Well-posedness for mean-field evolutions arising in superconductivity

Mitia Duerinckx

Well-posedness for mean-field evolutions arising in superconductivity

Mitia Duerinckx

We establish the existence of a global solution to a family of equations, which are obtained in certain regimes in [19] as the mean-field evolution of the supercurrent density in a (2D section of a) type-II superconductor with pinning and with imposed electric current. We also consider general vortex-sheet initial data, and investigate the uniqueness and regularity properties of the solution.

1 Introduction

We study the well-posedness of the following two evolution models coming from the mean-field limit equations of Ginzburg-Landau vortices: first, for $\alpha \geq 0$, $\beta \in \mathbb{R}$, we consider the "incompressible" flow

$$\partial_t v = \nabla P - \alpha(\Psi + v)\operatorname{curl} v + \beta(\Psi + v)^{\perp} \operatorname{curl} v, \qquad \operatorname{div}(av) = 0, \qquad \text{in } \mathbb{R}^+ \times \mathbb{R}^2, \tag{1.1}$$

and second, for $\lambda \geq 0$, $\alpha > 0$, $\beta \in \mathbb{R}$, we consider the "compressible" flow

$$\partial_t v = \lambda \nabla(a^{-1} \operatorname{div}(av)) - \alpha(\Psi + v)\operatorname{curl} v + \beta(\Psi + v)^{\perp} \operatorname{curl} v, \qquad \text{in } \mathbb{R}^+ \times \mathbb{R}^2, \tag{1.2}$$

with $v : \mathbb{R}^+ \times \mathbb{R}^2 := [0, \infty) \times \mathbb{R}^2 \to \mathbb{R}^2$, where $\Psi : \mathbb{R}^2 \to \mathbb{R}^2$ is a given forcing vector field, and where $a := e^h$ is determined by a given "pinning potential" $h : \mathbb{R}^2 \to \mathbb{R}$. More precisely, we investigate existence, uniqueness and regularity, both locally and globally in time, for the associated Cauchy problems; we also consider vortex-sheet initial data, and we study the degenerate case $\lambda = 0$ as well. As shown in a companion paper [19] with Serfaty, these equations are obtained in certain regimes as the mean-field evolution of the supercurrent density in a (2D section of a) type-II superconductor described by the 2D Ginzburg-Landau equation with pinning and with imposed electric current — but without gauge and in whole space, for simplicity.

Brief discussion of the model

Superconductors are materials that in certain circumstances lose their resistivity, which allows permanent supercurrents to circulate without loss of energy. In the case of type-II superconductors, if the external magnetic field is not too strong, it is expelled from the material (Meissner effect), while, if it is much too strong, the material returns to a normal state. Between these two critical values of the external field, these materials are in a mixed state, allowing a partial penetration of the external field through "vortices", which are accurately described by the (mesoscopic) Ginzburg-Landau theory. Restricting ourselves to a 2D section of a superconducting material, it is standard to study for simplicity the 2D Ginzburg-Landau equation on the whole plane (to avoid boundary issues) and without gauge (although the gauge is expected to bring only minor difficulties). We refer e.g. to [41, 40] for further reference on these models, and to [35] for a mathematical introduction. In this framework, in the asymptotic regime of a large Ginzburg-Landau parameter (which is indeed typically the case in real-life superconductors), vortices are known to become point-like, and to interact with one another according to a Coulomb pair potential. In the mean-field limit of a large number of vortices, the evolution of the (macroscopic) suitably normalized mean-field density $\omega : \mathbb{R}^+ \times \mathbb{R}^2 \to \mathbb{R}$ of the vortex liquid was then naturally conjectured to satisfy the following Chapman-Rubinstein-Schatzman-E equation [20, 12]

$$\partial_t \omega = \operatorname{div}(|\omega| \nabla(-\triangle)^{-1}\omega), \qquad \text{in } \mathbb{R}^+ \times \mathbb{R}^2, \tag{1.3}$$

where $(-\triangle)^{-1}\omega$ is indeed the Coulomb potential generated by the vortices. Although the vortex density ω is a priori a signed measure, we restrict here (and throughout this paper) to positive measures, $|\omega| = \omega$, so that the above is replaced by

$$\partial_t \omega = \operatorname{div}(\omega \nabla(-\triangle)^{-1}\omega). \tag{1.4}$$

More precisely, the mean-field supercurrent density $v : \mathbb{R}^+ \times \mathbb{R}^2 \to \mathbb{R}^2$ (linked to the vortex density through the relation $\omega = \operatorname{curl} v$) was conjectured to satisfy

$$\partial_t v = \nabla P - v \operatorname{curl} v, \qquad \operatorname{div} v = 0.$$

(Taking the curl of this equation indeed formally yields (1.4), noting that the incompressibility constraint $\operatorname{div} v = 0$ allows to write $v = \nabla^\perp \triangle^{-1}\omega$.)

On the other hand, in the context of superfluidity, a conservative counterpart of the usual parabolic Ginzburg-Landau equation is used as a mesoscopic model: this counterpart is given by the Gross-Pitaevskii equation, which is a particular instance of a nonlinear Schrödinger equation. At the level of the mean-field evolution of the corresponding vortices, we then need to replace (1.3)–(1.4) by their conservative versions, thus replacing $\nabla(-\triangle)^{-1}\omega$ by $\nabla^\perp(-\triangle)^{-1}\omega$. As argued e.g. in [4], there is also physical interest in rather starting from the "mixed-flow" (or "complex") Ginzburg-Landau equation, which is a mix between the usual Ginzburg-Landau equation describing superconductivity ($\alpha = 1$, $\beta = 0$, below), and its conservative counterpart given by the Gross-Pitaevskii equation ($\alpha = 0$, $\beta = 1$, below). The above mean-field equation for the supercurrent density v is then replaced by the following, for $\alpha \geq 0$, $\beta \in \mathbb{R}$,

$$\partial_t v = \nabla P - \alpha v \operatorname{curl} v + \beta v^\perp \operatorname{curl} v, \qquad \operatorname{div} v = 0. \tag{1.5}$$

Note that in the conservative case $\alpha = 0$, this equation is equivalent to the 2D Euler equation, as becomes clear from the identity $v^\perp \operatorname{curl} v = (v \cdot \nabla)v - \frac{1}{2}\nabla|v|^2$.

The first rigorous deductions of these (macroscopic) mean-field limit models from the (mesoscopic) Ginzburg-Landau equation are due to [28, 24], and to [36] for much more general regimes. As discovered by Serfaty [36], in some regimes with $\alpha > 0$, this limiting equation (1.5) is no longer correct, and must be replaced by the following compressible flow

$$\partial_t v = \lambda \nabla(\operatorname{div} v) - \alpha v \operatorname{curl} v + \beta v^\perp \operatorname{curl} v, \tag{1.6}$$

for some $\lambda > 0$. There is some interest in the degenerate case $\lambda = 0$ as well, since it is formally expected (although not proven) to be the correct mean-field evolution in some other regimes.

When an electric current is applied to a type-II superconductor, it flows through the material, inducing a Lorentz-like force that makes the vortices move, dissipates energy, and disrupts the permanent supercurrents. As most technological applications of superconducting materials occur in the mixed state, it is crucial to design ways to reduce this energy dissipation, by preventing vortices from moving. For that purpose a common attempt consists in introducing in the material inhomogeneities (e.g. impurities, or dislocations), which are indeed meant to destroy superconductivity locally and therefore "pin down" the vortices. This is usually modeled by correcting the Ginzburg-Landau equations with a non-uniform equilibrium density $a : \mathbb{R}^2 \to [0, 1]$, which locally lowers the energy penalty associated with the vortices (see e.g. [11, 8] for further details). As formally predicted by Chapman and Richardson [11], and first completely proven by [25, 37] (see also [23, 27] for the conservative case), in the asymptotic regime of a large Ginzburg-Landau parameter, this non-uniform density a translates at the level of the vortices into an effective "pinning potential" $h = \log a$, indeed attracting the vortices to the minima of a. As shown in our companion paper [19], the mean-field equations (1.5)–(1.6) are then replaced by (1.1)–(1.2), where the forcing Ψ can be decomposed as $\Psi := F^\perp - \nabla^\perp h$, in terms of the pinning force $-\nabla h$, and of some vector field $F : \mathbb{R}^2 \to \mathbb{R}^2$ related to the imposed electric current (see also [39, 37]).

Relation to previous works

The simplified model (1.4) describes the mean-field limit of the gradient-flow evolution of any particle system with Coulomb interactions [18]. As such, it is related to nonlocal aggregation and swarming models, which have attracted a lot of mathematical interest in recent years (see e.g. [7, 10] and the references therein); they consist in replacing the Coulomb potential $(-\triangle)^{-1}$ by a convolution with a more general kernel corresponding to an attractive (rather than repulsive) nonlocal interaction. Equation (1.4) was first studied by Lin and Zhang [29], who established global existence for vortex-sheet initial data $\omega|_{t=0} \in \mathcal{P}(\mathbb{R}^2)$, and uniqueness in some Zygmund space. To prove global existence for such rough initial data, they proceed by regularization of the data, then passing to the limit in the equation using the compactness given by some very strong a priori estimates obtained by ODE type arguments. As our main source of inspiration, their approach is described in more detail in the sequel. When viewing (1.4) as a mean-field model for the motion of the Ginzburg-Landau vortices in a superconductor, there is also interest in changing sign solutions and the correct model is then rather (1.3), for which global existence and uniqueness have been investigated in [17, 32]. In [3, 2], using an energy approach where the equation is seen as a formal gradient flow in the Wasserstein space of probability measures (à la Otto [34]), made rigorous by the minimizing movement approach of Ambrosio, Gigli and Savaré [1], analogues of equations (1.3)–(1.4) were studied in a 2D bounded domain, taking into account the possibility of mass entering or exiting the domain. In the case of nonnegative vorticity $\omega \ge 0$, essentially the same existence and uniqueness results are established in that setting in [3] as for (1.4). In the case $\omega \ge 0$ on the whole plane, still a different approach was developed by Serfaty and Vázquez [38], where equation (1.4) is obtained as a limit of nonlocal diffusions, and where uniqueness is further established for bounded solutions using transport arguments à la Loeper [31]. Note that no uniqueness is expected to hold for general measure solutions of (1.4) (see [3, Section 8]). In the present paper, we focus on the case $\omega \ge 0$ on the whole plane $\mathbb{R}^2$.

The model (1.5) is a linear combination of the gradient-flow equation (1.4) (obtained for $\alpha = 1$, $\beta = 0$), and of its conservative counterpart that is nothing but the 2D Euler equation (obtained for $\alpha = 0$, $\beta = 1$). The theory for the 2D Euler equation has been well-developed for a long time: global existence for vortex-sheet data is due to Delort [16], while the only known uniqueness result, due to Yudovich [42], holds in the class of bounded vorticity (see also [6] and the references therein). As far as the general model (1.5) is concerned, global existence and uniqueness results for smooth solutions are easily obtained by standard methods (see e.g. [13]). Although not surprising, global existence for this model is further proven here for vortex-sheet initial data.

On the other hand, the compressible model (1.6), first introduced in [36], is completely new in the literature. In [36, Appendix B], only local-in-time existence and uniqueness of smooth solutions are proven in the non-degenerate case $\lambda > 0$, using a standard iterative method. In the present paper, a similar local-in-time existence result is obtained for the degenerate parabolic case $\alpha = 1$, $\beta = 0$, $\lambda = 0$, which requires a more careful analysis of the iterative scheme, and global existence with vortex-sheet data is further proven in the non-degenerate parabolic case $\alpha = 1$, $\beta = 0$, $\lambda > 0$.

The general models (1.1)–(1.2), introduced in our companion paper [19], are inhomogeneous versions of (1.5)–(1.6) with forcing. Since these are new in the literature, the present paper aims at providing a detailed discussion of local and global existence, uniqueness, and regularity issues. Note that in the conservative regime $\alpha = 0$, $\beta = 1$, the incompressible model (1.1) takes the form of an "inhomogeneous" 2D Euler equation with "forcing": using the identity $v^\perp \operatorname{curl} v = (v \cdot \nabla)v - \frac{1}{2}\nabla|v|^2$, and setting $\tilde{P} := P - \frac{1}{2}|v|^2$, we indeed find

$$\partial_t v = \nabla \tilde{P} + \Psi^\perp \operatorname{curl} v + (v \cdot \nabla)v, \qquad \operatorname{div}(av) = 0.$$

We are aware of no work on this modified Euler equation, which seems to have no obvious interpretation in terms of fluid mechanics. As far as global existence issues are concerned, it should be clear from the Delort type identity (1.9) below that inhomogeneities give rise to important difficulties: indeed, for h non-constant, the first term $-\frac{1}{2}|v|^2\nabla^\perp h$ in (1.9) does not vanish and is clearly not weakly continuous as a function of v (although the second term is, as in the classical theory [16]). Because of that, we obtain no result for vortex-sheet initial data in that case, and only manage to prove global existence for initial vorticity in $\mathrm{L}^q(\mathbb{R}^2)$ for some $q > 1$.

Notions of weak solutions for (1.1) and (1.2)

We first introduce the vorticity formulation of equations (1.1) and (1.2), which will be more convenient to work with. Setting $\omega := \operatorname{curl} v$ and $\zeta := \operatorname{div}(av)$, each of these equations may be rewritten as a nonlinear nonlocal transport equation for the vorticity ω,

$$\partial_t \omega = \operatorname{div}(\omega(\alpha(\Psi + v)^\perp + \beta(\Psi + v))), \quad \operatorname{curl} v = \omega, \quad \operatorname{div}(av) = \zeta, \tag{1.7}$$

where in the incompressible case (1.1) we have $\zeta := 0$, while in the compressible case (1.2) ζ is the solution of the following transport-diffusion equation (which is highly degenerate as $\lambda = 0$),

$$\partial_t\zeta - \lambda\triangle\zeta + \lambda\operatorname{div}(\zeta\nabla h) = \operatorname{div}(a\omega(-\alpha(\Psi + v) + \beta(\Psi + v)^\perp)). \tag{1.8}$$

Let us now precisely define our notions of weak solutions for (1.1) and (1.2). (We denote by $\mathcal{M}^+_{\mathrm{loc}}(\mathbb{R}^2)$ the convex cone of locally finite non-negative Borel measures on $\mathbb{R}^2$, and by $\mathcal{P}(\mathbb{R}^2)$ the convex subset of probability measures, endowed with the usual weak-* topology.)

Definition 1.1. Let $h, \Psi \in \mathrm{L}^\infty(\mathbb{R}^2)$, $T > 0$, and set $a := e^h$.

(a) Given $v^\circ \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^2)^2$ with $\omega^\circ = \operatorname{curl} v^\circ \in \mathcal{M}^+_{\mathrm{loc}}(\mathbb{R}^2)$ and $\zeta^\circ := \operatorname{div}(av^\circ) \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^2)$, we say that v is a *weak solution of* (1.2) on $[0,T) \times \mathbb{R}^2$ with initial data v°, if $v \in \mathrm{L}^2_{\mathrm{loc}}([0,T) \times \mathbb{R}^2)^2$ satisfies $\omega := \operatorname{curl} v \in \mathrm{L}^1_{\mathrm{loc}}([0,T); \mathcal{M}^+_{\mathrm{loc}}(\mathbb{R}^2))$, $\zeta := \operatorname{div}(av) \in \mathrm{L}^2_{\mathrm{loc}}([0,T); \mathrm{L}^2(\mathbb{R}^2))$, $|v|^2\omega \in \mathrm{L}^1_{\mathrm{loc}}([0,T); \mathrm{L}^1(\mathbb{R}^2))$ (hence also $\omega v \in \mathrm{L}^1_{\mathrm{loc}}([0,T) \times \mathbb{R}^2)^2$), and satisfies (1.2) in the distributional sense, that is, for all $\psi \in C^\infty_c([0,T) \times \mathbb{R}^2)^2$,

$$\int \psi(0,\cdot) \cdot v^\circ + \iint v \cdot \partial_t\psi = \lambda \iint a^{-1}\zeta \operatorname{div}\psi + \iint \psi \cdot (\alpha(\Psi + v) - \beta(\Psi + v)^\perp)\omega.$$

(b) Given $v^\circ \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^2)^2$ with $\omega^\circ := \operatorname{curl} v^\circ \in \mathcal{M}^+_{\mathrm{loc}}(\mathbb{R}^2)$ and $\operatorname{div}(av^\circ) = 0$, we say that v is a *weak solution of* (1.1) on $[0,T)\times\mathbb{R}^2$ with initial data v°, if $v \in \mathrm{L}^2_{\mathrm{loc}}([0,T)\times\mathbb{R}^2)^2$ satisfies $\omega := \operatorname{curl} v \in \mathrm{L}^1_{\mathrm{loc}}([0,T); \mathcal{M}^+_{\mathrm{loc}}(\mathbb{R}^2))$, $|v|^2\omega \in \mathrm{L}^1_{\mathrm{loc}}([0,T); \mathrm{L}^1(\mathbb{R}^2)^2)$ (hence also $\omega v \in \mathrm{L}^1_{\mathrm{loc}}([0,T) \times \mathbb{R}^2)^2$), $\operatorname{div}(av) = 0$ in the distributional sense, and satisfies the vorticity formulation (1.7) in the distributional sense, that is, for all $\psi \in C^\infty_c([0,T) \times \mathbb{R}^2)$,

$$\int \psi(0,\cdot)\omega^\circ + \iint \omega\partial_t\psi = \iint \nabla\psi \cdot (\alpha(\Psi + v)^\perp + \beta(\Psi + v))\omega.$$

(c) Given $v^\circ \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^2)^2$ with $\omega^\circ := \operatorname{curl} v^\circ \in \mathcal{M}^+_{\mathrm{loc}}(\mathbb{R}^2)$ and $\operatorname{div}(av^\circ) = 0$, we say that v is a *very weak solution of* (1.1) on $[0,T)\times\mathbb{R}^2$ with initial data v°, if $v \in \mathrm{L}^2_{\mathrm{loc}}([0,T)\times\mathbb{R}^2)^2$ satisfies $\omega := \operatorname{curl} v \in \mathrm{L}^1_{\mathrm{loc}}([0,T); \mathcal{M}^+_{\mathrm{loc}}(\mathbb{R}^2))$, $\operatorname{div}(av) = 0$ in the distributional sense, and satisfies, for all $\psi \in C^\infty_c([0,T) \times \mathbb{R}^2)$,

$$\int \psi(0,\cdot)\omega^\circ + \iint \omega\partial_t\psi = \iint \nabla\psi \cdot (\alpha\Psi^\perp + \beta\Psi)\omega + \iint (\alpha\nabla\psi + \beta\nabla^\perp\psi) \cdot \Big(\frac{1}{2}|v|^2\nabla h + a^{-1}\operatorname{div}(aS_v)\Big),$$

in terms of the stress-energy tensor $S_v := v \otimes v - \frac{1}{2}\operatorname{Id}|v|^2$.

Remarks 1.2.

(i) Weak solutions of (1.2) are defined directly from (1.2), and satisfy in particular the vorticity formulation (1.7)–(1.8) in the distributional sense. As far as weak solutions of (1.1) are concerned, they are rather defined in terms of the vorticity formulation (1.7), in order to avoid compactness and regularity issues related to the pressure p. Nevertheless, if v is a weak solution of (1.1) in the above sense, then under mild regularity assumptions we may use the formula $v = a^{-1}\nabla^\perp(\operatorname{div} a^{-1}\nabla)^{-1}\omega$ to deduce that v actually satisfies (1.1) in the distributional sense on $[0,T) \times \mathbb{R}^2$ for some distribution p (cf. Lemma 2.8 below for details).

(ii) The definition (c) of a very weak solution of (1.1) is motivated as follows (see also the definition of "general weak solutions" of (1.4) in [29]). In the purely conservative case $\alpha = 0$, there are too few a priori estimates to make sense of the product ωv. As is now common in 2D fluid mechanics (see e.g. [13]), the idea is to reinterpret this product in terms of the stress-energy tensor S_v, using the following identity: given $\operatorname{div}(av) = 0$, we have for smooth enough fields

$$\omega v = -\frac{1}{2}|v|^2\nabla^\perp h - a^{-1}(\operatorname{div}(aS_v))^\perp, \tag{1.9}$$

where the right-hand side now makes sense in $\mathrm{L}^1_{\mathrm{loc}}([0,T); W^{-1,1}_{\mathrm{loc}}(\mathbb{R}^2)^2)$ whenever $v \in \mathrm{L}^2_{\mathrm{loc}}([0,T) \times \mathbb{R}^2)^2$. In particular, if $\omega \in \mathrm{L}^p_{\mathrm{loc}}([0,T) \times \mathbb{R}^2)$ and $v \in \mathrm{L}^{p'}_{\mathrm{loc}}([0,T) \times \mathbb{R}^2)$ for some $1 \le p \le \infty$, $1/p + 1/p' = 1$, then the product ωv makes perfect sense and the above identity (1.9) holds in the distributional sense, hence in that case v is a weak solution of (1.1) whenever it is a very weak solution. In reference to [16], identity (1.9) is henceforth called an "(inhomogeneous) Delort type identity".

Statement of the main results

Global existence and regularity results are summarized in the following theorem. Our approach relies on proving a priori estimates for the vorticity ω in $\mathrm{L}^q(\mathbb{R}^2)$ for some $q>1$. For the compressible model (1.2), such estimates are only obtained in the parabolic regime, hence our limitation to that regime. In parabolic cases, particularly strong estimates are available, and existence is then established even for vortex-sheet data, thus completely extending the known theory for (1.4) (see [29, 38]). Note that the additional exponential growth in the dispersive estimate (1.10) below is only due to the forcing Ψ. In the conservative incompressible case, the situation is the most delicate because of a lack of strong enough a priori estimates, and only existence of very weak solutions is expected and proven. As is standard in 2D fluid mechanics (see e.g. [13]), the natural space for the solution v is $\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+;\bar v^\circ+\mathrm{L}^2(\mathbb{R}^2)^2)$ for a given smooth reference field $\bar v^\circ:\mathbb{R}^2\to\mathbb{R}^2$.

Theorem 1 (Global existence)**.** *Let $\lambda>0$, $\alpha\ge 0$, $\beta\in\mathbb{R}$, $h,\Psi\in W^{1,\infty}(\mathbb{R}^2)^2$, and set $a:=e^h$. Let $\bar v^\circ\in W^{1,\infty}(\mathbb{R}^2)^2$ be some reference map with $\bar\omega^\circ:=\operatorname{curl}\bar v^\circ\in\mathcal{P}\cap H^{s_0}(\mathbb{R}^2)$ for some $s_0>1$, and with either $\operatorname{div}(a\bar v^\circ)=0$ in the case (1.1), or $\bar\zeta^\circ:=\operatorname{div}(a\bar v^\circ)\in H^{s_0}(\mathbb{R}^2)$ in the case (1.2). Let $v^\circ\in\bar v^\circ+\mathrm{L}^2(\mathbb{R}^2)^2$, with $\omega^\circ:=\operatorname{curl}v^\circ\in\mathcal{P}(\mathbb{R}^2)$, and with either $\operatorname{div}(av^\circ)=0$ in the case (1.1), or $\zeta^\circ:=\operatorname{div}(av^\circ)\in\mathrm{L}^2(\mathbb{R}^2)$ in the case (1.2). Denoting by $C>0$ any constant depending only on an upper bound on α, $|\beta|$ and $\|(h,\Psi)\|_{W^{1,\infty}}$, the following hold:*

(i) Parabolic compressible case (that is, (1.2) with $\alpha>0$, $\beta=0$):
There exists a weak solution $v\in\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+;\bar v^\circ+\mathrm{L}^2(\mathbb{R}^2)^2)$ on $\mathbb{R}^+\times\mathbb{R}^2$ with initial data v°, with $\omega:=\operatorname{curl}v\in\mathrm{L}^\infty(\mathbb{R}^+;\mathcal{P}(\mathbb{R}^2))$ and $\zeta:=\operatorname{div}(av)\in\mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^+;\mathrm{L}^2(\mathbb{R}^2))$, and with

$$\|\omega^t\|_{\mathrm{L}^\infty}\le(\alpha t)^{-1}+C\alpha^{-1}e^{Ct},\qquad \textit{for all } t>0. \tag{1.10}$$

Moreover, if $\omega^\circ\in\mathrm{L}^q(\mathbb{R}^2)$ for some $q>1$, then $\omega\in\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+;\mathrm{L}^q(\mathbb{R}^2))$.

(ii) Parabolic incompressible case (that is, (1.1) with $\alpha>0$, $\beta=0$, or with $\alpha>0$, $\beta\in\mathbb{R}$, h constant):
There exists a weak solution $v\in\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+;\bar v^\circ+\mathrm{L}^2(\mathbb{R}^2)^2)$ on $\mathbb{R}^+\times\mathbb{R}^2$ with initial data v°, with $\omega:=\operatorname{curl}v\in\mathrm{L}^\infty(\mathbb{R}^+;\mathcal{P}(\mathbb{R}^2))$, and with the dispersive estimate (1.10). Moreover, if $\omega^\circ\in\mathrm{L}^q(\mathbb{R}^2)$ for some $q>1$, then $\omega\in\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+;\mathrm{L}^q(\mathbb{R}^2))\cap\mathrm{L}^{q+1}_{\mathrm{loc}}(\mathbb{R}^+;\mathrm{L}^{q+1}(\mathbb{R}^2))$.

(iii) Mixed-flow incompressible case (that is, (1.1) with $\alpha>0$, $\beta\in\mathbb{R}$):
If $\omega^\circ\in\mathrm{L}^q(\mathbb{R}^2)$ for some $q>1$, there exists a weak solution $v\in\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+;\bar v^\circ+\mathrm{L}^2(\mathbb{R}^2)^2)$ on $\mathbb{R}^+\times\mathbb{R}^2$ with initial data v°, and with $\omega:=\operatorname{curl}v\in\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+;\mathcal{P}\cap\mathrm{L}^q(\mathbb{R}^2))\cap\mathrm{L}^{q+1}_{\mathrm{loc}}(\mathbb{R}^+;\mathrm{L}^{q+1}(\mathbb{R}^2))$.

(iv) Conservative incompressible case (that is, (1.1) with $\alpha=0$, $\beta\in\mathbb{R}$):
If $\omega^\circ\in\mathrm{L}^q(\mathbb{R}^2)$ for some $q>1$, there exists a very weak solution $v\in\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+;\bar v^\circ+\mathrm{L}^2(\mathbb{R}^2)^2)$ on $\mathbb{R}^+\times\mathbb{R}^2$ with initial data v°, and with $\omega:=\operatorname{curl}v\in\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+;\mathcal{P}\cap\mathrm{L}^q(\mathbb{R}^2))$. This is a weak solution whenever $q\ge 4/3$.

We set $\zeta^\circ,\bar\zeta^\circ,\zeta:=0$ in the incompressible case (1.1). If in addition ω°, $\zeta^\circ\in\mathrm{L}^\infty(\mathbb{R}^2)$, then we further have $v\in\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+;\mathrm{L}^\infty(\mathbb{R}^2)^2)$, $\omega\in\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+;\mathrm{L}^1\cap\mathrm{L}^\infty(\mathbb{R}^2))$, and $\zeta\in\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+;\mathrm{L}^2\cap\mathrm{L}^\infty(\mathbb{R}^2)^2)$. If h, Ψ, $\bar v^\circ\in W^{s+1,\infty}(\mathbb{R}^2)^2$ and ω°, $\bar\omega^\circ$, ζ°, $\bar\zeta^\circ\in H^s(\mathbb{R}^2)$ for some $s>1$, then $v\in\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+;\bar v^\circ+H^{s+1}(\mathbb{R}^2)^2)$ and $\omega,\zeta\in\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+;H^s(\mathbb{R}^2)^2)$. If h, Ψ, $v^\circ\in C^{s+1}(\mathbb{R}^2)^2$ for some non-integer $s>0$, then $v\in\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+;C^{s+1}(\mathbb{R}^2)^2)$.

For the regimes that are not described in the above — i.e., the degenerate compressible case $\lambda=0$, and the mixed-flow compressible case (as well as the a priori unphysical case $\alpha<0$) —, only local-in-time existence is proven for smooth enough initial data. Note that in the degenerate case v and ω are on the same footing in terms of regularity.

Theorem 2 (Local existence)**.** *Given some $s>1$, let $h,\Psi,\bar v^\circ\in W^{s+1,\infty}(\mathbb{R}^2)^2$, set $a:=e^h$, and let $v^\circ\in\bar v^\circ+H^{s+1}(\mathbb{R}^2)^2$ with $\omega^\circ:=\operatorname{curl}v^\circ$, $\bar\omega^\circ:=\operatorname{curl}\bar v^\circ\in H^s(\mathbb{R}^2)$, and with either $\operatorname{div}(av^\circ)=\operatorname{div}(a\bar v^\circ)=0$ in the case (1.1), or $\zeta^\circ:=\operatorname{div}(av^\circ)$, $\bar\zeta^\circ:=\operatorname{div}(a\bar v^\circ)\in H^s(\mathbb{R}^2)$ in the case (1.2). The following hold:*

(i) Incompressible case (that is, (1.1) with $\alpha,\beta\in\mathbb{R}$):
There exists $T>0$ and a weak solution $v\in\mathrm{L}^\infty_{\mathrm{loc}}([0,T);\bar v^\circ+H^{s+1}(\mathbb{R}^2)^2)$ on $[0,T)\times\mathbb{R}^2$ with initial data v°.

(ii) Non-degenerate compressible case (that is, (1.2) with $\alpha,\beta\in\mathbb{R}$, $\lambda>0$):
There exists $T>0$ and a weak solution $v\in\mathrm{L}^\infty_{\mathrm{loc}}([0,T);\bar v^\circ+H^{s+1}(\mathbb{R}^2)^2)$ on $[0,T)\times\mathbb{R}^2$ with initial data v°.

(iii) Degenerate parabolic compressible case (that is, (1.2) with $\alpha\in\mathbb{R}$, $\beta=\lambda=0$):
If Ψ, $\bar v^\circ\in W^{s+2,\infty}(\mathbb{R}^2)^2$ and ω°, $\bar\omega^\circ\in H^{s+1}(\mathbb{R}^2)$, there exists $T>0$ and a weak solution $v\in\mathrm{L}^\infty_{\mathrm{loc}}([0,T);\bar v^\circ+H^{s+1}(\mathbb{R}^2)^2)$ on $[0,T)\times\mathbb{R}^2$ with initial data v°, and with $\omega:=\operatorname{curl}v\in\mathrm{L}^\infty_{\mathrm{loc}}([0,T);H^{s+1}(\mathbb{R}^2))$.

Let us finally turn to uniqueness issues. No uniqueness is expected to hold for general weak measure solutions of (1.1), as it is already known to fail for the 2D Euler equation (see e.g. [6] and the references therein), and as it is also expected to fail for equation (1.4) (see [3, Section 8]). In both cases, as already explained, the only known uniqueness results are in the class of bounded vorticity. For the general incompressible model (1.1), similar arguments as for (1.4) are still available and the same uniqueness result holds, while for the non-degenerate compressible model (1.2) the result is slightly weaker. In the degenerate parabolic case, the result is even worse since v and ω must then be on the same footing in terms of regularity.

Theorem 3 (Uniqueness)**.** *Let $\lambda \geq 0$, $\alpha, \beta \in \mathbb{R}$, $T > 0$, $h, \Psi \in W^{1,\infty}(\mathbb{R}^2)$, and set $a := e^h$. Let $v^\circ : \mathbb{R}^2 \to \mathbb{R}^2$ with $\omega^\circ := \operatorname{curl} v^\circ \in \mathcal{P}(\mathbb{R}^2)$, and with either $\operatorname{div}(av^\circ) = 0$ in the case (1.1), or $\operatorname{div}(av^\circ) \in \mathrm{L}^2(\mathbb{R}^2)$ in the case (1.2).*

(i) Incompressible case (that is, (1.1) with $\alpha, \beta \in \mathbb{R}$):
There exists at most a unique weak solution v on $[0, T) \times \mathbb{R}^2$ with initial data v°, in the class of all w's such that $\operatorname{curl} w \in \mathrm{L}^\infty_{\mathrm{loc}}([0, T); \mathrm{L}^\infty(\mathbb{R}^2))$.

(ii) Non-degenerate compressible case (that is, (1.2) with $\alpha, \beta \in \mathbb{R}$, $\lambda > 0$):
There exists at most a unique weak solution v on $[0, T) \times \mathbb{R}^2$ with initial data v°, in the class in the class $\mathrm{L}^2_{\mathrm{loc}}([0, T); v^\circ + \mathrm{L}^2(\mathbb{R}^2)^2) \cap \mathrm{L}^\infty_{\mathrm{loc}}([0, T); W^{1,\infty}(\mathbb{R}^2)^2)$.

(iii) Degenerate parabolic compressible case (that is, (1.2) with $\alpha \in \mathbb{R}$, $\lambda = \beta = 0$):
There exists at most a unique weak solution v on $[0, T) \times \mathbb{R}^2$ with initial data v°, in the class of all w's in $\mathrm{L}^2_{\mathrm{loc}}([0, T); v^\circ + \mathrm{L}^2(\mathbb{R}^2)^2) \cap \mathrm{L}^\infty_{\mathrm{loc}}([0, T); \mathrm{L}^\infty(\mathbb{R}^2)^2)$ with $\operatorname{curl} w \in \mathrm{L}^2_{\mathrm{loc}}([0, T); \mathrm{L}^2(\mathbb{R}^2)) \cap \mathrm{L}^\infty_{\mathrm{loc}}([0, T); W^{1,\infty}(\mathbb{R}^2))$.

Roadmap to the proof of the main results

We begin in Section 3 with the local existence of smooth solutions, summarized in Theorem 2 above. In the non-degenerate case $\lambda > 0$, the proof follows from a standard iterative scheme as in [36, Appendix B]. It is performed here in Sobolev spaces, but could be done in Hölder spaces as well. In the degenerate parabolic case $\alpha = 1$, $\beta = 0$, $\lambda = 0$, a similar argument holds, but requires a more careful analysis of the iterative scheme.

We then turn to global existence in Section 4. In order to pass from local to global existence, we prove estimates for the Sobolev (and Hölder) norms of solutions through the norm of their initial data. As shown in Section 4.2, arguing quite similarly as in the work by Lin and Zhang [29] on the simpler model (1.4), such estimates on Sobolev norms essentially follow from an a priori estimate for the vorticity in $\mathrm{L}^\infty(\mathbb{R}^2)$. In [29] such an a priori estimate for the vorticity was achieved by a simple ODE type argument, using that for (1.4) the evolution of the vorticity along characteristics can be explicitly integrated. This ODE argument can still be somehow adapted to the more sophisticated models (1.1) and (1.2) in the parabolic case (cf. Lemma 4.3(iii)). This yields the nice dispersive estimate (1.10) for the L^∞-norm of the vorticity (through its initial mass $\int \omega^\circ = 1$ only), which of course differs from [29] by the additional exponential growth due to the forcing Ψ. For the incompressible model (1.1) in the mixed-flow case, such arguments are no longer available, and only a weaker estimate is obtained, controlling the L^q-norm of the solution (as well as its space-time L^{q+1}-norm if $\alpha > 0$) by the L^q-norm of the data, for all $1 < q \leq \infty$ (cf. Lemma 4.2). This is instead proven by a careful energy type argument.

In order to handle rougher initial data, we regularize the data and then pass to the limit in the equation, using the compactness given by the available a priori estimates. The simplest energy estimates only give bounds for v in $\bar{v}^\circ + \mathrm{L}^2(\mathbb{R}^2)^2$ and for ζ in $\mathrm{L}^2(\mathbb{R}^2)$. To pass to the limit in the nonlinear term ωv, the additional estimates for the vorticity in $\mathrm{L}^q(\mathbb{R}^2)$, $q > 1$, then again turn out to be crucial. To get to vortex-sheet initial data in parabolic cases, as in [29] we make use of some compactness result due to Lions [30] in the context of the compressible Navier-Stokes equations. The model (1.1) in the conservative case $\alpha = 0$ is however more subtle because of a lack of strong enough a priori estimates. Only very weak solutions are then expected and obtained (for initial vorticity in $\mathrm{L}^q(\mathbb{R}^2)$ with $q > 1$), and compactness is in that case proven by hand.

