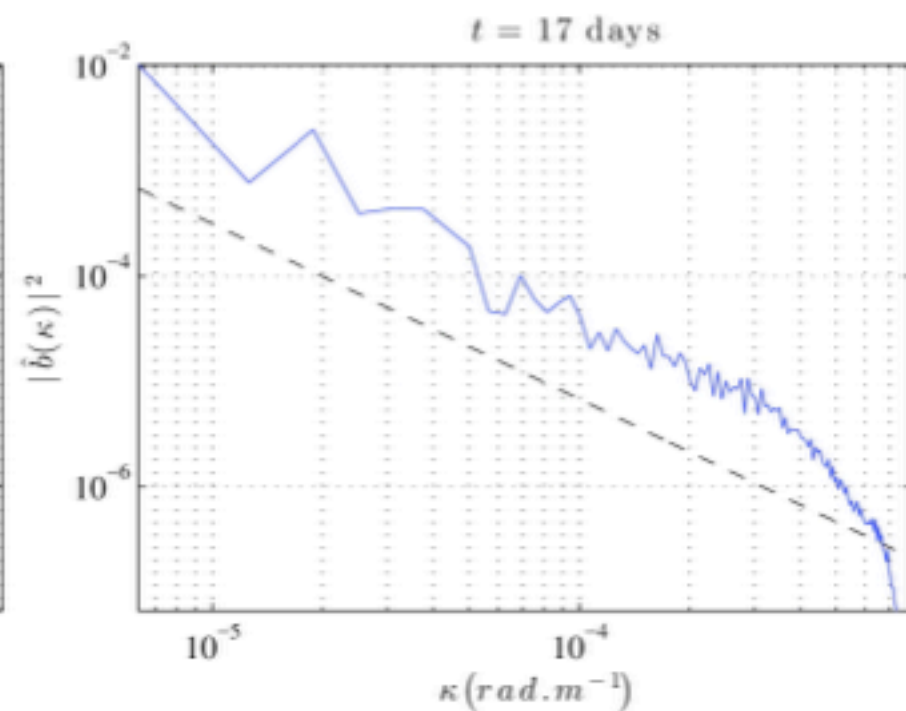
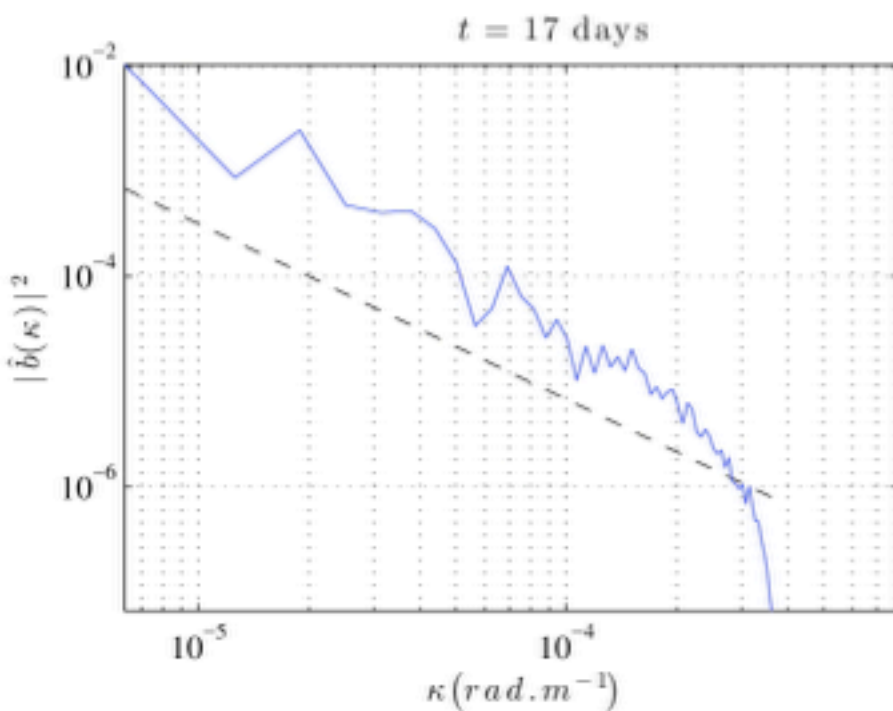
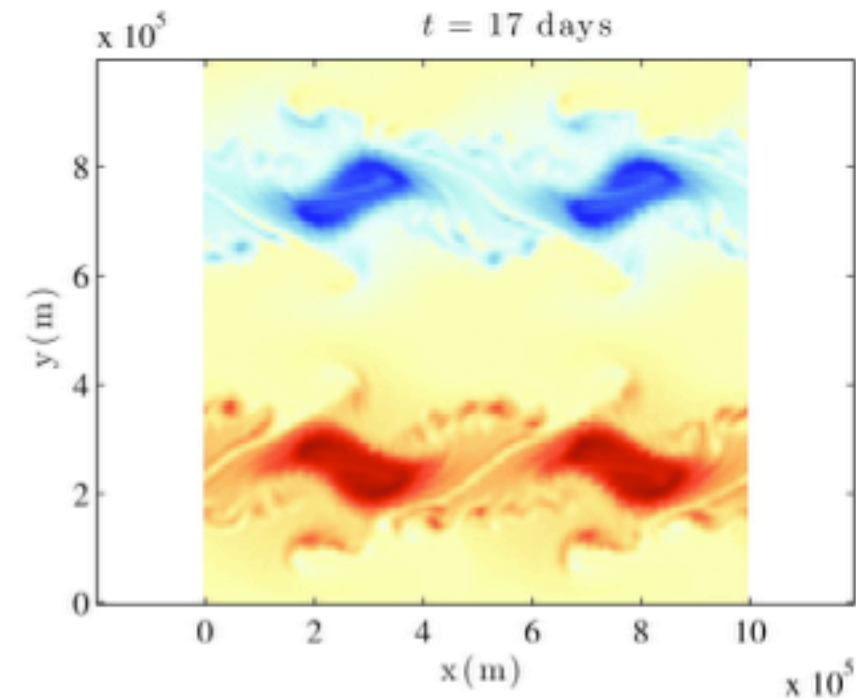
