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Abstract: In this article, motivated by a recent work of R. Brockett Brockett (2013), we study a robust synchronization problem for multistable Brockett oscillators within an Input-to-State Stability (ISS) framework. Based on a recent generalization of the classical ISS theory to multistable systems and its application to the synchronization of multistable systems, a synchronization protocol is designed with respect to compact invariant sets of the unperturbed Brockett oscillator. The invariant sets are assumed to admit a decomposition without cycles (*i.e.* with neither homoclinic nor heteroclinic orbits). Contrarily to the local analysis of Brockett (2013), the conditions obtained in our work are global and applicable for family of non-identical oscillators. Numerical simulation examples illustrate our theoretical results.

Keywords: Input-to-State Stability, synchronization, multistability, Brockett oscillator

1. INTRODUCTION

During centuries, oscillators have attracted attention of researchers in various scientific disciplines. An oscillating behavior is pervasive in nature, technology and human society Blekhman (1988); Osipov et al. (2007); Pikovsky and Kurths (2003); Izhikevich (2007), representing repetitive or periodic processes and having several remarkable features. In this context, an important issue is the collective behavior of networked/coupled oscillators and their ability for *synchronization.* Synchronization has several potential application domains, for instance, smooth operations of micro-grids Efimov et al. (2016), cooperative multitasking and formation control Wei and Beard (2008), and so on. The core of synchronization is the collective objective of agents in a network to reach a consensus about certain variables of interest.

The existing literature on the synchronization problem is very vast and covers many areas. Interested readers may consult Gazi and Passino (2011); Shamma (2008); Lewis et al. (2014); Efimov (2015). In the context of the synchronization of oscillators, R. Brockett has recently introduced the following model Brockett (2013):

$$\ddot{x} + \varepsilon\dot{x}\left(\dot{x}^2 + x^2 - 1\right) + x = \varepsilon^2 u, x \in \mathbb{R}^n, \varepsilon > 0. \qquad (1)$$

In Brockett (2013), a centralized synchronization protocol has been proposed for the model (1), such that the conventional averaging theory does not predict the existence of a periodic (almost periodic) solution for small ϵ. However, qualitative synchronization together with small amplitude irregular motion can be seen through numerical studies. Next, for $|\varepsilon|$ sufficiently small, but non-zero, let us consider the set

$$S_\varepsilon = \left\{(x,\dot{x}) \mid \left(\dot{x}^2 + x^2 - 1\right) + 2\varepsilon^2 x\dot{x}\,\mathrm{sign}\left(\dot{x}^2 + x^2 - 1\right) = \varepsilon\right\},$$

which contains two smooth closed contours: Γ_ε^+ lies outside the unit circle in the $(x,\dot{x})$-space and Γ_ε^- lies inside the unit circle. Both curves approach the unit circle as ε goes to zero. Then the main result of Brockett (2013) is given below.

Theorem 1. Let $\Gamma_\varepsilon^\pm$ be as before. Then there exist $\varepsilon_0 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$, the solutions of (1) beginning in the annulus bounded by Γ_ε^+ and Γ_ε^- remain in this annulus for all time, provided that $|u| \leq \sqrt{x^2 + \dot{x}^2}$.

Theorem 1 provides a local synchronization result which depends on a small parameter $\varepsilon \neq 0$. Moreover, the result is applicable to the synchronization of identical oscillators only.

The goal of our work is to extend the result of Brockett (2013) and to develop a protocol of global synchronization in the network of (1), for the case of identical and non-identical models of the agents. It is assumed that the oscillators are connected through a N-cycle graph Pemmaraju and Skiena (2003). The proposed solution is based on the framework of ISS for multistable systems Angeli and Efimov (2015, 2013).

The ISS property provides a natural framework of stability analysis with respect to input perturbations (see Dashkovskiy et al. (2011) and references therein). The classical definition allows the stability properties with respect to arbitrary compact invariant sets (and not simply equilibria) to be formulated and characterized. Nevertheless, the implicit requirement is that these sets should be simultaneously Lyapunov stable and globally attractive, which makes the basic theory not applicable for a global analysis of many dynamical behaviors of

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interest, having multistability Angeli et al. (2004); Angeli and Sontag (2004); Gelig et al. (1978) or periodic oscillations Stan and Sepulchre (2007), just to name a few, and only local analysis remains possible Chaves et al. (2008). Some attempts were made to overcome such limitations by introducing the notions of almost global stability Rantzer (2001) and almost input-to-state stability Angeli (2004), *etc.*