Uniqueness issues are finally addressed in Section 5. Following Serfaty [36, Appendix B], a weak-strong uniqueness principle for both (1.1) and (1.2) is proven by energy methods in the non-degenerate case $\lambda > 0$. Note that this uniqueness principle is the key to the mean-field limit results for the Ginzburg-Landau vortices in our companion paper [19], following the strategy developed by Serfaty [36]. A much weaker weak-strong uniqueness principle is also obtained in the degenerate parabolic case $\beta = \lambda = 0$. For the compressible model (1.1), uniqueness in the class of bounded vorticity is easily obtained using the approach by Serfaty and Vázquez [38] for equation (1.4), which consists in adapting the corresponding uniqueness result for the 2D Euler equation due to Yudovich [42] together with a transport argument à la Loeper [31].

To ease the presentation, the various independent PDE results that are needed in the proofs are isolated in Section 2, including general a priori estimates for transport and transport-diffusion equations, some global elliptic regularity results, as well as critical potential theory estimates. The interest of such estimates for our purposes should be already clear from the form of the equations in the vorticity formulation (1.7)–(1.8).

Notation

For any vector field $F = (F_1, F_2)$ on $\mathbb{R}^2$, we denote $F^\perp = (-F_2, F_1)$, $\operatorname{curl} F = \partial_1 F_2 - \partial_2 F_1$, and also as usual $\operatorname{div} F = \partial_1 F_1 + \partial_2 F_2$. Given two linear operators A, B on some function space, we denote by $[A, B] := AB - BA$ their commutator. For any exponent $1 \le p \le \infty$, we denote its Hölder conjugate by $p' := p/(p-1)$. Denote by $B(x, r)$ the ball of radius r centered at x in $\mathbb{R}^d$, and set $B_r := B(0, r)$ and $B(x) := B(x, 1)$. We use the notation $a \wedge b = \min\{a, b\}$ and $a \vee b = \max\{a, b\}$ for all $a, b \in \mathbb{R}$. Given a function $f : \mathbb{R}^d \to \mathbb{R}$, we denote its positive and negative parts by $f^+(x) := 0 \vee f(x)$ and $f^-(x) := 0 \vee (-f)(x)$, respectively. We write $\lesssim$ and $\simeq$ for $\le$ and $=$ up to (unless explicitly stated) universal constants. The space of Lebesgue-measurable functions on $\mathbb{R}^d$ is denoted by $\operatorname{Mes}(\mathbb{R}^d)$, the set of Borel probability measures on $\mathbb{R}^d$ is denoted by $\mathcal{P}(\mathbb{R}^d)$, and for all $\sigma > 0$, $C^\sigma(\mathbb{R}^d)$ stands as usual for the Hölder space $C^{\lfloor \sigma \rfloor, \sigma - \lfloor \sigma \rfloor}(\mathbb{R}^d)$. For $\sigma \in (0, 1)$, we denote by $|\cdot|_{C^\sigma}$ the usual Hölder seminorm, and by $\|\cdot\|_{C^\sigma} := |\cdot|_{C^\sigma} + \|\cdot\|_{\mathrm{L}^\infty}$ the corresponding norm. We denote by $\mathrm{L}^p_{\mathrm{uloc}}(\mathbb{R}^d)$ the Banach space of functions that are uniformly locally L^p-integrable, with norm $\|f\|_{\mathrm{L}^p_{\mathrm{uloc}}} := \sup_x \|f\|_{\mathrm{L}^p(B(x))}$. Given a Banach space $X \subset \operatorname{Mes}(\mathbb{R}^d)$ and $t > 0$, we use the notation $\|\cdot\|_{\mathrm{L}^p_t X}$ for the usual norm in $\mathrm{L}^p([0, t]; X)$.

2 Preliminary results

In this section, we establish various PDE results that are needed in the sequel and are of independent interest. As most of them do not depend on the choice of space dimension 2, they are stated here in general dimension d. We first recall the following useful proxy for a fractional Leibniz rule, which is essentially due to Kato and Ponce (see e.g. [21, Theorem 1.4]).

Lemma 2.1 (Kato-Ponce inequality). *Let $d \ge 1$, $s \ge 0$, $p \in (1, \infty)$, and $1/p_i + 1/q_i = 1/p$ with $i = 1, 2$ and $p_1, q_1, p_2, q_2 \in (1, \infty]$. Then, for $f, g \in C_c^\infty(\mathbb{R}^d)$, we have*

$$\|fg\|_{W^{s,p}} \lesssim \|f\|_{\mathrm{L}^{p_1}} \|g\|_{W^{s,q_1}} + \|g\|_{\mathrm{L}^{p_2}} \|f\|_{W^{s,q_2}}.$$

The following gives a general estimate for the evolution of the Sobolev norms of the solutions of transport equations (see also [29, equation (7)] for a simpler version), which will be useful in the sequel since the vorticity ω indeed satisfies an equation of this form (1.7).

Lemma 2.2 (A priori estimate for transport equations). *Let $d \ge 1$, $s \ge 0$, $T > 0$. Given a vector field $w \in \mathrm{L}^\infty_{\mathrm{loc}}([0, T); W^{1,\infty}(\mathbb{R}^d)^d)$, let $\rho \in \mathrm{L}^\infty_{\mathrm{loc}}([0, T); H^s(\mathbb{R}^d))$ satisfy the transport equation $\partial_t \rho = \operatorname{div}(\rho w)$ in the distributional sense in $[0, T) \times \mathbb{R}^d$. Further assume $w - W \in \mathrm{L}^\infty_{\mathrm{loc}}([0, T); H^{s+1}(\mathbb{R}^d)^d)$ for some reference map $W \in W^{s+1,\infty}(\mathbb{R}^d)^d$. Then for all $t \in [0, T)$,*

$$\begin{aligned} \partial_t \|\rho^t\|_{H^s} \le 2\|(\nabla w^t, \nabla W)\|_{\mathrm{L}^\infty} \|\rho^t\|_{H^s} + \|\rho^t\|_{\mathrm{L}^\infty} \|\operatorname{div}(w^t - W)\|_{H^s} \\ + \|\rho^t\|_{\mathrm{L}^2} \|\operatorname{div} W\|_{W^{s,\infty}} + \frac{1}{2} \|\operatorname{div} w^t\|_{\mathrm{L}^\infty} \|\rho^t\|_{H^s}, \end{aligned} \tag{2.1}$$

where we use the notation $\|(\nabla w^t, \nabla W)\|_{\mathrm{L}^\infty} := \|\nabla w^t\|_{\mathrm{L}^\infty} \vee \|\nabla W\|_{\mathrm{L}^\infty}$. Also, for all $t \in [0, T)$,

$$\|\rho^t - \rho^\circ\|_{\dot{H}^{-1}} \le \|\rho\|_{\mathrm{L}^\infty_t \mathrm{L}^2} \|w\|_{\mathrm{L}^1_t \mathrm{L}^\infty}. \tag{2.2}$$

Proof. We split the proof into two steps: we first prove (2.1) as a corollary of the celebrated Kato-Ponce commutator estimate, and then we check estimate (2.2), which is but a straightforward observation.

Step 1: proof of (2.1). Let $s \ge 0$. The time-derivative of the H^s-norm of the solution ρ can be computed as follows, using the notation $\langle\nabla\rangle := (1+|\nabla|^2)^{1/2}$,

$$\begin{aligned}\partial_t\|\rho^t\|_{H^s}^2 = 2\int(\langle\nabla\rangle^s\rho^t)(\langle\nabla\rangle^s \operatorname{div}(\rho^t w^t)) &= 2\int(\langle\nabla\rangle^s\rho^t)[\langle\nabla\rangle^s \operatorname{div}, w^t]\rho^t + 2\int(\langle\nabla\rangle^s\rho^t)(w^t\cdot\nabla\langle\nabla\rangle^s\rho^t)\\ &= 2\int(\langle\nabla\rangle^s\rho^t)[\langle\nabla\rangle^s \operatorname{div}, w^t]\rho^t - \int|\langle\nabla\rangle^s\rho^t|^2\operatorname{div} w^t\\ &\le 2\|\rho^t\|_{H^s}\|[\langle\nabla\rangle^s \operatorname{div}, w^t]\rho^t\|_{\mathrm{L}^2} + \|(\operatorname{div} w^t)^-\|_{\mathrm{L}^\infty}\|\rho^t\|_{H^s}^2,\end{aligned}$$

which we may further bound by

$$\partial_t\|\rho^t\|_{H^s} \le \|[\langle\nabla\rangle^s \operatorname{div}, w^t - W]\rho^t\|_{\mathrm{L}^2} + \|[\langle\nabla\rangle^s \operatorname{div}, W]\rho^t\|_{\mathrm{L}^2} + \frac12\|(\operatorname{div} w^t)^-\|_{\mathrm{L}^\infty}\|\rho^t\|_{H^s}.$$

Now we recall the following general form of the Kato-Ponce commutator estimate [26, Lemma X1] (which follows by replacing the use of [14] by the later work [15] in the proof of [26, Lemma X1]; see also [21, Theorem 1.4]): given $p\in(1,\infty)$, and $1/p_i + 1/q_i = 1/p$ with $i=1,2$ and $p_1,q_1,p_2,q_2\in(1,\infty]$, we have for any $f,g\in C_c^\infty(\mathbb{R}^d)$,

$$\|[\langle\nabla\rangle^s \operatorname{div}, f]g\|_{\mathrm{L}^p} \lesssim \|\nabla f\|_{\mathrm{L}^{q_1}}\|g\|_{W^{s,p_1}} + \|g\|_{\mathrm{L}^{q_2}}\|\operatorname{div} f\|_{W^{s,p_2}}.$$

This estimate yields

$$\begin{aligned}\partial_t\|\rho^t\|_{H^s} &\le \|\rho^t\|_{H^s}\|\nabla(w^t - W)\|_{\mathrm{L}^\infty} + \|\rho^t\|_{\mathrm{L}^\infty}\|\operatorname{div}(w^t-W)\|_{H^s}\\ &\quad + \|\rho^t\|_{H^s}\|\nabla W\|_{\mathrm{L}^\infty} + \|\rho^t\|_{\mathrm{L}^2}\|\operatorname{div} W\|_{W^{s,\infty}} + \frac12\|(\operatorname{div} w^t)^-\|_{\mathrm{L}^\infty}\|\rho^t\|_{H^s},\end{aligned}$$

and the result (2.1) follows.

Step 2: proof of (2.2). Let $\epsilon>0$. We denote by $\hat u$ the Fourier transform of a function u on $\mathbb{R}^d$. Set $G^t := \rho^t w^t$, so that the equation for ρ takes the form $\partial_t\rho^t = \operatorname{div} G^t$. Rewriting this equation in Fourier space and testing it against $(\epsilon+|\xi|)^{-2}(\hat\rho^t - \hat\rho^\circ)(\xi)$, we find

$$\begin{aligned}\partial_t\int(\epsilon+|\xi|)^{-2}|\hat\rho^t(\xi)-\hat\rho^\circ(\xi)|^2d\xi &= 2i\int(\epsilon+|\xi|)^{-2}\xi\cdot\hat G^t(\xi)(\overline{\hat\rho^t(\xi)-\hat\rho^\circ(\xi)})d\xi\\ &\le 2\int(\epsilon+|\xi|)^{-1}|\hat\rho^t(\xi)-\hat\rho^\circ(\xi)||\hat G^t(\xi)|d\xi,\end{aligned}$$

and hence, by the Cauchy-Schwarz inequality,

$$\partial_t\Big(\int(\epsilon+|\xi|)^{-2}|\hat\rho^t(\xi)-\hat\rho^\circ(\xi)|^2d\xi\Big)^{1/2} \le \Big(\int|\hat G^t(\xi)|^2d\xi\Big)^{1/2}.$$

Integrating in time and letting $\epsilon\downarrow 0$, we obtain

$$\|\rho^t-\rho^\circ\|_{\dot H^{-1}} \le \|G\|_{\mathrm{L}^1_t\mathrm{L}^2} \le \|\rho\|_{\mathrm{L}^\infty_t\mathrm{L}^2}\|w\|_{\mathrm{L}^1_t\mathrm{L}^\infty},$$

that is, (2.2). ☐

As the evolution of the divergence ζ in the compressible model (1.2) is given by the transport-diffusion equation (1.8), the following parabolic regularity results will be needed. Note that a variant of item (ii) below can be found e.g. in [5, Section 3.4]. Item (iii) could be substantially refined (weakening the norm of g in time, at the price of a stronger norm in space), but the statement below is already more than enough for our purposes.

Lemma 2.3 (A priori estimates for transport-diffusion equations)**.** *Let $d\ge1$, $T>0$. Let $g\in\mathrm{L}^1_{\mathrm{loc}}([0,T)\times\mathbb{R}^d)^d$, and let w satisfy $\partial_t w - \triangle w + \operatorname{div}(w\nabla h) = \operatorname{div} g$ in the distributional sense in $[0,T)\times\mathbb{R}^d$ with initial data w°. The following hold:*

(i) *for all* $s \geq 0$*, if* $\nabla h \in W^{s,\infty}(\mathbb{R}^d)^d$*,* $w \in \mathrm{L}^\infty_{\mathrm{loc}}([0,T);H^s(\mathbb{R}^d))$*, and* $g \in \mathrm{L}^2_{\mathrm{loc}}([0,T);H^s(\mathbb{R}^d)^d)$*, then we have for all* $t \in [0,T)$*,*

$$\|w^t\|_{H^s} \leq Ce^{Ct}(\|w^\circ\|_{H^s} + \|g\|_{\mathrm{L}^2_t H^s}),$$

where the constants C*'s depend only on an upper bound on* s *and* $\|\nabla h\|_{W^{s,\infty}}$*;*

(ii) *if* $\nabla h \in \mathrm{L}^\infty(\mathbb{R}^d)$*,* $w^\circ \in \mathrm{L}^2(\mathbb{R}^d)$*,* $w \in \mathrm{L}^\infty_{\mathrm{loc}}([0,T);\mathrm{L}^2(\mathbb{R}^d))$*, and* $g \in \mathrm{L}^2_{\mathrm{loc}}([0,T);\mathrm{L}^2(\mathbb{R}^d))$*, then we have for all* $t \in [0,T)$*,*

$$\|w^t - w^\circ\|_{\dot H^{-1}\cap \mathrm{L}^2} \leq Ce^{Ct}(\|w^\circ\|_{\mathrm{L}^2} + \|g\|_{\mathrm{L}^2_t \mathrm{L}^2}),$$

where the constants C*'s depend only on an upper bound on* $\|\nabla h\|_{\mathrm{L}^\infty}$*;*

(iii) *for all* $1 \leq p,q \leq \infty$*, and all* $\frac{dq}{d+q} < s \leq q$*,* $s \geq 1$*, if* $\nabla h \in \mathrm{L}^\infty(\mathbb{R}^d)$*,* $w \in \mathrm{L}^p_{\mathrm{loc}}([0,T);\mathrm{L}^q(\mathbb{R}^d))$*, and* $g \in \mathrm{L}^p_{\mathrm{loc}}([0,T);\mathrm{L}^s(\mathbb{R}^d))$*, then we have for all* $t \in [0,T)$*,*

$$\|w\|_{\mathrm{L}^p_t \mathrm{L}^q} \lesssim (\|w^\circ\|_{\mathrm{L}^q} + \kappa^{-1}t^\kappa \|g\|_{\mathrm{L}^p_t \mathrm{L}^s}) \exp\Big(\inf_{2<r<\infty} r^{-1}\big(1+(r-2)^{-r/2}\big)(Ct)^{r/2}\Big).$$

where $\kappa := \frac{d}{2}(\frac{1}{d}+\frac{1}{q}-\frac{1}{s}) > 0$*, and where the constant* C*'s depend only on* $\|\nabla h\|_{\mathrm{L}^\infty}$*.*

Proof. We split the proof into three steps, proving items (i), (ii) and (iii) separately.

Step 1: proof of (i). Denote $G := g - w\nabla h$, so that w satisfies $\partial_t w - \triangle w = \operatorname{div} G$. Let $\langle \xi\rangle := (1+|\xi|^2)^{1/2}$, and let $\hat u$ denote the Fourier transform of a function u on $\mathbb{R}^d$. Let $s \geq 0$ be fixed, and assume that $\nabla h, w, g$ are as in the statement of (i) (which implies $G \in \mathrm{L}^2_{\mathrm{loc}}([0,T);H^s(\mathbb{R}^d))$ as shown below). In this step, we use the notation $\lesssim$ for $\leq$ up to a constant C as in the statement. For all $\epsilon > 0$, rewriting the equation for w in Fourier space and then testing it against $(\epsilon+|\xi|)^{-2}\langle\xi\rangle^{2s}\partial_t\hat w(\xi)$, we obtain

$$\int (\epsilon+|\xi|)^{-2}\langle\xi\rangle^{2s}|\partial_t \hat w^t(\xi)|^2 d\xi + \frac12\int \frac{|\xi|^2}{(\epsilon+|\xi|)^2}\langle\xi\rangle^{2s}\partial_t|\hat w^t(\xi)|^2 d\xi = i\int (\epsilon+|\xi|)^{-2}\langle\xi\rangle^{2s}\xi\cdot \hat G^t(\xi)\overline{\partial_t \hat w^t(\xi)}d\xi,$$

and hence, integrating over $[0,t]$, and using the inequality $2xy \leq x^2+y^2$,

$$\begin{aligned}
&\int_0^t \int (\epsilon+|\xi|)^{-2}\langle\xi\rangle^{2s}|\partial_u \hat w^u(\xi)|^2 d\xi du + \frac12\int \frac{|\xi|^2}{(\epsilon+|\xi|)^2}\langle\xi\rangle^{2s}|\hat w^t(\xi)|^2 d\xi\\
&= \frac12\int \frac{|\xi|^2}{(\epsilon+|\xi|)^2}\langle\xi\rangle^{2s}|\hat w^\circ(\xi)|^2 d\xi + i\int_0^t\int (\epsilon+|\xi|)^{-2}\langle\xi\rangle^{2s}\xi\cdot\hat G^u(\xi)\overline{\partial_u\hat w^u(\xi)}d\xi du\\
&\leq \frac12\int\langle\xi\rangle^{2s}|\hat w^\circ(\xi)|^2d\xi + \frac12\int_0^t\int\langle\xi\rangle^{2s}|\hat G^u(\xi)|^2 d\xi du + \frac12\int_0^t\int(\epsilon+|\xi|)^{-2}\langle\xi\rangle^{2s}|\partial_u\hat w^u(\xi)|^2d\xi du.
\end{aligned}$$

Absorbing in the left-hand side the last right-hand side term, and letting $\epsilon \downarrow 0$, it follows that

$$\int\langle\xi\rangle^{2s}|\hat w^t(\xi)|^2d\xi \leq \int\langle\xi\rangle^{2s}|\hat w^\circ(\xi)|^2 d\xi + \int_0^t\int\langle\xi\rangle^{2s}|\hat G^u(\xi)|^2d\xi du,$$

or equivalently

$$\|w^t\|_{H^s} \leq \|w^\circ\|_{H^s} + \|G\|_{\mathrm{L}^2_t H^s}.$$

Lemma 2.1 yields

$$\begin{aligned}
\|G\|_{\mathrm{L}^2_t H^s} \leq \|g\|_{\mathrm{L}^2_t H^s} + \|w\nabla h\|_{\mathrm{L}^2_t H^s} &\lesssim \|g\|_{\mathrm{L}^2_t H^s} + \|\nabla h\|_{W^{s,\infty}}\|w\|_{\mathrm{L}^2_t\mathrm{L}^2} + \|\nabla h\|_{\mathrm{L}^\infty}\|w\|_{\mathrm{L}^2_t H^s}\\
&\lesssim \|g\|_{\mathrm{L}^2_t H^s} + \|w\|_{\mathrm{L}^2_t H^s},
\end{aligned}$$

so that we obtain

$$\|w^t\|^2_{H^s} \lesssim \|w^\circ\|^2_{H^s} + \|g\|^2_{\mathrm{L}^2_t H^s} + \int_0^t \|w^u\|^2_{H^s}du,$$

and item (i) now follows from the Grönwall inequality.

Step 2: proof of (ii). Let $G^t := g^t - w^t\nabla h$, and let $\nabla h, w^\circ, w, g$ be as in the statement of (ii). For all $\epsilon > 0$, rewriting the equation for w in Fourier space and then integrating it against $(\epsilon+|\xi|)^{-2}(\hat w^t - \hat w^\circ)$, we may estimate

$$\partial_t \int (\epsilon+|\xi|)^{-2}|(\hat w^t-\hat w^\circ)(\xi)|^2 d\xi = 2\int (\epsilon+|\xi|)^{-2}\overline{(\hat w^t-\hat w^\circ)(\xi)}\partial_t \hat w^t(\xi)d\xi$$
$$\le -2\int \frac{|\xi|^2}{(\epsilon+|\xi|)^2}|(\hat w^t-\hat w^\circ)(\xi)|^2 + 2\int \frac{|\xi|^2}{(\epsilon+|\xi|)^2}|(\hat w^t-\hat w^\circ)(\xi)||\hat w^\circ(\xi)| + 2\int (\epsilon+|\xi|)^{-1}|(\hat w^t-\hat w^\circ)(\xi)||\hat G^t(\xi)|d\xi$$
$$\le \int \frac{|\xi|^2}{(\epsilon+|\xi|)^2}|\hat w^\circ(\xi)|^2 + \int (\epsilon+|\xi|)^{-2}|(\hat w^t-\hat w^\circ)(\xi)|^2 d\xi + \int (1+|\xi|^2)^{-1}|\hat G^t(\xi)|^2 d\xi,$$

that is

$$\partial_t \int (\epsilon+|\xi|)^{-2}|(\hat w^t-\hat w^\circ)(\xi)|^2 d\xi \le \int (\epsilon+|\xi|)^{-2}|(\hat w^t-\hat w^\circ)(\xi)|^2 d\xi + \|w^\circ\|_{\mathrm{L}^2}^2 + \|G^t\|_{H^{-1}}^2,$$

and hence by the Grönwall inequality

$$\int (\epsilon+|\xi|)^{-2}|(\hat w^t-\hat w^\circ)(\xi)|^2 d\xi \le e^t\big(\|w^\circ\|_{\mathrm{L}^2}^2 + \|G\|_{\mathrm{L}^2_t H^{-1}}^2\big).$$

Letting $\epsilon \downarrow 0$, it follows that $w^t - w^\circ \in \dot H^{-1}(\mathbb{R}^2)$ with

$$\|w^t - w^\circ\|_{\dot H^{-1}} \le e^t(\|w^\circ\|_{\mathrm{L}^2} + \|G\|_{\mathrm{L}^2_t H^{-1}}) \le e^t(\|w^\circ\|_{\mathrm{L}^2} + \|g\|_{\mathrm{L}^2_t H^{-1}} + \|\nabla h\|_{\mathrm{L}^\infty}\|w\|_{\mathrm{L}^2_t \mathrm{L}^2}).$$

Combining this with (i) for $s=0$, item (ii) follows.

Step 3: proof of (iii). Let $1 \le p,q \le \infty$, and assume that $w \in \mathrm{L}^p([0,T];\mathrm{L}^q(\mathbb{R}^d))$, $\nabla h \in \mathrm{L}^\infty(\mathbb{R}^d)$, and $g \in \mathrm{L}^p([0,T];\mathrm{L}^q(\mathbb{R}^d))$. In this step, we use the notation $\lesssim$ for $\le$ up to a constant C as in the statement. Denoting by $\Gamma^t(x) := Ct^{-d/2}e^{-|x|^2/(2t)}$ the heat kernel, Duhamel's representation formula yields

$$w^t(x) = \Gamma^t * w^\circ(x) + \phi_g^t(x) - \int_0^t \int \nabla\Gamma^u(y)\cdot\nabla h(y) w^{t-u}(x-y)dydu,$$

where we have set

$$\phi_g^t(x) := \int_0^t \int \nabla\Gamma^u(y)\cdot g^{t-u}(x-y)dydu.$$

We find by the triangle inequality

$$\|w^t\|_{\mathrm{L}^q} \le \|w^\circ\|_{\mathrm{L}^q}\int |\Gamma^t(y)|dy + \|\phi_g^t\|_{\mathrm{L}^q} + \|\nabla h\|_{\mathrm{L}^\infty}\int_0^t \|w^{t-u}\|_{\mathrm{L}^q}\int |\nabla\Gamma^u(y)|dydu,$$

hence by a direct computation

$$\|w^t\|_{\mathrm{L}^q} \lesssim \|w^\circ\|_{\mathrm{L}^q} + \|\phi_g^t\|_{\mathrm{L}^q} + \int_0^t \|w^{t-u}\|_{\mathrm{L}^q} u^{-1/2}du.$$

Integrating with respect to t, the triangle and the Hölder inequalities yield

$$\|w\|_{\mathrm{L}^p_t \mathrm{L}^q} \lesssim t^{1/p}\|w^\circ\|_{\mathrm{L}^q} + \|\phi_g\|_{\mathrm{L}^p_t \mathrm{L}^q} + \bigg(\int_0^t \Big(\int_0^t \mathbb{1}_{u<v}\|w^{v-u}\|_{\mathrm{L}^q}u^{-1/2}du\Big)^p dv\bigg)^{1/p}$$
$$\lesssim t^{1/p}\|w^\circ\|_{\mathrm{L}^q} + \|\phi_g\|_{\mathrm{L}^p_t \mathrm{L}^q} + \int_0^t \|w\|_{\mathrm{L}^p_u \mathrm{L}^q}(t-u)^{-1/2}du$$
$$\lesssim t^{1/p}\|w^\circ\|_{\mathrm{L}^q} + \|\phi_g\|_{\mathrm{L}^p_t \mathrm{L}^q} + (1-r'/2)^{-1/r'}t^{\frac12-\frac1r}\bigg(\int_0^t \|w\|_{\mathrm{L}^p_u \mathrm{L}^q}^r du\bigg)^{1/r},$$

for all $r>2$. Noting that $(1-r'/2)^{-1/r'}\lesssim 1+(r-2)^{-1/2}$, and optimizing in r, the Grönwall inequality then gives

$$\|w\|_{\mathrm{L}^p_t\mathrm{L}^q}\lesssim (t^{1/p}\|w^\circ\|_{\mathrm{L}^q}+\|\phi_g\|_{\mathrm{L}^p_t\mathrm{L}^q})\exp\Big(\inf_{2<r<\infty}\frac{C^r}{r}(1+(r-2)^{-r/2})t^{r/2}\Big).\tag{2.3}$$

Now it remains to estimate the norm of ϕ_g. A similar computation as above yields $\|\phi_g\|_{\mathrm{L}^p_t\mathrm{L}^q}\lesssim t^{1/2}\|g\|_{\mathrm{L}^p_t\mathrm{L}^q}$, but a more careful estimate is needed. For $1\le s\le q$, we may estimate by the Hölder inequality

$$|\phi_g^t(x)|\le\int_0^t\Big(\int|\nabla\Gamma^u|^{s'/2}\Big)^{1/s'}\Big(\int|\nabla\Gamma^u(x-y)|^{s/2}|g^{t-u}(y)|^s dy\Big)^{1/s}du,$$

and hence, by the triangle inequality,

$$\|\phi_g^t\|_{\mathrm{L}^q}\le\int_0^t\Big(\int|\nabla\Gamma^u|^{s'/2}\Big)^{1/s'}\Big(\int|\nabla\Gamma^u|^{q/2}\Big)^{1/q}\Big(\int|g^{t-u}|^s\Big)^{1/s}du.$$

Assuming that $\kappa:=\frac{d}{2}\big(\frac1d+\frac1q-\frac1s\big)>0$ (note that $\kappa\le 1/2$ follows from the choice $s\le q$), a direct computation then yields

$$\|\phi_g^t\|_{\mathrm{L}^q}\lesssim\int_0^t u^{\kappa-1}\|g^{t-u}\|_{\mathrm{L}^s}du.$$

Integrating with respect to t, we find by the triangle inequality

$$\|\phi_g\|_{\mathrm{L}^p_t\mathrm{L}^q}\lesssim\int_0^t u^{\kappa-1}\Big(\int_0^{t-u}\|g^v\|^p_{\mathrm{L}^s}dv\Big)^{1/p}du\ \lesssim\kappa^{-1}t^\kappa\|g^v\|_{\mathrm{L}^p_t\mathrm{L}^s},$$

and the result (iii) follows from this together with (2.3). □

Another ingredient that we need is the following string of critical potential theory estimates. The Sobolev embedding for $W^{1,d}(\mathbb{R}^d)$ gives that $\|\nabla\triangle^{-1}w\|_{\mathrm{L}^\infty}$ is *almost* bounded by the $\mathrm{L}^d(\mathbb{R}^d)$-norm of w, while the Calderón-Zygmund theory gives that $\|\nabla^2\triangle^{-1}w\|_{\mathrm{L}^\infty}$ is *almost* bounded by the $\mathrm{L}^\infty(\mathbb{R}^d)$-norm of w. The following result makes these assertions precise in a quantitative way, somehow in the spirit of [9]. Item (iii) can be found e.g. in [29, Appendix] in a slightly different form, but we were unable to find items (i) and (ii) in the literature. (By $(-\triangle)^{-1}$ we henceforth mean the convolution with the Coulomb kernel, that is, given $w\in C^\infty_c(\mathbb{R}^d)$, $v=(-\triangle)^{-1}w$ is meant as the decaying solution of $-\triangle v=w$.)

Lemma 2.4 (Potential estimates in L^∞)**.** *Let $d\ge2$. For all $w\in C^\infty_c(\mathbb{R}^d)$ the following hold:*[1]

(i) for all $1\le p<d<q\le\infty$, choosing $\theta\in(0,1)$ such that $\frac1d=\frac\theta p+\frac{1-\theta}{q}$, we have

$$\|\nabla\triangle^{-1}w\|_{\mathrm{L}^\infty}\lesssim\big((1-d/q)\wedge(1-p/d)\big)^{-1+1/d}\|w\|_{\mathrm{L}^d}\bigg(1+\log\frac{\|w\|^\theta_{\mathrm{L}^p}\|w\|^{1-\theta}_{\mathrm{L}^q}}{\|w\|_{\mathrm{L}^d}}\bigg)^{1-1/d};$$

(ii) if $w=\operatorname{div}\xi$ for $\xi\in C^\infty_c(\mathbb{R}^d)^d$, then, for all $d<q\le\infty$ and $1\le p<\infty$, we have

$$\|\nabla\triangle^{-1}w\|_{\mathrm{L}^\infty}\lesssim(1-d/q)^{-1+1/d}\|w\|_{\mathrm{L}^d}\bigg(1+\log^+\frac{\|w\|_{\mathrm{L}^q}}{\|w\|_{\mathrm{L}^d}}\bigg)^{1-1/d}+p\|\xi\|_{\mathrm{L}^p};$$

(iii) for all $0<s\le1$ and $1\le p<\infty$, we have

$$\|\nabla^2\triangle^{-1}w\|_{\mathrm{L}^\infty}\lesssim s^{-1}\|w\|_{\mathrm{L}^\infty}\bigg(1+\log\frac{\|w\|_{C^s}}{\|w\|_{\mathrm{L}^\infty}}\bigg)+p\|w\|_{\mathrm{L}^p}.$$

1. A direct adaptation of the proof shows that in parts (i) and (ii) the L^∞-norms in the left-hand sides could be replaced by Hölder C^ϵ-norms with $\epsilon\in[0,1)$: the exponents d in the right-hand sides then need to be replaced by $d/(1-\epsilon)>d$, and an additional multiplicative prefactor $(1-\epsilon)^{-1}$ is further needed.

Proof. Recall that $-\triangle^{-1}w = g_d * w$, where $g_d(x) = c_d|x|^{2-d}$ if $d > 2$ and $g_2(x) = -c_2 \log|x|$ if $d = 2$. The stated result is based on suitable decompositions of this Green integral. We split the proof into three steps, separately proving items (i), (ii) and (iii).

Step 1: proof of (i). Let $0 < \gamma \le \Gamma < \infty$. The obvious estimate $|\nabla\triangle^{-1}w(x)| \lesssim \int |x-y|^{1-d}|w(y)|dy$ may be decomposed as

$$|\nabla\triangle^{-1}w(x)| \lesssim \int_{|x-y|<\gamma} |x-y|^{1-d}|w(y)|dy + \int_{\gamma<|x-y|<\Gamma} |x-y|^{1-d}|w(y)|dy + \int_{|x-y|>\Gamma} |x-y|^{1-d}|w(y)|dy.$$

Let $1 \le p < d < q \le \infty$. We use the Hölder inequality with exponents $(q/(q-1), q)$ for the first term, $(d/(d-1), d)$ for the second, and $(p/(p-1), p)$ for the third, which yields after straightforward computations

$$|\nabla\triangle^{-1}w(x)| \lesssim (q'(1-d/q))^{-1/q'}\gamma^{1-d/q}\|w\|_{\mathrm{L}^q} + (\log(\Gamma/\gamma))^{(d-1)/d}\|w\|_{\mathrm{L}^d} + (p'(d/p-1))^{-1/p'}\Gamma^{1-d/p}\|w\|_{\mathrm{L}^p}.$$

Item (i) now easily follows, choosing $\gamma^{1-d/q} = \|w\|_{\mathrm{L}^d}/\|w\|_{\mathrm{L}^q}$ and $\Gamma^{d/p-1} = \|w\|_{\mathrm{L}^p}/\|w\|_{\mathrm{L}^d}$, noting that $\gamma \le \Gamma$ follows from interpolation of L^d between L^p and L^∞, and observing that

$$(q'(1-d/q))^{-1/q'} \lesssim (1-d/q)^{-1+1/d}, \qquad (p'(d/p-1))^{-1/p'} \lesssim (1-p/d)^{-1+1/d}.$$

Step 2: proof of (ii). Let $0 < \gamma \le 1 \le \Gamma < \infty$, and let χ_Γ denote a cut-off function with $\chi_\Gamma = 0$ on B_Γ, $\chi_\Gamma = 1$ outside $B_{\Gamma+1}$, and $|\nabla\chi_\Gamma| \le 2$. We may then decompose

$$\begin{aligned}\nabla\triangle^{-1}w(x) = &\int_{|x-y|<\gamma} \nabla g_d(x-y)w(y)dy + \int_{\gamma\le|x-y|\le\Gamma} \nabla g_d(x-y)w(y)dy \\ &+ \int_{\Gamma\le|x-y|\le\Gamma+1} \nabla g_d(x-y)(1-\chi_\Gamma(x-y))w(y)dy + \int_{|x-y|\ge\Gamma} \nabla g_d(x-y)\chi_\Gamma(x-y)w(y)dy.\end{aligned}$$

Using $w = \operatorname{div}\xi$ and integrating by parts, the last term becomes

$$\int \nabla g_d(x-y)\chi_\Gamma(x-y)w(y)dy = \int \nabla g_d(x-y)\otimes\nabla\chi_\Gamma(x-y)\cdot\xi(y)dy + \int \chi_\Gamma(x-y)\nabla^2 g_d(x-y)\cdot\xi(y)dy.$$

Choosing $\Gamma = 1$, we may then estimate

$$|\nabla\triangle^{-1}w(x)| \lesssim \int_{|x-y|<\gamma} |x-y|^{1-d}|w(y)|dy + \int_{\gamma\le|x-y|\le2} |x-y|^{1-d}|w(y)|dy + \int_{|x-y|\ge1} |x-y|^{-d}|\xi(y)|dy.$$

Using the Hölder inequality just as in the proof of (i) above for the first two terms, with $d < q \le \infty$, and using the Hölder inequality with exponents $(p/(p-1), p)$ for the last line, we obtain, for any $1 \le p < \infty$,

$$|\nabla\triangle^{-1}w(x)| \lesssim (q'(1-d/q))^{-1/q'}\gamma^{1-d/q}\|w\|_{\mathrm{L}^q} + (\log(2/\gamma))^{(d-1)/d}\|w\|_{\mathrm{L}^d} + (d(p'-1))^{-1/p'}\|\xi\|_{\mathrm{L}^p},$$

so that item (ii) follows from the choice $\gamma^{1-d/q} = 1 \wedge (\|w\|_{\mathrm{L}^d}/\|w\|_{\mathrm{L}^q})$, observing that $(d(p'-1))^{-1/p'} \le p$.