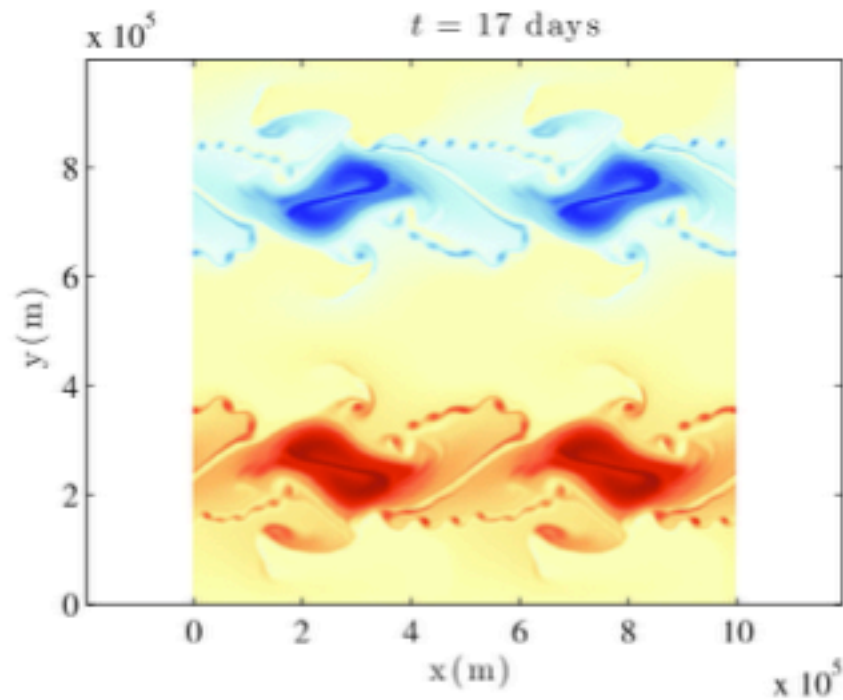
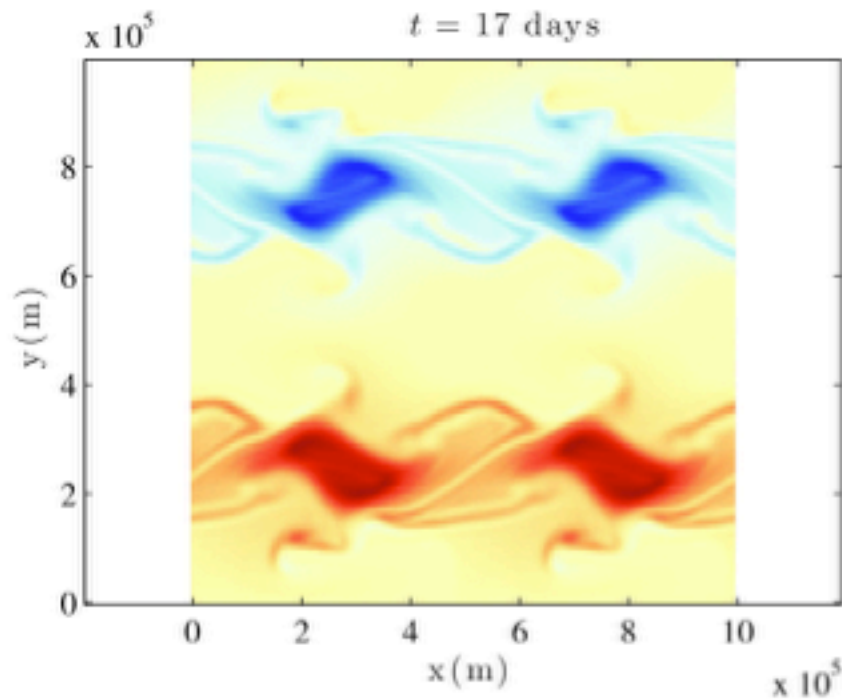
Deterministic 128x128