Recently, the authors in Angeli and Efimov (2015, 2013) have found that a natural way of developing ISS theory for systems with multiple invariant sets consists in relaxing the Lyapunov stability requirement Efimov (2012) (rather than the global nature of the attractivity property). Using this relatively mild condition, the ISS theory has been generalized in Angeli and Efimov (2015, 2013), as well as the related literature on time invariant autonomous dynamical systems on compact spaces Nitecki and Shub (1975) for multistable systems. Multistability accounts for the possible coexistence of various oscillatory regimes or equilibria in the state space of the system for the same set of parameters. Any system that exhibits multistability is called a multistable system. For a multistable system, it is frequently very difficult to predict the asymptotic regime which it will attain asymptotically for the given set of initial conditions and inputs. Following the results of Angeli and Efimov (2015, 2013), the authors in Ahmed et al. (2015) have provided conditions for the robust synchronization of multistable systems in the presence of external inputs.

In our current work, the results presented in Angeli and Efimov (2013) and Ahmed et al. (2015) are applied to provide sufficient conditions for the existence of robust synchronization for identical/non-identical Brockett oscillators in the presence of external inputs. In opposite to the local results of Brockett (2013), the conditions obtained in this work are global.

The rest of this paper is organized as follows. Section 2 introduces some preliminaries about decomposable sets, notions of robustness and the conditions of robust synchronization of multistable systems. More details about Brockett oscillators and the synchronization of a family of oscillators can be found in Section 3 and 4, respectively. In Section 5, a numerical simulation example is given to illustrate these results. Concluding remarks in Section 6 close this article.

2. PRELIMINARIES

2.1 Preliminaries on input-to-stability of multistable systems

This section has been taken from Ahmed et al. (2015); Angeli and Efimov (2015). Let M be an n-dimensional $\mathcal{C}^2$ connected and orientable Riemannian manifold without a boundary and $x \in M$. Let $f : M \times \mathbb{R}^m \to T_xM$ be a map of class $\mathcal{C}^1$. Throughout this work, we assume that all manifolds are embedded in a Euclidean space of dimension n, so they contain 0. Consider a nonlinear system of the following form:

$$\dot{x}(t) = f\left(x(t), d(t)\right), \tag{2}$$

where the state $x(t) \in M$ and $d(t) \in \mathbb{R}^m$ (the input $d(\cdot)$ is a locally essentially bounded and measurable signal) for $t \geq 0$. We denote by $X\left(t, x; d(\cdot)\right)$ the uniquely defined solution of (2) at time t satisfying $X(0, x; d(\cdot)) = x$. Together with (2), we will analyze its unperturbed version:

$$\dot{x}(t) = f\left(x(t), 0\right). \tag{3}$$

A set $S \subset M$ is invariant for the unperturbed system (3) if $X\left(t, x; 0\right) \in S$, for all $t \in \mathbb{R}$ and for all $x \in S$. For a set $S \subset M$, define the distance to S from a point $x \in M$ by $|x|_S = \inf_{a\in S}\delta(x, a)$ where the $\delta(x_1, x_2)$ denotes the Riemannian distance between x_1 and x_2 in M. We have $|x| = |x|_{\{0\}}$ for $x \in M$, the usual Euclidean norm of a vector $x \in \mathbb{R}^n$. For a signal $d : \mathbb{R} \to \mathbb{R}^m$, the essential supremum norm is defined as $\|d\|_\infty = \operatorname{ess\,sup}_{t\geq 0} |d(t)|$.

A function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ is said to belong to class $\mathcal{K}$, *i.e.* $\alpha \in \mathcal{K}$, if it is continuous, strictly increasing and $\alpha(0) = 0$. Furthermore, $\alpha \in \mathcal{K}_\infty$ if $\alpha \in \mathcal{K}$ and it is unbounded em i.e. $\lim_{s\to\infty} \alpha(s) = \infty$. For any $x \in M$, the $\alpha-$ and $\omega-$ limit sets for (3) can be defined as follows:

$$\alpha(x) := \left\{ y \in M \mid y = \lim_{n\to-\infty} X(x, t_n) \text{ with } t_n \searrow -\infty \right\},$$
$$\omega(x) := \left\{ y \in M \mid y = \lim_{n\to\infty} X(x, t_n) \text{ with } t_n \nearrow \infty \right\}.$$

2.2 Decomposable sets

Let $\Lambda \subset M$ be a compact invariant set for (3).