Step 3: proof of (iii). Given $0 < \gamma \le 1$, using the integration by parts

$$\int_{|x-y|<\gamma} \nabla^2 g_d(x-y)dy = \int_{|x-y|=\gamma} n\otimes\nabla g_d(x-y)dy,$$

and using the notation $(x-y)^{\otimes2} := (x-y)\otimes(x-y)$, we may decompose

$$\begin{aligned}|\nabla^2\triangle^{-1}w(x)| &\lesssim \left|\int_{|x-y|<\gamma} \frac{(x-y)^{\otimes2}}{|x-y|^{d+2}}w(y)dy\right| + \left|\int_{\gamma\le|x-y|<1} \frac{(x-y)^{\otimes2}}{|x-y|^{d+2}}w(y)dy\right| + \left|\int_{|x-y|\ge1} \frac{(x-y)^{\otimes2}}{|x-y|^{d+2}}w(y)dy\right| \\ &\lesssim \left|\int_{|x-y|<\gamma} \frac{(x-y)^{\otimes2}}{|x-y|^{d+2}}(w(x)-w(y))dy\right| + |w(x)|\left|\int_{|x-y|=\gamma} \frac{x-y}{|x-y|^d}dy\right| \\ &\qquad + \left|\int_{\gamma\le|x-y|<1} \frac{(x-y)^{\otimes2}}{|x-y|^{d+2}}w(y)dy\right| + \left|\int_{|x-y|\ge1} \frac{(x-y)^{\otimes2}}{|x-y|^{d+2}}w(y)dy\right|.\end{aligned}$$

Let $0 < s \le 1$ and $1 \le p < \infty$. Using the inequality $|w(x) - w(y)| \le |x-y|^s |w|_{C^s}$, and then applying the Hölder inequality with exponents $(1,\infty)$ for the first three terms, and $(p/(p-1), p)$ for the last one, we obtain after straightforward computations

$$|\nabla^2 \triangle^{-1} w(x)| \lesssim s^{-1}\gamma^s |w|_{C^s} + \|w\|_{\mathrm{L}^\infty} + |\log\gamma| \|w\|_{\mathrm{L}^\infty} + (d(p'-1))^{-1/p'} \|w\|_{\mathrm{L}^p}.$$

Item (iii) then follows for the choice $\gamma^s = \|w\|_{\mathrm{L}^\infty}/\|w\|_{C^s} \le 1$. □

In addition to the Sobolev regularity of solutions, the Hölder regularity is also studied in the sequel, in the framework of the usual Besov spaces $C^s_*(\mathbb{R}^d) := B^s_{\infty,\infty}(\mathbb{R}^d)$ (see e.g. [5]). These spaces actually coincide with the usual Hölder spaces $C^s(\mathbb{R}^d)$ only for non-integer $s \ge 0$ (for integer $s \ge 0$, they are strictly larger than $W^{s,\infty}(\mathbb{R}^d) \supset C^s(\mathbb{R}^d)$ and coincide with the corresponding Zygmund spaces). The following potential theory estimates are then needed both in Sobolev and in Hölder-Zygmund spaces.

Lemma 2.5 (Potential estimates in Sobolev and in Hölder-Zygmund spaces)**.** *Let $d \ge 2$. For all $w \in C^\infty_c(\mathbb{R}^d)$, the following hold:*

(i) for all $s \ge 0$,

$$\|\nabla \triangle^{-1} w\|_{H^s} \lesssim \|w\|_{\dot H^{-1} \cap H^{s-1}}, \qquad \|\nabla^2 \triangle^{-1} w\|_{H^s} \lesssim \|w\|_{H^s};$$

(ii) for all $s \in \mathbb{R}$,

$$\|\nabla \triangle^{-1} w\|_{C^s_*} \lesssim_s \|w\|_{\dot H^{-1} \cap C^{s-1}_*}, \qquad \|\nabla^2 \triangle^{-1} w\|_{C^s_*} \lesssim_s \|w\|_{\dot H^{-1} \cap C^s_*},$$

and for all $1 \le p < d$ and $1 \le q < \infty$,

$$\|\nabla \triangle^{-1} w\|_{C^s_*} \lesssim_{p,s} \|w\|_{\mathrm{L}^p \cap \mathrm{L}^\infty \cap C^{s-1}_*}, \qquad \|\nabla^2 \triangle^{-1} w\|_{C^s_*} \lesssim_{q,s} \|w\|_{\mathrm{L}^q \cap C^s_*},$$

where the subscripts s, p, q indicate the additional dependence of the multiplicative constants on an upper bound on s, $(d-p)^{-1}$, and q, respectively.

Proof. As item (i) is obvious via Fourier transform, we focus on item (ii). Let $s \in \mathbb{R}$, let $\chi \in C^\infty_c(\mathbb{R}^d)$ be fixed with $\chi = 1$ in a neighborhood of the origin, and let $\chi(\nabla)$ denote the corresponding pseudo-differential operator. Applying [5, Proposition 2.78] to the operator $(1-\chi(\nabla))\nabla\triangle^{-1}$, we find

$$\|\nabla \triangle^{-1} w\|_{C^s_*} \le \|(1-\chi(\nabla))\nabla\triangle^{-1} w\|_{C^s_*} + \|\chi(\nabla)\nabla\triangle^{-1} w\|_{C^s_*} \lesssim \|w\|_{C^{s-1}_*} + \|\chi(\nabla)\nabla\triangle^{-1} w\|_{C^s_*}.$$

Let k denote the smallest integer $\ge s \vee 0$. Noting that $\|v\|_{C^s_*} \lesssim \sum_{j=0}^k \|\nabla^j v\|_{\mathrm{L}^\infty}$ holds for all v, we deduce

$$\|\nabla \triangle^{-1} w\|_{C^s_*} \lesssim \|w\|_{C^{s-1}_*} + \sum_{j=0}^k \|\nabla^j \chi(\nabla)\nabla\triangle^{-1} w\|_{\mathrm{L}^\infty},$$

and similarly

$$\|\nabla^2 \triangle^{-1} w\|_{C^s_*} \lesssim \|w\|_{C^s_*} + \sum_{j=0}^k \|\nabla^j \chi(\nabla)\nabla^2\triangle^{-1} w\|_{\mathrm{L}^\infty}.$$

Writing $\nabla^j\chi(\nabla)\nabla\triangle^{-1} w = \nabla^j\chi * \nabla\triangle^{-1} w$, we find

$$\|\nabla^j\chi(\nabla)\nabla\triangle^{-1} w\|_{\mathrm{L}^\infty} \le \|\nabla^j\chi\|_{\mathrm{L}^2} \|\nabla\triangle^{-1} w\|_{\mathrm{L}^2} = \|\nabla^j\chi\|_{\mathrm{L}^2} \|w\|_{\dot H^{-1}},$$

and the first two estimates in item (ii) follow. Rather writing $\nabla^j\chi(\nabla)\nabla\triangle^{-1} w = \nabla\triangle^{-1}(\nabla^j\chi * w)$, and using the obvious estimate $|\nabla\triangle^{-1} v(x)| \lesssim \int |x-y|^{1-d} |v(y)| dy$ as in the proof of Lemma 2.4, we find for all $1 \le p < d$,

$$\begin{aligned} \|\nabla^j\chi(\nabla)\nabla\triangle^{-1} w\|_{\mathrm{L}^\infty} &\lesssim \sup_x \int_{|x-y|\le 1} |x-y|^{1-d} |\nabla^j\chi * w(y)| dy + \sup_x \int_{|x-y|>1} |x-y|^{1-d} |\nabla^j\chi * w(y)| dy \\ &\lesssim_p \|\nabla^j\chi * w\|_{\mathrm{L}^p \cap \mathrm{L}^\infty} \lesssim \|\nabla^j\chi\|_{\mathrm{L}^1} \|w\|_{\mathrm{L}^p \cap \mathrm{L}^\infty}, \end{aligned}$$

and the third estimate in item (ii) follows. The last estimate in (ii) is now easily obtained, arguing similarly as in the proof of Lemma 2.4(iii). □

We now recall some global elliptic regularity results for the operator $-\operatorname{div}(b\nabla)$ on the whole plane $\mathbb{R}^2$; as no reference was found in the literature, a detailed proof is included.

Lemma 2.6 (Global elliptic regularity)**.** *Let $b \in W^{1,\infty}(\mathbb{R}^2)^{2\times 2}$ be uniformly elliptic,* $\mathrm{Id} \le b \le \Lambda\,\mathrm{Id}$*, for some $\Lambda < \infty$. Given $f : \mathbb{R}^2 \to \mathbb{R}$, $g : \mathbb{R}^2 \to \mathbb{R}^2$, we consider the following equations in $\mathbb{R}^2$,*

$$-\operatorname{div}(b\nabla u) = f, \qquad \text{and} \qquad -\operatorname{div}(b\nabla v) = \operatorname{div} g.$$

The following properties hold.

(i) Meyers type estimates: *There exists $2 < p_0, q_0, r_0 < \infty$ (depending only on an upper bound on Λ) such that for all $2 < p \le p_0$, all $q_0 \le q < \infty$, and all $r_0' \le r \le r_0$,*

$$\|\nabla u\|_{\mathrm{L}^p} \le C_p \|f\|_{\mathrm{L}^{2p/(p+2)}}, \qquad \|v\|_{\mathrm{L}^q} \le C_q \|g\|_{\mathrm{L}^{2q/(q+2)}}, \qquad \text{and} \qquad \|\nabla v\|_{\mathrm{L}^r} \le C \|g\|_{\mathrm{L}^r},$$

for some constant C depending only on an upper bound on Λ, and for constants C_p and C_q further depending on an upper bound on $(p-2)^{-1}$ and q, respectively.

(ii) Sobolev regularity: *For all $s \ge 0$, we have $\|\nabla u\|_{H^s} \le C_s \|f\|_{\dot H^{-1}\cap H^{s-1}}$ and $\|\nabla v\|_{H^s} \le C_s \|g\|_{H^s}$, where the constant C_s depends only on an upper bound on s and on $\|b\|_{W^{s,\infty}}$.*

(iii) Schauder type estimate: *For all $s \in (0,1)$, we have $|\nabla u|_{C^s} \le C_s \|f\|_{\mathrm{L}^{2/(1-s)}}$ and $|v|_{C^s} \le C_s' \|g\|_{\mathrm{L}^{2/(1-s)}}$, where the constant C_s (resp. C_s') depends only on s and on an upper bound on $\|b\|_{W^{s,\infty}}$ (resp. on s and on the modulus of continuity of b).*

In particular, we have $\|\nabla u\|_{\mathrm{L}^\infty} \le C\|f\|_{\mathrm{L}^1\cap\mathrm{L}^\infty}$ and $\|v\|_{\mathrm{L}^\infty} \le C'\|g\|_{\mathrm{L}^1\cap\mathrm{L}^\infty}$, where the constant C (resp. C') depends only on an upper bound on $\|b\|_{W^{1,\infty}}$ (resp. Λ).

Proof. We split the proof into three steps, first proving (i) as a consequence of Meyers' perturbative argument, then turning to the Sobolev regularity (ii), and finally to the Schauder type estimate (iii). The additional L^∞-estimate for v directly follows from item (i) and the Sobolev embedding, while the corresponding estimate for ∇u follows from items (i) and (iii) by interpolation: for $2 < p \le p_0$ and $s \in (0,1)$, we indeed find

$$\|\nabla u\|_{\mathrm{L}^\infty} \lesssim \|\nabla u\|_{\mathrm{L}^p} + |\nabla u|_{C^s} \le C_p\|f\|_{\mathrm{L}^{2p/(p+2)}} + C_s\|f\|_{\mathrm{L}^{2/(1-s)}} \le C_{p,s}\|f\|_{\mathrm{L}^1\cap\mathrm{L}^\infty}.$$

In the proof below, we use the notation $\lesssim$ for $\le$ up to a constant $C > 0$ that depends only on an upper bound on Λ, and we add subscripts to indicate dependence on further parameters.

Step 1: proof of (i). We first consider the norm of v. By Meyers' perturbative argument [33], there exists some $1 < r_0 < 2$ (depending only on Λ) such that $\|\nabla v\|_{\mathrm{L}^r} \lesssim \|g\|_{\mathrm{L}^r}$ holds for all $r_0 \le r \le r_0'$. On the other hand, decomposing the equation for v as

$$-\triangle v = \operatorname{div}(g + (b-1)\nabla v),$$

we deduce from Riesz potential theory that for all $1 < r < 2$

$$\|v\|_{\mathrm{L}^{2r/(2-r)}} \lesssim_r \|g + (b-1)\nabla v\|_{\mathrm{L}^r} \lesssim \|g\|_{\mathrm{L}^r} + \|\nabla v\|_{\mathrm{L}^r},$$

and hence $\|v\|_{\mathrm{L}^{2r/(2-r)}} \lesssim_r \|g\|_{\mathrm{L}^r}$ for all $r_0 \le r < 2$, that is, $\|v\|_{\mathrm{L}^q} \lesssim_q \|g\|_{\mathrm{L}^{2q/(q+2)}}$ for all $\frac{2r_0}{2-r_0} \le q < \infty$.

We now turn to the norm of ∇u. The proof follows from a suitable adaptation of Meyers' perturbative argument [33], again combined with Riesz potential theory; for the reader's convenience, a complete proof is included. First recall that the Calderón-Zygmund theory yields $\|\nabla^2\triangle w\|_{\mathrm{L}^p} \le K_p\|w\|_{\mathrm{L}^p}$ for all $1 < p < \infty$ and all $w \in C_c^\infty(\mathbb{R}^2)$, where the constants K_p's moreover satisfy $\limsup_{p\to 2} K_p \le K_2$, while a simple energy estimate allows to choose $K_2 = 1$. Now rewriting the equation for u as

$$-\triangle u = \frac{2}{\Lambda+1} f + \operatorname{div}\left(\frac{2}{\Lambda+1}\Big(b - \frac{\Lambda+1}{2}\Big)\nabla u\right),$$

we deduce from Riesz potential theory and from the Calderón-Zygmund theory (applied to the first and to the second right-hand side term, respectively), for all $2 < p < \infty$,

$$\begin{aligned}\|\nabla u\|_{\mathrm{L}^p} &\le \frac{2}{\Lambda+1}\|\nabla\triangle^{-1}f\|_{\mathrm{L}^p} + \Big\|\nabla\triangle^{-1}\operatorname{div}\Big(\frac{2}{\Lambda+1}\Big(b-\frac{\Lambda+1}{2}\Big)\nabla u\Big)\Big\|_{\mathrm{L}^p}\\ &\le \frac{2C_p}{\Lambda+1}\|f\|_{\mathrm{L}^{2p/(p+2)}} + \frac{2K_p}{\Lambda+1}\Big\|\Big(b-\frac{\Lambda+1}{2}\Big)\nabla u\Big\|_{\mathrm{L}^p}\\ &\le \frac{2C_p}{\Lambda+1}\|f\|_{\mathrm{L}^{2p/(p+2)}} + \frac{K_p(\Lambda-1)}{\Lambda+1}\|\nabla u\|_{\mathrm{L}^p},\end{aligned}$$

where the last inequality follows from $\mathrm{Id} \le b \le \Lambda\,\mathrm{Id}$. Since we have $\frac{\Lambda-1}{\Lambda+1} < 1$ and $\limsup_{p\to 2} K_p \le K_2 = 1$, we may choose $p_0 > 2$ close enough to 2 such that $\frac{K_p(\Lambda-1)}{\Lambda+1} < 1$ holds for all $2 \le p \le p_0$. This allows to absorb the last term of the above right-hand side, and to conclude $\|\nabla u\|_{\mathrm{L}^p} \lesssim_p \|f\|_{\mathrm{L}^{2p/(p+2)}}$ for all $2 < p \le p_0$.

Step 2: proof of (ii). We focus on the result for u, as the argument for v is very similar. A simple energy estimate yields

$$\int |\nabla u|^2 \le \int \nabla u\cdot b\nabla u = \int fu \le \|f\|_{\dot H^{-1}}\|\nabla u\|_{\mathrm{L}^2},$$

hence $\|\nabla u\|_{\mathrm{L}^2} \le \|f\|_{\dot H^{-1}}$, that is, (ii) with $s = 0$. The result (ii) for any integer $s \ge 0$ is then deduced by induction, successively differentiating the equation. It remains to consider the case of fractional values $s \ge 0$. We only display the argument for $0 < s < 1$, while the other cases are similarly obtained after differentiation of the equation. Let $0 < s < 1$ be fixed. We use the following finite difference characterization of the fractional Sobolev space $H^s(\mathbb{R}^2)$: a function $w \in \mathrm{L}^2(\mathbb{R}^2)$ belongs to $H^s(\mathbb{R}^2)$, if and only if it satisfies $\|w - w(\cdot + h)\|_{\mathrm{L}^2} \le K|h|^s$ for all $h \in \mathbb{R}^2$, for some $K > 0$, and we then have $\|w\|_{\dot H^s} \le K$. This characterization is easily checked, using e.g. the identity $\|w - w(\cdot+h)\|^2_{\mathrm{L}^2} \simeq \int |1 - e^{i\xi\cdot h}|^2|\hat w(\xi)|^2 d\xi$, where $\hat w$ denotes the Fourier transform of w, and noting that $|1 - e^{ia}| \le 2 \wedge |a|$ holds for all $a \in \mathbb{R}$. Now applying finite difference to the equation for u, we find for all $h \in \mathbb{R}^2$,

$$-\operatorname{div}(b(\cdot+h)(\nabla u - \nabla u(\cdot+h))) = \operatorname{div}((b - b(\cdot+h))\nabla u) + f - f(\cdot+h),$$

and hence, testing against $u - u(\cdot + h)$,

$$\begin{aligned}\int |\nabla u - \nabla u(\cdot+h)|^2 &\le -\int (\nabla u - \nabla u(\cdot+h))\cdot(b - b(\cdot+h))\nabla u + \int (u - u(\cdot+h))(f - f(\cdot+h))\\ &\le |h|^s|b|_{C^s}\|\nabla u\|_{\mathrm{L}^2}\|\nabla u - \nabla u(\cdot+h)\|_{\mathrm{L}^2} + \|f - f(\cdot+h)\|_{\dot H^{-1}}\|\nabla u - \nabla u(\cdot+h)\|_{\mathrm{L}^2},\end{aligned}$$

where we compute by means of Fourier transforms

$$\|f - f(\cdot+h)\|^2_{\dot H^{-1}} \simeq \int |\xi|^{-2}|1 - e^{i\xi\cdot h}|^2|\hat f(\xi)|^2 d\xi \lesssim \int |\xi|^{-2}|\xi\cdot h|^{2s}|\hat f(\xi)|^2 d\xi \lesssim |h|^{2s}\|f\|^2_{\dot H^{-1}\cap H^{s-1}}.$$

Further combining this with the L^2-estimate for ∇u proven at the beginning of this step, we conclude

$$\|\nabla u - \nabla u(\cdot+h)\|_{\mathrm{L}^2} \lesssim |h|^s(|b|_{C^s}\|\nabla u\|_{\mathrm{L}^2} + \|f\|_{\dot H^{-1}\cap H^{s-1}}) \lesssim |h|^s(1 + |b|_{C^s})\|f\|_{\dot H^{-1}\cap H^{s-1}},$$

and the result follows from the above stated characterization of $H^s(\mathbb{R}^2)$.

Step 3: proof of (iii). We focus on the result for u, while that for v is easily obtained as an adaptation of [22, Theorem 3.8]. Let $x_0 \in \mathbb{R}^2$ be fixed. The equation for u may be rewritten as

$$-\operatorname{div}(b(x_0)\nabla u) = f + \operatorname{div}((b - b(x_0))\nabla u).$$

For all $r > 0$, let $w_r \in u + H^1_0(B(x_0, r))$ be the unique solution of $-\operatorname{div}(b(x_0)\nabla w_r) = 0$ in $B(x_0, r)$. The difference $v_r := u - w_r \in H^1_0(B(x_0, r))$ then satisfies in $B(x_0, r)$

$$-\operatorname{div}(b(x_0)\nabla v_r) = f + \operatorname{div}((b - b(x_0))\nabla u).$$

Testing this equation against v_r itself, we obtain

$$\int |\nabla v_r|^2 \leq \Big|\int_{B(x_0,r)} f v_r\Big| + \int_{B(x_0,r)} |b-b(x_0)||\nabla u||\nabla v_r| \leq \Big|\int_{B(x_0,r)} f v_r\Big| + r^s |b|_{C^s} \|\nabla u\|_{\mathrm{L}^2(B(x_0,r))} \|\nabla v_r\|_{\mathrm{L}^2}.$$

We estimate the first term as follows

$$\Big|\int_{B(x_0,r)} f v_r\Big| = \Big|\int_{B(x_0,r)} \nabla v_r \cdot \nabla \triangle^{-1}(\mathbb{1}_{B(x_0,r)} f)\Big| \leq \|\nabla v_r\|_{\mathrm{L}^{p'}(B(x_0,r))} \|\nabla \triangle^{-1}(\mathbb{1}_{B(x_0,r)} f)\|_{\mathrm{L}^p},$$

and hence by Riesz potential theory, for all $2 < p < \infty$,

$$\Big|\int_{B(x_0,r)} f v_r\Big| \lesssim_p \|\nabla v_r\|_{\mathrm{L}^{p'}(B(x_0,r))} \|f\|_{\mathrm{L}^{2p/(p+2)}(B(x_0,r))}.$$

The Hölder inequality then yields, choosing $q := \frac{2}{1-s} > 2$,

$$\Big|\int_{B(x_0,r)} f v_r\Big| \lesssim_p r^{\frac{2}{p'}-1} \|\nabla v_r\|_{\mathrm{L}^2}\ r^{1+\frac{2}{p}-\frac{2}{q}} \|f\|_{\mathrm{L}^q} = r^{2(1-\frac{1}{q})} \|\nabla v_r\|_{\mathrm{L}^2} \|f\|_{\mathrm{L}^q} = r^{1+s} \|\nabla v_r\|_{\mathrm{L}^2} \|f\|_{\mathrm{L}^{2/(1-s)}}.$$

Combining the above estimates, and using the inequality $2xy \leq x^2 + y^2$ to absorb the norms $\|\nabla v_r\|_{\mathrm{L}^2}$ appearing in the right-hand side, we find

$$\int |\nabla v_r|^2 \lesssim r^{2(1+s)} \|f\|^2_{\mathrm{L}^{2/(1-s)}} + r^{2s} |b|^2_{C^s} \|\nabla u\|^2_{\mathrm{L}^2(B(x_0,r))}.$$

We are now in position to conclude exactly as in the classical proof of the Schauder estimates (see e.g. [22, Theorem 3.13]). ☐

The interaction force v in equation (1.7) is defined by the values of $\operatorname{curl} v$ and $\operatorname{div}(av)$. The following result shows how v is controlled by such specifications.

Lemma 2.7. *Let $a, a^{-1} \in \mathrm{L}^\infty(\mathbb{R}^2)$. For all $\delta\omega, \delta\zeta \in \dot H^{-1}(\mathbb{R}^2)$, there exists a unique $\delta v \in \mathrm{L}^2(\mathbb{R}^2)^2$ such that $\operatorname{curl} \delta v = \delta\omega$ and $\operatorname{div}(a\delta v) = \delta\zeta$. Moreover, for all $s \geq 0$, if $a, a^{-1} \in W^{s,\infty}(\mathbb{R}^2)$ and $\delta\omega, \delta\zeta \in \dot H^{-1} \cap H^{s-1}(\mathbb{R}^2)$, we have*

$$\|\delta v\|_{H^s} \leq C\|\delta\omega\|_{\dot H^{-1}\cap H^{s-1}} + C\|\delta\zeta\|_{\dot H^{-1}\cap H^{s-1}},$$

where the constants C's depend only on an upper bound on s and $\|(a, a^{-1})\|_{W^{s,\infty}}$.

Proof. We split the proof into two steps.

Step 1: uniqueness. We prove that at most one function $\delta v \in \mathrm{L}^2(\mathbb{R}^2)^2$ can be associated with a given couple $(\delta\omega, \delta\zeta)$. For that purpose, we assume that $\delta v \in \mathrm{L}^2(\mathbb{R}^2)^2$ satisfies $\operatorname{curl} \delta v = 0$ and $\operatorname{div}(a\delta v) = 0$, and we deduce $\delta v = 0$. By the Hodge decompositions in $\mathrm{L}^2(\mathbb{R}^2)^2$, there exist functions $\phi, \psi \in H^1_{\mathrm{loc}}(\mathbb{R}^2)$ such that $a\delta v = \nabla\phi + \nabla^\perp\psi$, with $\nabla\phi, \nabla\psi \in \mathrm{L}^2(\mathbb{R}^2)^2$. Now note that $\triangle\phi = \operatorname{div}(a\delta v) = 0$ and $\operatorname{div}(a^{-1}\nabla\psi) + \operatorname{curl}(a^{-1}\nabla\phi) = \operatorname{curl} \delta v = 0$, which implies $\nabla\phi = 0$ and $\nabla\psi = 0$, hence $\delta v = 0$.

Step 2: existence. Given $\delta\omega, \delta\zeta \in \dot H^{-1}(\mathbb{R}^2)$, we deduce that $\nabla(\operatorname{div} a^{-1}\nabla)^{-1}\delta\omega$ and $\nabla(\operatorname{div} a\nabla)^{-1}\delta\zeta$ are well-defined in $\mathrm{L}^2(\mathbb{R}^2)^2$. The vector field

$$\delta v := a^{-1}\nabla^\perp(\operatorname{div} a^{-1}\nabla)^{-1}\delta\omega + \nabla(\operatorname{div} a\nabla)^{-1}\delta\zeta$$

is thus well-defined in $\mathrm{L}^2(\mathbb{R}^2)^2$, and trivially satisfies $\operatorname{curl} \delta v = \delta\omega$, $\operatorname{div}(a\delta v) = \delta\zeta$. The additional estimate follows from Lemmas 2.1 and 2.6(ii). ☐

As emphasized in Remark 1.2(i), weak solutions of the incompressible model (1.1) are rather defined via the vorticity formulation (1.7) in order to avoid compactness issues related to the pressure P. Although this will not be used in the sequel, we quickly check that under mild regularity assumptions a weak solution v of (1.1) automatically also satisfies equation (1.1) in the distributional sense on $[0, T) \times \mathbb{R}^2$ for some pressure $P : \mathbb{R}^2 \to \mathbb{R}$.

Lemma 2.8 (Control on the pressure)**.** *Let $\alpha, \beta \in \mathbb{R}$, $T > 0$, $h \in W^{1,\infty}(\mathbb{R}^2)$, and $\Psi, \bar v^\circ \in \mathrm{L}^\infty(\mathbb{R}^2)^2$. There exists $2 < q_0 \lesssim 1$ (depending only on an upper bound on $\|h\|_{\mathrm{L}^\infty}$) such that the following holds: If $v \in \mathrm{L}^\infty_{\mathrm{loc}}([0,T); \bar v^\circ + \mathrm{L}^2(\mathbb{R}^2)^2)$ is a weak solution of (1.1) on $[0,T) \times \mathbb{R}^2$ with $\omega := \operatorname{curl} v \in \mathrm{L}^\infty_{\mathrm{loc}}([0,T); \mathcal{P} \cap \mathrm{L}^{q_0}(\mathbb{R}^2))$, then v actually satisfies (1.1) in the distributional sense on $[0,T) \times \mathbb{R}^2$ for some pressure $P \in \mathrm{L}^\infty_{\mathrm{loc}}([0,T); \mathrm{L}^{q_0}(\mathbb{R}^2))$.*

Proof. In this proof, we use the notation $\lesssim$ for $\le$ up to a constant C depending only on an upper bound on $\|(h, \Psi, \bar v^\circ)\|_{\mathrm{L}^\infty}$. Let $2 < p_0, q_0 \lesssim 1$ and $r_0 = p_0$ be as in Lemma 2.6(i) (with b replaced by a or a^{-1}), and note that q_0 can be chosen large enough so that $\frac{1}{p_0} + \frac{1}{q_0} \le \frac{1}{2}$. Assume that $\omega \in \mathrm{L}^\infty_{\mathrm{loc}}([0,T); \mathcal{P} \cap \mathrm{L}^{q_0}(\mathbb{R}^2))$ holds for this choice of the exponent q_0. By Lemma 2.6(i), the function

$$P := (-\operatorname{div} a \nabla)^{-1} \operatorname{div}(a\omega(-\alpha(\Psi + v) + \beta(\Psi + v)^\perp))$$

is well-defined in $\mathrm{L}^\infty_{\mathrm{loc}}([0,T); \mathrm{L}^{q_0}(\mathbb{R}^2))$ and satisfies for all t,

$$\begin{aligned} \|P^t\|_{\mathrm{L}^{q_0}} &\lesssim \|a\omega^t(-\alpha(\Psi + v^t) + \beta(\Psi + v^t)^\perp)\|_{\mathrm{L}^{2q_0/(2+q_0)}} \\ &\lesssim \|\Psi + \bar v^\circ\|_{\mathrm{L}^\infty} \|\omega^t\|_{\mathrm{L}^{2q_0/(2+q_0)}} + \|v^t - \bar v^\circ\|_{\mathrm{L}^2} \|\omega^t\|_{\mathrm{L}^{q_0}} \\ &\lesssim (1 + \|v^t - \bar v^\circ\|_{\mathrm{L}^2}) \|\omega^t\|_{\mathrm{L}^1 \cap \mathrm{L}^{q_0}}. \end{aligned}$$

Now note that the following Helmholtz-Leray type identity follows from the proof of Lemma 2.7: for any vector field $F \in C^\infty_c(\mathbb{R}^2)^2$,

$$F = a^{-1} \nabla^\perp (\operatorname{div} a^{-1} \nabla)^{-1} \operatorname{curl} F + \nabla (\operatorname{div} a \nabla)^{-1} \operatorname{div}(aF). \tag{2.4}$$

This implies in particular, for the choice $F = \omega(-\alpha(\Psi + v) + \beta(\Psi + v)^\perp)$,

$$\begin{aligned} &a^{-1} \nabla^\perp (\operatorname{div} a^{-1} \nabla)^{-1} \operatorname{div}(\omega(\alpha(\Psi + v)^\perp + \beta(\Psi + v))) \\ &= a^{-1} \nabla^\perp (\operatorname{div} a^{-1} \nabla)^{-1} \operatorname{curl}(\omega(-\alpha(\Psi + v) + \beta(\Psi + v)^\perp)) \\ &= \omega(-\alpha(\Psi + v) + \beta(\Psi + v)^\perp) + \nabla P. \end{aligned} \tag{2.5}$$

For $\phi \in C^\infty_c([0,T) \times \mathbb{R}^2)^2$, we know by Lemma 2.6(i) that $(\operatorname{div} a^{-1} \nabla)^{-1} \operatorname{curl}(a^{-1}\phi) \in C^\infty_c([0,T); \mathrm{L}^{q_0}(\mathbb{R}^2))$ and $\nabla(\operatorname{div} a^{-1} \nabla)^{-1} \operatorname{curl}(a^{-1}\phi) \in C^\infty_c([0,T); \mathrm{L}^2 \cap \mathrm{L}^{p_0}(\mathbb{R}^2))$. With the choice $\frac{1}{p_0} + \frac{1}{q_0} \le \frac{1}{2}$, the L^{q_0}-regularity of ω then allows to test the weak formulation of (1.7) (which defines weak solutions of (1.1), cf. Definition 1.1(b)) against $(\operatorname{div} a^{-1} \nabla)^{-1} \operatorname{curl}(a^{-1}\phi)$, to the effect of

$$\begin{aligned} \int \omega^\circ (\operatorname{div} a^{-1} \nabla)^{-1} \operatorname{curl}(a^{-1}\phi(0,\cdot)) &+ \iint \omega (\operatorname{div} a^{-1} \nabla)^{-1} \operatorname{curl}(a^{-1}\partial_t \phi) \\ &= \iint \omega(\alpha(\Psi + v)^\perp + \beta(\Psi + v)) \cdot \nabla (\operatorname{div} a^{-1} \nabla)^{-1} \operatorname{curl}(a^{-1}\phi). \end{aligned}$$

As by (2.4) the constraint $\operatorname{div}(av) = 0$ implies $v = a^{-1} \nabla^\perp (\operatorname{div} a^{-1} \nabla)^{-1} \omega$ and $v^\circ = a^{-1} \nabla^\perp (\operatorname{div} a^{-1} \nabla)^{-1} \omega^\circ$, and as by definition $\omega \in \mathrm{L}^\infty_{\mathrm{loc}}([0,T); \mathrm{L}^1 \cap \mathrm{L}^2(\mathbb{R}^2))$, we deduce $v \in \mathrm{L}^\infty_{\mathrm{loc}}([0,T); \mathrm{L}^{p_0}(\mathbb{R}^2)^2)$ from Lemma 2.6(i). We may then integrate by parts in the weak formulation above, which yields

$$\int \phi(0,\cdot) \cdot v^\circ + \iint \partial_t \phi \cdot v = -\iint a^{-1}\phi \cdot \nabla^\perp (\operatorname{div} a^{-1} \nabla)^{-1} \operatorname{div}(\omega(\alpha(\Psi + v)^\perp + \beta(\Psi + v))),$$

and the result now directly follows from the decomposition (2.5). ∎

3 Local-in-time existence of smooth solutions

In this section, we prove the local-in-time existence of smooth solutions of (1.1) and of (1.2), as summarized in Theorem 2. Note that we choose to work here in the framework of Sobolev spaces, but the results could easily be adapted to Hölder spaces (compare indeed with Lemma 4.7). We first study the non-degenerate case $\lambda > 0$.