Deterministic 512x512

Stochastic 128x128



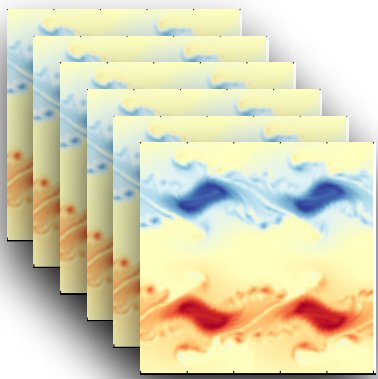
One realization



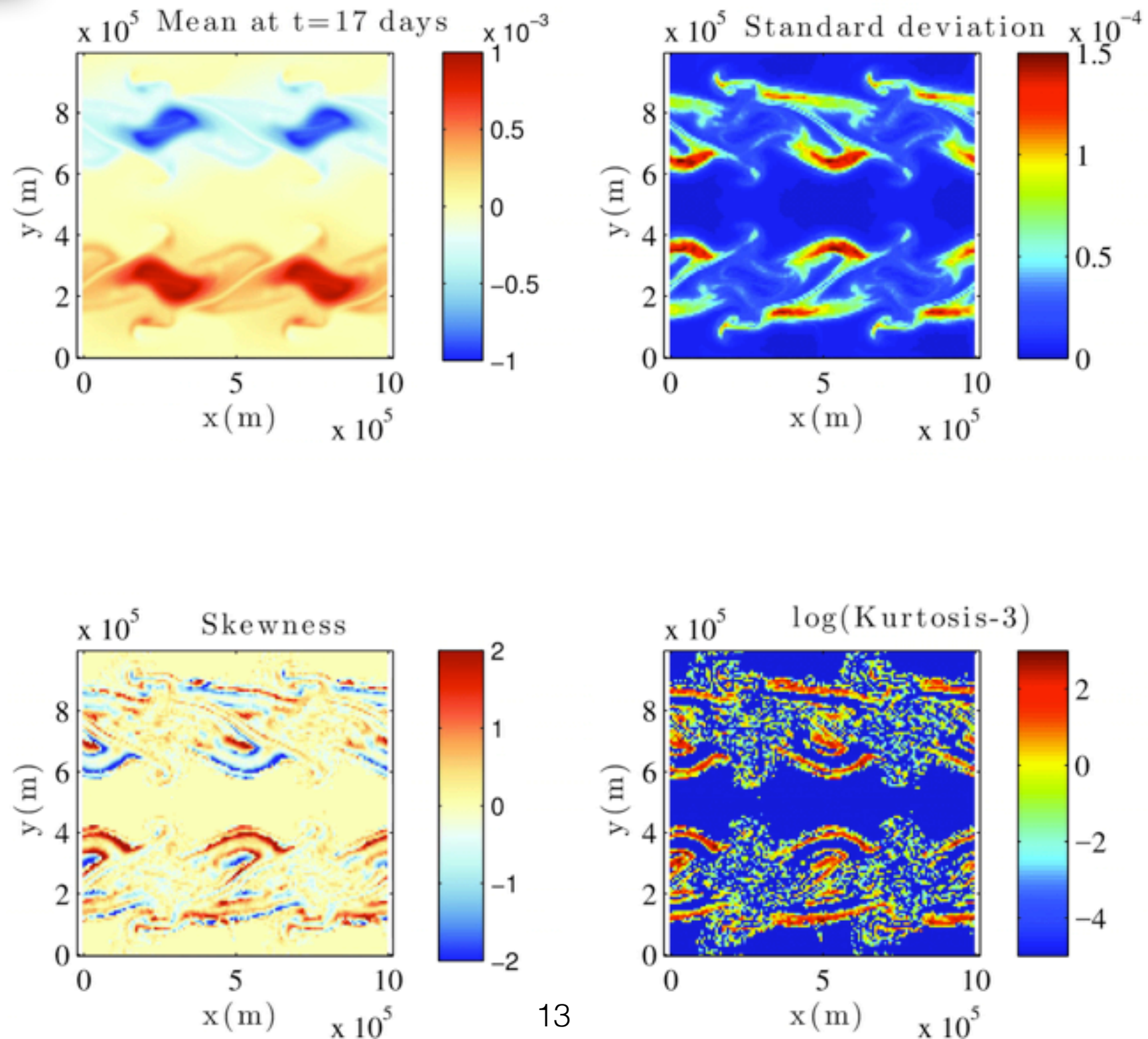
Deterministic 128x128

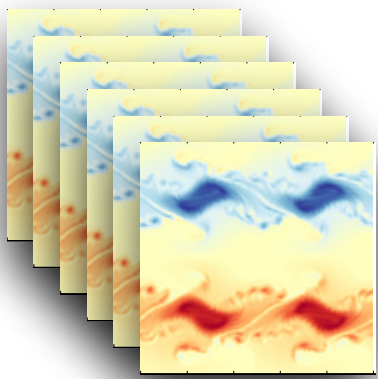