Definition 1. Nitecki and Shub (1975) A decomposition of Λ is a finite and disjoint family of compact invariant sets $\Lambda_1, \ldots, \Lambda_k$ such that $\Lambda = \bigcup_{i=1}^{k} \Lambda_i$.

For an invariant set Λ, its attracting and repulsing subsets are defined as follows:

$$\mathfrak{A}(\Lambda) = \{x \in M \mid |X(t, x, 0)|_\Lambda \to 0 \text{ as } t \to +\infty\},$$
$$\mathfrak{R}(\Lambda) = \{x \in M \mid |X(t, x, 0)|_\Lambda \to 0 \text{ as } t \to -\infty\}.$$

Define a relation on invariant sets in M: for $\mathcal{W} \subset M$ and $\mathcal{D} \subset M$, we write $\mathcal{W} \prec \mathcal{D}$ if $\mathfrak{A}(\mathcal{W}) \cap \mathfrak{R}(\mathcal{D}) \neq \emptyset$.

Definition 2. Nitecki and Shub (1975) Let $\Lambda_1, \ldots, \Lambda_k$ be a decomposition of Λ, then

(1) An r-cycle ($r \geq 2$) is an ordered r-tuple of distinct indices $i_1, \ldots, i_r$ such that $\Lambda_{i_1} \prec \ldots \prec \Lambda_{i_r} \prec \Lambda_{i_1}$.
(2) A 1-cycle is an index i such that $(\mathfrak{R}(\Lambda_i) \cap \mathfrak{A}(\Lambda_i)) \setminus \Lambda_i \neq \emptyset$.
(3) A filtration ordering is a numbering of the Λ_i so that $\Lambda_i \prec \Lambda_j \Rightarrow i \leq j$.

As we can conclude from Definition 2, the existence of an r-cycle with $r \geq 2$ is equivalent to the existence of a heteroclinic cycle for (3) Guckenheimer and Holmes (1988). Moreover, the existence of a 1-cycle implies the existence of a homoclinic cycle for (3) Guckenheimer and Holmes (1988).

Definition 3. Let $\mathcal{W} \subset M$ be a compact set containing all $\alpha-$ and $\omega-$limit sets of (3). We say that $\mathcal{W}$ is decomposable if it admits a finite decomposition without cycles, $\mathcal{W} = \bigcup_{i=1}^{k} \mathcal{W}_i$, for some non-empty disjoint compact sets $\mathcal{W}_i$, forming a filtration ordering of $\mathcal{W}$.

This definition of the compact set $\mathcal{W}$ will be used all through the article.

2.3 Robustness notions

The following robustness notions for systems in (2) have been introduced in Angeli and Efimov (2013).

Definition 4. We say that the system (2) has the practical asymptotic gain (pAG) property if there exist $\eta \in \mathcal{K}_\infty$ and $q \in \mathbb{R}$, $q \geq 0$, such that for all $x \in M$ and all measurable essentially bounded inputs $d(\cdot)$, the solutions are defined for all $t \geq 0$ and

$$\limsup_{t \to +\infty} |X(t,x;d)|_{\mathcal{W}} \leq \eta\left(\|d\|_\infty\right) + q. \qquad (4)$$

If $q = 0$, then we say that the asymptotic gain (AG) property holds.

Definition 5. We say that the system (2) has the limit property (LIM) with respect to $\mathcal{W}$ if there exists $\mu \in \mathcal{K}_\infty$ such that for all $x \in M$ and all measurable essentially bounded inputs $d(\cdot)$, the solutions are defined for all $t \geq 0$ and the following holds:

$$\inf_{t \geq 0} |X(t,x;d)|_{\mathcal{W}} \leq \mu\left(\|d\|_\infty\right).$$

Definition 6. We say that the system (2) has the practical global stability (pGS) property with respect to $\mathcal{W}$ if there exist $\beta \in \mathcal{K}_\infty$ and $q \geq 0$ such that for all $x \in M$ and all measurable essentially bounded inputs $d(\cdot)$, the following holds for all $t \geq 0$:

$$|X(t,x;d)|_{\mathcal{W}} \leq q + \beta\left(\max\{|x|_{\mathcal{W}}, \|d\|_\infty\}\right).$$

To characterize (4) in terms of Lyapunov functions, it has been shown in Angeli and Efimov (2013) that the following notion is suitable:

Definition 7. We say that a $\mathcal{C}^1$ function $V : M \to \mathbb{R}$ is a practical ISS-Lyapunov function for (2) if there exists $\mathcal{K}_\infty$ functions $\alpha_1, \alpha_2, \alpha$ and γ, and scalars $q, c \geq 0$ such that

$$\alpha_1(|x|_{\mathcal{W}}) \leq V(x) \leq \alpha_2(|x|_{\mathcal{W}} + c),$$

the function V is constant on each $\mathcal{W}_i$ and the dissipation inequality below holds:

$$DV(x)f(x,d) \leq -\alpha(|x|_{\mathcal{W}}) + \gamma(|d|) + q.$$

If this latter holds for $q = 0$, then V is said to be an ISS-Lyapunov function.

Notice that the existence of α_2 and c follows (without any additional assumptions) by standard continuity arguments.

The main result of Angeli and Efimov (2013) connecting these robust stability properties is stated below:

Theorem 2. Consider a nonlinear system as in (2) and let a compact invariant set $\mathcal{W}$ containing all $\alpha-$ and $\omega-$limit sets of (3) be decomposable (in the sense of Definition 3). Then the following are equivalent:

(1) The system admits an ISS Lyapunov function;
(2) The system enjoys the AG property;
(3) The system admits a practical ISS Lyapunov function;
(4) The system enjoys the pAG property;
(5) The system enjoys the LIM property and the pGS.

A system in (2) that satisfies this list of equivalent properties is called ISS with respect to the set $\mathcal{W}$ Angeli and Efimov (2013).

2.4 Robust synchronization of multistable systems

This section summarizes the result on robust synchronization of multistable systems obtained in Ahmed et al. (2015). The following family of nonlinear systems is considered in this section:

$$\dot{x}_i(t) = f_i\left(x_i(t), u_i(t), d_i(t)\right),\ i = 1, \ldots, N,\ N > 1, \qquad (5)$$

where the state $x_i(t) \in M_i$, with M_i an n_i-dimensional $\mathcal{C}^2$ connected and orientable Riemannian manifold without a boundary, the control $u_i(t) \in \mathbb{R}^{m_i}$ and the external disturbance $d_i(t) \in \mathbb{R}^{p_i}$ ($u_i(\cdot)$ and $d_i(\cdot)$ are locally essentially bounded and measurable signals) for $t \geq 0$ and the map $f_i : M_i \times \mathbb{R}^{m_i} \times \mathbb{R}^{p_i} \to T_{x_i}M_i$ is $\mathcal{C}^1$, $f_i(0,0,0) = 0$. Denote the common state vector of (5) as $x = [x_1^T, \ldots, x_N^T]^T \in M = \prod_{i=1}^N M_i$, so M is the corresponding Riemannian manifold of dimension $n = \sum_{i=1}^N n_i$ where the family (5) evolves and $d = [d_1^T, \ldots, d_N^T]^T \in \mathbb{R}^p$ with $p = \sum_{i=1}^N p_i$ is the total exogenous input.

Assumption 1. For all $i = 1, \ldots, N$, each system in (5) has a compact invariant set $\mathcal{W}_i$ containing all $\alpha-$ and $\omega-$limit sets of $\dot{x}_i(t) = f_i\left(x_i(t), 0, 0\right)$, $\mathcal{W}_i$ is decomposable in the sense of Definition 3, and the system enjoys the AG property with respect to inputs u_i and d_i as in Definition 4.

This assumption implies that family (5) is composed of robustly stable nonlinear systems. In this case, let us find a condition under which the existence of a global synchronization/consensus protocol for $d = 0$ implies robust synchronization in (5) for a bounded $d \neq 0$. Let a $\mathcal{C}^1$ function $\psi(x) : M \to \mathbb{R}^q$, $\psi(0) = 0$ be a synchronization measure for (5). We say that the family (5) is synchronized (or reached the consensus) if $\psi\left(x(t)\right) \equiv 0$ for all $t \geq 0$ on the solutions of the network under properly designed control actions

$$u_i(t) = \varphi_i\left(\psi\left(x(t)\right)\right) \qquad (6)$$

($\varphi_i : \mathbb{R}^q \to \mathbb{R}^{m_i}$ is a $\mathcal{C}^1$ function, $\varphi_i(0) = 0$) for $d(t) \equiv 0$, $t \geq 0$. In this case, the set $\mathcal{A} = \{x \in \mathcal{W} \mid \psi(x) = 0\}$ contains the synchronous solutions of the unperturbed family in (5) and the problem of synchronization of "natural" trajectories is considered since $\mathcal{A} \subset \mathcal{W}$. Due to the condition $\varphi_i(0) = 0$, the convergence of ψ (synchronization/consensus) implies that the solutions of the interconnection belong to $\mathcal{W}$.