Proposition 3.1 (Local existence, non-degenerate case)**.** *Let $\alpha,\beta\in\mathbb{R}$, $\lambda>0$. Let $s>1$, and let $h,\Psi,\bar v^\circ\in W^{s+1,\infty}(\mathbb{R}^2)^2$. Let $v^\circ\in\bar v^\circ+H^{s+1}(\mathbb{R}^2)^2$ with $\omega^\circ:=\operatorname{curl}v^\circ$, $\bar\omega^\circ:=\operatorname{curl}\bar v^\circ\in H^s(\mathbb{R}^2)$, and with either $\operatorname{div}(av^\circ)=\operatorname{div}(a\bar v^\circ)=0$ in the case (1.1), or $\zeta^\circ:=\operatorname{div}(av^\circ)$, $\bar\zeta^\circ:=\operatorname{div}(a\bar v^\circ)\in H^s(\mathbb{R}^2)$ in the case (1.2). Then, there exists $T>0$ and a weak solution $v\in\mathrm{L}^\infty([0,T);\bar v^\circ+H^{s+1}(\mathbb{R}^2)^2)$ of (1.1) or of (1.2) on $[0,T)\times\mathbb{R}^2$ with initial data v°. Moreover, T depends only on an upper bound on $|\alpha|$, $|\beta|$, λ, λ^{-1}, s, $(s-1)^{-1}$, $\|(h,\Psi,\bar v^\circ)\|_{W^{s+1,\infty}}$, $\|v^\circ-\bar v^\circ\|_{H^{s+1}}$, $\|(\omega^\circ,\bar\omega^\circ,\zeta^\circ,\bar\zeta^\circ)\|_{H^s}$.*

Proof. We focus on the compressible case (1.2), the situation being similar and simpler in the incompressible case (1.1). Let $s>1$. We set up the following iterative scheme: let $v_0:=v^\circ$, $\omega_0:=\omega^\circ=\operatorname{curl}v^\circ$ and $\zeta_0:=\zeta^\circ=\operatorname{div}(av^\circ)$, and, for all $n\ge0$, given v_n, $\omega_n:=\operatorname{curl}v_n$, and $\zeta_n:=\operatorname{div}(av_n)$, we let ω_{n+1} and ζ_{n+1} solve on $\mathbb{R}^+\times\mathbb{R}^2$ the linear PDEs

$$\partial_t\omega_{n+1}=\operatorname{div}(\omega_{n+1}(\alpha(\Psi+v_n)^\perp+\beta(\Psi+v_n))),\quad \omega_{n+1}|_{t=0}=\omega^\circ,\tag{3.1}$$

$$\partial_t\zeta_{n+1}=\lambda\triangle\zeta_{n+1}-\lambda\operatorname{div}(\zeta_{n+1}\nabla h)+\operatorname{div}(a\omega_n(-\alpha(\Psi+v_n)+\beta(\Psi+v_n)^\perp)),\quad \zeta_{n+1}|_{t=0}=\zeta^\circ,\tag{3.2}$$

and we let v_{n+1} satisfy $\operatorname{curl}v_{n+1}=\omega_{n+1}$ and $\operatorname{div}(av_{n+1})=\zeta_{n+1}$. For all $n\ge0$, let also

$$t_n:=\sup\Big\{t\ge0:\|(\omega_n^t,\zeta_n^t)\|_{H^s}+\|v_n^t-\bar v^\circ\|_{H^{s+1}}\le C_0\Big\},$$

for some $C_0\ge1$ to be suitably chosen (depending on the initial data), and let $T_0:=\inf_n t_n$. We show that this iterative scheme is well-defined, that $T_0>0$, and that it converges to a solution of equation (1.2) on $[0,T_0)\times\mathbb{R}^2$.

We split the proof into two steps. In this proof, we use the notation $\lesssim$ for $\le$ up to a constant $C>0$ that depends only on an upper bound on $|\alpha|$, $|\beta|$, λ, λ^{-1}, s, $(s-1)^{-1}$, $\|(h,\Psi,\bar v^\circ)\|_{W^{s+1,\infty}}$, $\|v^\circ-\bar v^\circ\|_{H^{s+1}}$, $\|(\zeta^\circ,\bar\zeta^\circ)\|_{H^s}$, and $\|(\omega^\circ,\bar\omega^\circ)\|_{H^s}$.

Step 1: the iterative scheme is well-defined. In this step, we show that for all $n\ge0$ the system (3.1)–(3.2) admits a unique solution $(\omega_{n+1},\zeta_{n+1},v_{n+1})$ with $\omega_{n+1}\in\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+;H^s(\mathbb{R}^2))$, $\zeta_{n+1}\in\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+;H^s(\mathbb{R}^2))$, and $v_{n+1}\in\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+;\bar v^\circ+H^{s+1}(\mathbb{R}^2)^2)$, and that moreover for a suitable choice of $1\le C_0\lesssim1$ we have $T_0\ge C_0^{-4}>0$. We argue by induction. Let $n\ge0$ be fixed, and assume that (ω_n,ζ_n,v_n) is well-defined with $\omega_n\in\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+;H^s(\mathbb{R}^2))$, $\zeta_n\in\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+;H^s(\mathbb{R}^2))$, and $v_n\in\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+;\bar v^\circ+H^{s+1}(\mathbb{R}^2)^2)$. (For $n=0$, this is indeed trivial by assumption.)

We first study the equation for ω_{n+1}. By the Sobolev embedding with $s>1$, v_n is Lipschitz-continuous, and by assumption Ψ is also Lipschitz-continuous, hence the transport equation (3.1) admits a unique continuous solution ω_{n+1}, which automatically belongs to $\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+;\omega^\circ+\dot H^{-1}\cap H^s(\mathbb{R}^2))$ by Lemma 2.2. More precisely, for all $t\ge0$, Lemma 2.2 together with the Sobolev embedding for $s>1$ yields

$$\begin{aligned}\partial_t\|\omega_{n+1}^t\|_{H^s}&\le C(1+\|\nabla v_n^t\|_{\mathrm{L}^\infty})\|\omega_{n+1}^t\|_{H^s}+C\|\omega_{n+1}^t\|_{\mathrm{L}^\infty}\|\nabla(v_n^t-\bar v^\circ)\|_{H^s}\\&\le C(1+\|v_n^t-\bar v^\circ\|_{H^{s+1}})\|\omega_{n+1}^t\|_{H^s}.\end{aligned}$$

Hence, for all $t\in[0,t_n]$, we obtain $\partial_t\|\omega_{n+1}^t\|_{H^s}\le CC_0\|\omega_{n+1}^t\|_{H^s}$, which proves

$$\|\omega_{n+1}^t\|_{H^s}\le e^{CC_0t}\|\omega^\circ\|_{H^s}\le Ce^{CC_0t}.$$

Noting that

$$\|\omega^\circ-\bar\omega^\circ\|_{\dot H^{-1}}\le\|v^\circ-\bar v^\circ\|_{\mathrm{L}^2}\le C,$$

Lemma 2.2 together with the Sobolev embedding for $s>1$ also gives for all $t\ge0$,

$$\begin{aligned}\|\omega_{n+1}^t-\bar\omega^\circ\|_{\dot H^{-1}}\le C+\|\omega_{n+1}^t-\omega^\circ\|_{\dot H^{-1}}&\le C+Ct\|\omega_{n+1}\|_{\mathrm{L}^\infty_t\mathrm{L}^2}(1+\|v_n\|_{\mathrm{L}^\infty_t\mathrm{L}^\infty})\\&\le C+Ct\|\omega_{n+1}\|_{\mathrm{L}^\infty_tH^s}(1+\|v_n-\bar v^\circ\|_{\mathrm{L}^\infty_tH^s}),\end{aligned}$$

and hence, for all $t\in[0,t_n]$,

$$\|\omega_{n+1}^t-\bar\omega^\circ\|_{\dot H^{-1}}\le C(1+tC_0)e^{CC_0t}.$$

We now turn to ζ_{n+1}. Equation (3.2) (with $\lambda>0$) is a transport-diffusion equation and admits a unique solution ζ_{n+1}, which belongs to $\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+;(\zeta^\circ+\dot H^{-1}(\mathbb{R}^d))\cap H^s(\mathbb{R}^d))$ by Lemma 2.3(i)–(ii). More precisely, for all

$t \geq 0$, Lemma 2.3(i) yields for $s > 1$

$$\begin{aligned}\|\zeta_{n+1}^t\|_{H^s} &\leq Ce^{Ct}\big(\|\zeta^\circ\|_{H^s} + \|a\omega_n(\alpha(\Psi+v_n)^\perp + \beta(\Psi+v_n))\|_{\mathrm{L}^2_t H^s}\big)\\ &\leq Ce^{Ct}\big(1 + t^{1/2}\|\omega_n\|_{\mathrm{L}^\infty_t H^s}(1 + \|v_n - \bar v^\circ\|_{\mathrm{L}^\infty_t H^s})\big),\end{aligned} \tag{3.3}$$

where we have used Lemma 2.1 together with the Sobolev embedding to estimate the terms. Noting that

$$\|\zeta^\circ - \bar\zeta^\circ\|_{\dot H^{-1}} \leq \|av^\circ - a\bar v^\circ\|_{\mathrm{L}^2} \leq C,$$

Lemma 2.3(ii) together with the Sobolev embedding for $s > 1$ also gives for all $t \geq 0$,

$$\begin{aligned}\|\zeta_{n+1}^t - \bar\zeta^\circ\|_{\dot H^{-1}} \leq C + \|\zeta_{n+1}^t - \zeta^\circ\|_{\dot H^{-1}} &\leq C + Ce^{Ct}(\|\zeta^\circ\|_{\mathrm{L}^2} + \|a\omega_n(\alpha(\Psi+v_n)^\perp + \beta(\Psi+v_n))\|_{\mathrm{L}^2_t \mathrm{L}^2})\\ &\leq Ce^{Ct}(1 + t^{1/2}\|\omega_n\|_{\mathrm{L}^\infty_t H^s}(1 + \|v_n - \bar v^\circ\|_{\mathrm{L}^\infty_t H^s}).\end{aligned}$$

Combining this with (3.3) yields, for all $t \in [0, t_n]$,

$$\|\zeta_{n+1}^t\|_{H^s} + \|\zeta_{n+1}^t - \bar\zeta^\circ\|_{\dot H^{-1}} \leq Ce^{Ct}\big(1 + t^{1/2}C_0(1+C_0)\big) \leq C(1 + t^{1/2}C_0^2)e^{Ct}.$$

We finally turn to v_{n+1}. By the above properties of ω_{n+1} and ζ_{n+1}, Lemma 2.7 ensures that v_{n+1} is uniquely defined in $\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; \bar v^\circ + H^{s+1}(\mathbb{R}^2)^2)$ with $\operatorname{curl}(v_{n+1}^t - \bar v^\circ) = \omega_{n+1}^t - \bar\omega^\circ$ and $\operatorname{div}(a(v_{n+1}^t - \bar v^\circ)) = \zeta_{n+1}^t - \bar\zeta^\circ$ for all $t \geq 0$. More precisely, Lemma 2.7 gives for all $t \in [0, t_n]$

$$\begin{aligned}\|v_{n+1}^t - \bar v^\circ\|_{H^{s+1}} &\leq C\|\omega_{n+1}^t - \bar\omega^\circ\|_{\dot H^{-1}\cap H^s} + C\|\zeta_{n+1}^t - \bar\zeta^\circ\|_{\dot H^{-1}\cap H^s}\\ &\leq C + C\|\omega_{n+1}^t - \bar\omega^\circ\|_{\dot H^{-1}} + C\|\omega_{n+1}^t\|_{H^s} + C\|\zeta_{n+1}^t - \bar\zeta^\circ\|_{\dot H^{-1}} + C\|\zeta_{n+1}^t\|_{H^s}\\ &\leq C(1 + tC_0 + t^{1/2}C_0^2)e^{CC_0 t}.\end{aligned}$$

Hence, we have proven that $(\omega_{n+1}, \zeta_{n+1}, v_{n+1})$ is well-defined in the correct space, and moreover, combining all the previous estimates, we find for all $t \in [0, t_n]$

$$\|(\omega_{n+1}^t, \zeta_{n+1}^t)\|_{H^s} + \|v_{n+1}^t - \bar v^\circ\|_{H^{s+1}} \leq C(1 + tC_0 + t^{1/2}C_0^2)e^{CC_0 t}.$$

Therefore, choosing $C_0 = 1 + 3Ce^C \lesssim 1$, we obtain for all $t \leq t_n \wedge C_0^{-4}$

$$\|(\omega_{n+1}^t, \zeta_{n+1}^t)\|_{H^s} + \|v_{n+1}^t - \bar v^\circ\|_{H^{s+1}} \leq C_0,$$

and thus $t_{n+1} \geq t_n \wedge C_0^{-4}$. The result follows by induction.

Step 2: passing to the limit in the scheme. In this step, we show that up to an extraction the iterative scheme (ω_n, ζ_n, v_n) converges to a weak solution of equation (1.2) on $[0, T_0) \times \mathbb{R}^2$.

By Step 1, the sequences $(\omega_n)_n$ and $(\zeta_n)_n$ are bounded in $\mathrm{L}^\infty([0,T_0]; H^s(\mathbb{R}^2)^2)$, and the sequence $(v_n)_n$ is bounded in $\mathrm{L}^\infty([0,T_0]; \bar v^\circ + H^{s+1}(\mathbb{R}^2)^2)$. Up to an extraction, we thus have $\omega_n \overset{*}{\rightharpoonup} \omega$, $\zeta_n \overset{*}{\rightharpoonup} \zeta$ in $\mathrm{L}^\infty([0,T_0]; H^s(\mathbb{R}^2))$, and $v_n \overset{*}{\rightharpoonup} v$ in $\mathrm{L}^\infty([0,T_0]; \bar v^\circ + H^{s+1}(\mathbb{R}^2)^2)$. Comparing with equation (3.1), we deduce that $(\partial_t\omega_n)_n$ is bounded in $\mathrm{L}^\infty([0,T_0]; H^{s-1}(\mathbb{R}^2))$. Since by the Rellich theorem the space $H^s(U)$ is compactly embedded in $H^{s-1}(U)$ for any bounded domain $U \subset \mathbb{R}^2$, the Aubin-Simon lemma ensures that we have $\omega_n \to \omega$ strongly in $C^0([0,T_0]; H^{s-1}_{\mathrm{loc}}(\mathbb{R}^2))$. This implies in particular $\omega_n v_n \to \omega v$ in the distributional sense, and hence we may pass to the limit in the weak formulation of equations (3.1)–(3.2), which yields $\operatorname{curl} v = \omega$, $\operatorname{div}(av) = \zeta$, with ω and ζ satisfying, in the distributional sense on $[0, T_0) \times \mathbb{R}^2$,

$$\partial_t\omega = \operatorname{div}(\omega(\alpha(\Psi+v)^\perp + \beta(\Psi+v))), \quad \omega|_{t=0} = \omega^\circ,$$
$$\partial_t\zeta = \lambda\triangle\zeta - \lambda\operatorname{div}(\zeta\nabla h) + \operatorname{div}(a\omega(-\alpha(\Psi+v) + \beta(\Psi+v)^\perp)), \quad \zeta|_{t=0} = \zeta^\circ,$$

that is, the vorticity formulation (1.7)–(1.8). Let us quickly deduce that v is a weak solution of (1.2). From the above equations, we deduce $\partial_t\omega \in \mathrm{L}^\infty([0,T_0]; \dot H^{-1} \cap H^{s-1}(\mathbb{R}^2))$ and $\partial_t\zeta \in \mathrm{L}^\infty([0,T_0]; \dot H^{-1} \cap H^{s-2}(\mathbb{R}^2))$. Lemma 2.7 then implies $\partial_t v \in \mathrm{L}^\infty([0,T_0]; H^{s-1}(\mathbb{R}^2)^2)$. We may then deduce that the quantity $V := \partial_t v - \lambda\nabla(a^{-1}\zeta) + \alpha(\Psi+v)\omega - \beta(\Psi+v)^\perp\omega$ belongs to $\mathrm{L}^\infty([0,T_0]; \mathrm{L}^2(\mathbb{R}^2)^2)$ and satisfies $\operatorname{curl} V = \operatorname{div}(aV) = 0$ in the distributional sense. Using the Hodge decomposition in $\mathrm{L}^2(\mathbb{R}^2)^2$, we easily conclude $V = 0$, hence $v \in \mathrm{L}^\infty([0,T_0]; \bar v^\circ + H^{s+1}(\mathbb{R}^2)^2)$ is indeed a weak solution of (1.2) on $[0, T_0) \times \mathbb{R}^2$. □

We now turn to the local-in-time existence of smooth solutions of (1.2) in the degenerate case $\lambda = 0$; note that the proof only works in the parabolic regime $\beta = 0$.

Proposition 3.2 (Local existence, degenerate case). *Let $\alpha \in \mathbb{R}$, $\beta = \lambda = 0$. Let $s > 2$, and let $h \in W^{s,\infty}(\mathbb{R}^2)$, $\Psi, \bar v^\circ \in W^{s+1,\infty}(\mathbb{R}^2)^2$. Let $v^\circ \in \bar v^\circ + H^s(\mathbb{R}^2)^2$ with $\omega^\circ := \operatorname{curl} v^\circ$, $\bar\omega^\circ := \operatorname{curl} \bar v^\circ \in H^s(\mathbb{R}^2)$ and $\zeta^\circ := \operatorname{div}(av^\circ)$, $\bar\zeta^\circ := \operatorname{div}(a\bar v^\circ) \in H^{s-1}(\mathbb{R}^2)$. Then, there exists $T > 0$ and a weak solution $v \in \mathrm{L}^\infty([0,T); \bar v^\circ + H^s(\mathbb{R}^2)^2)$ of (1.2) on $[0,T) \times \mathbb{R}^2$, with initial data v°. Moreover, T depends only on an upper bound on $|\alpha|$, s, $(s-2)^{-1}$, $\|h\|_{W^{s,\infty}}$, $\|(\Psi, \bar v^\circ)\|_{W^{s+1,\infty}}$, $\|v^\circ - \bar v^\circ\|_{H^s}$, $\|(\omega^\circ, \bar\omega^\circ)\|_{H^s}$, and $\|(\zeta^\circ, \bar\zeta^\circ)\|_{H^{s-1}}$.*

Proof. We consider the same iterative scheme (ω_n, ζ_n, v_n) as in the proof of Proposition 3.1, but with $\lambda = \beta = 0$. Let $s > 2$. For all $n \ge 0$, let

$$t_n := \sup\Big\{t \ge 0 : \|\omega_n^t\|_{H^s} + \|\zeta_n^t\|_{H^{s-1}} + \|v_n^t - \bar v^\circ\|_{H^s} \le C_0\Big\},$$

for some $C_0 \ge 1$ to be suitably chosen (depending on initial data), and let $T_0 := \inf_n t_n$. In this proof, we use the notation $\lesssim$ for $\le$ up to a constant $C > 0$ that depends only on an upper bound on $|\alpha|$, s, $(s-2)^{-1}$, $\|h\|_{W^{s,\infty}}$, $\|(\Psi, \bar v^\circ)\|_{W^{s+1,\infty}}$, $\|v^\circ - \bar v^\circ\|_{H^s}$, $\|(\zeta^\circ, \bar\zeta^\circ)\|_{H^{s-1}}$, and $\|(\omega^\circ, \bar\omega^\circ)\|_{H^s}$.

Just as in the proof of Proposition 3.1, we first need to show that this iterative scheme is well-defined and that $T_0 > 0$. We proceed by induction: let $n \ge 0$ be fixed, and assume that (ω_n, ζ_n, v_n) is well-defined with $\omega_n \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; H^s(\mathbb{R}^2))$, $\zeta_n \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; H^{s-1}(\mathbb{R}^2))$, and $v_n \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; \bar v^\circ + H^s(\mathbb{R}^2)^2)$. (For $n = 0$ this is indeed trivial by assumption.)

We first study ζ_{n+1}. As $\lambda = 0$, equation (3.2) takes the form $\partial_t \zeta_{n+1} = -\alpha \operatorname{div}(a\omega_n(\Psi + v_n))$. Integrating this equation in time then yields

$$\|\zeta_{n+1}^t\|_{H^{s-1}} \le \|\zeta^\circ\|_{H^{s-1}} + |\alpha| \int_0^t \|\omega_n^u(\Psi + v_n^u)\|_{H^s} du \lesssim 1 + t(1 + \|v_n - \bar v^\circ\|_{\mathrm{L}^\infty_t H^s})\|\omega_n\|_{\mathrm{L}^\infty_t H^s}.$$

where we have used Lemma 2.1 together with the Sobolev embedding to estimate the last term. Similarly, noting that $\|\zeta^\circ - \bar\zeta^\circ\|_{\dot H^{-1}} \le \|av^\circ - a\bar v^\circ\|_{\mathrm{L}^2} \le C$, we find for $s > 1$

$$\begin{aligned}\|\zeta_{n+1}^t - \bar\zeta^\circ\|_{\dot H^{-1}} \le C + \|\zeta_{n+1}^t - \zeta^\circ\|_{\dot H^{-1}} &\le \|\zeta^\circ\|_{H^{s-1}} + |\alpha| \int_0^t \|\omega_n^u(\Psi + v_n^u)\|_{\mathrm{L}^2} du \\ &\lesssim 1 + t(1 + \|v_n - \bar v^\circ\|_{\mathrm{L}^\infty_t H^s})\|\omega_n\|_{\mathrm{L}^\infty_t H^s}.\end{aligned}$$

Hence we obtain for all $t \in [0, t_n]$

$$\|\zeta_{n+1}^t\|_{H^{s-1}} + \|\zeta_{n+1}^t - \bar\zeta^\circ\|_{\dot H^{-1}} \le C + Ct(1 + C_0)C_0 \le C(1 + tC_0^2).$$

We now turn to the study of ω_{n+1}. As $\beta = 0$, equation (3.1) takes the form $\partial_t \omega_{n+1} = \alpha \operatorname{div}(\omega_{n+1}(\Psi + v_n)^\perp)$. For all $t \ge 0$, Lemma 2.2 together with the Sobolev embedding then yields (here the choice $\beta = 0$ is crucial)

$$\begin{aligned}\partial_t \|\omega_{n+1}^t\|_{H^s} &\lesssim (1 + \|\nabla v_n^t\|_{\mathrm{L}^\infty})\|\omega_{n+1}^t\|_{H^s} + \|\omega_{n+1}^t\|_{\mathrm{L}^\infty} \|\operatorname{curl}(v_n^t - \bar v^\circ)\|_{H^s} \\ &\lesssim (1 + \|\omega_n^t\|_{H^s} + \|\nabla(v_n^t - \bar v^\circ)\|_{\mathrm{L}^\infty})\|\omega_{n+1}^t\|_{H^s}.\end{aligned}$$

By the Sobolev embedding for $s > 2$, we find, for all $t \in [0, t_n]$,

$$\partial_t \|\omega_{n+1}^t\|_{H^s} \le C(1 + \|\omega_n^t\|_{H^s} + \|v_n^t - \bar v^\circ\|_{H^s})\|\omega_{n+1}^t\|_{H^s} \le C(1 + 2C_0)\|\omega_{n+1}^t\|_{H^s},$$

and thus

$$\|\omega_{n+1}^t\|_{H^s} \le \|\omega^\circ\|_{H^s} e^{C(1+2C_0)t} \le Ce^{CC_0 t}.$$

Moreover, noting that $\|\omega^\circ - \bar\omega^\circ\|_{\dot H^{-1}} \le \|v^\circ - \bar v^\circ\|_{\mathrm{L}^2} \le C$, and applying Lemma 2.2 together with the Sobolev embedding, we obtain

$$\begin{aligned}\|\omega_{n+1}^t - \bar\omega^\circ\|_{\dot H^{-1}} &\le C + \|\omega_{n+1}^t - \omega^\circ\|_{\dot H^{-1}} \\ &\le C + Ct(1 + \|v_n\|_{\mathrm{L}^\infty_t \mathrm{L}^\infty})\|\omega_{n+1}\|_{\mathrm{L}^\infty_t \mathrm{L}^2} \\ &\le C + Ct(1 + \|v_n - \bar v^\circ\|_{\mathrm{L}^\infty_t H^s})\|\omega_{n+1}\|_{\mathrm{L}^\infty_t \mathrm{L}^2},\end{aligned}$$

hence for all $t \in [0, t_n]$

$$\|\omega_{n+1}^t - \bar{\omega}^\circ\|_{\dot{H}^{-1}} \le C + Ct(1 + C_0)\|\omega_{n+1}\|_{\mathrm{L}_t^\infty \mathrm{L}^2} \le C + CC_0 t e^{CC_0 t}.$$

We finally turn to v_{n+1}. By the above properties of ω_{n+1} and ζ_{n+1}, Lemma 2.7 ensures that v_{n+1} is uniquely defined in $\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; \bar{v}^\circ + H^s(\mathbb{R}^2)^2)$, and for all $t \in [0, t_n]$

$$\begin{aligned}
\|v_{n+1}^t - \bar{v}^\circ\|_{H^s} &\le C\|\omega_{n+1}^t - \bar{\omega}^\circ\|_{\dot{H}^{-1}\cap H^{s-1}} + C\|\zeta_{n+1}^t - \bar{\zeta}^\circ\|_{\dot{H}^{-1}\cap H^{s-1}} \\
&\le C + C\|\omega_{n+1}^t - \bar{\omega}^\circ\|_{\dot{H}^{-1}} + C\|\omega_{n+1}^t\|_{H^s} + C\|\zeta_{n+1}^t - \bar{\zeta}^\circ\|_{\dot{H}^{-1}} + C\|\zeta_{n+1}^t\|_{H^{s-1}} \\
&\le C(1 + tC_0^2)e^{CC_0 t}.
\end{aligned}$$

Hence, we have proven that $(\omega_{n+1}, \zeta_{n+1}, v_{n+1})$ is well-defined in the correct space, and moreover, combining all the previous estimates, we find for all $t \in [0, t_n]$

$$\|\omega_{n+1}^t\|_{H^s} + \|\zeta_{n+1}^t\|_{H^{s-1}} + \|v_{n+1}^t - \bar{v}^\circ\|_{H^s} \le C(1 + tC_0^2)e^{CC_0 t}.$$

Therefore, choosing $C_0 = 1 + 2Ce^C \lesssim 1$, we obtain for all $t \le t_n \wedge C_0^{-2}$

$$\|\omega_{n+1}^t\|_{H^s} + \|\zeta_{n+1}^t\|_{H^{s-1}} + \|v_{n+1}^t - \bar{v}^\circ\|_{H^s} \le C_0,$$

and thus $t_{n+1} \ge t_n \wedge C_0^{-2}$. The conclusion now follows just as in the proof of Proposition 3.1. ∎

4 Global existence

As local existence is proven in the framework of Sobolev spaces, the strategy for global existence consists in looking for a priori estimates on the Sobolev norms. Since we are also interested in Hölder regularity of solutions, we study a priori estimates on Hölder-Zygmund norms as well. As we will see, the main key for this is to prove a priori estimates for the vorticity ω in $\mathrm{L}^p(\mathbb{R}^2)$ for some $p > 1$.

4.1 A priori estimates

We begin with the following elementary energy estimates. Note that in the degenerate case $\lambda = 0$, the a priori estimate for ζ in $\mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^+; \mathrm{L}^2(\mathbb{R}^2))$ disappears, which is precisely the reason why we do not manage to prove any global result in that case. Although we stick in the sequel to the framework of item (iii), a priori estimates in slightly more general spaces are obtained in item (ii) for the compressible model (1.2).

Lemma 4.1 (Energy estimates)**.** *Let $\lambda \ge 0$, $\alpha \ge 0$, $\beta \in \mathbb{R}$, $T > 0$ and $\Psi \in W^{1,\infty}(\mathbb{R}^2)$. Let $v^\circ \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^2)^2$ be such that $\omega^\circ := \operatorname{curl} v^\circ \in \mathcal{P} \cap \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^2)$, and such that either $\operatorname{div}(av^\circ) = 0$ in the case (1.1), or $\zeta^\circ := \operatorname{div}(av^\circ) \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^2)$ in the case (1.2). Let $v \in \mathrm{L}^2_{\mathrm{loc}}([0,T) \times \mathbb{R}^2)^2$ be a weak solution of (1.1) or of (1.2) on $[0,T) \times \mathbb{R}^2$ with initial data v°. Set $\zeta := 0$ in the case (1.1). Then the following hold:*

(i) For all $t \in [0, T)$, we have $\omega^t \in \mathcal{P}(\mathbb{R}^2)$.

(ii) Localized energy estimate for (1.2): *if $v \in \mathrm{L}^2_{\mathrm{loc}}([0,T); \mathrm{L}^2_{\mathrm{uloc}}(\mathbb{R}^2)^2)$ is such that $\omega \in \mathrm{L}^\infty_{\mathrm{loc}}([0,T); \mathrm{L}^\infty(\mathbb{R}^2))$ and $\zeta \in \mathrm{L}^2_{\mathrm{loc}}([0,T); \mathrm{L}^2_{\mathrm{uloc}}(\mathbb{R}^2))$, then we have for all $t \in [0, T)$*

$$\|v^t\|^2_{\mathrm{L}^2_{\mathrm{uloc}}} + \alpha\||v|^2\omega\|_{\mathrm{L}^1_t \mathrm{L}^1_{\mathrm{uloc}}} + \lambda\|\zeta\|^2_{\mathrm{L}^2_t \mathrm{L}^2_{\mathrm{uloc}}} \le \begin{cases} Ce^{C(1+\lambda^{-1})t}\|v^\circ\|^2_{\mathrm{L}^2_{\mathrm{uloc}}}, & \text{if } \alpha = 0,\ \lambda > 0; \\ C\alpha^{-1}\lambda^{-1}(e^{\lambda t} - 1) + Ce^{\lambda t}\|v^\circ\|^2_{\mathrm{L}^2_{\mathrm{uloc}}}, & \text{if } \alpha > 0,\ \lambda > 0; \\ C\alpha^{-1}t + C\|v^\circ\|^2_{\mathrm{L}^2_{\mathrm{uloc}}}, & \text{if } \alpha > 0,\ \lambda = 0; \end{cases}$$

where the constants C's depend only on an upper bound on α, $|\beta|$, λ, $\|h\|_{W^{1,\infty}}$, $\|\Psi\|_{\mathrm{L}^\infty}$, and additionally on $\|\nabla\Psi\|_{\mathrm{L}^\infty}$ in the case $\alpha = 0$.

(iii) Relative energy estimate for (1.1) and (1.2): *if there is some* $\bar v^\circ \in W^{1,\infty}(\mathbb{R}^2)^2$ *such that* $v^\circ \in \bar v^\circ + \mathrm{L}^2(\mathbb{R}^2)^2$, $\bar\omega^\circ := \operatorname{curl} \bar v^\circ \in \mathrm{L}^2(\mathbb{R}^2)$, *and such that either* $\operatorname{div}(a\bar v^\circ) = 0$ *in the case* (1.1), *or* $\bar\zeta^\circ := \operatorname{div}(a\bar v^\circ) \in \mathrm{L}^2(\mathbb{R}^2)$ *in the case* (1.2), *and if* $v \in \mathrm{L}^\infty_{\mathrm{loc}}([0,T); \bar v^\circ + \mathrm{L}^2(\mathbb{R}^2))$, $\omega \in \mathrm{L}^\infty_{\mathrm{loc}}([0,T); \mathrm{L}^\infty(\mathbb{R}^2))$, $\zeta \in \mathrm{L}^2_{\mathrm{loc}}([0,T); \mathrm{L}^2(\mathbb{R}^2))$, *then we have for all* $t \in [0,T)$

$$\int_{\mathbb{R}^2} a|v^t - \bar v^\circ|^2 + \alpha \int_0^t du \int_{\mathbb{R}^2} a|v^u - \bar v^\circ|^2 \omega^u + \lambda \int_0^t du \int_{\mathbb{R}^2} a^{-1}|\zeta^u|^2$$
$$\le \begin{cases} Ct(1+\alpha^{-1}) + \int_{\mathbb{R}^2} a|v^\circ - \bar v^\circ|^2, & \text{in both cases (1.1) and (1.2), with } \alpha > 0; \\ e^{Ct}\big(1 + \int_{\mathbb{R}^2} a|v^\circ - \bar v^\circ|^2\big), & \text{in the case (1.1), with } \alpha = 0 \\ C(e^{C(1+\lambda^{-1})t} - 1) + e^{C(1+\lambda^{-1})t} \int_{\mathbb{R}^2} a|v^\circ - \bar v^\circ|^2, & \text{in the case (1.2), with } \alpha = 0,\ \lambda > 0; \end{cases}$$

where the constants C*'s depend only on an upper bound on* α, $|\beta|$, λ, $\|h\|_{W^{1,\infty}}$, $\|(\Psi, \bar v^\circ)\|_{\mathrm{L}^\infty}$, $\|\bar\zeta^\circ\|_{\mathrm{L}^2}$, *and additionally on* $\|\bar\omega^\circ\|_{\mathrm{L}^2}$ *and* $\|(\nabla\Psi, \nabla\bar v^\circ)\|_{\mathrm{L}^\infty}$ *in the case* $\alpha = 0$.

Proof. Item (i) is a standard consequence of the fact that ω satisfies a transport equation (1.7). It thus remains to check items (ii) and (iii). We split the proof into three steps.

Step 1: proof of (ii). Let v be a weak solution of the compressible equation (1.2) as in the statement, and let also $C > 0$ denote any constant as in the statement. We prove more precisely, for all $t \in [0,T)$ and $x_0 \in \mathbb{R}^2$,

$$\int ae^{-|x-x_0|}|v^t|^2 + \alpha \int_0^t du \int ae^{-|x-x_0|}|v^u|^2\omega^u + \lambda \int_0^t du \int a^{-1}e^{-|x-x_0|}|\zeta^u|^2 \tag{4.1}$$
$$\le \begin{cases} e^{C(1+\lambda^{-1})t} \int ae^{-|x-x_0|}|v^\circ|^2, & \text{if } \alpha = 0,\ \lambda > 0; \\ C\alpha^{-1}\lambda^{-1}(e^{\lambda t} - 1) + e^{\lambda t} \int ae^{-|x-x_0|}|v^\circ|^2, & \text{if } \alpha > 0,\ \lambda > 0; \\ C\alpha^{-1}t + \int ae^{-|x-x_0|}|v^\circ|^2, & \text{if } \alpha > 0,\ \lambda = 0. \end{cases}$$

Item (ii) directly follows from this, noting that

$$\|f\|^p_{\mathrm{L}^p_{\mathrm{uloc}}} \simeq \sup_{x_0 \in \mathbb{R}^2} \int e^{-|x-x_0|}|f(x)|^p dx$$

holds for all $1 \le p < \infty$. So it suffices to prove (4.1). Let $x_0 \in \mathbb{R}^2$ be fixed, and denote by $\chi(x) := e^{-|x-x_0|}$ the exponential cut-off function centered at x_0. From equation (1.2), we compute the following time-derivative

$$\partial_t \int a\chi|v^t|^2 = 2\int a\chi(\lambda\nabla(a^{-1}\zeta^t) - \alpha(\Psi + v^t)\omega^t + \beta(\Psi + v^t)^\perp\omega^t) \cdot v^t,$$

and hence, by integration by parts with $|\nabla\chi| \le \chi$,

$$\begin{aligned} \partial_t \int a\chi|v^t|^2 &= -2\lambda \int a^{-1}\chi|\zeta^t|^2 - 2\lambda \int \nabla\chi \cdot v^t\zeta^t - 2\alpha \int a\chi|v^t|^2\omega^t + 2\int a\chi(-\alpha\Psi + \beta\Psi^\perp) \cdot v^t\omega^t \\ &\le -2\lambda \int a^{-1}\chi|\zeta^t|^2 + 2\lambda \int \chi|\zeta^t||v^t| - 2\alpha \int a\chi|v^t|^2\omega^t + 2\int a\chi(-\alpha\Psi + \beta\Psi^\perp) \cdot v^t\omega^t. \end{aligned} \tag{4.2}$$

First consider the case $\alpha > 0$. We may then bound the terms as follows, using the inequality $2xy \le x^2 + y^2$,

$$\begin{aligned} \partial_t \int a\chi|v^t|^2 &\le -2\lambda \int a^{-1}\chi|\zeta^t|^2 + 2\lambda \int \chi|\zeta^t||v^t| - 2\alpha \int a\chi|v^t|^2\omega^t + 2C \int a\chi|v^t|\omega^t \\ &\le -\lambda \int a^{-1}\chi|\zeta^t|^2 + \lambda \int a\chi|v^t|^2 - \alpha \int a\chi|v^t|^2\omega^t + C\alpha^{-1} \underbrace{\int a\chi\omega^t}_{\le C}. \end{aligned}$$

As ω^t is nonnegative by item (i), the first and third right-hand side terms are nonpositive, and the Grönwall inequality yields $\int a\chi|v^t|^2 \le C\alpha^{-1}\lambda^{-1}(e^{\lambda t} - 1) + e^{\lambda t} \int a\chi|v^\circ|^2$ (or $\int a\chi|v^t|^2 \le C\alpha^{-1}t + \int a\chi|v^\circ|^2$ if $\lambda = 0$). The above estimate may then be rewritten as follows

$$\alpha\int a\chi|v^t|^2\omega^t+\lambda\int a^{-1}\chi|\zeta^t|^2\le C\alpha^{-1}+\lambda\int a\chi|v^t|^2-\partial_t\int a\chi|v^t|^2$$
$$\le C\alpha^{-1}e^{\lambda t}+\lambda e^{\lambda t}\int a\chi|v^\circ|^2-\partial_t\int a\chi|v^t|^2.$$

Integrating in time yields

$$\alpha\int_0^T dt\int a\chi|v^t|^2\omega^t+\lambda\int_0^T dt\int a^{-1}\chi|\zeta^t|^2\le C\alpha^{-1}\lambda^{-1}(e^{-\lambda T}-1)+e^{\lambda T}\int a\chi|v^\circ|^2-\int a\chi|v^T|^2,$$

so that (4.1) is proven for $\alpha>0$. We now turn to the case $\alpha=0$, $\lambda>0$. In that case, using the following Delort type identity, which holds here in $\mathrm{L}^\infty_{\mathrm{loc}}([0,T);W^{-1,1}_{\mathrm{loc}}(\mathbb{R}^2)^2)$,

$$\omega v=a^{-1}\zeta v^\perp-\frac12|v|^2\nabla^\perp h-a^{-1}(\operatorname{div}(aS_v))^\perp,\qquad S_v:=v\otimes v-\frac12|v|^2\,\mathrm{Id},$$

the estimate (4.2) becomes, by integration by parts with $|\nabla\chi|\le\chi$,

$$\partial_t\int a\chi|v^t|^2\le-2\lambda\int a^{-1}\chi|\zeta^t|^2+2\lambda\int\chi|\zeta^t||v^t|-2\alpha\int a\chi|v^t|^2\omega^t+2\int\chi(-\alpha\Psi+\beta\Psi^\perp)\cdot(v^t)^\perp\zeta^t$$
$$-\int a\chi(-\alpha\Psi+\beta\Psi^\perp)\cdot\nabla^\perp h|v^t|^2+2\int a\chi(\alpha\nabla\Psi^\perp+\beta\nabla\Psi):S_{v^t}+2\int a\chi|\alpha\Psi^\perp+\beta\Psi||S_{v^t}|,$$

and hence, noting that $|S_{v^t}|\le C|v^t|^2$, and using the inequality $2xy\le x^2+y^2$,

$$\partial_t\int a\chi|v^t|^2\le-2\lambda\int a^{-1}\chi|\zeta^t|^2+2C\int\chi|\zeta^t||v^t|-2\alpha\int a\chi|v^t|^2\omega^t+C\int a\chi|v^t|^2$$
$$\le-\lambda\int a^{-1}\chi|\zeta^t|^2+C(1+\lambda^{-1})\int a\chi|v^t|^2.$$

The Grönwall inequality yields $\int a\chi|v^t|^2\le e^{C(1+\lambda^{-1})t}\int a\chi|v^\circ|^2$. The above estimate may then be rewritten as follows:

$$\lambda\int a^{-1}\chi|\zeta^t|^2\le C(1+\lambda^{-1})\int a\chi|v^t|^2-\partial_t\int a\chi|v^t|^2$$
$$\le C(1+\lambda^{-1})e^{C(1+\lambda^{-1})t}\int a\chi|v^\circ|^2-\partial_t\int a\chi|v^t|^2.$$

Integrating in time, the result (4.1) is proven for $\alpha=0$. (Note that this proof cannot be adapted to the incompressible case (1.1), due to the lack of a sufficiently good control on the pressure p in (1.1) in general.)