Deterministic 512x512

Stochastic 128x128

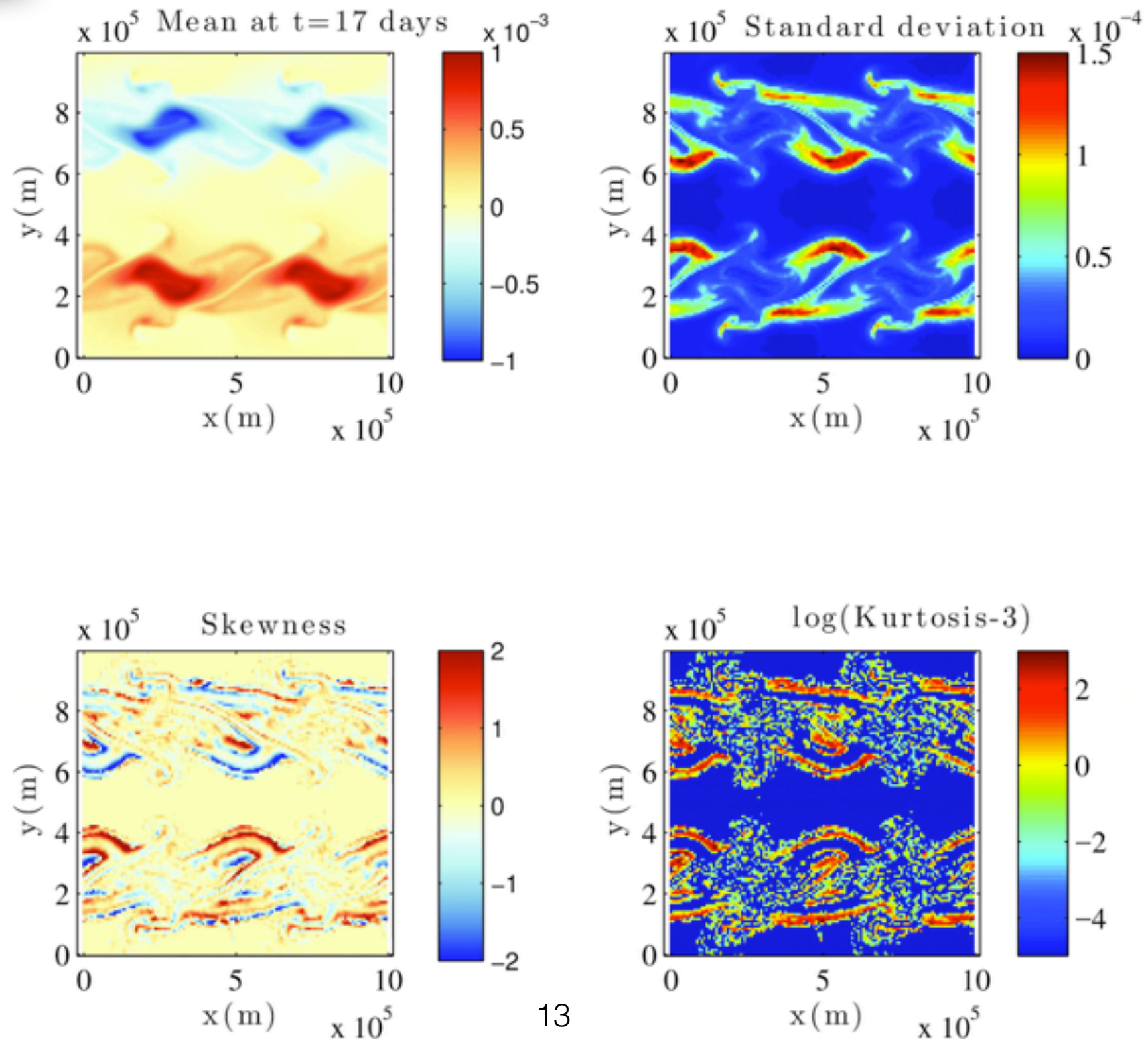


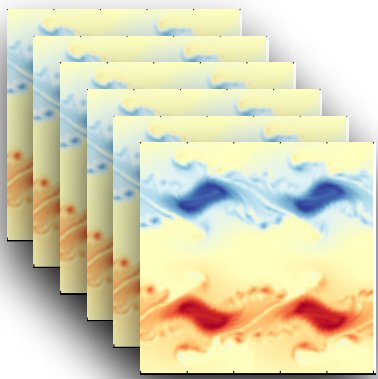
Ensemble



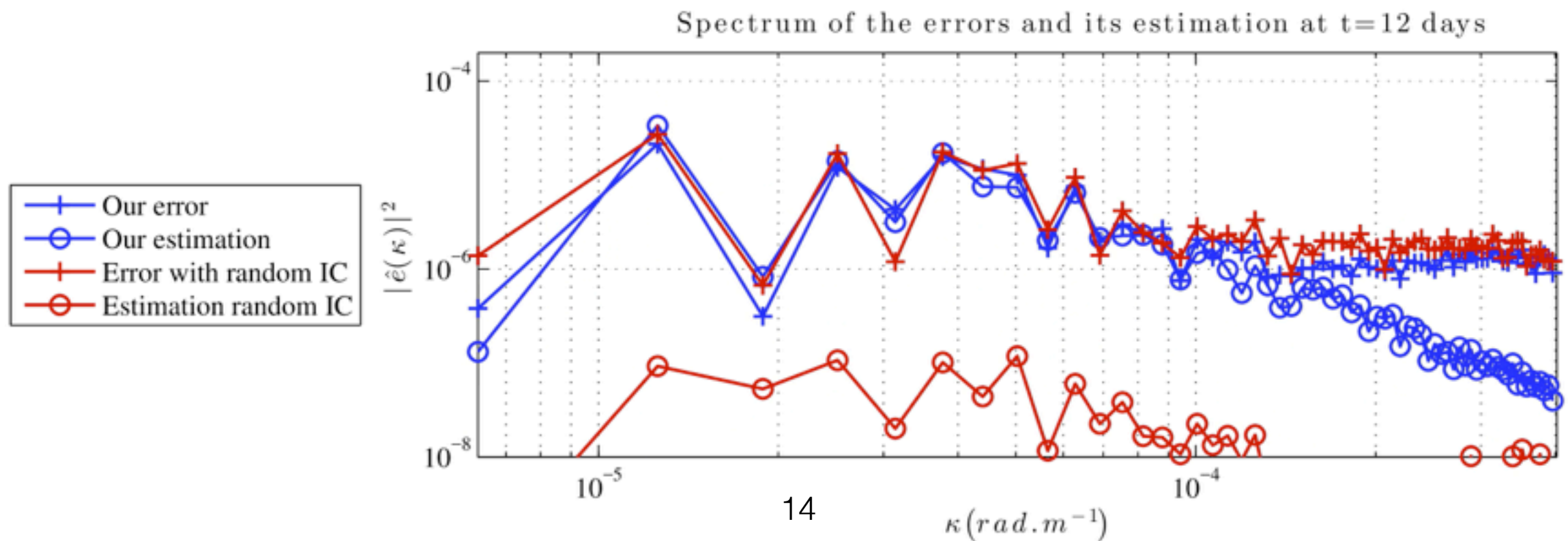
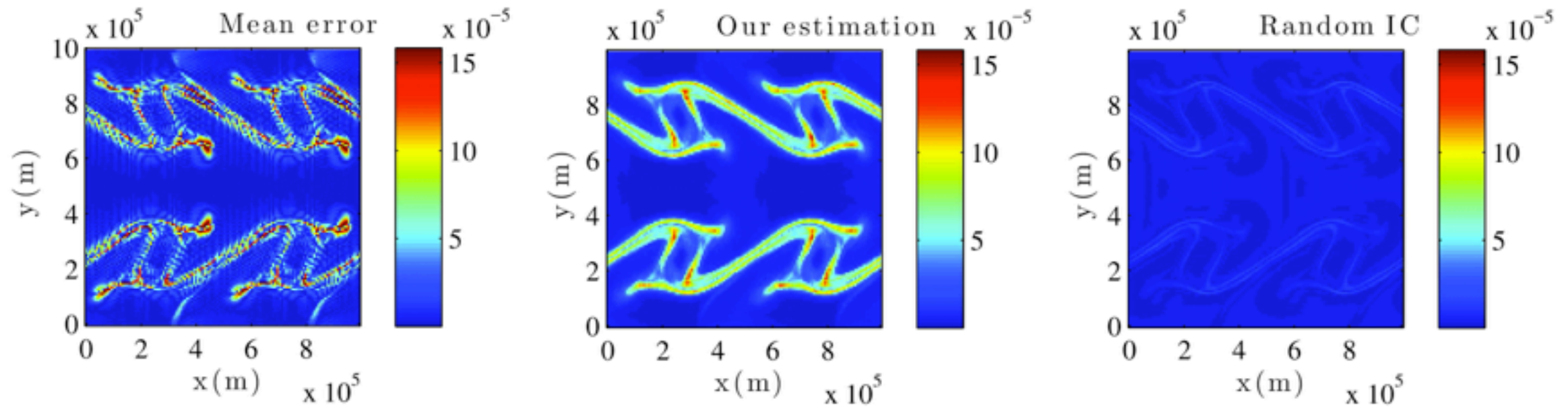