Assumption 2. The set $\mathcal{A}$ is compact, it contains all $\alpha-$ and $\omega-$limit sets of (5), (6) for $d = 0$, and it is decomposable.

Therefore, it is assumed that the controls $\varphi_i(\psi)$ ensure the network global synchronization, while decomposability in general follows from Assumption 1. Through the setup as above, by selecting the shapes of φ_i, it is possible to guarantee robust synchronization of (5) for any measurable and essentially bounded input d. This fact can be summarized by the following two results.

In this paper proofs are omitted due to space limitation.

Proposition 3. Let Assumption 1 be satisfied for (5). Then there exist φ_i, $i = 1, \ldots, N$ in (6) such that the interconnection (5), (6) has pGS property with respect to the set $\mathcal{W}$.

Theorem 4. Let assumptions 1 and 2 be satisfied for (5), (6). Then there exist φ_i, $i = 1, \ldots, N$ in (6) such that the interconnection (5), (6) has AG property with respect to $\mathcal{A}$.

3. THE BROCKETT OSCILLATOR

Let us consider the Brockett oscillator Brockett (2013):

$$\ddot{\xi} + b\dot{\xi}\left(\dot{\xi}^2 + \xi^2 - 1\right) + \xi = au, \qquad (7)$$

where $\xi \in \mathbb{R}$, $\dot{\xi} \in \mathbb{R}$ are the states variables, $a, b > 0$ are parameters and u is the control input. By considering $x_1 = \xi$, $\dot{x}_1 = x_2 = \dot{\xi}$, $x = [x_1, x_2]^T$ and $|x| = \sqrt{x_1^2 + x_2^2}$ equation (7) can be written in the state-space form as:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 + au - bx_2\left(|x|^2 - 1\right), \end{aligned} \qquad (8)$$

where the states of the system (8), *i.e.* x, evolve in the manifold $M = \mathbb{R}^2$. By analyzing equation (8) it can be seen that the unperturbed system admits two invariant sets: namely, the origin $\mathcal{W}_1 = \{0\}$ and the limit cycle $\mathcal{W}_2 = \Gamma = \{x \in M : |x|^2 = 1\}$. So, the invariant set for the trajectories of (8) can be defined as:

$$\mathcal{W} := \mathcal{W}_1 \cup \mathcal{W}_2 = \{0\} \cup \Gamma. \tag{9}$$

In order to verify the decomposability of the invariant set $\mathcal{W}$, we need to know the nature of the equilibrium $\mathcal{W}_1$ and the limit cycle $\mathcal{W}_2 = \Gamma$. This information can be obtained by analyzing the Lyapunov stability of the unperturbed system (8).

3.1 Stability of the autonomous Brockett oscillator

Since, $\mathcal{W}$ is invariant for the trajectories of (9), then the following proposition provides the stability of the unforced Brockett oscillator w.r.t. $\mathcal{W}$.

Proposition 5. For the unperturbed Brockett oscillator defined in (8) with $u = 0$, the limit cycle Γ is almost globally asymptotically stable and the origin is unstable.

As a result, it can be concluded that the origin is unstable.

3.2 Stability of the non-autonomous Brockett oscillator

In the previous section, we have proved the stability of the unperturbed system. In this section, we will analyze the stability of the Brockett oscillator in the presence of input. As it was shown in the previous section, the limit cycle Γ is almost globally asymptotically stable. So, any solution of the unperturbed Brockett oscillator converges to Γ, except for the one initiated at 0, which is unstable. So, it can be conclude that $\mathcal{W}$ contains all $\alpha-$ and $\omega-$limit sets of the unperturbed systems of (8) and it admits a decomposition without cycles. Consequently the result of Angeli and Efimov (2015, 2013) can be applied for our case to show the robust stability of the Brockett oscillator in (8) with respect to $\mathcal{W}$:

Proposition 6. The system (8) is ISS with respect to the set $\mathcal{W}$.

Remark 1. It is straightforward to check that there exists a function $\alpha \in \mathcal{K}_\infty$ such that for all $x \in M$ and $u = 0$ we have $\dot{V} \leq -\alpha(|x|_{\mathcal{W}})$. Thus V is a global Lyapunov function establishing multistability of (8) with respect to $\mathcal{W}$ for $u = 0$.