Step 2: proof of (iii) for (1.2). We denote by C any positive constant as in the statement (iii). From equation (1.2), we compute the following time-derivative

$$\partial_t\int a|v^t-\bar v^\circ|^2=2\int a(\lambda\nabla(a^{-1}\zeta^t)-\alpha(\Psi+v^t)\omega^t+\beta(\Psi+v^t)^\perp\omega^t)\cdot(v^t-\bar v^\circ),$$

or equivalently, integrating by parts and suitably regrouping the terms,

$$\partial_t\int a|v^t-\bar v^\circ|^2=-2\lambda\int a^{-1}|\zeta^t|^2+2\lambda\int a^{-1}\zeta^t\bar\zeta^\circ-2\alpha\int a|v^t-\bar v^\circ|^2\omega^t\qquad(4.3)$$
$$+2\int a(-\alpha(\Psi+\bar v^\circ)+\beta(\Psi+\bar v^\circ)^\perp)\cdot(v^t-\bar v^\circ)\omega^t.$$

First consider the case $\alpha > 0$. We may then bound the terms as follows, using the inequality $2xy \le x^2 + y^2$,

$$\partial_t \int a|v^t - \bar v^\circ|^2 \le -2\lambda \int a^{-1}|\zeta^t|^2 + 2\lambda \int a^{-1}\zeta^t\bar\zeta^\circ - 2\alpha \int a|v^t - \bar v^\circ|^2\omega^t + 2C \int a|v^t - \bar v^\circ|\omega^t$$
$$\le -\lambda \int a^{-1}|\zeta^t|^2 + \lambda \int a^{-1}|\bar\zeta^\circ|^2 - \alpha \int a|v^t - \bar v^\circ|^2\omega^t + C\alpha^{-1}.$$

Applying the Grönwall inequality as in Step 1, item (iii) easily follows from this in the case $\alpha > 0$. We now turn to the case $\alpha = 0$, $\lambda > 0$. In that case, we rather rewrite (4.3) in the form

$$\partial_t \int a|v^t - \bar v^\circ|^2 = -2\lambda \int a^{-1}|\zeta^t|^2 + 2\lambda \int a^{-1}\zeta^t\bar\zeta^\circ - 2\alpha \int a|v^t - \bar v^\circ|^2\omega^t$$
$$+ 2 \int a(-\alpha(\Psi + \bar v^\circ) + \beta(\Psi + \bar v^\circ)^\perp) \cdot (v^t - \bar v^\circ)(\omega^t - \bar\omega^\circ) + 2 \int a(-\alpha(\Psi + \bar v^\circ) + \beta(\Psi + \bar v^\circ)^\perp) \cdot (v^t - \bar v^\circ)\bar\omega^\circ,$$

so that, using the following Delort type identity, which holds here in $\mathrm{L}^\infty_{\mathrm{loc}}([0,T); W^{-1,1}_{\mathrm{loc}}(\mathbb{R}^2)^2)$,

$$(\omega - \bar\omega^\circ)(v - \bar v^\circ) = a^{-1}(\zeta - \bar\zeta^\circ)(v - \bar v^\circ)^\perp - \frac{1}{2}|v - \bar v^\circ|^2\nabla^\perp h - a^{-1}(\operatorname{div}(aS_{v-\bar v^\circ}))^\perp,$$

we find by integration by parts

$$\partial_t \int a|v^t - \bar v^\circ|^2 = -2\lambda \int a^{-1}|\zeta^t|^2 + 2\lambda \int a^{-1}\zeta^t\bar\zeta^\circ - 2\alpha \int a|v^t - \bar v^\circ|^2\omega^t$$
$$+ 2 \int (-\alpha(\Psi + \bar v^\circ) + \beta(\Psi + \bar v^\circ)^\perp) \cdot (v^t - \bar v^\circ)^\perp(\zeta^t - \bar\zeta^\circ) - \int a(-\alpha(\Psi + \bar v^\circ) + \beta(\Psi + \bar v^\circ)^\perp) \cdot \nabla^\perp h|v^t - \bar v^\circ|^2$$
$$+ 2 \int a\nabla(\alpha(\Psi + \bar v^\circ)^\perp + \beta(\Psi + \bar v^\circ)) : S_{v^t - \bar v^\circ} + 2 \int a(-\alpha(\Psi + \bar v^\circ) + \beta(\Psi + \bar v^\circ)^\perp) \cdot (v^t - \bar v^\circ)\bar\omega^\circ.$$

We may then bound the terms as follows, using the inequality $2xy \le x^2 + y^2$,

$$\partial_t \int a|v^t - \bar v^\circ|^2 \le -2\lambda \int a^{-1}|\zeta^t|^2 + 2\lambda \int a^{-1}|\zeta^t||\bar\zeta^\circ| - 2\alpha \int a|v^t - \bar v^\circ|^2\omega^t$$
$$+ C \int |v^t - \bar v^\circ|\,|\zeta^t| + C \int |v^t - \bar v^\circ|\,|\bar\zeta^\circ| + C \int a|v^t - \bar v^\circ|^2 + C \int a|v^t - \bar v^\circ|\bar\omega^\circ$$
$$\le -\lambda \int a^{-1}|\zeta^t|^2 + C \int a^{-1}|\bar\zeta^\circ|^2 + C \int |\bar\omega^\circ|^2 + C(1 + \lambda^{-1}) \int a|v^t - \bar v^\circ|^2.$$

Item (iii) in the case $\alpha = 0$ then easily follows from the Grönwall inequality.

Step 3: proof of (iii) for (1.1). We denote by C any positive constant as in the statement (iii). As the identity $v - \bar v^\circ = a^{-1}\nabla^\perp(\operatorname{div} a^{-1}\nabla)^{-1}(\omega - \bar\omega^\circ)$ follows from (2.4) together with the constraint $\operatorname{div}(av) = 0$, and as by assumption $v - \bar v^\circ \in \mathrm{L}^2_{\mathrm{loc}}([0,T); \mathrm{L}^2(\mathbb{R}^2)^2)$, we deduce $\omega - \bar\omega^\circ \in \mathrm{L}^2_{\mathrm{loc}}([0,T); \dot H^{-1}(\mathbb{R}^2))$ and $(\operatorname{div} a^{-1}\nabla)^{-1}(\omega - \bar\omega^\circ) \in \mathrm{L}^2_{\mathrm{loc}}([0,T); \dot H^1(\mathbb{R}^2))$. In particular, this implies by integration by parts

$$\int a|v - \bar v^\circ|^2 = \int a^{-1}|\nabla(\operatorname{div} a^{-1}\nabla)^{-1}(\omega - \bar\omega^\circ)|^2 = \int (\omega - \bar\omega^\circ)(-\operatorname{div} a^{-1}\nabla)^{-1}(\omega - \bar\omega^\circ). \tag{4.4}$$

From equation (1.7), we compute the following time-derivative

$$\partial_t \int (\omega - \bar\omega^\circ)(-\operatorname{div} a^{-1}\nabla)^{-1}(\omega - \bar\omega^\circ)$$
$$= 2 \int \nabla(\operatorname{div} a^{-1}\nabla)^{-1}(\omega - \bar\omega^\circ) \cdot (\alpha(\Psi + v)^\perp + \beta(\Psi + v))\omega$$
$$= -2 \int a(v - \bar v^\circ)^\perp \cdot \Big(\alpha(v - \bar v^\circ)^\perp + \beta(v - \bar v^\circ) + \alpha(\Psi + \bar v^\circ)^\perp + \beta(\Psi + \bar v^\circ)\Big)\omega$$
$$= -2\alpha \int a|v - \bar v^\circ|^2\omega - 2 \int a\omega(v - \bar v^\circ)^\perp \cdot (\alpha(\Psi + \bar v^\circ)^\perp + \beta(\Psi + \bar v^\circ)).$$

Combining this with identity (4.4), we are now in position to conclude exactly as in Step 2 after equation (4.3) (but with $\zeta, \bar\zeta^\circ \equiv 0$). ☐

The energy estimates given by Lemma 4.1 above are not strong enough to prove global existence. The key is to find an additional a priori L^p-estimate for the vorticity ω. We begin with the following L^p-estimate. Note that unfortunately the same kind of argument does not work in the mixed-flow compressible case (that is, (1.2) with $\alpha \ge 0$, $\beta \neq 0$), as it would require a too strong additional control on the norm $\|\zeta^t\|_{\mathrm{L}^{p+1}}$. This is why this case is excluded from our global results.

Lemma 4.2 (L^p-estimates for vorticity). *Let $\lambda, \alpha \ge 0$, $\beta \in \mathbb{R}$, $T > 0$, $h, \Psi \in W^{1,\infty}(\mathbb{R}^2)$, $\bar v^\circ \in \mathrm{L}^\infty(\mathbb{R}^2)^2$, and $v^\circ \in \bar v^\circ + \mathrm{L}^2(\mathbb{R}^2)^2$, with $\omega^\circ := \operatorname{curl} v^\circ \in \mathcal{P}(\mathbb{R}^2)$, $\bar\omega^\circ := \operatorname{curl} \bar v^\circ \in \mathcal{P} \cap \mathrm{L}^\infty(\mathbb{R}^2)$. In the case (1.1), also assume $\operatorname{div}(av^\circ) = \operatorname{div}(a\bar v^\circ) = 0$. Let $v \in \mathrm{L}^\infty_{\mathrm{loc}}([0,T); \bar v^\circ + \mathrm{L}^2 \cap \mathrm{L}^\infty(\mathbb{R}^2)^2)$ be a weak solution of (1.1) or of (1.2) in $[0,T) \times \mathbb{R}^2$ with initial data v°, and with $\omega := \operatorname{curl} v \in \mathrm{L}^\infty_{\mathrm{loc}}([0,T); \mathcal{P} \cap \mathrm{L}^\infty(\mathbb{R}^2))$. For all $1 < p \le \infty$ and $t \in [0,T)$*

(i) in the case (1.1) with $\alpha > 0$, $\beta \in \mathbb{R}$, we have

$$\left(\frac{\alpha(p-1)}{2}\right)^{1/p} \|\omega\|^{1+1/p}_{\mathrm{L}^{p+1}_t \mathrm{L}^{p+1}} + \|\omega^t\|_{\mathrm{L}^p} \le \|\omega^\circ\|_{\mathrm{L}^p} + C_p, \tag{4.5}$$

where the constant C_p depends only on an upper bound on $(p-1)^{-1}$, α, α^{-1}, $|\beta|$, T, $\|(h,\Psi)\|_{W^{1,\infty}}$, $\|(\bar v^\circ, \bar\omega^\circ)\|_{\mathrm{L}^\infty}$, and on $\|v^\circ - \bar v^\circ\|_{\mathrm{L}^2}$;

(ii) in both cases (1.1) and (1.2) with $\alpha \ge 0$, $\beta = 0$, $\lambda \ge 0$, the same estimate (4.5) holds, where the constant $C_p = C$ depends only on an upper bound on α, T, and on $\|(\operatorname{curl}\Psi)^-\|_{\mathrm{L}^\infty}$.

Proof. It is sufficient to prove the result for all $1 < p < \infty$. In this proof, we use the notation $\lesssim$ for $\le$ up to a constant $C > 0$ as in the statement but independent of p. As explained at the end of Step 1, we may focus on item (i), the other being much simpler. We split the proof into three steps. Set $\bar\theta^\circ := \operatorname{div} \bar v^\circ$, $\theta := \operatorname{div} v$. In the sequel, we repeatedly use the a priori estimate of Lemma 4.1(i) in the following interpolated form: for all $s \le q$ and $t \in [0,T)$

$$\|\omega^t\|_{\mathrm{L}^s} \le \|\omega^t\|^{q'/s'}_{\mathrm{L}^q} \|\omega^t\|^{1-q'/s'}_{\mathrm{L}^1} = \|\omega^t\|^{q'/s'}_{\mathrm{L}^q}. \tag{4.6}$$

Step 1: preliminary estimate for ω. In this step, we prove that for all $1 < p < \infty$ and all $t \in [0,T)$

$$\alpha(p-1)\|\omega\|^{p+1}_{\mathrm{L}^{p+1}_t \mathrm{L}^{p+1}} + \|\omega^t\|^p_{\mathrm{L}^p} \le \|\omega^\circ\|^p_{\mathrm{L}^p} + C(p-1)(t^{1/p} + \|v\|_{\mathrm{L}^p_t \mathrm{L}^\infty}) \|\omega\|^{p-1/p}_{\mathrm{L}^{p+1}_t \mathrm{L}^{p+1}}. \tag{4.7}$$

Using equation (1.7) and integrating by parts we may compute

$$\begin{aligned}
\partial_t \int (\omega^t)^p &= p \int (\omega^t)^{p-1} \operatorname{div}(\omega^t(\alpha(\Psi + v^t)^\perp + \beta(\Psi + v^t))) \\
&= -p(p-1) \int (\omega^t)^{p-1} \nabla\omega^t \cdot (\alpha(\Psi + v^t)^\perp + \beta(\Psi + v^t)) \\
&= -(p-1) \int \nabla(\omega^t)^p \cdot (\alpha(\Psi + v^t)^\perp + \beta(\Psi + v^t)) \\
&= (p-1) \int (\omega^t)^p \operatorname{div}(\alpha(\Psi + v^t)^\perp + \beta(\Psi + v^t)).
\end{aligned}$$

Using the constraint $\operatorname{div}(av) = 0$ to compute $\operatorname{div}(\alpha v^\perp + \beta v) = -\alpha\omega + \beta \operatorname{div} v = -\alpha\omega - \beta\nabla h \cdot v$, we find

$$\begin{aligned}
(p-1)^{-1}\partial_t \int (\omega^t)^p &\le -\alpha \int (\omega^t)^{p+1} + C \int (\omega^t)^p (1 + |v^t|) \\
&\le -\alpha \int (\omega^t)^{p+1} + C(1 + \|v^t\|_{\mathrm{L}^\infty}) \int (\omega^t)^p.
\end{aligned}$$

By interpolation (4.6), we obtain

$$\alpha\int(\omega^t)^{p+1}+(p-1)^{-1}\partial_t\int(\omega^t)^p\le C(1+\|v^t\|_{\mathrm{L}^\infty})\|\omega^t\|_{\mathrm{L}^{p+1}}^{p-1/p},$$

and the result (4.7) directly follows by integration with respect to t and by the Hölder inequality. Note that in the case of item (ii) we rather have

$$\alpha\int(\omega^t)^{p+1}+(p-1)^{-1}\partial_t\int(\omega^t)^p\le\alpha\|(\operatorname{curl}\Psi)^-\|_{\mathrm{L}^\infty}\int(\omega^t)^p\le\alpha\|(\operatorname{curl}\Psi)^-\|_{\mathrm{L}^\infty}\Big(\int(\omega^t)^{p+1}\Big)^{1-1/p},$$

from which the conclusion (ii) already follows.

Step 2: preliminary estimate for v. In this step, we prove for all $2<q\le\infty$ and $t\in[0,T)$

$$\|v^t\|_{\mathrm{L}^\infty}\lesssim 1+(1-2/q)^{-1/2}\|\omega^t\|_{\mathrm{L}^q}^{q'/2}\log^{1/2}(2+\|\omega^t\|_{\mathrm{L}^q}).\tag{4.8}$$

Let $2<q\le\infty$. Note that $v^t-\bar v^\circ=\nabla^\perp\triangle^{-1}(\omega^t-\omega^\circ)+\nabla\triangle^{-1}(\theta^t-\bar\theta^\circ)$. By Lemma 2.4(i) for $w:=\omega^t-\bar\omega^\circ$ and Lemma 2.4(ii) for $w:=\theta^t-\bar\theta^\circ=\operatorname{div}(v^t-\bar v^\circ)$, we find

$$\begin{aligned}\|v^t\|_{\mathrm{L}^\infty}&\le\|\bar v^\circ\|_{\mathrm{L}^\infty}+\|\nabla\triangle^{-1}(\omega^t-\bar\omega^\circ)\|_{\mathrm{L}^\infty}+\|\nabla\triangle^{-1}(\theta^t-\bar\theta^\circ)\|_{\mathrm{L}^\infty}\\&\lesssim 1+(1-2/q)^{-1/2}\|\omega^t-\bar\omega^\circ\|_{\mathrm{L}^2}\log^{1/2}(2+\|\omega^t-\bar\omega^\circ\|_{\mathrm{L}^1\cap\mathrm{L}^q})\\&\qquad+\|\theta^t-\bar\theta^\circ\|_{\mathrm{L}^2}\log^{1/2}(2+\|\theta^t-\bar\theta^\circ\|_{\mathrm{L}^2\cap\mathrm{L}^\infty})+\|v^t-\bar v^\circ\|_{\mathrm{L}^2}.\end{aligned}$$

Noting that $\theta^t-\bar\theta^\circ=-\nabla h\cdot(v^t-\bar v^\circ)$, using interpolation (4.6) in the form $\|\omega^t\|_{\mathrm{L}^2}\lesssim\|\omega^t\|_{\mathrm{L}^q}^{q'/2}$, and using the a priori estimates of Lemma 4.1 in the form $\|v^t-\bar v^\circ\|_{\mathrm{L}^2}+\|\omega^t\|_{\mathrm{L}^1}\lesssim 1$, we obtain

$$\|v^t\|_{\mathrm{L}^\infty}\lesssim(1-2/q)^{-1/2}\|\omega^t\|_{\mathrm{L}^q}^{q'/2}\log^{1/2}(2+\|\omega^t\|_{\mathrm{L}^q})+\log^{1/2}(2+\|v^t-\bar v^\circ\|_{\mathrm{L}^\infty}),$$

and the result follows, absorbing in the left-hand side the last norm of v.

Step 3: conclusion. Let $1<p<\infty$. From (4.8) with $q=p+1$, we deduce in particular

$$\|v^t\|_{\mathrm{L}^\infty}\lesssim 1+(1-1/p)^{-1/2}\|\omega^t\|_{\mathrm{L}^{p+1}}^{\frac12(1+1/p)}\log^{1/2}(2+\|\omega^t\|_{\mathrm{L}^{p+1}})\lesssim(1-1/p)^{-1/2}\big(1+\|\omega^t\|_{\mathrm{L}^{p+1}}^{\frac34(1+1/p)}\big),$$

and hence, integrating with respect to t and combining with (4.7),

$$\begin{aligned}\alpha(p-1)\|\omega\|_{\mathrm{L}^{p+1}_t\mathrm{L}^{p+1}}^{p+1}+\|\omega^t\|_{\mathrm{L}^p}^p&\le\|\omega^\circ\|_{\mathrm{L}^p}^p+Cp\big(1+\|\omega\|_{\mathrm{L}^{p+1}_t\mathrm{L}^{p+1}}^{\frac34(1+1/p)}\big)\|\omega\|_{\mathrm{L}^{p+1}_t\mathrm{L}^{p+1}}^{p-1/p}\\&\le\|\omega^\circ\|_{\mathrm{L}^p}^p+Cp\|\omega\|_{\mathrm{L}^{p+1}_t\mathrm{L}^{p+1}}^{p-1/p}+Cp\|\omega\|_{\mathrm{L}^{p+1}_t\mathrm{L}^{p+1}}^{p+\frac34}.\end{aligned}$$

We may now absorb in the left-hand side the last two terms, to the effect of

$$\frac{\alpha(p-1)}{2}\|\omega\|_{\mathrm{L}^{p+1}_t\mathrm{L}^{p+1}}^{p+1}+\|\omega^t\|_{\mathrm{L}^p}^p\le\|\omega^\circ\|_{\mathrm{L}^p}^p+C_p^p,$$

where the constant C_p further depends on an upper bound on $(p-1)^{-1}$, and the conclusion follows. ☐

Inspired by the work of Lin and Zhang [29] on the simplified equation (1.4), we now exploit the very particular structure of the transport equation (1.7) to deduce in the parabolic case an a priori L^p-estimate for the vorticity ω through its initial L^1-norm only. Note that the same estimate holds in the mixed-flow incompressible case with a constant. This strengthening of Lemma 4.2 is the key for global existence results with vortex-sheet data. In items (i)–(ii) below, we further display what can be obtained from this ODE method in other regimes, but the conclusion is then weaker than that of Lemma 4.2.

Lemma 4.3 (L^p-estimates for vorticity, cont'd)**.** *Let $\lambda \ge 0$, $\alpha \ge 0$, $\beta \in \mathbb{R}$, $T > 0$, and $h, \Psi, v^\circ \in W^{1,\infty}(\mathbb{R}^2)^2$, with $\omega^\circ := \operatorname{curl} v^\circ \in \mathcal{P} \cap C^0(\mathbb{R}^2)$. Set $\zeta^\circ := \operatorname{div}(av^\circ)$, and in the case (1.1) assume that $\operatorname{div}(av^\circ) = 0$. Let $v \in W^{1,\infty}_{\mathrm{loc}}([0,T); W^{1,\infty}(\mathbb{R}^2)^2)$ be a weak solution of (1.1) or of (1.2) on $[0,T) \times \mathbb{R}^2$ with initial data v°. For all $1 \le p \le \infty$ and $t \in [0,T)$, the following hold:*

(i) in both cases (1.1) and (1.2), without restriction on the parameters,

$$\|\omega^t\|_{\mathrm{L}^p} \le \|\omega^\circ\|_{\mathrm{L}^p} \min\bigg\{ \exp\Big[\frac{p-1}{p}\big(Ct + C|\beta|\|\zeta\|_{\mathrm{L}^1_t\mathrm{L}^\infty} + C|\beta|\|\nabla h\|_{\mathrm{L}^\infty}\|v\|_{\mathrm{L}^1_t\mathrm{L}^\infty}\big)\Big];$$
$$\exp\Big[\frac{p-1}{p}\big(C + Ct + C|\beta|\|\zeta\|_{\mathrm{L}^1_t\mathrm{L}^\infty} + C\alpha\|\nabla h\|_{\mathrm{L}^\infty}\|v\|_{\mathrm{L}^1_t\mathrm{L}^\infty}\big)\Big]\bigg\};$$

(ii) in the case (1.1) with either $\beta = 0$ or $\alpha = 0$ or h constant, and in the case (1.2) with $\beta = 0$, we have

$$\|\omega^t\|_{\mathrm{L}^p} \le Ce^{Ct}\|\omega^\circ\|_{\mathrm{L}^p};$$

(iii) given $\alpha > 0$, in the case (1.1) with either $\beta = 0$ or h constant, and in the case (1.2) with $\beta = 0$, we have

$$\|\omega^t\|_{\mathrm{L}^p} \le \Big((\alpha t)^{-1} + C\alpha^{-1}e^{Ct}\Big)^{1-1/p};$$

where the constants C's depend only on an upper bound on α, $|\beta|$, and on $\|(h,\Psi)\|_{W^{1,\infty}}$.

Remark 4.4. In the context of item (iii), if we further assume $\Psi \equiv 0$ (i.e. no forcing), then the constant C in Step 2 of the proof below may then be set to 0, so that we simply obtain, for all $1 \le p < \infty$ and all $t > 0$,

$$\|\omega^t\|_{\mathrm{L}^p} \le \Big(\int |\omega^\circ|^p (1 + \alpha t\omega^\circ)^{1-p}\Big)^{1/p} \le (\alpha t)^{-(1-1/p)},$$

without additional exponential growth.

Proof. We split the proof into two steps, and we use the notation $\lesssim$ for $\le$ up to a constant $C > 0$ as in the statement.

Step 1: general bounds. In this step, we prove (i) (from which (ii) directly follows, noting that a constant implies $\nabla h \equiv 0$). Let us consider the flow

$$\partial_t\psi^t(x) = -\alpha(\Psi + v^t)^\perp(\psi^t(x)) - \beta(\Psi + v^t)(\psi^t(x)), \qquad \psi^t(x)|_{t=0} = x.$$

The Lipschitz assumptions ensure that ψ is well-defined in $W^{1,\infty}_{\mathrm{loc}}([0,T); W^{1,\infty}(\mathbb{R}^2)^2)$. As ω satisfies the transport equation (1.7) with initial data $\omega^\circ \in C^0(\mathbb{R}^2)$, the method of propagation along characteristics yields

$$\omega^t(x) = \omega^\circ((\psi^t)^{-1}(x))|\det \nabla(\psi^t)^{-1}(x)| = \omega^\circ((\psi^t)^{-1}(x))|\det \nabla\psi^t((\psi^t)^{-1}(x))|^{-1},$$

and hence, for any $1 \le p < \infty$, we have

$$\int |\omega^t|^p = \int |\omega^\circ((\psi^t)^{-1}(x))|^p |\det \nabla\psi^t((\psi^t)^{-1}(x))|^{-p} dx = \int |\omega^\circ(x)|^p |\det \nabla\psi^t(x)|^{1-p} dx, \tag{4.9}$$

while, for $p = \infty$,

$$\|\omega^t\|_{\mathrm{L}^\infty} \le \|\omega^\circ\|_{\mathrm{L}^\infty}\|(\det \nabla\psi^t)^{-1}\|_{\mathrm{L}^\infty}.$$

Now let us examine this determinant more closely. By the Liouville-Ostrogradski formula,

$$|\det \nabla\psi^t(x)|^{-1} = \exp\bigg(\int_0^t \operatorname{div}\Big(\alpha(\Psi + v^u)^\perp + \beta(\Psi + v^u)\Big)(\psi^u(x))du\bigg). \tag{4.10}$$

A simple computation gives

$$\operatorname{div}(\alpha(v^t)^\perp + \beta v^t) = -\alpha \operatorname{curl} v^t + \beta \operatorname{div} v^t = -\alpha\omega^t + \beta a^{-1}\zeta^t - \beta\nabla h \cdot v^t, \tag{4.11}$$

hence, by non-negativity of ω,

$$\operatorname{div}(\alpha(v^t)^\perp + \beta v^t) \le |\beta| \|a^{-1}\|_{\mathrm{L}^\infty} \|\zeta^t\|_{\mathrm{L}^\infty} + |\beta| \|\nabla h\|_{\mathrm{L}^\infty} \|v^t\|_{\mathrm{L}^\infty}.$$

We then find

$$|\det \nabla \psi^t(x)|^{-1} \le \exp\big(t\alpha \|\operatorname{curl}\Psi\|_{\mathrm{L}^\infty} + t|\beta| \|\operatorname{div}\Psi\|_{\mathrm{L}^\infty} + |\beta| \|a^{-1}\|_{\mathrm{L}^\infty} \|\zeta\|_{\mathrm{L}^1_t\mathrm{L}^\infty} + |\beta| \|\nabla h\|_{\mathrm{L}^\infty} \|v\|_{\mathrm{L}^1_t\mathrm{L}^\infty}\big),$$

and thus, for all $1 \le p \le \infty$,

$$\begin{aligned} \|\omega^t\|_{\mathrm{L}^p} \le \|\omega^\circ\|_{\mathrm{L}^p} \exp\Big[\frac{p-1}{p}\big(&t\alpha \|\operatorname{curl}\Psi\|_{\mathrm{L}^\infty} + t|\beta| \|\operatorname{div}\Psi\|_{\mathrm{L}^\infty} \\ &+ |\beta| \|a^{-1}\|_{\mathrm{L}^\infty} \|\zeta\|_{\mathrm{L}^1_t\mathrm{L}^\infty} + |\beta| \|\nabla h\|_{\mathrm{L}^\infty} \|v\|_{\mathrm{L}^1_t\mathrm{L}^\infty}\big)\Big]. \end{aligned} \tag{4.12}$$

On the other hand, noting that

$$\partial_t h(\psi^t(x)) = -\nabla h(\psi^t(x)) \cdot (\alpha(\Psi + v^t)^\perp + \beta(\Psi + v^t))(\psi^t(x)),$$

we may alternatively rewrite

$$\begin{aligned} \operatorname{div}(\alpha(v^t)^\perp + \beta v^t)(\psi^t(x)) &= \big(-\alpha\omega^t + \beta a^{-1}\zeta^t - \beta \nabla h \cdot v^t\big)(\psi^t(x)) \\ &= \partial_t h(\psi^t(x)) + \big(-\alpha\omega^t + \beta a^{-1}\zeta^t - \alpha\nabla^\perp h \cdot v^t + \nabla h \cdot (\alpha\Psi^\perp + \beta\Psi)\big)(\psi^t(x)). \end{aligned} \tag{4.13}$$

Integrating this identity with respect to t and using again the same formula for $|\det \nabla\psi^t|^{-1}$, we obtain

$$\begin{aligned} \|\omega^t\|_{\mathrm{L}^p} \le \|\omega^\circ\|_{\mathrm{L}^p} \exp\Big[\frac{p-1}{p}\big(&t\alpha \|\operatorname{curl}\Psi\|_{\mathrm{L}^\infty} + t|\beta| \|\operatorname{div}\Psi\|_{\mathrm{L}^\infty} + |\beta| \|a^{-1}\|_{\mathrm{L}^\infty} \|\zeta\|_{\mathrm{L}^1_t\mathrm{L}^\infty} \\ &+ 2\|h\|_{\mathrm{L}^\infty} + t(\alpha + |\beta|) \|\nabla h\|_{\mathrm{L}^\infty} \|\Psi\|_{\mathrm{L}^\infty} + \alpha \|\nabla h\|_{\mathrm{L}^\infty} \|v\|_{\mathrm{L}^1_t\mathrm{L}^\infty}\big)\Big]. \end{aligned} \tag{4.14}$$

Combining (4.12) and (4.14), the conclusion (i) follows.

Step 2: proof of (iii). It suffices to prove the result for any $1 < p < \infty$. Let such a p be fixed. Assuming either $\beta = 0$, or $\zeta \equiv 0$ and a constant, we deduce from (4.9), (4.10) and (4.11)

$$\begin{aligned} \int |\omega^t|^p &= \int |\omega^\circ(x)|^p \exp\Big((p-1)\int_0^t \operatorname{div}\big(\alpha(\Psi + v^u)^\perp + \beta(\Psi + v^u)\big)(\psi^u(x))du\Big)dx \\ &\le e^{C(p-1)t} \int |\omega^\circ(x)|^p \exp\Big(-\alpha(p-1)\int_0^t \omega^u(\psi^u(x))du\Big)dx. \end{aligned} \tag{4.15}$$

Let x be momentarily fixed, and set $f_x(t) := \omega^t(\psi^t(x))$. We need to estimate the integral $\int_0^t f_x(u)du$. For that purpose, we first compute $\partial_t f_x$: again using (4.11) (with either $\beta = 0$, or $\zeta \equiv 0$ and a constant), we find

$$\begin{aligned} \partial_t f_x(t) &= \operatorname{div}(\omega^t(\alpha(\Psi + v^t)^\perp + \beta(\Psi + v^t)))(\psi^t(x)) - \nabla\omega^t(\psi^t(x)) \cdot (\alpha(\Psi + v^t)^\perp + \beta(\Psi + v^t))(\psi^t(x)) \\ &= \omega^t(\psi^t(x)) \operatorname{div}(\alpha(\Psi + v^t)^\perp + \beta(\Psi + v^t)))(\psi^t(x)) \\ &= -\alpha(\omega^t(\psi^t(x)))^2 + (-\alpha\omega^t \operatorname{curl}\Psi + \beta\omega^t \operatorname{div}\Psi)(\psi^t(x)), \end{aligned}$$

and hence

$$\partial_t f_x \ge -\alpha f_x^2 - C f_x.$$

We may then deduce $f_x \ge g_x$ pointwise, where g_x satisfies

$$\partial_t g_x = -\alpha g_x^2 - C g_x, \qquad g_x(0) = f_x(0) = \omega^\circ(x).$$

A direct computation yields

$$g_x(t) = \frac{Ce^{-Ct}\omega^\circ(x)}{C + \alpha(1 - e^{-Ct})\omega^\circ(x)},$$

and hence

$$\int_0^t f_x(u)du \geq \int_0^t g_x(u)du = \alpha^{-1}\log\Big(1+\alpha C^{-1}(1-e^{-Ct})\omega^\circ(x)\Big).$$

Inserting this into (4.15), we obtain for all $t>0$

$$\begin{aligned}\int|\omega^t|^p &\leq e^{C(p-1)t}\int|\omega^\circ(x)|^p\Big(1+\alpha C^{-1}(1-e^{-Ct})\omega^\circ(x)\Big)^{1-p}dx\\ &\leq\bigg(\frac{C\alpha^{-1}e^{Ct}}{1-e^{-Ct}}\bigg)^{p-1}\int|\omega^\circ(x)|dx=\bigg(\frac{C\alpha^{-1}e^{Ct}}{1-e^{-Ct}}\bigg)^{p-1}.\end{aligned}$$

The result (iii) then follows from the obvious inequality $e^{Ct}(1-e^{-Ct})^{-1}\leq e^{Ct}+1+(Ct)^{-1}$ for all $t>0$. ☐

In the following result, we examine the regularity of v and of the divergence ζ that follows from the boundedness of the vorticity ω.