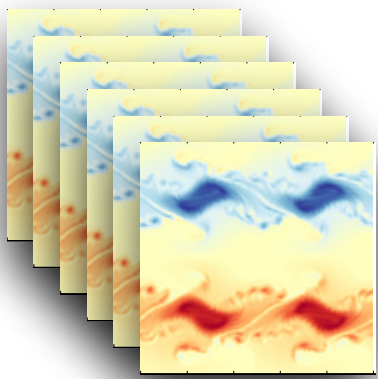
Ensemble



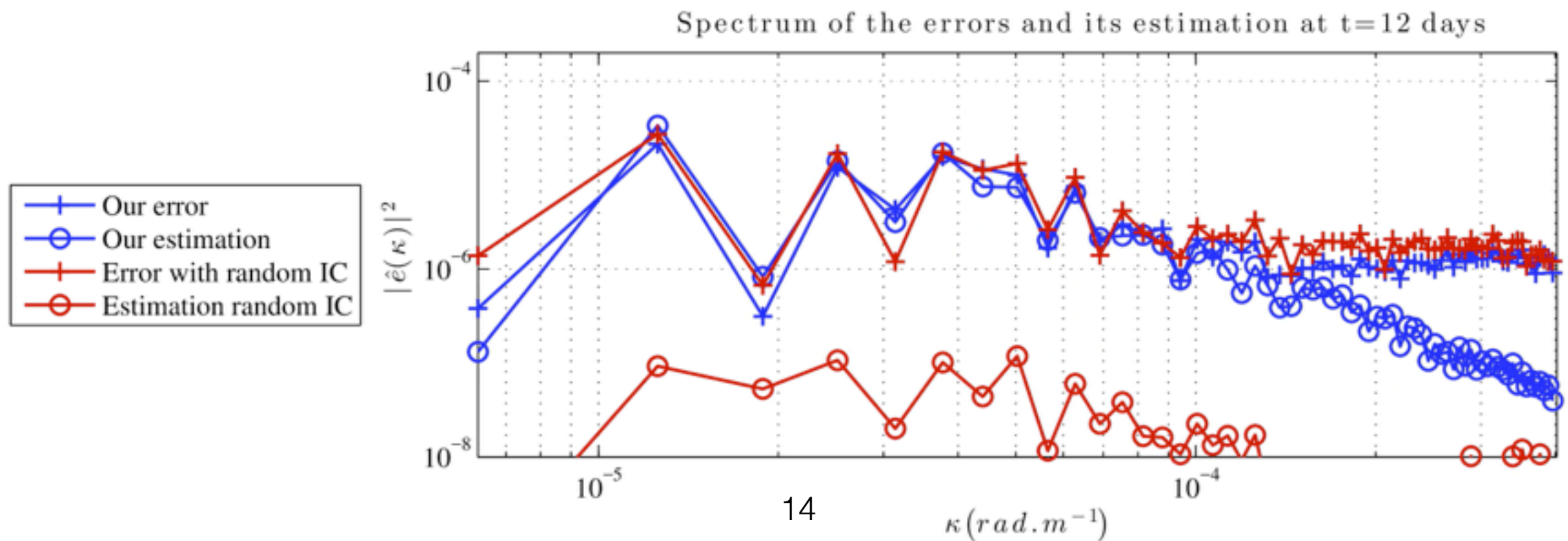
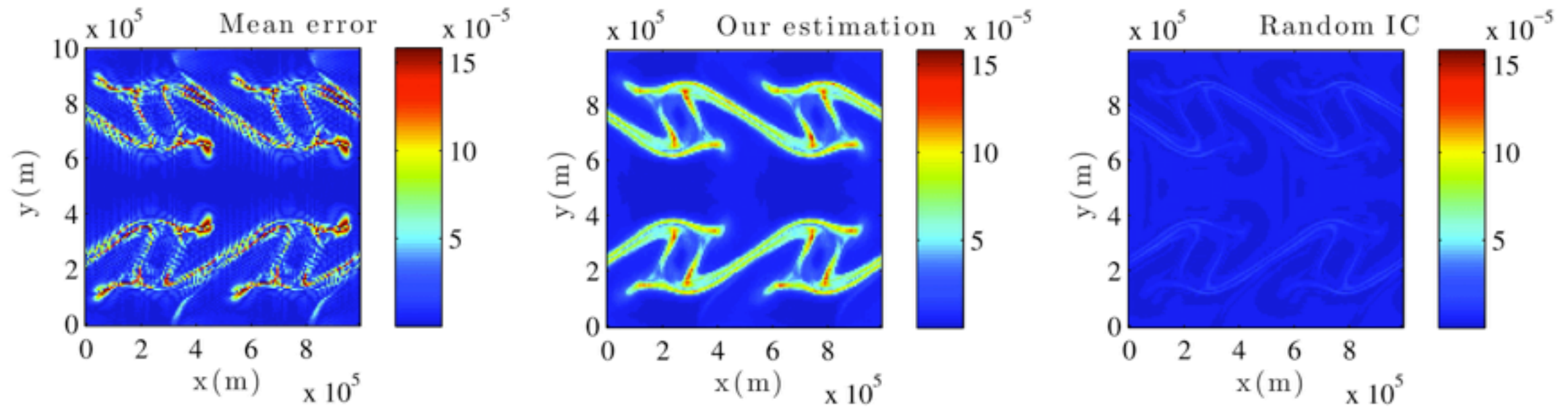


Ensemble





Ensemble



Random reduced order model from stochastic Navier-Stokes equation

Why a reduced order model?