4. SYNCHRONIZATION OF BROCKETT OSCILLATORS

The following family of Brockett oscillators is considered in this section for some $N > 1$:

$$\begin{aligned} \dot{x}_{1i} &= x_{2i}, \\ \dot{x}_{2i} &= a_i u_i - x_{1i} - b_i x_{2i}\left(|x_i|^2 - 1\right), i = \overline{1, N}, \end{aligned} \tag{10}$$

where $a_i, b_i > 0$ are the parameters of an individual oscillator, the state $x_i = [x_{1i}\ x_{2i}]^T \in M_i = \mathbb{R}^2$, the control $u_i \in \mathbb{R}$ ($u_i : \mathbb{R}_+ \to \mathbb{R}$ is locally essentially bounded and measurable signal). Denote the common state vector of (5) as $x = [x_1^T, \ldots, x_N^T]^T \in M = \prod_{i=1}^N M_i$, so M is the corresponding Riemannian manifold of dimension $n = 2N$ where the family (5) behaves and $u = [u_1, \ldots, u_N]^T \in \mathbb{R}^N$ is the common input. Through propositions 5 and 6, it has been shown that each member of family (10) is robustly stable with respect to the set $\mathcal{W}_i = \{x_i \in M_i : |x_i|^2 = 1\} \cup \{0\}$. Consequently, the family (10) is a robustly stable nonlinear system. As a result, Assumption 1 is satisfied for the case of the family of Brockett oscillators (10). The synchronization problem is then the problem of finding a protocol u that makes the family (10) synchronized. There are several works devoted to synchronization and design of consensus protocols for such a family or oscillatory network Li et al. (2010); Pogromsky (2008).

Let a $\mathcal{C}^1$ function $\psi : M \to \mathbb{R}^q$, $\psi(0) = 0$ be a synchronization measure for (10). We say that the family (10) is synchronized (or reached the consensus) if $\psi(x(t)) \equiv 0$ for all $t \geq 0$ on the solutions of the network under properly designed control actions

$$u_i(t) = \varphi_i\left[\psi(x(t))\right], \tag{11}$$

where $\varphi_i : \mathbb{R}^q \to \mathbb{R}$ is a $\mathcal{C}^1$ function, $\varphi_i(0) = 0$. Due to the condition $\varphi_i(0) = 0$, the convergence of ψ (synchronization/consensus) implies that the solutions of the interconnection belong to $\mathcal{W} = \prod_{i=1}^N \mathcal{W}_i$. In this case the set $\mathcal{A} = \{x \in \mathcal{W} \mid \psi(x) = 0\}$ contains the synchronous solutions of the family in (11) and the problem of synchronization of "natural" trajectories is considered since $\mathcal{A} \subset \mathcal{W}$.

In this work we begin with the following synchronization measure:

$$\begin{aligned} \psi &= [\psi_1, \ldots, \psi_N]^T, \\ \psi_i &= \begin{cases} (x_{2(i+1)} - x_{2i}), & i = \overline{1, N-1} \\ x_{21} - x_{2N}, & i = N \end{cases}. \end{aligned}$$

From a graph theory point of view, the oscillators are connected through a N-cycle graph Pemmaraju and Skiena (2003) (each oscillator needs only the information of its next neighbor), *i.e.*

$$\psi = M \begin{bmatrix} x_{21} \\ \vdots \\ x_{2N} \end{bmatrix}, \ M = \begin{bmatrix} -1 & 1 & 0 & \cdots & 0 \\ 0 & -1 & 1 & \ddots & \\ & \ddots & \ddots & \ddots & \\ & & & & 1 \\ 1 & & & & -1 \end{bmatrix},$$

and any other connection type can be studied similarly. Moreover, the interconnection matrix M has Metzler form since all off-diagonal elements are positive. Next, let us define the synchronization error among the various states of the oscillators as follows:

$$\begin{aligned} e_1 &= x_{11} - x_{12} \\ \dot{e}_1 &= x_{21} - x_{22} = e_2 \\ e_3 &= x_{12} - x_{13} \\ \dot{e}_3 &= x_{22} - x_{23} = e_4 \\ &\vdots \\ e_{2N-1} &= x_{1N} - x_{11} \\ \dot{e}_{2N-1} &= x_{2N} - x_{21} = e_{2N} \end{aligned}$$