Lemma 4.5 (Relative L^p-estimates)**.** *Let $\lambda>0$, $\alpha\geq 0$, $\beta\in\mathbb{R}$, $T>0$, $h,\Psi,\bar v^\circ\in W^{1,\infty}(\mathbb{R}^2)^2$, and $v^\circ\in\bar v^\circ+\mathrm{L}^2(\mathbb{R}^2)^2$, with $\omega^\circ:=\operatorname{curl}v^\circ\in\mathcal{P}(\mathbb{R}^2)$, $\bar\omega^\circ:=\operatorname{curl}\bar v^\circ\in\mathcal{P}\cap\mathrm{L}^\infty(\mathbb{R}^2)$, and with either $\operatorname{div}(av^\circ)=\operatorname{div}(a\bar v^\circ)=0$ in the case (1.1), or $\zeta^\circ:=\operatorname{div}(av^\circ)$, $\bar\zeta^\circ:=\operatorname{div}(a\bar v^\circ)\in\mathrm{L}^2\cap\mathrm{L}^\infty(\mathbb{R}^2)$ in the case (1.2). Let $v\in\mathrm{L}^\infty_{\mathrm{loc}}([0,T);\bar v^\circ+\mathrm{L}^2(\mathbb{R}^2)^2)$ be a weak solution of (1.1) or of (1.2) on $[0,T)\times\mathbb{R}^2$ with initial data v°, and with $\omega:=\operatorname{curl}v\in\mathrm{L}^\infty([0,T];\mathrm{L}^\infty(\mathbb{R}^2))$. Then we have for all $t\in[0,T)$*

$$\|\zeta^t\|_{\mathrm{L}^2\cap\mathrm{L}^\infty}\leq C,\qquad\|\operatorname{div}(v^t-\bar v^\circ)\|_{\mathrm{L}^2\cap\mathrm{L}^\infty}\leq C,\qquad\|v^t\|_{\mathrm{L}^\infty}\leq C,$$

where the constants C's depend only on an upper bound on α, $|\beta|$, λ, λ^{-1}, T, $\|h\|_{W^{1,\infty}}$, $\|(\Psi,\bar v^\circ)\|_{\mathrm{L}^\infty}$, $\|v^\circ-\bar v^\circ\|_{\mathrm{L}^2}$, $\|\bar\omega^\circ\|_{\mathrm{L}^1\cap\mathrm{L}^\infty}$, $\|(\zeta^\circ,\bar\zeta^\circ)\|_{\mathrm{L}^2\cap\mathrm{L}^\infty}$, $\|\omega\|_{\mathrm{L}^\infty_T\mathrm{L}^\infty}$, and additionally on $\|(\nabla\Psi,\nabla\bar v^\circ)\|_{\mathrm{L}^\infty}$ (resp. on α^{-1}) in the case $\alpha=0$ (resp. $\alpha>0$).

Proof. In this proof, we use the notation $\lesssim$ for $\leq$ up to a constant $C>0$ as in the statement, and we also set $\theta:=\operatorname{div}v$, $\bar\theta^\circ:=\operatorname{div}\bar v^\circ$. We may focus on the case of the compressible equation (1.2), the other case being similar and simpler. We split the proof into three steps.

Step 1: preliminary estimate for v. In this step, we prove for all $t\in[0,T)$,

$$\|v^t\|_{\mathrm{L}^\infty}\lesssim 1+\|\theta^t-\bar\theta^\circ\|_{\mathrm{L}^2}\log^{1/2}(2+\|\theta^t-\bar\theta^\circ\|_{\mathrm{L}^2\cap\mathrm{L}^\infty}).\tag{4.16}$$

Note that $v^t-\bar v^\circ=\nabla^\perp\triangle^{-1}(\omega^t-\bar\omega^\circ)+\nabla\triangle^{-1}(\theta^t-\bar\theta^\circ)$. By Lemma 2.4(i)–(ii), we may then estimate

$$\begin{aligned}\|v^t-\bar v^\circ\|_{\mathrm{L}^\infty}&\leq\|\nabla\triangle^{-1}(\omega^t-\bar\omega^\circ)\|_{\mathrm{L}^\infty}+\|\nabla\triangle^{-1}(\theta^t-\bar\theta^\circ)\|_{\mathrm{L}^\infty}\\&\lesssim\|\omega^t-\bar\omega^\circ\|_{\mathrm{L}^2}\log^{1/2}(2+\|\omega^t-\bar\omega^\circ\|_{\mathrm{L}^1\cap\mathrm{L}^\infty})\\&\qquad+\|\theta^t-\bar\theta^\circ\|_{\mathrm{L}^2}\log^{1/2}(2+\|\theta^t-\bar\theta^\circ\|_{\mathrm{L}^2\cap\mathrm{L}^\infty})+\|v^t-\bar v^\circ\|_{\mathrm{L}^2},\end{aligned}$$

so that (4.16) follows from the a priori estimates of Lemma 4.1 (in the form $\|v^t-\bar v^\circ\|_{\mathrm{L}^2}+\|\omega^t\|_{\mathrm{L}^1}\lesssim 1$) and the boundedness assumption on ω.

Step 2: boundedness of θ. In this step, we prove $\|\theta^t-\bar\theta^\circ\|_{\mathrm{L}^2\cap\mathrm{L}^\infty}\lesssim 1$ for all $t\in[0,T)$. We begin with the L^2-estimate. As ζ satisfies the transport-diffusion equation (1.8), Lemma 2.3(i) with $s=0$ gives

$$\begin{aligned}\|\zeta^t\|_{\mathrm{L}^2}&\lesssim\|\zeta^\circ\|_{\mathrm{L}^2}+\|a\omega(-\alpha(\Psi+v)+\beta(\Psi+v)^\perp)\|_{\mathrm{L}^2_t\mathrm{L}^2}\\&\lesssim 1+\|\omega\|_{\mathrm{L}^2_t\mathrm{L}^\infty}\|v-\bar v^\circ\|_{\mathrm{L}^\infty_t\mathrm{L}^2}+\|\omega\|_{\mathrm{L}^2_t\mathrm{L}^2}\|(\Psi,\bar v^\circ)\|_{\mathrm{L}^\infty},\end{aligned}$$

and hence $\|\zeta^t\|_{\mathrm{L}^2}\lesssim 1$ follows from the a priori estimates of Lemma 4.1 (in the form $\|v^t-\bar v^\circ\|_{\mathrm{L}^2}+\|\omega^t\|_{\mathrm{L}^1}\lesssim 1$) and the boundedness assumption for ω. Similarly, for $\theta^t=a^{-1}\zeta^t-\nabla h\cdot v^t$, we deduce $\|\theta^t-\bar\theta^\circ\|_{\mathrm{L}^2}\lesssim 1$. We now turn to the L^∞-estimate. Lemma 2.3(iii) with $p=q=s=\infty$ gives

$$\begin{aligned}\|\zeta^t\|_{\mathrm{L}^\infty}&\lesssim\|\zeta^\circ\|_{\mathrm{L}^\infty}+\|a\omega(-\alpha(\Psi+v)+\beta(\Psi+v)^\perp)\|_{\mathrm{L}^\infty_t\mathrm{L}^\infty}\\&\lesssim 1+\|\omega\|_{\mathrm{L}^\infty_t\mathrm{L}^\infty}(1+\|v\|_{\mathrm{L}^\infty_t\mathrm{L}^\infty}),\end{aligned}\tag{4.17}$$

or alternatively, for $\theta^t = a^{-1}\zeta^t - \nabla h \cdot v^t$,

$$\|\theta^t\|_{\mathrm{L}^\infty} \lesssim 1 + \|v^t\|_{\mathrm{L}^\infty} + \|\omega\|_{\mathrm{L}^\infty_t \mathrm{L}^\infty}(1 + \|v\|_{\mathrm{L}^\infty_t \mathrm{L}^\infty}).$$

Combining this estimate with the result of Step 1 yields

$$\begin{aligned}\|\theta^t\|_{\mathrm{L}^\infty} \lesssim 1 &+ \|\theta^t - \bar\theta^\circ\|_{\mathrm{L}^2} \log^{1/2}(2 + \|\theta^t - \bar\theta^\circ\|_{\mathrm{L}^2 \cap \mathrm{L}^\infty}) \\ &+ \|\omega\|_{\mathrm{L}^\infty_t \mathrm{L}^\infty}(1 + \|\theta - \bar\theta^\circ\|_{\mathrm{L}^\infty_t \mathrm{L}^2} \log^{1/2}(2 + \|\theta - \bar\theta^\circ\|_{\mathrm{L}^\infty_t(\mathrm{L}^2 \cap \mathrm{L}^\infty)})).\end{aligned}$$

Now the boundedness assumption on ω and the L^2-estimate for θ proven above reduce this expression to

$$\|\theta^t\|_{\mathrm{L}^\infty} \lesssim 1 + \log^{1/2}(2 + \|\theta\|_{\mathrm{L}^\infty_t \mathrm{L}^\infty}).$$

Taking the supremum with respect to t, we may then conclude $\|\theta^t\|_{\mathrm{L}^\infty} \lesssim 1$ for all $t \in [0, T)$.

Step 3: conclusion. By the result of Step 2, the estimate (4.16) of Step 1 takes the form $\|v^t\|_{\mathrm{L}^\infty} \lesssim 1$. The estimate (4.17) of Step 2 then yields $\|\zeta^t\|_{\mathrm{L}^\infty} \lesssim 1$, while the L^2-estimate for ζ has already been proven in Step 2. ☐

4.2 Propagation of regularity

As local existence is established in Proposition 3.1 only for smooth enough data, it is necessary for the global existence result to first prove propagation of regularity along the flow. We prove it here as a consequence of the boundedness of the vorticity ω. We begin with Sobolev H^s-regularity, and then turn to Hölder regularity.

Lemma 4.6 (Conservation of Sobolev norms)**.** *Let $s > 1$. Let $\lambda > 0$, $\alpha \ge 0$, $\beta \in \mathbb{R}$, $T > 0$, $h, \Psi, \bar v^\circ \in W^{s+1,\infty}(\mathbb{R}^2)^2$, and $v^\circ \in \bar v^\circ + \mathrm{L}^2(\mathbb{R}^2)^2$, with $\omega^\circ := \operatorname{curl} v^\circ, \bar\omega^\circ := \operatorname{curl} \bar v^\circ \in \mathcal{P} \cap H^s(\mathbb{R}^2)$, and with either $\operatorname{div}(av^\circ) = \operatorname{div}(a\bar v^\circ) = 0$ in the case (1.1), or $\zeta^\circ := \operatorname{div}(av^\circ), \bar\zeta^\circ := \operatorname{div}(a\bar v^\circ) \in H^s(\mathbb{R}^2)$ in the case (1.2). Let $v \in \mathrm{L}^\infty([0,T]; \bar v^\circ + H^{s+1}(\mathbb{R}^2)^2)$ be a weak solution of (1.1) or of (1.2) on $[0, T) \times \mathbb{R}^2$ with initial data v°. Then we have for all $t \in [0, T)$*

$$\|\omega^t\|_{H^s} \le C, \qquad \|\zeta^t\|_{H^s} \le C, \qquad \|v^t - \bar v^\circ\|_{H^{s+1}} \le C, \qquad \|\nabla v^t\|_{\mathrm{L}^\infty} \le C,$$

where the constants C's depend only on an upper bound on s, $(s-1)^{-1}$, α, $|\beta|$, λ, λ^{-1}, T, $\|(h, \Psi, \bar v^\circ)\|_{W^{s+1,\infty}}$, $\|v^\circ - \bar v^\circ\|_{\mathrm{L}^2}$, $\|(\omega^\circ, \bar\omega^\circ, \zeta^\circ, \bar\zeta^\circ)\|_{H^s}$, $\|\omega\|_{\mathrm{L}^\infty_T \mathrm{L}^\infty}$, and additionally on α^{-1} in the case $\alpha > 0$.

Proof. We set $\theta := \operatorname{div} v$, $\bar\theta^\circ := \operatorname{div} \bar v^\circ$. In this proof, we use the notation $\lesssim$ for $\le$ up to a constant $C > 0$ as in the statement. We may focus on the compressible case (1.2), the other case being similar and simpler. We split the proof into four steps.

Step 1: time-derivative of $\|\omega\|_{H^s}$. In this step, we prove for all $s \ge 0$ and $t \in [0, T)$

$$\partial_t \|\omega^t\|_{H^s} \lesssim (1 + \|\nabla v^t\|_{\mathrm{L}^\infty})(1 + \|\omega^t\|_{H^s}) + \|\theta^t - \bar\theta^\circ\|_{H^s}.$$

Lemma 2.2 with $\rho = \omega$, $w = \alpha(\Psi + v)^\perp + \beta(\Psi + v)$ and $W = \alpha(\Psi + \bar v^\circ)^\perp + \beta(\Psi + \bar v^\circ)$ yields

$$\begin{aligned}\partial_t \|\omega^t\|_{H^s} \lesssim (1 + \|\nabla v^t\|_{\mathrm{L}^\infty})\|\omega^t\|_{H^s} &+ (\alpha\|\omega^t - \bar\omega^\circ\|_{H^s} + |\beta| \|\theta^t - \bar\theta^\circ\|_{H^s})\|\omega^t\|_{\mathrm{L}^\infty} \\ &+ \|\omega^t\|_{\mathrm{L}^2} + (1 + \alpha\|\omega^t\|_{\mathrm{L}^\infty} + |\beta| \|\theta^t\|_{\mathrm{L}^\infty})\|\omega^t\|_{H^s}.\end{aligned}$$

and the claim then follows from Lemma 4.5 and the boundedness assumption on ω.

Step 2: Lipschitz estimate for v. In this step, we prove for all $s > 1$ and $t \in [0, T)$

$$\|\nabla v^t\|_{\mathrm{L}^\infty} \lesssim \log(2 + \|\omega^t\|_{H^s} + \|\theta^t - \bar\theta^\circ\|_{H^s}). \tag{4.18}$$

Since $v^t - \bar v^\circ = \nabla^\perp \triangle^{-1}(\omega^t - \bar\omega^\circ) + \nabla \triangle^{-1}(\theta^t - \bar\theta^\circ)$, Lemma 2.4(iii) yields, together with the Sobolev embedding of H^s into a Hölder space for any $s > 1$,

$$\begin{aligned}\|\nabla(v^t - \bar v^\circ)\|_{\mathrm{L}^\infty} &\le \|\nabla^2 \triangle^{-1}(\omega^t - \bar\omega^\circ)\|_{\mathrm{L}^\infty} + \|\nabla^2 \triangle^{-1}(\theta^t - \bar\theta^\circ)\|_{\mathrm{L}^\infty} \\ &\lesssim \|\omega^t - \bar\omega^\circ\|_{\mathrm{L}^\infty} \log(2 + \|\omega^t - \bar\omega^\circ\|_{H^s}) + \|\omega^t - \bar\omega^\circ\|_{\mathrm{L}^1} \\ &\quad + \|\theta^t - \bar\theta^\circ\|_{\mathrm{L}^\infty} \log(2 + \|\theta^t - \bar\theta^\circ\|_{H^s}) + \|\theta^t - \bar\theta^\circ\|_{\mathrm{L}^2},\end{aligned}$$

and the claim (4.18) then follows from Lemma 4.1(i), Lemma 4.5, and the boundedness assumption on ω.

Step 3: Sobolev estimate for θ. In this step, we prove for all $s \geq 0$ and $t \in [0, T)$

$$\|\theta^t - \bar{\theta}^\circ\|_{H^s} \lesssim 1 + \|\omega\|_{\mathrm{L}^\infty_t H^s}. \tag{4.19}$$

As ζ satisfies the transport-diffusion equation (1.8), Lemma 2.3(i)–(ii) gives, for all $s \geq 0$,

$$\|\zeta^t\|_{H^s} \lesssim \|\zeta^\circ\|_{H^s} + \|a\omega(-\alpha(\Psi + v) + \beta(\Psi + v)^\perp)\|_{\mathrm{L}^2_t H^s}.$$

Using Lemma 2.1 to estimate the right-hand side, we find, for all $s \geq 0$,

$$\begin{aligned}\|\zeta^t\|_{H^s} &\lesssim 1 + \|a\omega(-\alpha(v - \bar{v}^\circ) + \beta(v - \bar{v}^\circ)^\perp)\|_{\mathrm{L}^2_t H^s} + \|a\omega(-\alpha(\Psi + \bar{v}^\circ) + \beta(\Psi + \bar{v}^\circ)^\perp)\|_{\mathrm{L}^2_t H^s} \\ &\lesssim 1 + \|\omega\|_{\mathrm{L}^\infty_t \mathrm{L}^\infty}\|v - \bar{v}^\circ\|_{\mathrm{L}^2_t H^s} + \|\omega\|_{\mathrm{L}^2_t H^s}\|v - \bar{v}^\circ\|_{\mathrm{L}^\infty_t \mathrm{L}^\infty} \\ &\qquad + \|\omega\|_{\mathrm{L}^2_t \mathrm{L}^2}(1 + \|\bar{v}^\circ\|_{W^{s,\infty}}) + \|\omega\|_{\mathrm{L}^2_t H^s}(1 + \|\bar{v}^\circ\|_{\mathrm{L}^\infty}),\end{aligned}$$

and hence, by Lemma 4.5 and the boundedness assumption on ω,

$$\|\zeta^t\|_{H^s} \lesssim 1 + \|\omega\|_{\mathrm{L}^\infty_t H^s} + \|v - \bar{v}^\circ\|_{\mathrm{L}^\infty_t H^s}. \tag{4.20}$$

Lemma 2.7 then yields for all $s \geq 0$,

$$\|\zeta^t\|_{H^s} \lesssim 1 + \|\omega\|_{\mathrm{L}^\infty_t H^s} + \|\omega - \bar{\omega}^\circ\|_{\mathrm{L}^\infty_t(\dot{H}^{-1} \cap H^{s-1})} + \|\zeta - \bar{\zeta}^\circ\|_{\mathrm{L}^\infty_t(\dot{H}^{-1} \cap H^{s-1})}.$$

Noting that $\|(\omega - \bar{\omega}^\circ, \zeta - \bar{\zeta}^\circ)\|_{\dot{H}^{-1}} \lesssim \|v - \bar{v}^\circ\|_{\mathrm{L}^2}$, and using Lemma 4.1(iii) in the form $\|v - \bar{v}^\circ\|_{\mathrm{L}^2} \lesssim 1$, we deduce

$$\|\zeta^t\|_{H^s} \lesssim 1 + \|\omega\|_{\mathrm{L}^\infty_t H^s} + \|\zeta\|_{\mathrm{L}^\infty_t H^{s-1}}.$$

Taking the supremum in time, we find by induction $\|\zeta\|_{\mathrm{L}^\infty_t H^s} \lesssim 1 + \|\omega\|_{\mathrm{L}^\infty_t H^s} + \|\zeta\|_{\mathrm{L}^\infty_t \mathrm{L}^2}$ for all $s \geq 0$. Recalling that Lemma 4.5 gives $\|\theta^t - \bar{\theta}^\circ\|_{\mathrm{L}^2} \lesssim 1$, and using the identity $\theta^t = a^{-1}\zeta^t - \nabla h \cdot v^t$, the claim (4.19) directly follows.

Step 4: conclusion. Combining the results of the three previous steps yields, for all $s > 1$,

$$\begin{aligned}\partial_t\|\omega^t\|_{H^s} &\lesssim (1 + \|\omega^t\|_{H^s})\log(2 + \|\omega^t\|_{H^s} + \|\theta^t - \bar{\theta}^\circ\|_{H^s}) + \|\theta^t - \bar{\theta}^\circ\|_{H^s} \\ &\lesssim (1 + \|\omega^t\|_{H^s})\log(2 + \|\omega\|_{\mathrm{L}^\infty_t H^s}) + \|\omega\|_{\mathrm{L}^\infty_t H^s},\end{aligned}$$

hence

$$\partial_t\|\omega\|_{\mathrm{L}^\infty_t H^s} \leq \sup_{[0,t]} \partial_t\|\omega\|_{H^s} \lesssim (1 + \|\omega\|_{\mathrm{L}^\infty_t H^s})\log(2 + \|\omega\|_{\mathrm{L}^\infty_t H^s}),$$

and the Grönwall inequality then gives $\|\omega\|_{\mathrm{L}^\infty_t H^s} \lesssim 1$. Combining this with (4.18), (4.19) and (4.20), and recalling the identity $v^t - \bar{v}^\circ = \nabla^\perp\triangle^{-1}(\omega^t - \bar{\omega}^\circ) + \nabla\triangle^{-1}(\theta^t - \bar{\theta}^\circ)$, the conclusion follows. □

We now turn to the propagation of Hölder regularity. More precisely, we consider the Besov spaces $C^s_*(\mathbb{R}^2) := B^s_{\infty,\infty}(\mathbb{R}^2)$. Recall that these spaces coincide with the usual Hölder spaces $C^s(\mathbb{R}^2)$ only for non-integer $s \geq 0$ (for integer $s > 0$, they are strictly larger and coincide with the corresponding Zygmund spaces).

Lemma 4.7 (Conservation of Hölder-Zygmund norms). *Let $s > 0$. Let $\lambda > 0$, $\alpha \geq 0$, $\beta \in \mathbb{R}$, $T > 0$, and $h, \Psi, v^\circ \in C^{s+1}_*(\mathbb{R}^2)^2$ with $\omega^\circ := \operatorname{curl} v^\circ \in \mathcal{P}(\mathbb{R}^2)$, and with either $\operatorname{div}(av^\circ) = 0$ in the case (1.1), or $\zeta^\circ := \operatorname{div}(av^\circ) \in \mathrm{L}^2(\mathbb{R}^2)$ in the case (1.2). Let $v \in \mathrm{L}^\infty([0,T]; C^{s+1}_*(\mathbb{R}^2)^2)$ be a weak solution of (1.1) or of (1.2) on $[0,T) \times \mathbb{R}^2$ with initial data v°. Then we have for all $t \in [0,T)$*

$$\|\omega^t\|_{C^s_*} \leq C, \qquad \|\zeta^t\|_{C^s_*} \leq C, \qquad \|v^t\|_{C^{s+1}_*} \leq C,$$

where the constants C's depend only on an upper bound on s, s^{-1}, α, $|\beta|$, λ, λ^{-1}, T, $\|(h, \Psi, v^\circ)\|_{C^{s+1}_*}$, $\|\zeta^\circ\|_{\mathrm{L}^2}$, $\|\omega\|_{\mathrm{L}^\infty_T \mathrm{L}^\infty}$, and additionally on α^{-1} in the case $\alpha > 0$.

Proof. We set $\theta := \operatorname{div} v$. In this proof, we use the notation $\lesssim$ for $\le$ up to a constant $C>0$ as in the statement. We may focus on the compressible equation (1.2), the other case being similar and simpler. We split the proof into four steps, and make a systematic use of the standard Besov machinery as presented in [5].

Step 1: time-derivative of $\|\omega^t\|_{C_*^s}$. In this step, we prove, for all $s\ge 0$ and $t\in[0,T)$,

$$\partial_t\|\omega^t\|_{C_*^s} \lesssim (1+\|\omega^t\|_{C_*^s})(1+\|\nabla v^t\|_{\mathrm{L}^\infty\cap C_*^{s-1}}) + \|\theta^t\|_{C_*^s}.$$

The transport equation (1.7) has the form $\partial_t\omega^t = \operatorname{div}(\omega^t w^t)$ with $w^t = \alpha(\Psi+v^t)^\perp + \beta(\Psi+v^t)$. Arguing as in [5, Chapter 3.2], we obtain, for all $s\ge -1$,

$$\partial_t\|\omega^t\|_{C_*^s} \lesssim \|\omega^t\|_{C_*^s}\|\nabla w^t\|_{\mathrm{L}^\infty\cap C_*^{s-1}} + \|\omega^t \operatorname{div} w^t\|_{C_*^s}.$$

Using the usual product rules [5, Corollary 2.86], for all $s>0$,

$$\begin{aligned}\partial_t\|\omega^t\|_{C_*^s} &\lesssim \|\omega^t\|_{C_*^s}\|\nabla w^t\|_{\mathrm{L}^\infty\cap C_*^{s-1}} + \|\omega^t\|_{\mathrm{L}^\infty}\|\operatorname{div} w^t\|_{C_*^s} + \|\omega^t\|_{C_*^s}\|\operatorname{div} w^t\|_{\mathrm{L}^\infty}\\ &\lesssim \|\omega^t\|_{C_*^s}(1+\|\nabla v^t\|_{\mathrm{L}^\infty\cap C_*^{s-1}}) + \|\omega^t\|_{\mathrm{L}^\infty}(1+\|\omega^t\|_{C_*^s}+\|\theta^t\|_{C_*^s}),\end{aligned}$$

and the result follows from the boundedness assumption on ω.

Step 2: estimate for ∇v^t. In this step, we prove, for all $s>0$ and $t\in[0,T)$,

$$\|\nabla v^t\|_{\mathrm{L}^\infty\cap C_*^{s-1}} \lesssim \|\omega^t\|_{C_*^{s-1}} + \|\theta^t\|_{C_*^{s-1}} + \log(2+\|\omega^t\|_{C_*^s}+\|\theta^t\|_{C_*^s}).$$

Since $v^t - v^\circ = \nabla^\perp\triangle^{-1}(\omega^t-\omega^\circ) + \nabla\triangle^{-1}(\theta^t-\theta^\circ)$, Lemma 2.5(ii) yields for all $s\in\mathbb{R}$,

$$\|\nabla v^t\|_{C_*^{s-1}} \lesssim 1 + \|\omega^t-\omega^\circ\|_{\dot H^{-1}\cap C_*^{s-1}} + \|\theta^t-\theta^\circ\|_{\dot H^{-1}\cap C_*^{s-1}},$$

and thus, noting that $\|(\omega-\omega^\circ,\theta-\theta^\circ)\|_{\dot H^{-1}} \lesssim \|v-v^\circ\|_{\mathrm{L}^2}$, and using Lemma 4.1(iii) in the form $\|v-v^\circ\|_{\mathrm{L}^2}\lesssim 1$,

$$\|\nabla v^t\|_{C_*^{s-1}} \lesssim 1 + \|\omega^t\|_{C_*^{s-1}} + \|\theta^t\|_{C_*^{s-1}}.$$

Arguing as in Step 2 of the proof of Lemma 4.6 further yields, for all $s>0$,

$$\|\nabla v^t\|_{\mathrm{L}^\infty} \lesssim \log(2+\|\omega^t\|_{C_*^s}+\|\theta^t-\theta^\circ\|_{C_*^s}),$$

so the result follows.

Step 3: estimate for θ^t. In this step, we prove, for all $s\ge -1$ and $t\in[0,T)$,

$$\|\theta^t\|_{C_*^s} \lesssim 1 + \|\omega\|_{\mathrm{L}_t^\infty C_*^{s-1}}.$$

As ζ satisfies the transport-diffusion equation (1.8), we obtain, for all $s\ge -1$, arguing as in [5, Chapter 3.4],

$$\|\zeta^t\|_{C_*^s} \lesssim \|\zeta^\circ\|_{C_*^s} + \|a\omega(-\alpha(\Psi+v)+\beta(\Psi+v)^\perp)\|_{\mathrm{L}_t^\infty C_*^{s-1}},$$

and thus, by the usual product rules [5, Corollary 2.86], the boundedness assumption on ω, and Lemma 4.5, we deduce, for all $s\ge -1$,

$$\begin{aligned}\|\zeta^t\|_{C_*^s} &\lesssim 1 + \|\omega\|_{\mathrm{L}_t^\infty(\mathrm{L}^\infty\cap C_*^{s-1})}(1+\|v\|_{\mathrm{L}_t^\infty \mathrm{L}^\infty}) + \|\omega\|_{\mathrm{L}_t^\infty \mathrm{L}^\infty}(1+\|v\|_{\mathrm{L}_t^\infty(\mathrm{L}^\infty\cap C_*^{s-1})})\\ &\lesssim 1 + \|\omega\|_{\mathrm{L}_t^\infty C_*^{s-1}} + \|v\|_{\mathrm{L}_t^\infty C_*^{s-1}},\end{aligned}$$

or alternatively, in terms of $\theta^t = a^{-1}\zeta^t - \nabla h\cdot v^t$,

$$\|\theta^t\|_{C_*^s} \lesssim \|\zeta^t\|_{\mathrm{L}^\infty\cap C_*^s} + \|v^t\|_{\mathrm{L}^\infty\cap C_*^s} \lesssim 1 + \|\omega\|_{\mathrm{L}_t^\infty C_*^{s-1}} + \|v\|_{\mathrm{L}_t^\infty C_*^s}.$$

Decomposing $v^t - v^\circ = \nabla^\perp \triangle^{-1}(\omega^t - \omega^\circ) + \nabla \triangle^{-1}(\theta^t - \theta^\circ)$, using Lemma 2.5(ii), and again Lemma 4.1(iii) in the form $\|(\omega - \omega^\circ, \theta - \theta^\circ)\|_{\dot H^{-1}} \lesssim \|v - v^\circ\|_{\mathrm{L}^2} \lesssim 1$, we find

$$\|v^t\|_{C^s_*} \lesssim 1 + \|\omega^t - \omega^\circ\|_{\dot H^{-1} \cap C^{s-1}_*} + \|\theta^t - \theta^\circ\|_{\dot H^{-1} \cap C^{s-1}_*} \lesssim 1 + \|\omega^t\|_{C^{s-1}_*} + \|\theta^t\|_{C^{s-1}_*},$$

and hence

$$\|\theta\|_{\mathrm{L}^\infty_t C^s_*} \lesssim 1 + \|\omega\|_{\mathrm{L}^\infty_t C^{s-1}_*} + \|\theta\|_{\mathrm{L}^\infty_t C^{s-1}_*}.$$

If $s \le 1$, then we have $\|\cdot\|_{C^{s-1}_*} \lesssim \|\cdot\|_{\mathrm{L}^\infty}$, so that the above estimate, the boundedness assumption on ω, and Lemma 4.5 yield $\|\theta\|_{\mathrm{L}^\infty_t C^s_*} \lesssim 1$. The result for $s > 1$ then follows by induction.

Step 4: conclusion. Combining the results of the three previous steps yields, for all $s > 0$,

$$\begin{aligned}\partial_t \|\omega\|_{\mathrm{L}^\infty_t C^s_*} \le \sup_{[0,t]} \partial_t \|\omega\|_{C^s_*} &\lesssim (1 + \|\omega\|_{\mathrm{L}^\infty_t C^s_*})(\|\omega\|_{\mathrm{L}^\infty_t C^{s-1}_*} + \|\theta\|_{\mathrm{L}^\infty_t C^{s-1}_*} + \log(2 + \|\omega^t\|_{C^s_*} + \|\theta^t\|_{C^s_*})) + \|\theta\|_{\mathrm{L}^\infty_t C^s_*}\\ &\lesssim (1 + \|\omega\|_{\mathrm{L}^\infty_t C^s_*})(\|\omega\|_{\mathrm{L}^\infty_t C^{s-1}_*} + \log(2 + \|\omega\|_{\mathrm{L}^\infty_t C^s_*})).\end{aligned}$$

If $s \le 1$, then we have $\|\cdot\|_{C^{s-1}_*} \lesssim \|\cdot\|_{\mathrm{L}^\infty}$, so that the above estimate and the boundedness assumption on ω yield $\partial_t \|\omega\|_{\mathrm{L}^\infty_t C^s_*} \lesssim (1 + \|\omega\|_{\mathrm{L}^\infty_t C^s_*}) \log(2 + \|\omega\|_{\mathrm{L}^\infty_t C^s_*})$, hence $\|\omega\|_{\mathrm{L}^\infty_t C^s_*} \lesssim 1$ by the Grönwall inequality. The conclusion for $s > 1$ then follows by induction. ∎

4.3 Global existence of solutions

With Lemmas 4.6 and 4.7 at hand, together with the a priori bounds of Lemmas 4.2 and 4.3, it is now straightforward to deduce the following global existence result from the local existence statement of Proposition 3.1.

Corollary 4.8 (Global existence of smooth solutions)**.** *Let $s > 1$. Let $\lambda > 0$, $\alpha \ge 0$, $\beta \in \mathbb{R}$, $h, \Psi, \bar v^\circ \in W^{s+1,\infty}(\mathbb{R}^2)^2$, and $v^\circ \in \bar v^\circ + \mathrm{L}^2(\mathbb{R}^2)^2$, with $\omega^\circ := \operatorname{curl} v^\circ, \bar\omega^\circ := \operatorname{curl} \bar v^\circ \in \mathcal{P} \cap H^s(\mathbb{R}^2)$, and with either $\operatorname{div}(av^\circ) = \operatorname{div}(a\bar v^\circ) = 0$ in the case (1.1), or $\zeta^\circ := \operatorname{div}(av^\circ), \bar\zeta^\circ := \operatorname{div}(a\bar v^\circ) \in H^s(\mathbb{R}^2)$ in the case (1.2). Then,*

(i) there exists a global weak solution $v \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; \bar v^\circ + H^{s+1}(\mathbb{R}^2)^2)$ of (1.1) on $\mathbb{R}^+ \times \mathbb{R}^2$ with initial data v°, and with $\omega = \operatorname{curl} v \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; \mathcal{P} \cap H^s(\mathbb{R}^2))$;

(ii) if $\beta = 0$, there exists a global weak solution $v \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; \bar v^\circ + H^{s+1}(\mathbb{R}^2)^2)$ of (1.2) on $\mathbb{R}^+ \times \mathbb{R}^2$ with initial data v°, and with $\omega := \operatorname{curl} v \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; \mathcal{P} \cap H^s(\mathbb{R}^2))$ and $\zeta := \operatorname{div}(av) \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; H^s(\mathbb{R}^2))$.

Proof. We may focus on item (ii), the first item being completely similar. In this proof we use the notation $\simeq$ and $\lesssim$ for $=$ and $\le$ up to positive constants that depend only on an upper bound on α, α^{-1}, $|\beta|$, λ, λ^{-1}, s, $(s-1)^{-1}$, $\|(h, \Psi, \bar v^\circ)\|_{W^{s+1,\infty}}$, $\|v^\circ - \bar v^\circ\|_{\mathrm{L}^2}$, $\|(\omega^\circ, \bar\omega^\circ, \zeta^\circ, \bar\zeta^\circ)\|_{H^s}$.

Given $\bar v^\circ \in W^{s+1,\infty}(\mathbb{R}^2)^2$ and $v^\circ \in \bar v^\circ + \mathrm{L}^2(\mathbb{R}^2)^2$ with $\omega^\circ, \bar\omega^\circ \in \mathcal{P} \cap H^s(\mathbb{R}^2)$ and $\zeta^\circ, \bar\zeta^\circ \in H^s(\mathbb{R}^2)$, Proposition 3.1 gives a time $T > 0$, $T \simeq 1$, such that there exists a weak solution $v \in \mathrm{L}^\infty([0,T); \bar v^\circ + H^s(\mathbb{R}^2)^2)$ of (1.2) on $[0,T) \times \mathbb{R}^2$ with initial data v°. For all $t \in [0,T)$, Lemma 4.3(ii) (with $\beta = 0$) then gives $\|\omega^t\|_{\mathrm{L}^\infty} \lesssim 1$, which implies by Lemma 4.6

$$\|\omega^t\|_{H^s} + \|\zeta^t\|_{H^s} + \|v^t - \bar v^\circ\|_{H^{s+1}} \lesssim 1,$$

and moreover by Lemma 4.1(i) we have $\omega^t \in \mathcal{P}(\mathbb{R}^2)$ for all $t \in [0,T)$. These a priori estimates show that the solution v can be extended globally in time. ∎

We now extend this global existence result beyond the setting of smooth initial data. We start with the following result for L^2-data, which is easily deduced by approximation. (Note that, for $a = e^h$ smooth, it is not clear at all whether smooth functions are dense in the set $\{(w, z) \in W^{1,\infty}(\mathbb{R}^2)^2 \times \mathrm{L}^2(\mathbb{R}^2) : z = \operatorname{div}(aw)\}$, which thus causes troubles when regularizing the initial data; this problem is solved e.g. by assuming that the reference map $\bar v^\circ$ further satisfies $\bar\omega^\circ, \bar\zeta^\circ \in H^s(\mathbb{R}^2)$ for some $s > 1$, as we do here.)