- Very fast simulation of very complex system (e.g. for industrial application)

What is a reduced order model?

- Physical model (PDE) simplified using observations

PCA on data to reduce the state-space dimension

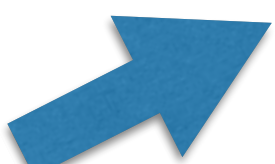
$$\boldsymbol{v}(\boldsymbol{x}, t) = \overline{\boldsymbol{v}}(\boldsymbol{x}) + \sum_{i=0}^n b_i(t) \boldsymbol{\phi}_i(\boldsymbol{x}) + \sum_{i=n+1}^N b_i(t) \boldsymbol{\phi}_i(\boldsymbol{x})$$

PCA on data to reduce the state-space dimension

Resolved modes

$$v(x, t) = \bar{v}(x) + \sum_{i=0}^n b_i(t) \phi_i(x) + \sum_{i=n+1}^N b_i(t) \phi_i(x)$$

w



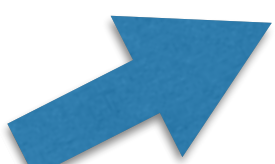
PCA on data to reduce the state-space dimension

Resolved modes

Unresolved modes

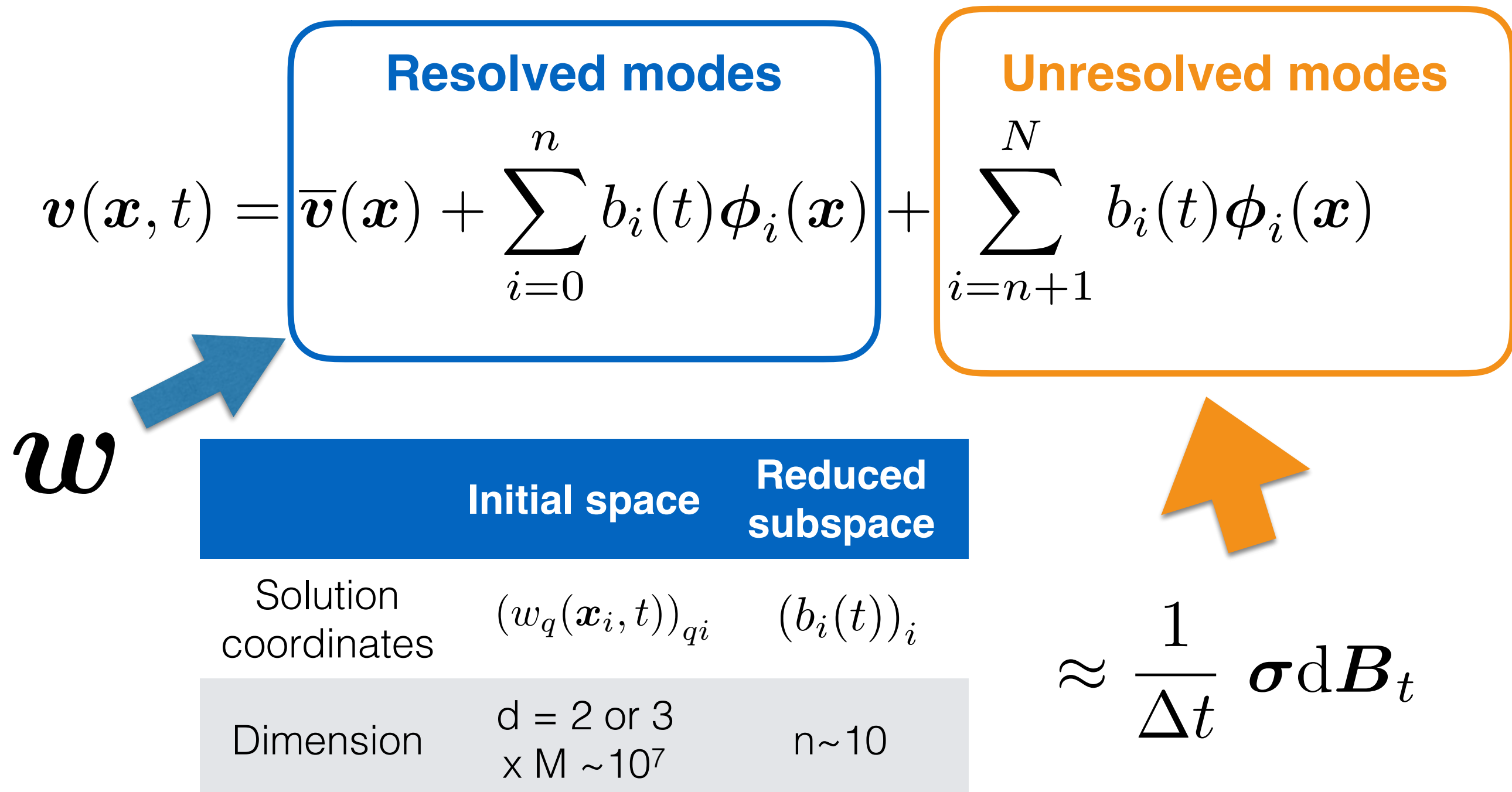
$$v(x, t) = \bar{v}(x) + \sum_{i=0}^n b_i(t) \phi_i(x) + \sum_{i=n+1}^N b_i(t) \phi_i(x)$$

w




$$\approx \frac{1}{\Delta t} \sigma dB_t$$

PCA on data to reduce the state-space dimension



Assumptions of finite variations for $\mathbf{w}$ and Galerkin projection gives **ODEs** for resolved modes:

$$\int_{\Omega} \phi_i \cdot (\text{stochastic Navier-Stokes})$$

$$db_i = F_i(\mathbf{b})dt$$

Assumptions of finite variations for $\mathbf{w}$ and Galerkin projection gives **ODEs** for resolved modes:

$$\int_{\Omega} \phi_i \cdot (\text{stochastic Navier-Stokes})$$

$$db_i = \boxed{F_i(\mathbf{b})} dt$$


2nd order polynomial:
coefficients given by physics,

$$(\phi_j)_j \text{ and } \mathbf{a}(\mathbf{x}, \mathbf{x}, t) = \frac{1}{t} < (\boldsymbol{\sigma}(\mathbf{x}, t) \mathbf{B})_{obs}, (\boldsymbol{\sigma}(\mathbf{x}, t) \mathbf{B})_{obs}^T$$