when

$$\begin{aligned} \psi_i &= -e_{2i} \quad i = \overline{1, N}, \\ \psi_N &= -\sum_{i=1}^{N-1} e_{2i} \end{aligned}$$

and the quantity $e = 0$ implies that $\psi = 0$ (the synchronization state is reached). For $y_i = |x_i|^2 - 1$ the error dynamics can be written in the form:

$$\begin{aligned}\dot{e}_{2i-1} &= e_{2i},\ i = \overline{1, N-1}, \\ \dot{e}_{2i} &= -e_{2i-1} + (a_i u_i - a_{i+1} u_{i+1}) - b_i x_{2i} y_i \\ &\quad + b_{i+1} x_{2(i+1)} y_{i+1},\ i = \overline{1, N-1}.\end{aligned} \tag{12}$$

Since $e_{2N-j} = \sum_{i=1}^{N-1} e_{2i-j}$ for $j = 0, 1$, then only $N - 1$ errors can be considered in (12).

Let $N = 2$, take

$$u = k\psi, \tag{13}$$

e.g. $\varphi(\psi) = k\psi$ in (11), and for the closed loop system consider the following Lyapunov function

$$V(x) = \sum_{i=1}^{N} \frac{b_i}{4a_i k} y_i^2 + \frac{1}{2} \sum_{i=1}^{2N-2} e_i^2. \tag{14}$$

Notice that $V(x) = 0$ for all $x \in \mathcal{A} \cap \prod_{i=1}^{N} \mathcal{W}_{2i}$ and positive otherwise. By taking the total derivative of $V(x)$ along the solutions of (10) and (12) with (13), we obtain the following:

$$\begin{aligned}\dot{V} &= \sum_{i=1}^{2} \frac{b_i}{2a_i k} y_i \dot{y}_i + \sum_{i=1}^{2} e_i \dot{e}_i \\ &= \sum_{i=1}^{2} \frac{b_i}{2a_i k} y_i \left(2a_i x_{2i} u_i - 2b_i x_{2i} y_i\right) + e_1 \dot{e}_1 + e_2 \dot{e}_2 \\ &= -\frac{b_1^2}{a_1 k} x_{21}^2 y_1^2 - \frac{b_2^2}{a_2 k} x_{22}^2 y_2^2 - 2b_1 x_{21} y_1 e_2 \\ &\quad + 2b_2 x_{22} y_2 e_2 - (a_1 k + a_2 k)\, e_2^2 \\ &= X^T Q X,\end{aligned}$$

where $X = [\, x_{21} y_1\ x_{22} y_2\ e_2 \,]^T$, $Q = \left[\begin{array}{c|c} A & B \\ \hline B^T & C \end{array}\right]$, $A = \operatorname{diag}\left(-\frac{b_1^2}{a_1 k}, -\frac{b_2^2}{a_2 k}\right)$, $B = [\, -b_1\ b_2 \,]^T$, $C = -k\,(a_1 + a_2)$. Next, the Schur complement of Q in C is:

$$\begin{aligned}Q/C &= A - BC^{-1}B^T \\ &= \begin{bmatrix} \dfrac{-b_1^2}{a_1 k} + \dfrac{b_1^2}{(a_1 + a_2)\, k} & -\dfrac{b_1 b_2}{k\,(a_1 + a_2)} \\ \star & \dfrac{-b_2^2}{a_2 k} + \dfrac{b_2^2}{(a_1 + a_2)\, k} \end{bmatrix}\end{aligned}$$

and this matrix has eigenvalues

$$\lambda_{Q/C} = \left\{ 0, -\frac{(a_2 b_1)^2 + (a_1 b_2)^2}{a_1 a_2\, (a_1 + a_2)\, k} \right\},$$

then, according to Schur complement lemma Boyd and Vandenberghe (2004), $Q \preceq 0$ for any $a_i, b_i > 0$. As a result, $\dot{V} \le 0$. Since the largest invariant set where $\dot{V} = 0$ belongs to the sub-manifold with $e_2 = 0$, then $\psi = 0$ there. On the basis of LaSalle's invariance principle LaSalle (1960), it can be concluded that the closed loop system (10) and (15) for $N = 2$, is globally asymptotically synchronized (globally asymptotically stable with respect to the set $\mathcal{A}$). The following result has been proven:

Theorem 7. Consider the family of Brockett oscillators (10) with $N = 2$ and (13) for $k > 0$. Then it is synchronized *i.e.* the system is globally asymptotically stable with respect to the set $\mathcal{A}$.