Corollary 4.9 (Global existence for L^2-data)**.** *Let $\lambda > 0$, $\alpha \ge 0$, $\beta \in \mathbb{R}$, $h, \Psi \in W^{1,\infty}(\mathbb{R}^2)^2$. Let $\bar v^\circ \in W^{1,\infty}(\mathbb{R}^2)^2$ be some reference map with $\bar\omega^\circ := \operatorname{curl} \bar v^\circ \in \mathcal{P} \cap H^s(\mathbb{R}^2)$ for some $s > 1$, and with either $\operatorname{div}(a\bar v^\circ) = 0$ in the case (1.1), or $\bar\zeta^\circ := \operatorname{div}(a\bar v^\circ) \in H^s(\mathbb{R}^2)$ in the case (1.2). Let $v^\circ \in \bar v^\circ + \mathrm{L}^2(\mathbb{R}^2)^2$, with $\omega^\circ := \operatorname{curl} v^\circ \in \mathcal{P} \cap \mathrm{L}^2(\mathbb{R}^2)$, and with either $\operatorname{div}(av^\circ) = 0$ in the case (1.1), or $\zeta^\circ := \operatorname{div}(av^\circ) \in \mathrm{L}^2(\mathbb{R}^2)$ in the case (1.2). Then,*

(i) *there exists a global weak solution* $v \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; \bar v^\circ + \mathrm{L}^2(\mathbb{R}^2)^2)$ *of* (1.1) *on* $\mathbb{R}^+ \times \mathbb{R}^2$ *with initial data* v°*, and with* $v \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^+; \bar v^\circ + H^1(\mathbb{R}^2)^2)$ *and* $\omega := \operatorname{curl} v \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; \mathcal{P} \cap \mathrm{L}^2(\mathbb{R}^2))$*;*

(ii) *if* $\beta = 0$*, there exists a global weak solution* $v \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; \bar v^\circ + \mathrm{L}^2(\mathbb{R}^2)^2)$ *of* (1.2) *on* $\mathbb{R}^+ \times \mathbb{R}^2$ *with initial data* v°*, and with* $v \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^+; \bar v^\circ + H^1(\mathbb{R}^2)^2)$*,* $\omega := \operatorname{curl} v \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; \mathcal{P} \cap \mathrm{L}^2(\mathbb{R}^2))$ *and* $\zeta := \operatorname{div}(av) \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^+; \mathrm{L}^2(\mathbb{R}^2))$*.*

Proof. We may focus on the case (ii) (with $\beta = 0$), the other case being exactly similar. In this proof we use the notation $\lesssim$ for $\le$ up to a positive constant that depends only on an upper bound on α, α^{-1}, λ, $(s-1)^{-1}$, $\|(h, \Psi, \bar v^\circ)\|_{W^{1,\infty}}$, $\|(\bar\omega^\circ, \bar\zeta^\circ)\|_{H^s}$, $\|v^\circ - \bar v^\circ\|_{\mathrm{L}^2}$, and $\|(\omega^\circ, \zeta^\circ)\|_{\mathrm{L}^2}$. We use the notation $\lesssim_t$ if it further depends on an upper bound on time t.

Let $\rho \in C^\infty_c(\mathbb{R}^2)$ with $\rho \ge 0$, $\int \rho = 1$, and $\rho(0) = 1$, define $\rho_\epsilon(x) := \epsilon^{-d}\rho(x/\epsilon)$ for all $\epsilon > 0$, and set $\omega^\circ_\epsilon := \rho_\epsilon * \omega^\circ$, $\bar\omega^\circ_\epsilon := \rho_\epsilon * \bar\omega^\circ$, $\zeta^\circ_\epsilon := \rho_\epsilon * \zeta^\circ$, $\bar\zeta^\circ_\epsilon := \rho_\epsilon * \bar\zeta^\circ$, $a_\epsilon := \rho_\epsilon * a$ and $\Psi_\epsilon := \rho_\epsilon * \Psi$. For all $\epsilon > 0$, we have ω°_ϵ, $\bar\omega^\circ_\epsilon \in \mathcal{P} \cap H^\infty(\mathbb{R}^2)$, ζ°_ϵ, $\bar\zeta^\circ_\epsilon \in H^\infty(\mathbb{R}^2)$, and a_ϵ, a_ϵ^{-1}, $\Psi_\epsilon \in C^\infty(\mathbb{R}^2)^2$. By construction, we have $a_\epsilon \to a$, $a_\epsilon^{-1} \to a^{-1}$, $\Psi_\epsilon \to \Psi$ in $W^{1,\infty}(\mathbb{R}^2)$, $\bar\omega^\circ_\epsilon - \bar\omega^\circ$, $\bar\zeta^\circ_\epsilon - \bar\zeta^\circ \to 0$ in $\dot H^{-1} \cap H^s(\mathbb{R}^2)$, and $\omega^\circ_\epsilon - \omega^\circ$, $\zeta^\circ_\epsilon - \zeta^\circ$ in $\dot H^{-1} \cap \mathrm{L}^2(\mathbb{R}^2)$. The additional convergence in $\dot H^{-1}(\mathbb{R}^2)$ indeed follows from the following computation with Fourier transforms,

$$\|\omega^\circ_\epsilon - \omega^\circ\|^2_{\dot H^{-1}} = \int |\xi|^{-2}|\hat\rho(\epsilon\xi) - 1|^2 |\hat\omega^\circ(\xi)|^2 d\xi \le \epsilon^2 \|\nabla\hat\rho\|^2_{\mathrm{L}^\infty} \int |\hat\omega^\circ|^2 = \epsilon^2\|\nabla\hat\rho\|^2_{\mathrm{L}^\infty}\|\omega^\circ\|^2_{\mathrm{L}^2},$$

and similarly for $\bar\omega^\circ_\epsilon$, ζ°_ϵ, and $\bar\zeta^\circ_\epsilon$. Lemma 2.7 then gives a unique $v^\circ_\epsilon \in v^\circ + H^1(\mathbb{R}^2)^2$ and a unique $\bar v^\circ_\epsilon \in \bar v^\circ + H^{s+1}(\mathbb{R}^2)^2$ such that $\operatorname{curl} v^\circ_\epsilon = \omega^\circ_\epsilon$, $\operatorname{curl} \bar v^\circ_\epsilon = \bar\omega^\circ_\epsilon$, $\operatorname{div}(a_\epsilon v^\circ_\epsilon) = \zeta^\circ_\epsilon$, $\operatorname{div}(a_\epsilon \bar v^\circ_\epsilon) = \bar\zeta^\circ_\epsilon$, and we have $v^\circ_\epsilon - v^\circ \to 0$ in $H^1(\mathbb{R}^2)^2$ and $\bar v^\circ_\epsilon - \bar v^\circ \to 0$ in $H^{s+1}(\mathbb{R}^2)^2$. In particular, the assumption $\bar v^\circ \in W^{1,\infty}(\mathbb{R}^2)^2$ yields by the Sobolev embedding with $s > 1$,

$$\|\bar v^\circ_\epsilon\|_{W^{1,\infty}} \lesssim \|\bar v^\circ_\epsilon - \bar v^\circ\|_{H^{s+1}} + \|\bar v^\circ\|_{W^{1,\infty}} \lesssim 1,$$

and the assumption $v^\circ - \bar v^\circ \in \mathrm{L}^2(\mathbb{R}^2)^2$ implies

$$\|v^\circ_\epsilon - \bar v^\circ_\epsilon\|_{\mathrm{L}^2} \le \|v^\circ_\epsilon - v^\circ\|_{\mathrm{L}^2} + \|v^\circ - \bar v^\circ\|_{\mathrm{L}^2} + \|\bar v^\circ_\epsilon - \bar v^\circ\|_{\mathrm{L}^2} \lesssim 1.$$

Corollary 4.8 then gives a solution $v_\epsilon \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; \bar v^\circ_\epsilon + H^\infty(\mathbb{R}^2)^2)$ of (1.2) on $\mathbb{R}^+ \times \mathbb{R}^2$ with initial data v°_ϵ, and with (a, Ψ) replaced by $(a_\epsilon, \Psi_\epsilon)$. Lemma 4.1(iii) and Lemma 4.3(ii) (with $\beta = 0$) give for all $t \ge 0$

$$\|v_\epsilon - \bar v^\circ_\epsilon\|_{\mathrm{L}^\infty_t \mathrm{L}^2} + \|\zeta_\epsilon\|_{\mathrm{L}^2_t \mathrm{L}^2} + \|\omega_\epsilon\|_{\mathrm{L}^\infty_t \mathrm{L}^2} \lesssim_t 1,$$

hence by Lemma 2.7, together with the obvious estimate $\|(\omega_\epsilon - \bar\omega^\circ_\epsilon, \zeta_\epsilon - \bar\zeta^\circ_\epsilon)\|_{\dot H^{-1}} \lesssim \|v_\epsilon - \bar v^\circ_\epsilon\|_{\mathrm{L}^2}$,

$$\|v_\epsilon - \bar v^\circ_\epsilon\|_{\mathrm{L}^2_t H^1} \lesssim \|v_\epsilon - \bar v^\circ_\epsilon\|_{\mathrm{L}^2_t \mathrm{L}^2} + \|\zeta_\epsilon - \bar\zeta^\circ_\epsilon\|_{\mathrm{L}^2_t \mathrm{L}^2} + \|\omega_\epsilon - \bar\omega^\circ_\epsilon\|_{\mathrm{L}^2_t \mathrm{L}^2} \lesssim_t 1.$$

As $\bar v^\circ_\epsilon$ is bounded in $H^1_{\mathrm{loc}}(\mathbb{R}^2)^2$, we deduce up to an extraction $v_\epsilon \rightharpoonup v$ in $\mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^+; H^1_{\mathrm{loc}}(\mathbb{R}^2)^2)$, and also $\omega_\epsilon \rightharpoonup \omega$, $\zeta_\epsilon \rightharpoonup \zeta$ in $\mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^+; \mathrm{L}^2(\mathbb{R}^2))$, for some functions v, ω, ζ. Comparing equation (1.7) with the above estimates, we deduce that $(\partial_t \omega_\epsilon)_\epsilon$ is bounded in $\mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^+; W^{-1,1}_{\mathrm{loc}}(\mathbb{R}^2))$. Since by the Rellich theorem the space $\mathrm{L}^2(U)$ is compactly embedded in $H^{-1}(U) \subset W^{-1,1}(U)$ for any bounded domain $U \subset \mathbb{R}^2$, the Aubin-Simon lemma ensures that we have $\omega_\epsilon \to \omega$ strongly in $\mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^+; H^{-1}_{\mathrm{loc}}(\mathbb{R}^2))$. This implies $\omega_\epsilon v_\epsilon \to \omega v$ in the distributional sense. We may then pass to the limit in the weak formulation of equation (1.2), and the result follows. ∎

We now investigate the case of rougher initial data. Using the a priori estimates of Lemmas 4.2 and 4.3(ii), we prove global existence for L^q-data, $q > 1$. In the parabolic regime $\alpha > 0$, $\beta = 0$, we have at our disposal the much finer a priori estimates of Lemma 4.3(iii), which then allow to deduce global existence for vortex-sheet data $\omega^\circ \in \mathcal{P}(\mathbb{R}^2)$. As in [29] we make here a crucial use of some compactness result due to Lions [30] in the context of the compressible Navier-Stokes equations. In the conservative regime (iv) below, however, this result is not enough and compactness is proven by hand. Theorem 1 directly follows from this proposition, together with the various a priori estimates of Sections 4.1 and 4.2.

Proposition 4.10 (Global existence for general data)**.** *Let* $\lambda > 0$*,* $\alpha \ge 0$*,* $\beta \in \mathbb{R}$*, and* $h, \Psi \in W^{1,\infty}(\mathbb{R}^2)^2$*. Let* $\bar v^\circ \in W^{1,\infty}(\mathbb{R}^2)^2$ *be some reference map with* $\bar\omega^\circ := \operatorname{curl} \bar v^\circ \in \mathcal{P} \cap H^s(\mathbb{R}^2)$ *for some* $s > 1$*, and with either* $\operatorname{div}(a\bar v^\circ) = 0$ *in the case* (1.1)*, or* $\bar\zeta^\circ := \operatorname{div}(a\bar v^\circ) \in H^s(\mathbb{R}^2)$ *in the case* (1.2)*. Let* $v^\circ \in \bar v^\circ + \mathrm{L}^2(\mathbb{R}^2)^2$*, with* $\omega^\circ = \operatorname{curl} v^\circ \in \mathcal{P}(\mathbb{R}^2)$*, and with either* $\operatorname{div}(av^\circ) = 0$ *in the case* (1.1)*, or* $\zeta^\circ := \operatorname{div}(av^\circ) \in \mathrm{L}^2(\mathbb{R}^2)$ *in the case* (1.2)*. Then,* *(i)* Case (1.2) with $\alpha > 0$, $\beta = 0$: *There exists a weak solution $v \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; \bar v^\circ + \mathrm{L}^2(\mathbb{R}^2)^2)$ on $\mathbb{R}^+ \times \mathbb{R}^2$ with initial data v°, and with $\omega = \operatorname{curl} v \in \mathrm{L}^\infty(\mathbb{R}^+; \mathcal{P}(\mathbb{R}^2))$ and $\zeta = \operatorname{div}(av) \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^+; \mathrm{L}^2(\mathbb{R}^2))$.*

(ii) Case (1.1) with $\alpha > 0$, and either $\beta = 0$ or a constant: *There exists a weak solution $v \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; \bar v^\circ + \mathrm{L}^2(\mathbb{R}^2)^2)$ on $\mathbb{R}^+ \times \mathbb{R}^2$ with initial data v°, and with $\omega = \operatorname{curl} v \in \mathrm{L}^\infty(\mathbb{R}^+; \mathcal{P}(\mathbb{R}^2))$.*

(iii) Case (1.1) with $\alpha > 0$: *If $\omega^\circ \in \mathrm{L}^q(\mathbb{R}^2)$ for some $q > 1$, there exists a weak solution $v \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; \bar v^\circ + \mathrm{L}^2(\mathbb{R}^2)^2)$ on $\mathbb{R}^+ \times \mathbb{R}^2$ with initial data v°, and with $\omega = \operatorname{curl} v \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; \mathcal{P} \cap \mathrm{L}^q(\mathbb{R}^2))$.*

(iv) Case (1.1) with $\alpha = 0$: *If $\omega^\circ \in \mathrm{L}^q(\mathbb{R}^2)$ for some $q > 1$, there exists a very weak solution $v \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; \bar v^\circ + \mathrm{L}^2(\mathbb{R}^2)^2)$ on $\mathbb{R}^+ \times \mathbb{R}^2$ with initial data v°, and with $\omega = \operatorname{curl} v \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; \mathcal{P} \cap \mathrm{L}^q(\mathbb{R}^2))$. This is a weak solution whenever $q \geq 4/3$.*

Proof. We split the proof into three steps, first proving item (i), then explaining how the argument has to be adapted to prove items (ii) and (iii), and finally turning to item (iv).

Step 1: proof of (i). In this step, we use the notation $\lesssim$ for $\leq$ up to a positive constant that depends only on an upper bound on α, α^{-1}, λ, $\|(h, \Psi, \bar v^\circ)\|_{W^{1,\infty}}$, $\|(\bar\omega^\circ, \bar\zeta^\circ)\|_{H^s}$, $\|v^\circ - \bar v^\circ\|_{\mathrm{L}^2}$, and $\|\zeta^\circ\|_{\mathrm{L}^2}$. We use the notation $\lesssim_t$ (resp. $\lesssim_{t,U}$) if it further depends on an upper bound on time t (resp. and on the size of $U \subset \mathbb{R}^2$).

Let $\rho \in C^\infty_c(\mathbb{R}^2)$ with $\rho \geq 0$, $\int \rho = 1$, $\rho(0) = 1$, and $\rho|_{\mathbb{R}^2 \setminus B_1} = 0$, define $\rho_\epsilon(x) := \epsilon^{-d}\rho(x/\epsilon)$ for all $\epsilon > 0$, and set $\omega^\circ_\epsilon := \rho_\epsilon * \omega^\circ$, $\bar\omega^\circ_\epsilon := \rho_\epsilon * \bar\omega^\circ$, $\zeta^\circ_\epsilon := \rho_\epsilon * \zeta^\circ$, $\bar\zeta^\circ_\epsilon := \rho_\epsilon * \bar\zeta^\circ$. For all $\epsilon > 0$, we have $\omega^\circ_\epsilon, \bar\omega^\circ_\epsilon \in \mathcal{P} \cap H^\infty(\mathbb{R}^2)$, ζ°_ϵ, $\bar\zeta^\circ_\epsilon \in H^\infty(\mathbb{R}^2)$. As in the proof of Corollary 4.9, we have by construction $\bar\omega^\circ_\epsilon - \bar\omega^\circ, \bar\zeta^\circ_\epsilon - \bar\zeta^\circ \to 0$ in $\dot H^{-1} \cap H^s(\mathbb{R}^2)$, and $\zeta^\circ_\epsilon - \zeta^\circ$ in $\dot H^{-1} \cap \mathrm{L}^2(\mathbb{R}^2)$. The assumption $v^\circ - \bar v^\circ \in \mathrm{L}^2(\mathbb{R}^2)^2$ further yields $\omega^\circ - \bar\omega^\circ \in \dot H^{-1}(\mathbb{R}^2)$, which implies $\omega^\circ_\epsilon - \bar\omega^\circ_\epsilon \to \omega^\circ - \bar\omega^\circ$, hence $\omega^\circ_\epsilon - \omega^\circ \to 0$, in $\dot H^{-1}(\mathbb{R}^2)$. Lemma 2.7 then gives a unique $v^\circ_\epsilon \in v^\circ + \mathrm{L}^2(\mathbb{R}^2)^2$ and a unique $\bar v^\circ_\epsilon \in \bar v^\circ + H^{s+1}(\mathbb{R}^2)^2$ such that $\operatorname{curl} v^\circ_\epsilon = \omega^\circ_\epsilon$, $\operatorname{curl} \bar v^\circ_\epsilon = \bar\omega^\circ_\epsilon$, $\operatorname{div}(a_\epsilon v^\circ_\epsilon) = \zeta^\circ_\epsilon$, $\operatorname{div}(a_\epsilon \bar v^\circ_\epsilon) = \bar\zeta^\circ_\epsilon$, and we have $v^\circ_\epsilon - v^\circ \to 0$ in $\mathrm{L}^2(\mathbb{R}^2)^2$ and $\bar v^\circ_\epsilon - \bar v^\circ \to 0$ in $H^{s+1}(\mathbb{R}^2)^2$. In particular, arguing as in the proof of Corollary 4.9, the assumption $\bar v^\circ \in W^{1,\infty}(\mathbb{R}^2)^2$ yields $\|\bar v^\circ_\epsilon\|_{W^{1,\infty}} \lesssim 1$ by the Sobolev embedding with $s > 1$, and the assumption $v^\circ - \bar v^\circ \in \mathrm{L}^2(\mathbb{R}^2)^2$ implies $\|v^\circ_\epsilon - \bar v^\circ_\epsilon\|_{\mathrm{L}^2} \lesssim 1$.

Corollary 4.9 then gives a global weak solution $v_\epsilon \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; \bar v^\circ_\epsilon + \mathrm{L}^2(\mathbb{R}^2)^2)$ of (1.2) on $\mathbb{R}^+ \times \mathbb{R}^2$ with initial data v°_ϵ, and Lemma 4.1(iii) yields for all $t \geq 0$

$$\|v_\epsilon - \bar v^\circ_\epsilon\|_{\mathrm{L}^\infty_t \mathrm{L}^2} + \|\zeta_\epsilon\|_{\mathrm{L}^2_t \mathrm{L}^2} \lesssim_t 1, \tag{4.21}$$

while Lemma 4.3(iii) with $\beta = 0$ yields after integration in time, for all $1 \leq p < 2$,

$$\|\omega_\epsilon\|_{\mathrm{L}^p_t \mathrm{L}^p} \lesssim \left(\int_0^t (u^{1-p} + e^{Cu}) du\right)^{1/p} \lesssim_t (2-p)^{-1/p}.$$

Using this last estimate for $p = 3/2$ and $11/6$, and combining it with Lemma 4.1(i) in the form $\|\omega_\epsilon\|_{\mathrm{L}^\infty_t \mathrm{L}^1} \leq 1$, we deduce by interpolation

$$\|\omega_\epsilon\|_{\mathrm{L}^2_t(\mathrm{L}^{4/3} \cap \mathrm{L}^{12/7})} \lesssim_t 1.$$

Now we need to prove more precise estimates on v_ϵ. First recall the identity

$$v_\epsilon = v_{\epsilon,1} + v_{\epsilon,2}, \qquad v_{\epsilon,1} := \nabla^\perp \triangle^{-1} \omega_\epsilon, \qquad v_{\epsilon,2} := \nabla \triangle^{-1} \operatorname{div} v_\epsilon. \tag{4.22}$$

On the one hand, as ω_ϵ is bounded in $\mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^+; \mathrm{L}^{4/3} \cap \mathrm{L}^{12/7}(\mathbb{R}^2))$, we deduce from Riesz potential theory that $v_{\epsilon,1}$ is bounded in $\mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^+; \mathrm{L}^4 \cap \mathrm{L}^{12}(\mathbb{R}^2)^2)$, and we deduce from the Calderón-Zygmund theory that $\nabla v_{\epsilon,1}$ is bounded in $\mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^+; \mathrm{L}^{4/3}(\mathbb{R}^2))$. On the other hand, decomposing

$$v_{\epsilon,2} = \nabla \triangle^{-1} \operatorname{div}(v_\epsilon - \bar v^\circ_\epsilon) + \bar v^\circ_\epsilon - \nabla^\perp \triangle^{-1} \bar\omega^\circ_\epsilon,$$

noting that $v_\epsilon - \bar v^\circ_\epsilon$ is bounded in $\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; \mathrm{L}^2(\mathbb{R}^2)^2)$ (cf. (4.21)), that $\bar v^\circ_\epsilon$ is bounded in $\mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^2)^2$, and that $\|\nabla \triangle^{-1} \bar\omega^\circ_\epsilon\|_{\mathrm{L}^2} \lesssim \|\bar\omega^\circ_\epsilon\|_{\mathrm{L}^1 \cap \mathrm{L}^\infty} \lesssim 1$ (cf. Lemma 2.4), we deduce that $v_{\epsilon,2}$ is bounded in $\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^2)^2)$. Further, decomposing

$$v_{\epsilon,2} = \nabla \triangle^{-1}(a^{-1}(\zeta_\epsilon - \bar\zeta^\circ_\epsilon)) - \nabla \triangle^{-1}(\nabla h \cdot (v_\epsilon - \bar v^\circ_\epsilon)) + \bar v^\circ_\epsilon - \nabla^\perp \triangle^{-1} \bar\omega^\circ_\epsilon,$$

we easily check that $\nabla v_{\epsilon,2}$ is bounded in $\mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^+;\mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^2)^2)$. We then conclude from the Sobolev embedding that $v_{\epsilon,2}$ is bounded in $\mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^+;\mathrm{L}^q_{\mathrm{loc}}(\mathbb{R}^2)^2)$ for all $q<\infty$. For our purposes it is enough to choose $q=4$ and 12. In particular, we have proven that, for all bounded subset $U\subset\mathbb{R}^2$,

$$\|\omega_\epsilon\|_{\mathrm{L}^2_t\mathrm{L}^{4/3}}+\|\zeta_\epsilon\|_{\mathrm{L}^2_t\mathrm{L}^2}+\|v_\epsilon\|_{\mathrm{L}^\infty_t\mathrm{L}^2(U)} \\ +\|v_{\epsilon,1}\|_{\mathrm{L}^2_t(\mathrm{L}^4\cap\mathrm{L}^{12})}+\|\nabla v_{\epsilon,1}\|_{\mathrm{L}^2_t\mathrm{L}^{4/3}}+\|v_{\epsilon,2}\|_{\mathrm{L}^2_t(\mathrm{L}^4\cap\mathrm{L}^{12}(U))}+\|\nabla v_{\epsilon,2}\|_{\mathrm{L}^2_t\mathrm{L}^2(U)}\lesssim_{t,U}1. \tag{4.23}$$

Therefore we have up to an extraction $\omega_\epsilon\rightharpoonup\omega$ in $\mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^+;\mathrm{L}^{4/3}(\mathbb{R}^2))$, $\zeta_\epsilon\rightharpoonup\zeta$ in $\mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^+;\mathrm{L}^2(\mathbb{R}^2))$, $v_{\epsilon,1}\rightharpoonup v_1$ in $\mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^+;\mathrm{L}^4(\mathbb{R}^2)^2)$, and $v_{\epsilon,2}\rightharpoonup v_2$ in $\mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^+;\mathrm{L}^4_{\mathrm{loc}}(\mathbb{R}^2)^2)$, for some functions ω,ζ,v_1,v_2. Comparing the above estimates with equation (1.7), we deduce that $(\partial_t\omega_\epsilon)_\epsilon$ is bounded in $\mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^+;W^{-1,1}_{\mathrm{loc}}(\mathbb{R}^2))$. Moreover, we find by interpolation, for any $|\xi|<1$ and any bounded domain $U\subset\mathbb{R}^2$, denoting by $U^1:=U+B_1$ its 1-fattening,

$$\begin{aligned}
&\|v_\epsilon-v_\epsilon(\cdot+\xi)\|_{\mathrm{L}^2_t\mathrm{L}^4(U)}\le\|v_{\epsilon,1}-v_{\epsilon,1}(\cdot+\xi)\|_{\mathrm{L}^2_t\mathrm{L}^4(U)}+\|v_{\epsilon,2}-v_{\epsilon,2}(\cdot+\xi)\|_{\mathrm{L}^2_t\mathrm{L}^4(U)}\\
&\le\|v_{\epsilon,1}-v_{\epsilon,1}(\cdot+\xi)\|^{1/4}_{\mathrm{L}^2_t\mathrm{L}^{4/3}(U)}\|v_{\epsilon,1}-v_{\epsilon,1}(\cdot+\xi)\|^{3/4}_{\mathrm{L}^2_t\mathrm{L}^{12}(U)}\\
&\qquad+\|v_{\epsilon,2}-v_{\epsilon,2}(\cdot+\xi)\|^{2/5}_{\mathrm{L}^2_t\mathrm{L}^2(U)}\|v_{\epsilon,2}-v_{\epsilon,2}(\cdot+\xi)\|^{3/5}_{\mathrm{L}^2_t\mathrm{L}^{12}(U)}\\
&\le2\|v_{\epsilon,1}-v_{\epsilon,1}(\cdot+\xi)\|^{1/4}_{\mathrm{L}^2_t\mathrm{L}^{4/3}(U)}\|v_{\epsilon,1}\|^{3/4}_{\mathrm{L}^2_t\mathrm{L}^{12}(U^1)}+2\|v_{\epsilon,2}-v_{\epsilon,2}(\cdot+\xi)\|^{2/5}_{\mathrm{L}^2_t\mathrm{L}^2(U)}\|v_{\epsilon,2}\|^{3/5}_{\mathrm{L}^2_t\mathrm{L}^{12}(U^1)}\\
&\le2|\xi|^{1/4}\|\nabla v_{\epsilon,1}\|^{1/4}_{\mathrm{L}^2_t\mathrm{L}^{4/3}(U^1)}\|v_{\epsilon,1}\|^{3/4}_{\mathrm{L}^2_t\mathrm{L}^{12}(U^1)}+2|\xi|^{2/5}\|\nabla v_{\epsilon,2}\|^{2/5}_{\mathrm{L}^2_t\mathrm{L}^2(U^1)}\|v_{\epsilon,2}\|^{3/5}_{\mathrm{L}^2_t\mathrm{L}^{12}(U^1)},
\end{aligned}$$

and hence by (4.23),

$$\|v_\epsilon-v_\epsilon(\cdot+\xi)\|_{\mathrm{L}^2_t\mathrm{L}^4(U)}\lesssim_{t,U}|\xi|^{1/4}+|\xi|^{2/5}.$$

Let us summarize the previous observations: up to an extraction, setting $v:=v_1+v_2$, we have

$$\begin{gathered}
\omega_\epsilon\rightharpoonup\omega\text{ in }\mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^+;\mathrm{L}^{4/3}(\mathbb{R}^2)),\ v_\epsilon\rightharpoonup v\text{ in }\mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^+;\mathrm{L}^4_{\mathrm{loc}}(\mathbb{R}^2)^2),\\
\partial_t\omega_\epsilon\text{ bounded in }\mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^+;W^{-1,1}_{\mathrm{loc}}(\mathbb{R}^2)),\\
\sup_{\epsilon>0}\|v_\epsilon-v_\epsilon(\cdot+\xi)\|_{\mathrm{L}^2_t\mathrm{L}^4(U)}\to0\text{ as }|\xi|\to0,\text{ for any }t\ge0\text{ and bounded subset }U\subset\mathbb{R}^2.
\end{gathered}$$

We may then apply [30, Lemma 5.1], which ensures that $\omega_\epsilon v_\epsilon\to\omega v$ holds in the distributional sense. This allows to pass to the limit in the weak formulation of equation (1.2), and the result follows.

Step 2: proof of (ii) and (iii). The proof of item (ii) is also based on Lemma 4.3(iii), and is completely analogous to the proof of item (i) in Step 1. As far as item (iii) is concerned, Lemma 4.3(iii) does no longer apply in that case, but, since we further assume $\omega^\circ\in\mathrm{L}^q(\mathbb{R}^2)$ for some $q>1$, Lemma 4.2 gives the following a priori estimate: for all $t\ge0$

$$\|\omega\|_{\mathrm{L}^{q+1}_t\mathrm{L}^{q+1}}+\|\omega\|_{\mathrm{L}^\infty_t\mathrm{L}^q}\lesssim_t1, \tag{4.24}$$

hence in particular by interpolation $\|\omega\|_{\mathrm{L}^p_t\mathrm{L}^p}\lesssim_t1$ for all $1\le p\le2$ (here we use the notation $\lesssim_t$ for $\le$ up to a constant that depends only on an upper bound on t, $(q-1)^{-1}$, α, α^{-1}, $|\beta|$, $\|(h,\Psi)\|_{W^{1,\infty}}$, $\|v^\circ-\bar v^\circ\|_{\mathrm{L}^2}$, and $\|\omega^\circ\|_{\mathrm{L}^q}$). The conclusion then follows from a similar argument as in Step 1.

Step 3: proof of (iv). We finally turn to the incompressible equation (1.1) in the conservative regime $\alpha=0$. Let $q>1$ be such that $\omega^\circ\in\mathrm{L}^q(\mathbb{R}^2)$. Lemmas 4.2 and 4.3(ii) ensure that ω_ϵ is bounded in $\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+;\mathrm{L}^1\cap\mathrm{L}^q(\mathbb{R}^2))$, and hence, for $q>4/3$, replacing the exponents $4/3$ and $12/7$ of Step 1 by $4/3$ and q, the argument of Step 1 can be immediately adapted to this case, for which we thus obtain global existence of a weak solution. In the remaining case $1<q<4/3$, the product $\omega\nabla\Delta^{-1}\omega$ (hence the product ωv, cf. (4.22)) does not make sense any more for $\omega\in\mathrm{L}^q(\mathbb{R}^2)$. Since in the conservative regime $\alpha=0$ no additional regularity is available (in particular, (4.24) does not hold), we do not expect the existence of a weak solution, and we need to turn to the notion of very weak solutions as defined in Definition 1.1(c), where the product ωv is reinterpreted à la Delort. Let us now focus on this case $1<q\le4/3$, and prove the global existence of a very weak solution. For the critical exponent

$q=4/3$, the integrability of v found below directly implies by Remark 1.2(ii) that the constructed very weak solution is automatically a weak solution. In this step, we use the notation $\lesssim$ for $\le$ up to a constant C that depends only on an upper bound on $(q-1)^{-1}$, $|\beta|$, $\|(h,\Psi,\bar v^\circ)\|_{W^{1,\infty}}$, $\|v^\circ-\bar v^\circ\|_{\mathrm{L}^2}$, $\|\bar\omega^\circ\|_{\mathrm{L}^2}$, and $\|\omega^\circ\|_{\mathrm{L}^q}$, and we use the notation $\lesssim_t$ (resp. $\lesssim_{t,U}$) if it further depends on an upper bound on time t (resp. on t and on the size of $U\subset\mathbb{R}^2$).

Let ω_ϵ°, $\bar\omega_\epsilon^\circ$, v_ϵ°, $\bar v_\epsilon^\circ$ be defined as in Step 1 (with of course $\zeta_\epsilon^\circ=\bar\zeta_\epsilon^\circ=0$), and let $v_\epsilon\in\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+;\bar v_\epsilon^\circ+\mathrm{L}^2(\mathbb{R}^2)^2)$ be a global weak solution of (1.1) on $\mathbb{R}^+\times\mathbb{R}^2$ with initial data v_ϵ°, as given by Corollary 4.9. Lemmas 4.1(iii) and 4.3(ii) then give for all $t\ge0$

$$\|\omega_\epsilon\|_{\mathrm{L}^\infty_t(\mathrm{L}^1\cap\mathrm{L}^q)}+\|v_\epsilon-\bar v_\epsilon^\circ\|_{\mathrm{L}^\infty_t\mathrm{L}^2}\lesssim_t 1. \tag{4.25}$$

As $\bar v_\epsilon^\circ$ is bounded in $\mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^2)^2$, we deduce in particular that v_ϵ is bounded in $\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+;\mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^2))$. Moreover, using the Delort type identity

$$\omega_\epsilon v_\epsilon=-\frac12|v_\epsilon|^2\nabla^\perp h-a^{-1}(\operatorname{div}(aS_{v_\epsilon}))^\perp,$$

we then deduce that $\omega_\epsilon v_\epsilon$ is bounded in $\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+;W^{-1,1}_{\mathrm{loc}}(\mathbb{R}^2)^2)$. Let us now recall the following useful decomposition:

$$v_\epsilon=v_{\epsilon,1}+v_{\epsilon,2},\quad v_{\epsilon,1}:=\nabla^\perp\triangle^{-1}\omega_\epsilon,\quad v_{\epsilon,2}:=\nabla\triangle^{-1}\operatorname{div}v_\epsilon. \tag{4.26}$$

By Riesz potential theory $v_{\epsilon,1}$ is bounded in $\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+;\mathrm{L}^p(\mathbb{R}^2)^2)$ for all $2<p\le\frac{2q}{2-q}$, while as in Step 1 we check that $v_{\epsilon,2}$ is bounded in $\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+;H^1_{\mathrm{loc}}(\mathbb{R}^2)^2)$. Hence by the Sobolev embedding, for all bounded domain $U\subset\mathbb{R}^2$ and all $t\ge0$,

$$\|(v_\epsilon,v_{\epsilon,1})\|_{\mathrm{L}^\infty_t\mathrm{L}^{2q/(2-q)}(U)}\lesssim_{t,U}1. \tag{4.27}$$

Up to an extraction we then have $v_\epsilon\overset{*}{\rightharpoonup}v$ in $\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+;\mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^2)^2)$ and $\omega_\epsilon\overset{*}{\rightharpoonup}\omega$ in $\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+;\mathrm{L}^q(\mathbb{R}^2))$, for some functions v,ω, with necessarily $\omega=\operatorname{curl}v$ and $\operatorname{div}(av)=0$.