Assumptions of finite variations for $\mathbf{w}$ and Galerkin projection gives **ODEs** for resolved modes:

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$$db_i = \boxed{F_i(\mathbf{b})} dt$$



2nd order polynomial:
coefficients given by physics,

$$(\phi_j)_j \text{ and } a(\mathbf{x}, \mathbf{x}, t) = \frac{1}{t} < (\boldsymbol{\sigma}(\mathbf{x}, t) \mathbf{B})_{obs}, (\boldsymbol{\sigma}(\mathbf{x}, t) \mathbf{B})_{obs}^T >$$

Model and estimation
(inspired by Genon-Catalot, Laredo, Picard 1992)

$$a(\mathbf{x}, \mathbf{x}, t) = z_0(\mathbf{x}) + \sum_{i=0}^n b_i(t) z_i(\mathbf{x})$$

$$z_i(\mathbf{x}) = \frac{1}{t} \int_0^t b_i(t) d < \boldsymbol{\sigma}(\mathbf{x}, t) \mathbf{B}, \boldsymbol{\sigma}(\mathbf{x}, t) \mathbf{B} >$$



Galerkin projection gives **SDEs** for resolved modes:

$$\int_{\Omega} \phi_i \cdot (\text{stochastic Navier-Stokes})$$

$$db_i = F_i(\mathbf{b})dt + (\boldsymbol{\alpha}_{\bullet i} d\mathbf{B}_t)^T \mathbf{b} + (\boldsymbol{\theta}_i d\mathbf{B}_t)$$

Galerkin projection gives **SDEs** for resolved modes:

$$\int_{\Omega} \phi_i \cdot (\text{stochastic Navier-Stokes})$$

$$\begin{matrix} n \times M & M \times 1 & n \times 1 & 1 \times M & M \times 1 \end{matrix}$$

$$db_i = F_i(\mathbf{b})dt + (\alpha_{\bullet i} dB_t)^T \mathbf{b} + (\theta_i dB_t)$$

Galerkin projection gives **SDEs** for resolved modes:

$$\int_{\Omega} \phi_i \cdot (\text{stochastic Navier-Stokes})$$

$$db_i = F_i(\mathbf{b})dt + \overbrace{(\underbrace{\alpha_{\bullet i}}_{n \times M} \underbrace{dB_t}_{M \times 1})^T \underbrace{\mathbf{b}}_{n \times 1}}_{\text{multiplicative noise}} + \overbrace{(\underbrace{\theta_i}_{1 \times M} \underbrace{dB_t}_{M \times 1})}_{\text{additive noise}}$$

Galerkin projection gives **SDEs** for resolved modes:

$$\int_{\Omega} \phi_i \cdot (\text{stochastic Navier-Stokes})$$

$$db_i = \boxed{F_i(\mathbf{b})}dt + \underbrace{\left(\boxed{\alpha_{\bullet i}} \boxed{dB_t} \right)^T \boxed{\mathbf{b}}}_{\text{multiplicative noise}} + \underbrace{\left(\boxed{\theta_i} \boxed{dB_t} \right)}_{\text{additive noise}}$$

$n \times M$ $M \times 1$ $n \times 1$ $1 \times M$ $M \times 1$

2nd order polynomial:
coefficients given by physics,

$$(\phi_j)_j \quad \text{and} \quad \mathbf{a}(\mathbf{x}, \mathbf{x}) = \frac{1}{t} < (\boldsymbol{\sigma}(\mathbf{x})\mathbf{B})_{obs}, (\boldsymbol{\sigma}(\mathbf{x})\mathbf{B})_{obs}^T >_t$$

Galerkin projection gives **SDEs** for resolved modes:

$$\int_{\Omega} \phi_i \cdot (\text{stochastic Navier-Stokes})$$

$$db_i = \boxed{F_i(\mathbf{b})}dt + \overbrace{\left(\overbrace{\alpha_{\bullet i}}^{n \times M} \overbrace{d\mathbf{B}_t}^{M \times 1} \right)^T \overbrace{\mathbf{b}}^{n \times 1}}^{\text{multiplicative noise}} + \overbrace{\left(\overbrace{\theta_i}^{1 \times M} \overbrace{d\mathbf{B}_t}^{M \times 1} \right)}^{\text{additive noise}}$$

Correlations to estimate

2nd order polynomial:
coefficients given by physics,

$$(\phi_j)_j \quad \text{and} \quad \mathbf{a}(\mathbf{x}, \mathbf{x}) = \frac{1}{t} < (\boldsymbol{\sigma}(\mathbf{x})\mathbf{B})_{obs}, (\boldsymbol{\sigma}(\mathbf{x})\mathbf{B})_{obs}^T >_t$$

Estimate of the correlation $\alpha_{pi} \alpha_{qj}^T$

SDE

$$db_i = F_i(\mathbf{b})dt + (\alpha_{\bullet i} d\mathbf{B}_t)^T \mathbf{b} + (\theta_i d\mathbf{B}_t)$$

multiplicative
noise

Classical estimate

$$\alpha_{pi} \alpha_{qj}^T = \frac{1}{t} \langle \alpha_{pi} \mathbf{B}, \alpha_{qj} \mathbf{B} \rangle_t$$