It has been observed in numerical experiments that for $N > 2$ and

$$u = k \begin{bmatrix} -1 & 1 & 0 & \cdots & 0 \\ 0 & -1 & 1 & \ddots & \\ & \ddots & \ddots & \ddots & \\ & & & & 1 \\ 1 & & & & -1 \end{bmatrix} \begin{bmatrix} x_{21} \\ \\ \vdots \\ \\ x_{2N} \end{bmatrix} \tag{15}$$

the synchronization persists, but the proof should be modified since (14) is not a Lyapunov function in such a case. To overcome this problem, based on the idea presented in Das and Lewis (2010), the following modification to the control law (15) can be proposed

$$u_i = k\psi_i + b_i x_{2i} y_i. \tag{16}$$

Since the modified control law (16) compensates the nonlinear part of (10), as a result the closed loop system becomes linear. In this case, it is trivial to show that the closed loop system (10) and (16) is globally asymptotically synchronized.

Theorem 7 guarantees global asymptotic stability of the synchronized behavior, but not the robustness. Note that the controls (15) and (16) are not bounded, then it is impossible to apply the result of Proposition 6 to prove robust stability of $\mathcal{W}$. Moreover, in many application areas, the control is bounded due to actuator limitations Hu and Lin (2001). With such a motivation, let us consider a bounded version of (11), then from propositions 3 and 6 the pGS property with respect to $\mathcal{W}$ immediately follows, and the next result summarizes the robust synchronization property:

Corollary 8. Let Assumption 2 be satisfied for given $\varphi_i, i = \overline{1, N}$, then the interconnection (10), (11) is ISS with respect to the set $\mathcal{A}$.

The proof follows from the result of Theorem 4 since Assumption 1 is satisfied due to Proposition 6.

5. EXAMPLES AND SIMULATIONS

To illustrate the theoretical results, let us consider 2 non-identical Brockett oscillators (10) with parameters as, $k = 5, a_1 = 0.5, b_1 = 3, a_2 = b_2 = 1.5$. For unbounded case, lets consider the following control:

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = k \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_{21} \\ x_{22} \end{bmatrix} \tag{17}$$

Control (17) has the form (15). The interconnection matrix has Metzler form since all the off-diagonal elements are positive. In this case, system (10) with control (15) is synchronized as shown in Theorem 7. Next, let us consider control (11):

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \operatorname{sat}\left(k \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_{21} \\ x_{22} \end{bmatrix}\right) \tag{18}$$

where the saturation function is defined as:

$$u_i = \operatorname{sat}(z) \iff u_i = \operatorname{sign}(z) \min(|z|, 1)$$

Lets consider, $\alpha = 0.05$. With this saturated control, the system (10) is robustly synchronized as shown in Corollary 8.

The simulation result with both types of control can be seen in Fig. 1. From the simulation it can be clearly concluded that in the case of saturated control, we have rapid convergence to synchronized state on the unit circle. In the unbounded

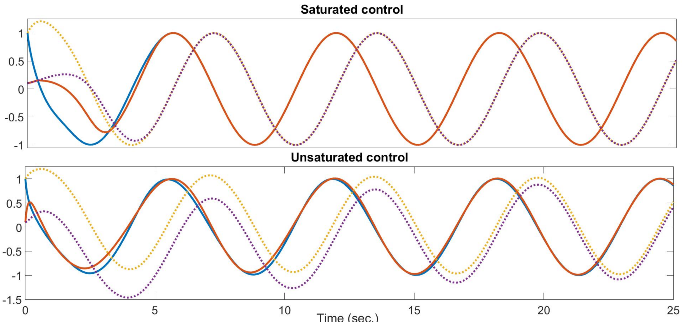

Figure 1. Synchronization result with saturated and unsaturated control for (10). Solid line - x_2, dashed line - x_1

case, synchronized state is achieved asymptotically. Simulation result shows the robust synchronization of the Brockett oscillator.

6. CONCLUSIONS

Robust synchronization of non-identical Brockett oscillators was studied in this paper. Sufficient conditions were derived for that purpose based on an extension of the ISS framework to systems evolving on a (non-compact) manifold and with multiple invariant sets. Global asymptotic stability and ISS stability analysis were done for individual oscillator followed by global stability analysis of the closed loop systems with respect to a decomposable invariant set $\mathcal{W}$. Numerical simulations demonstrated the effectiveness of our method to network of nonidentical Brockett oscillators.

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