Problem:

$$\underbrace{(\alpha_{pi} d\mathbf{B}_t)}_{1 \times M} \underbrace{)}_{M \times 1}_{obs} = \int_{\Omega} \underbrace{G_{pi}}_{\substack{\text{Known} \\ \text{but complex} \\ \text{linear operator}}} [\underbrace{(\sigma d\mathbf{B}_t)}_{\substack{d \times M \\ \text{(as a function} \\ \text{of space)}}}_{obs}] \rightarrow \text{Too complex to be computed for each time step}$$

Estimate of the correlation $\alpha_{pi} \alpha_{qj}^T$

SDE

$$db_i = F_i(\mathbf{b})dt + \underbrace{(\alpha_{\bullet i} d\mathbf{B}_t)^T \mathbf{b}}_{\text{multiplicative noise}} + (\theta_i d\mathbf{B}_t)$$

Classical estimate

$$\alpha_{pi} \alpha_{qj}^T = \frac{1}{t} \langle \alpha_{pi} \mathbf{B}, \alpha_{qj} \mathbf{B} \rangle_t$$

Problem:

- $$\underbrace{(\alpha_{pi} d\mathbf{B}_t)}_{1 \times M} \underbrace{d\mathbf{B}_t}_{M \times 1} = \int_{\Omega} \underbrace{G_{pi}}_{\substack{\text{Known} \\ \text{but complex} \\ \text{linear operator}}} [\underbrace{(\sigma d\mathbf{B}_t)}_{\substack{d \times M \\ \text{(as a function} \\ \text{of space)}}}]_{obs}$$

Too complex to be computed for each time step
- $$\underbrace{a(x, y)}_{d \times d \times M \times M} = \frac{1}{t} \langle (\sigma(x) \mathbf{B})_{obs}, (\sigma(y) \mathbf{B})_{obs}^T \rangle_t$$

Cannot be even memorized

Estimate of the correlation $\alpha_{pi}\alpha_{qj}^T$

SDE

$$db_i = F_i(\mathbf{b})dt + (\alpha_{\bullet i}d\mathbf{B}_t)^T \mathbf{b} + (\theta_i d\mathbf{B}_t)$$

multiplicative
noise

Solution:

$$\begin{aligned} \bullet \quad \frac{1}{t} \left\langle b_i, \int_0^t b_p (\alpha_{qj} d\mathbf{B}_t) \right\rangle_t &= \sum_k \alpha_{ki} \alpha_{qj}^T \underbrace{\frac{1}{t} \int_0^t b_k b_p}_{= \lambda_p \delta_{kp}} + \theta_i \alpha_{qj}^T \underbrace{\frac{1}{t} \int_0^t b_p}_{=0} \\ &= \lambda_p \alpha_{pi} \alpha_{qj}^T \end{aligned} \quad \text{(from the PCA)}$$

Estimate of the correlation $\alpha_{pi} \alpha_{qj}^T$

SDE

$$db_i = F_i(\mathbf{b})dt + (\alpha_{\bullet i} d\mathbf{B}_t)^T \mathbf{b} + (\theta_i d\mathbf{B}_t)$$

multiplicative
noise

Solution:

$$\begin{aligned} \bullet \quad \frac{1}{t} \left\langle b_i, \int_0^t b_p (\alpha_{qj} d\mathbf{B}_t) \right\rangle_t &= \sum_k \alpha_{ki} \alpha_{qj}^T \underbrace{\frac{1}{t} \int_0^t b_k b_p}_{=\lambda_p \delta_{kp}} + \theta_i \alpha_{qj}^T \underbrace{\frac{1}{t} \int_0^t b_p}_{=0} \\ &= \lambda_p \alpha_{pi} \alpha_{qj}^T \end{aligned} \quad \text{(from the PCA)}$$

$$\bullet \quad \frac{1}{t} \left\langle b_i, \int_0^t b_p (\alpha_{qj} d\mathbf{B}_{t'}) \right\rangle_t = \int_{\Omega} \mathbf{G}_{qj} \left[\frac{1}{t} \left\langle (b_i)_{obs}, \int_0^t (b_p \boldsymbol{\sigma} d\mathbf{B}_{t'})_{obs} \right\rangle_t \right]$$

By linearity of $\mathbf{G}_{qj}$

Estimate of the correlation $\alpha_{pi} \alpha_{qj}^T$

SDE

$$db_i = F_i(\mathbf{b})dt + (\alpha_{\bullet i} d\mathbf{B}_t)^T \mathbf{b} + (\theta_i d\mathbf{B}_t)$$

multiplicative
noise

Solution:

$$\begin{aligned} \bullet \quad \frac{1}{t} \left\langle b_i, \int_0^t b_p (\alpha_{qj} d\mathbf{B}_t) \right\rangle_t &= \sum_k \alpha_{ki} \alpha_{qj}^T \underbrace{\frac{1}{t} \int_0^t b_k b_p}_{=\lambda_p \delta_{kp}} + \theta_i \alpha_{qj}^T \underbrace{\frac{1}{t} \int_0^t b_p}_{=0} \\ &= \lambda_p \alpha_{pi} \alpha_{qj}^T \end{aligned}$$

(from the PCA)

$$\bullet \quad \frac{1}{t} \left\langle b_i, \int_0^t b_p (\alpha_{qj} d\mathbf{B}_{t'}) \right\rangle_t = \int_{\Omega} \mathbf{G}_{qj} \left[\frac{1}{t} \left\langle (b_i)_{obs}, \int_0^t (b_p \sigma d\mathbf{B}_{t'})_{obs} \right\rangle_t \right]$$

By linearity of $\mathbf{G}_{qj}$

Independent of time t !

$d \times M$

Conclusion

Conclusion

- Random transport applicable to any fluid dynamics models
- Better small scales
- Estimate position and amplitude of errors, extreme events, likely scenarios
- Possible applications for your previous and future estimation methods (e.g. MLE with $(\Theta(\mathbf{x}_i, t_j))_{ij}$)

Thank you for your attention

Code SQG MU:
link from Fluminance website - V. Resseguier