We now need to pass to the limit in the nonlinearity $\omega_\epsilon v_\epsilon$. For that purpose, for all $\eta>0$, we set $v_{\epsilon,\eta}:=\rho_\eta*v_\epsilon$ and $\omega_{\epsilon,\eta}:=\rho_\eta*\omega_\epsilon=\operatorname{curl}v_{\epsilon,\eta}$, where $\rho_\eta(x):=\eta^{-d}\rho(x/\eta)$ is the regularization kernel defined in Step 1, and we then decompose the nonlinearity as follows:

$$\omega_\epsilon v_\epsilon=(\omega_{\epsilon,\eta}-\omega_\epsilon)(v_{\epsilon,\eta}-v_\epsilon)-\omega_{\epsilon,\eta}v_{\epsilon,\eta}+\omega_{\epsilon,\eta}v_\epsilon+\omega_\epsilon v_{\epsilon,\eta}.$$

We consider separately each term in the right-hand side, and split the proof into four substeps.

Substep 3.1. We prove that $(\omega_{\epsilon,\eta}-\omega_\epsilon)(v_{\epsilon,\eta}-v_\epsilon)\to0$ in the distributional sense (and even strongly in $\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+;W^{-1,1}_{\mathrm{loc}}(\mathbb{R}^2)^2)$) as $\eta\downarrow0$, uniformly in $\epsilon>0$. For that purpose, we use the Delort type identity

$$(\omega_{\epsilon,\eta}-\omega_\epsilon)(v_{\epsilon,\eta}-v_\epsilon)=a^{-1}(v_{\epsilon,\eta}-v_\epsilon)\operatorname{div}(a(v_{\epsilon,\eta}-v_\epsilon))-\frac12|v_{\epsilon,\eta}-v_\epsilon|^2\nabla^\perp h-a^{-1}(\operatorname{div}(aS_{v_{\epsilon,\eta}-v_\epsilon}))^\perp.$$

Noting that the constraint $0=a^{-1}\operatorname{div}(av_\epsilon)=\nabla h\cdot v_\epsilon+\operatorname{div}v_\epsilon$ yields

$$a^{-1}\operatorname{div}(a(v_{\epsilon,\eta}-v_\epsilon))=\nabla h\cdot v_{\epsilon,\eta}+\operatorname{div}v_{\epsilon,\eta}=\nabla h\cdot(\rho_\eta*v_\epsilon)+\rho_\eta*\operatorname{div}v_\epsilon=\nabla h\cdot(\rho_\eta*v_\epsilon)-\rho_\eta*(\nabla h\cdot v_\epsilon),$$

the above identity becomes

$$\begin{aligned}(\omega_{\epsilon,\eta}-\omega_\epsilon)(v_{\epsilon,\eta}-v_\epsilon)=(v_{\epsilon,\eta}-v_\epsilon)\big(\nabla h\cdot(\rho_\eta*v_\epsilon)-\rho_\eta*(\nabla h\cdot v_\epsilon)\big)\\-\frac12|v_{\epsilon,\eta}-v_\epsilon|^2\nabla^\perp h-a^{-1}(\operatorname{div}(aS_{v_{\epsilon,\eta}-v_\epsilon}))^\perp.\end{aligned}$$

First, using the boundedness of v_ϵ (hence of $v_{\epsilon,\eta}$) in $\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+;\mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^2)^2)$, we may estimate, for any bounded domain $U\subset\mathbb{R}^2$, denoting by $U^\eta:=U+B_\eta$ its η-fattening,

$$\begin{aligned}&\int_U \big|(v_{\epsilon,\eta}-v_\epsilon)\big(\nabla h\cdot(\rho_\eta*v_\epsilon)-\rho_\eta*(\nabla h\cdot v_\epsilon)\big)\big|\\&\le \|(v_\epsilon,v_{\epsilon,\eta})\|_{\mathrm{L}^2(U)}\Big(\int_U\Big(\int\rho_\eta(y)|\nabla h(x)-\nabla h(x-y)||v_\epsilon(x-y)|dy\Big)^2dx\Big)^{1/2}\\&\lesssim \|(v_\epsilon,v_{\epsilon,\eta})\|^2_{\mathrm{L}^2(U^\eta)}\Big(\int\rho_\eta(y)\int_U|\nabla h(x)-\nabla h(x-y)|^2dxdy\Big)^{1/2},\end{aligned}$$

where the right-hand side converges to 0 as $\eta\downarrow 0$, uniformly in ϵ. Second, using the decomposition (4.26), and setting $v_{\epsilon,\eta,1}:=\rho_\eta*v_{\epsilon,1}$, $v_{\epsilon,\eta,2}:=\rho_\eta*v_{\epsilon,2}$, we may estimate by the Hölder inequality, for any bounded domain $U\subset\mathbb{R}^2$,

$$\begin{aligned}\int_U|(v_\epsilon-v_{\epsilon,\eta})\otimes(v_\epsilon-v_{\epsilon,\eta})|&\le\int_U|v_\epsilon-v_{\epsilon,\eta}||v_{\epsilon,1}-v_{\epsilon,\eta,1}|+\int_U|v_\epsilon-v_{\epsilon,\eta}||v_{\epsilon,2}-v_{\epsilon,\eta,2}|\\&\le\|(v_\epsilon,v_{\epsilon,\eta})\|_{\mathrm{L}^{2q/(2-q)}(U)}\|v_{\epsilon,1}-v_{\epsilon,\eta,1}\|_{\mathrm{L}^{2q/(3q-2)}(U)}+\|(v_\epsilon,v_{\epsilon,\eta})\|_{\mathrm{L}^2(U)}\|v_{\epsilon,2}-v_{\epsilon,\eta,2}\|_{\mathrm{L}^2(U)}.\end{aligned}$$

Recalling the choice $1<q\le 4/3$, we find by interpolation

$$\begin{aligned}\|v_{\epsilon,1}-v_{\epsilon,\eta,1}\|_{\mathrm{L}^{2q/(3q-2)}(U)}&\le\|v_{\epsilon,1}-v_{\epsilon,\eta,1}\|^{\frac{4-3q}{2-q}}_{\mathrm{L}^2(U)}\|v_{\epsilon,1}-v_{\epsilon,\eta,1}\|^{2\frac{q-1}{2-q}}_{\mathrm{L}^q(U)}\\&\le\eta^{2\frac{q-1}{2-q}}\|(v_{\epsilon,1},v_{\epsilon,\eta,1})\|^{\frac{4-3q}{2-q}}_{\mathrm{L}^2(U)}\|\nabla v_{\epsilon,1}\|^{2\frac{q-1}{2-q}}_{\mathrm{L}^q},\end{aligned}$$

and hence by the Calderón-Zygmund theory

$$\|v_{\epsilon,1}-v_{\epsilon,\eta,1}\|_{\mathrm{L}^{2q/(3q-2)}(U)}\lesssim\eta^{2\frac{q-1}{2-q}}\|(v_{\epsilon,1},v_{\epsilon,\eta,1})\|^{\frac{4-3q}{2-q}}_{\mathrm{L}^2(U)}\|\omega_\epsilon\|^{2\frac{q-1}{2-q}}_{\mathrm{L}^q},$$

while as in Step 1 we may estimate

$$\|v_{\epsilon,2}-v_{\epsilon,\eta,2}\|_{\mathrm{L}^2_t\mathrm{L}^2(U)}\le\eta\|\nabla v_{\epsilon,2}\|_{\mathrm{L}^2_t\mathrm{L}^2(U^\eta)}\lesssim_U\eta.$$

Combining this with the a priori estimates (4.27), we may conclude

$$\int_0^t\int_U|(v^i_\epsilon-v^i_{\epsilon,\eta})(v^j_\epsilon-v^j_{\epsilon,\eta})|\lesssim_{t,U}\eta^{2\frac{q-1}{2-q}}+\eta,$$

and the claim follows.

Substep 3.2. We set $v_\eta:=\rho_\eta*v$, $\omega_\eta:=\rho_\eta*\omega=\operatorname{curl}v_\eta$, and we prove that $-\omega_{\epsilon,\eta}v_{\epsilon,\eta}+\omega_{\epsilon,\eta}v_\epsilon+\omega_\epsilon v_{\epsilon,\eta}\to-\omega_\eta v_\eta+\omega_\eta v+\omega v_\eta$ in the distributional sense as $\epsilon\downarrow 0$, for any fixed $\eta>0$. As $q<2<q'$, the weak convergences $v_\epsilon\overset{*}{\rightharpoonup}v$ in $\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+;\mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^2)^2)$ and $\omega_\epsilon\overset{*}{\rightharpoonup}\omega$ in $\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+;\mathrm{L}^q(\mathbb{R}^2))$ imply for instance $v_{\epsilon,\eta}\overset{*}{\rightharpoonup}v_\eta$ in $\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+;W^{1,q'}_{\mathrm{loc}}(\mathbb{R}^2)^2)$ and $\omega_{\epsilon,\eta}\overset{*}{\rightharpoonup}\omega_\eta$ in $\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+;H^1(\mathbb{R}^2))$ for all $\eta>0$ (note that these are still only weak-* convergences because no regularization occurs with respect to the time-variable t). Moreover, examining equation (1.7) together with the a priori estimates obtained at the beginning of this step, we observe that $\partial_t\omega_\epsilon$ is bounded in $\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+;W^{-2,1}_{\mathrm{loc}}(\mathbb{R}^2))$, hence $\partial_t\omega_{\epsilon,\eta}=\rho_\eta*\partial_t\omega_\epsilon$ is also bounded in the same space. Since by the Rellich theorem the space $\mathrm{L}^q(U)$ is compactly embedded in $W^{-1,q}(U)\subset W^{-2,1}(U)$ for any bounded domain $U\subset\mathbb{R}^2$, the Aubin-Simon lemma ensures that we have $\omega_\epsilon\to\omega$ strongly in $\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+;W^{-1,q}_{\mathrm{loc}}(\mathbb{R}^2))$, and similarly, since $H^1(U)$ is compactly embedded in $\mathrm{L}^2(U)\subset W^{-2,1}(U)$, we also deduce $\omega_{\epsilon,\eta}\to\omega_\eta$ strongly in $\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+;\mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^2))$. This proves the claim.

Substep 3.3. We prove that $-\omega_\eta v_\eta+\omega_\eta v+\omega v_\eta\to-\frac12|v|^2\nabla^\perp h-a^{-1}(\operatorname{div}(aS_v))^\perp$ in the distributional sense as $\eta\downarrow 0$. For that purpose, we use the following Delort type identity:

$$\begin{aligned}-\omega_\eta v_\eta+\omega_\eta v+\omega v_\eta=-a^{-1}(v_\eta-v)\operatorname{div}(a(v_\eta-v))+\frac12|v_\eta-v|^2\nabla^\perp h+a^{-1}(\operatorname{div}(aS_{v_\eta-v}))^\perp\\+a^{-1}v\operatorname{div}(av)-\frac12|v|^2\nabla^\perp h-a^{-1}(\operatorname{div}(aS_v))^\perp.\end{aligned}$$

Noting that the limiting constraint $0 = a^{-1}\operatorname{div}(av) = \nabla h \cdot v + \operatorname{div} v$ gives

$$a^{-1}\operatorname{div}(a(v_\eta - v)) = \nabla h \cdot v_\eta + \operatorname{div} v_\eta = \nabla h \cdot (\rho_\eta * v) + \rho_\eta * \operatorname{div} v = \nabla h \cdot (\rho_\eta * v) - \rho_\eta * (\nabla h \cdot v),$$

the above identity takes the form

$$-\omega_\eta v_\eta + \omega_\eta v + \omega v_\eta = -a^{-1}(v_\eta - v)\big(\nabla h \cdot (\rho_\eta * v) - \rho_\eta * (\nabla h \cdot v)\big) + \frac{1}{2}|v_\eta - v|^2 \nabla^\perp h + a^{-1}(\operatorname{div}(aS_{v_\eta - v}))^\perp \\ - \frac{1}{2}|v|^2 \nabla^\perp h - a^{-1}(\operatorname{div}(aS_v))^\perp,$$

and it is thus sufficient to prove that the first three right-hand side terms tend to 0 in the distributional sense as $\eta \downarrow 0$. This is proven just as in Substep 3.1 above, with $v_{\epsilon,\eta}, v_\epsilon$ replaced by v_η, v.

Substep 3.4: conclusion. Combining the three previous substeps yields $\omega_\epsilon v_\epsilon \to -\frac{1}{2}|v|^2\nabla^\perp h - a^{-1}(\operatorname{div}(aS_v))^\perp$ in the distributional sense as $\epsilon \downarrow 0$. Passing to the limit in the weak formulation of equation (1.7), the conclusion follows. □

5 Uniqueness

We finally turn to the uniqueness results stated in Theorem 3. On the one hand, using similar energy arguments as in the proof of Lemma 4.1, in the spirit of [36, Appendix B], we prove a general weak-strong uniqueness principle (which only holds in a weaker form in the degenerate case $\lambda = 0$). In the incompressible case (1.1), we further prove uniqueness in the class of bounded vorticity, based on transport type arguments and on the Loeper inequality [31] (see also [38]), while these tools are no longer available in the compressible case.

Proposition 5.1 (Uniqueness)**.** *Let $\alpha, \beta \in \mathbb{R}$, $\lambda \geq 0$, $T > 0$, and $h, \Psi \in W^{1,\infty}(\mathbb{R}^2)^2$. Let $v^\circ : \mathbb{R}^2 \to \mathbb{R}^2$ with $\omega^\circ := \operatorname{curl} v^\circ \in \mathcal{P}(\mathbb{R}^2)$, and in the incompressible case (1.1) further assume that $\operatorname{div}(av^\circ) = 0$.*

(i) Weak-strong uniqueness principle for (1.1) and (1.2) in the non-degenerate case $\lambda > 0$, $\alpha \geq 0$:
If (1.1) or (1.2) admits a weak solution $v \in \mathrm{L}^2_{\mathrm{loc}}([0,T); v^\circ + \mathrm{L}^2(\mathbb{R}^2)^2) \cap \mathrm{L}^\infty_{\mathrm{loc}}([0,T); W^{1,\infty}(\mathbb{R}^2)^2)$ on $[0,T) \times \mathbb{R}^2$ with initial data v°, then it is the unique weak solution of (1.1) or of (1.2) on $[0,T) \times \mathbb{R}^2$ in the class $\mathrm{L}^2_{\mathrm{loc}}([0,T); v^\circ + \mathrm{L}^2(\mathbb{R}^2)^2)$ with initial data v°.

(ii) Weak-strong uniqueness principle for (1.2) in the degenerate parabolic case $\lambda = \beta = 0$, $\alpha \geq 0$:
Let E^2_{T,v° denote the class of all $w \in \mathrm{L}^2_{\mathrm{loc}}([0,T); v^\circ + \mathrm{L}^2(\mathbb{R}^2)^2)$ with $\operatorname{curl} w \in \mathrm{L}^2_{\mathrm{loc}}([0,T); \mathrm{L}^2(\mathbb{R}^2))$. If (1.2) admits a weak solution $v \in E^2_{T,v^\circ} \cap \mathrm{L}^\infty_{\mathrm{loc}}([0,T); \mathrm{L}^\infty(\mathbb{R}^2)^2)$ on $[0,T) \times \mathbb{R}^2$ with initial data v°, and with $\omega := \operatorname{curl} v \in \mathrm{L}^\infty_{\mathrm{loc}}([0,T); W^{1,\infty}(\mathbb{R}^2))$, then it is the unique weak solution of (1.2) on $[0,T) \times \mathbb{R}^2$ in the class E^2_{T,v° with initial data v°.

(iii) Uniqueness for (1.1) with bounded vorticity, $\alpha, \beta \in \mathbb{R}$:
There exists at most a unique weak solution v of (1.1) on $[0,T) \times \mathbb{R}^2$ with initial data v°, in the class of all w's such that $\operatorname{curl} w \in \mathrm{L}^\infty_{\mathrm{loc}}([0,T); \mathrm{L}^\infty(\mathbb{R}^2))$.

Moreover, in items (i)–(ii), the condition $\alpha \geq 0$ may be dropped if we further restrict to weak solutions v such that $\operatorname{curl} v \in \mathrm{L}^\infty_{\mathrm{loc}}([0,T); \mathrm{L}^\infty(\mathbb{R}^2))$.

Proof. In this proof, we use the notation $\lesssim$ for $\leq$ up to a constant $C > 0$ that depends only on an upper bound on α, $|\beta|$, λ, λ^{-1}, and $\|(h,\Psi)\|_{W^{1,\infty}}$, and we add subscripts to indicate dependence on further parameters. We split the proof into four steps, first proving item (i) in the case (1.1), then in the case (1.2), and finally turning to items (ii) and (iii).

Step 1: proof of (i) in the case (1.1). Let $\alpha \geq 0$, $\beta \in \mathbb{R}$, and let $v_1, v_2 \in \mathrm{L}^2_{\mathrm{loc}}([0,T); v^\circ + \mathrm{L}^2(\mathbb{R}^2)^2)$ be two weak solutions of (1.1) on $[0,T) \times \mathbb{R}^2$ with initial data v°, and assume $v_2 \in \mathrm{L}^\infty_{\mathrm{loc}}([0,T); W^{1,\infty}(\mathbb{R}^2)^2)$. Set $\delta v := v_1 - v_2$ and $\delta\omega := \omega_1 - \omega_2$. As the constraint $\operatorname{div}(a\delta v) = 0$ yields $\delta v = a^{-1}\nabla^\perp(\operatorname{div} a^{-1}\nabla)^{-1}\delta\omega$, and as by assumption $\delta v \in \mathrm{L}^2_{\mathrm{loc}}([0,T); \mathrm{L}^2(\mathbb{R}^2)^2)$, we deduce $\delta\omega \in \mathrm{L}^2_{\mathrm{loc}}([0,T); \dot H^{-1}(\mathbb{R}^2))$ and $(\operatorname{div} a^{-1}\nabla)^{-1}\delta\omega \in \mathrm{L}^2_{\mathrm{loc}}([0,T); \dot H^1(\mathbb{R}^2))$. Moreover, the definition of a weak solution ensures that $\omega_i := \operatorname{curl} v_i \in \mathrm{L}^\infty([0,T); \mathcal{P}(\mathbb{R}^2))$ (cf. Lemma 4.1(i)), and $|v_i|^2\omega_i \in \mathrm{L}^1_{\mathrm{loc}}([0,T);\mathrm{L}^1(\mathbb{R}^2))$, for $i=1,2$, so that all the integrations by parts below are directly justified. From equation (1.7), we compute the following time-derivative

$$\begin{aligned}\partial_t \int \delta\omega(-\operatorname{div} a^{-1}\nabla)^{-1}\delta\omega &= 2\int \nabla(\operatorname{div} a^{-1}\nabla)^{-1}\delta\omega\cdot\Big((\alpha(\Psi+v_1)^\perp+\beta(\Psi+v_1))\omega_1 \\ &\qquad\qquad -(\alpha(\Psi+v_2)^\perp+\beta(\Psi+v_2))\omega_2\Big) \\ &= -2\int a\delta v^\perp\cdot\Big((\alpha(\delta v)^\perp+\beta\delta v)\omega_1+(\alpha(\Psi+v_2)^\perp+\beta(\Psi+v_2))\delta\omega\Big) \\ &= -2\alpha\int a|\delta v|^2\omega_1-2\int a\delta\omega\delta v^\perp\cdot(\alpha(\Psi+v_2)^\perp+\beta(\Psi+v_2)). \end{aligned}\tag{5.1}$$

As v_2 is Lipschitz-continuous, and as the definition of a weak solution ensures that $\omega_1 v_1 \in \mathrm{L}^1_{\mathrm{loc}}([0,T);\mathrm{L}^1(\mathbb{R}^2)^2)$, the following Delort type identity holds in $\mathrm{L}^1_{\mathrm{loc}}([0,T);W^{-1,1}_{\mathrm{loc}}(\mathbb{R}^2)^2)$,

$$\delta\omega\delta v^\perp = \frac12|\delta v|^2\nabla h + a^{-1}\operatorname{div}(aS_{\delta v}).$$

Combining this with (5.1) and the non-negativity of $\alpha\omega_1$ yields

$$\begin{aligned}\partial_t\int \delta\omega(-\operatorname{div} a^{-1}\nabla)^{-1}\delta\omega &\le -\int a|\delta v|^2\nabla h\cdot(\alpha(\Psi+v_2)^\perp+\beta(\Psi+v_2)) + 2\int aS_{\delta v}:\nabla(\alpha(\Psi+v_2)^\perp+\beta(\Psi+v_2)) \\ &\le C(1+\|v_2\|_{W^{1,\infty}})\int a|\delta v|^2.\end{aligned}$$

The uniqueness result $\delta v = 0$ then follows from the Grönwall inequality, since by integration by parts

$$\int a|\delta v|^2 = \int a^{-1}|\nabla(\operatorname{div} a^{-1}\nabla)^{-1}\delta\omega|^2 = \int \delta\omega(-\operatorname{div} a^{-1}\nabla)^{-1}\delta\omega.$$

Note that if we further assume $\omega_1\in\mathrm{L}^\infty([0,T);\mathrm{L}^\infty(\mathbb{R}^2))$, then the non-negativity of α can be dropped: it indeed suffices to estimate in that case $-2\alpha\int a|\delta v|^2\omega_1 \le C\|\omega_1\|_{\mathrm{L}^\infty}\int a|\delta v|^2$, and the result then follows as above. A similar observation also holds in the context of item (ii).

Step 2: proof of (i) in the case (1.2). Let $\alpha\ge 0$, $\beta\in\mathbb{R}$, $\lambda>0$, and let $v_1,v_2\in\mathrm{L}^2_{\mathrm{loc}}([0,T);v^\circ+\mathrm{L}^2(\mathbb{R}^2)^2)$ be two weak solutions of (1.2) on $[0,T)\times\mathbb{R}^2$ with initial data v°, and assume $v_2\in\mathrm{L}^\infty_{\mathrm{loc}}([0,T);W^{1,\infty}(\mathbb{R}^2)^2)$. The definition of a weak solution ensures that $\omega_i := \operatorname{curl} v_i\in\mathrm{L}^\infty([0,T);\mathcal{P}(\mathbb{R}^2))$ (cf. Lemma 4.1(i)), $\zeta_i:=\operatorname{div}(av_i)\in\mathrm{L}^2_{\mathrm{loc}}([0,T);\mathrm{L}^2(\mathbb{R}^2))$, and $|v_i|^2\omega_i\in\mathrm{L}^1_{\mathrm{loc}}([0,T);\mathrm{L}^1(\mathbb{R}^2))$, for $i=1,2$, and hence the integrations by parts below are directly justified. Set $\delta v:=v_1-v_2$, $\delta\omega:=\omega_1-\omega_2$, and $\delta\zeta:=\zeta_1-\zeta_2$. From equation (1.2), we compute the following time-derivative

$$\begin{aligned}\partial_t\int a|\delta v|^2 &= 2\int a\delta v\cdot\Big(\lambda\nabla(a^{-1}\delta\zeta)-\alpha(\Psi+v_1)\omega_1+\beta(\Psi+v_1)^\perp\omega_1+\alpha(\Psi+v_2)\omega_2-\beta(\Psi+v_2)^\perp\omega_2\Big) \\ &= -2\lambda\int a^{-1}|\delta\zeta|^2-2\alpha\int a|\delta v|^2\omega_1+2\int a\delta\omega\delta v\cdot\big(\alpha(\Psi+v_2)-\beta(\Psi+v_2)^\perp\big).\end{aligned}$$

As v_2 is Lipschitz-continuous, and as $\omega_1 v_1\in\mathrm{L}^1_{\mathrm{loc}}([0,T)\times\mathbb{R}^2)^2$ follows from the definition of a weak solution, the following Delort type identity holds in $\mathrm{L}^1_{\mathrm{loc}}([0,T);W^{-1,1}_{\mathrm{loc}}(\mathbb{R}^2)^2)$,

$$\delta\omega\delta v = a^{-1}\delta\zeta\delta v^\perp - \frac12|\delta v|^2\nabla^\perp h - a^{-1}(\operatorname{div}(aS_{\delta v}))^\perp.$$

The above may then be estimated as follows, after integration by parts,

$$\partial_t\int a|\delta v|^2 \le -2\lambda\int a^{-1}|\delta\zeta|^2-2\alpha\int a|\delta v|^2\omega_1 + C(1+\|v_2\|_{\mathrm{L}^\infty})\int|\delta\zeta||\delta v| + C(1+\|v_2\|_{W^{1,\infty}})\int a|\delta v|^2,$$

and thus, using the choice $\lambda > 0$, the inequality $2xy \le x^2 + y^2$, and the non-negativity of $\alpha\omega_1$,

$$\partial_t \int a|\delta v|^2 \le C(1 + \|v_2\|^2_{W^{1,\infty}}) \int a|\delta v|^2.$$

The Grönwall inequality then implies uniqueness, $\delta v = 0$.

Step 3: proof of (ii). Let $\lambda = \beta = 0$, $\alpha = 1$, and let $v_1, v_2 \in \mathrm{L}^2_{\mathrm{loc}}([0,T); v^\circ + \mathrm{L}^2(\mathbb{R}^2)^2)$ be two weak solutions of (1.2) on $[0,T) \times \mathbb{R}^2$ with initial data v°, and with $\omega_i := \operatorname{curl} v_i \in \mathrm{L}^2_{\mathrm{loc}}([0,T); \mathrm{L}^2(\mathbb{R}^2))$ for $i = 1, 2$, and further assume $v_2 \in \mathrm{L}^\infty_{\mathrm{loc}}([0,T); \mathrm{L}^\infty(\mathbb{R}^2)^2)$ and $\omega_2 \in \mathrm{L}^\infty_{\mathrm{loc}}([0,T); W^{1,\infty}(\mathbb{R}^2))$. The definition of a weak solution ensures that $\omega_i := \operatorname{curl} v_i \in \mathrm{L}^\infty([0,T); \mathcal{P}(\mathbb{R}^2))$ (cf. Lemma 4.1(i)), $\zeta_i := \operatorname{div}(av_i) \in \mathrm{L}^2_{\mathrm{loc}}([0,T); \mathrm{L}^2(\mathbb{R}^2))$, and $|v_i|^2\omega_i \in \mathrm{L}^1_{\mathrm{loc}}([0,T); \mathrm{L}^1(\mathbb{R}^2))$, for $i = 1, 2$, and hence the integrations by parts below are directly justified. Denoting $\delta v := v_1 - v_2$ and $\delta\omega := \omega_1 - \omega_2$, equation (1.2) yields

$$\partial_t \delta v = -(\Psi + v_2)\delta\omega - \omega_1 \delta v, \tag{5.2}$$

while equation (1.7) takes the form

$$\begin{aligned} \partial_t \delta\omega &= \operatorname{div}((\Psi + v_2)^\perp \delta\omega) + \operatorname{div}(\omega_1 \delta v^\perp) \\ &= \operatorname{div}((\Psi + v_2)^\perp \delta\omega) + \nabla\omega_1 \cdot \delta v^\perp - \omega_1 \delta\omega \\ &= \operatorname{div}((\Psi + v_2)^\perp \delta\omega) + \nabla\omega_2 \cdot \delta v^\perp + \nabla\delta\omega \cdot \delta v^\perp - \omega_1 \delta\omega. \end{aligned} \tag{5.3}$$

Testing equation (5.2) against δv yields, by non-negativity of ω_1,

$$\partial_t \int |\delta v|^2 = -2\int |\delta v|^2 \omega_1 - 2\int \delta v \cdot (\Psi + v_2)\delta\omega \le C(1 + \|v_2\|_{\mathrm{L}^\infty}) \int |\delta v||\delta\omega|.$$

Testing equation (5.3) against $\delta\omega$ and integrating by parts yields, by non-negativity of ω_1 and ω_2,

$$\begin{aligned} \partial_t \int |\delta\omega|^2 &= -\int \nabla|\delta\omega|^2 \cdot (\Psi + v_2)^\perp + 2\int \delta\omega \nabla\omega_2 \cdot \delta v^\perp + \int \nabla|\delta\omega|^2 \cdot \delta v^\perp - 2\int |\delta\omega|^2 \omega_1 \\ &= -\int |\delta\omega|^2 (\operatorname{curl}\Psi + \omega_2) + 2\int \delta\omega \nabla\omega_2 \cdot \delta v^\perp + \int |\delta\omega|^2(\omega_1 - \omega_2) - 2\int |\delta\omega|^2 \omega_1 \\ &\le C\int |\delta\omega|^2 + 2\|\nabla\omega_2\|_{\mathrm{L}^\infty} \int |\delta v||\delta\omega|. \end{aligned}$$

Combining the above two estimates and using the inequality $2xy \le x^2 + y^2$, we find

$$\partial_t \int (|\delta v|^2 + |\delta\omega|^2) \le C(1 + \|(v_2, \nabla\omega_2)\|_{\mathrm{L}^\infty}) \int (|\delta v|^2 + |\delta\omega|^2),$$

and the uniqueness result follows from the Grönwall inequality.

Step 4: proof of (iii). Let $\alpha, \beta \in \mathbb{R}$, and let v_1, v_2 denote two solutions of (1.1) on $[0,T) \times \mathbb{R}^2$ with initial data v°, and with $\omega_1, \omega_2 \in \mathrm{L}^\infty_{\mathrm{loc}}([0,T); \mathrm{L}^\infty(\mathbb{R}^2))$. First we prove that v_1^t, v_2^t are log-Lipschitz for all $t \in [0,T)$ (compare with the easier situation in [38, Lemma 4.1]). Let $i = 1, 2$. Using the identity $v_i^t = \nabla^\perp \triangle^{-1} \omega_i^t + \nabla\triangle^{-1} \operatorname{div} v_i^t$ with $\operatorname{div} v_i^t = -\nabla h \cdot v_i^t$, we may decompose for all x, y,

$$|v_i^t(x) - v_i^t(y)| \le |\nabla\triangle^{-1}\omega_i^t(x) - \nabla\triangle^{-1}\omega_i^t(y)| + |\nabla\triangle^{-1}(\nabla h \cdot v_i^t)(x) - \nabla\triangle^{-1}(\nabla h \cdot v_i^t)(y)|.$$

By the embedding of the Zygmund space $C^1_*(\mathbb{R}^2) = B^1_{\infty,\infty}(\mathbb{R}^2)$ into the space of log-Lipschitz functions (see e.g. [5, Proposition 2.107]), we may estimate

$$|v_i^t(x) - v_i^t(y)| \lesssim \big(\|\nabla^2\triangle^{-1}\omega_i^t\|_{C^0_*} + \|\nabla^2\triangle^{-1}(\nabla h \cdot v_i^t)\|_{C^0_*}\big)|x - y|(1 + \log^-(|x - y|)),$$

and hence, for any $1 \le p < \infty$, by Lemma 2.5(ii), and recalling that $\mathrm{L}^\infty(\mathbb{R}^2)$ is embedded in $C^0_*(\mathbb{R}^2) = B^0_{\infty,\infty}(\mathbb{R}^2)$,

$$\begin{aligned} |v_i^t(x) - v_i^t(y)| &\lesssim_p \big(\|\omega_i^t\|_{\mathrm{L}^1 \cap C^0_*} + \|\nabla h \cdot v_i^t\|_{\mathrm{L}^p \cap C^0_*}\big)|x - y|(1 + \log^-(|x - y|)) \\ &\lesssim \big(\|\omega_i^t\|_{\mathrm{L}^1 \cap \mathrm{L}^\infty} + \|v_i^t\|_{\mathrm{L}^p \cap \mathrm{L}^\infty}\big)|x - y|(1 + \log^-(|x - y|)). \end{aligned}$$

Noting that $v_i^t = a^{-1}\nabla^\perp(\operatorname{div} a^{-1}\nabla)^{-1}\omega_i^t$, the elliptic estimates of Lemma 2.6 yield $\|v_i^t\|_{\mathrm{L}^{p_0}\cap\mathrm{L}^\infty} \lesssim \|\omega_i^t\|_{\mathrm{L}^1\cap\mathrm{L}^\infty}$ for some exponent $2 < p_0 \lesssim 1$. For the choice $p = p_0$, the above thus takes the following form,

$$|v_i^t(x) - v_i^t(y)| \lesssim \|\omega_i^t\|_{\mathrm{L}^1\cap\mathrm{L}^\infty}|x-y|(1+\log^-(|x-y|)) \le (1+\|\omega_i^t\|_{\mathrm{L}^\infty})|x-y|(1+\log^-(|x-y|)), \tag{5.4}$$

which proves the claim that v_1^t, v_2^t are log-Lipschitz for all $t \in [0,T)$.

For $i = 1,2$, as the vector field $\alpha(\Psi+v_i)+\beta(\Psi+v_i)^\perp$ is log-Lipschitz, the associated flow $\psi_i : [0,T)\times\mathbb{R}^2 \to \mathbb{R}^2$ is well-defined globally,

$$\partial_t\psi_i(x) = -(\alpha(\Psi+v_i)+\beta(\Psi+v_i)^\perp)(\psi_i(x)).$$

As the transport equation (1.7) ensures that $\omega_i^t = (\psi_i^t)_*\omega^\circ$ for $i = 1,2$, the 2-Wasserstein distance between the solutions $\omega_1^t, \omega_2^t \in \mathcal{P}(\mathbb{R}^2)$ is bounded by

$$W_2(\omega_1^t,\omega_2^t)^2 \le \int |\psi_1^t(x) - \psi_2^t(x)|^2\omega^\circ(x)dx =: Q^t. \tag{5.5}$$

Now the time-derivative of Q is estimated by

$$\begin{aligned}
\partial_t Q^t &= -2\int (\psi_1^t(x)-\psi_2^t(x))\cdot\big((\alpha\Psi+\beta\Psi^\perp)(\psi_1^t(x)) - (\alpha\Psi+\beta\Psi^\perp)(\psi_2^t(x))\big)\omega^\circ(x)dx \\
&\qquad - 2\int (\psi_1^t(x)-\psi_2^t(x))\cdot\big((\alpha v_1^t+\beta(v_1^t)^\perp)(\psi_1^t(x)) - (\alpha v_2^t+\beta(v_2^t)^\perp)(\psi_2^t(x))\big)\omega^\circ(x)dx \\
&\le CQ^t + C(Q^t)^{1/2}\Big(\int |v_1^t(\psi_1^t(x)) - v_2^t(\psi_2^t(x))|^2\omega^\circ(x)dx\Big)^{1/2} \\
&\le CQ^t + C(Q^t)^{1/2}(T_1^t+T_2^t)^{1/2},
\end{aligned}$$

where we have set

$$T_1^t := \int |(v_1^t - v_2^t)(\psi_2^t(x))|^2\omega^\circ(x)dx, \qquad T_2^t := \int |v_1^t(\psi_1^t(x)) - v_1^t(\psi_2^t(x))|^2\omega^\circ(x)dx.$$

We first study T_1. Using that $v_i = a^{-1}\nabla^\perp(\operatorname{div} a^{-1}\nabla)^{-1}\omega_i$, we find

$$\begin{aligned}
T_1^t = \int |v_1^t - v_2^t|^2\omega_2^t \le \|\omega_2^t\|_{\mathrm{L}^\infty}\int |v_1^t - v_2^t|^2 &= \|\omega_2^t\|_{\mathrm{L}^\infty}\int |\nabla(\operatorname{div} a^{-1}\nabla)^{-1}(\omega_1^t - \omega_2^t)|^2 \\
&\lesssim \|\omega_2^t\|_{\mathrm{L}^\infty}\int |\nabla\triangle^{-1}(\omega_1^t-\omega_2^t)|^2.
\end{aligned}$$

(Here, we use the fact that if $-\operatorname{div}(a^{-1}\nabla u_1) = -\triangle u_2$ with $u_1, u_2 \in H^1(\mathbb{R}^2)$, then $\int a^{-1}|\nabla u_1|^2 = \int \nabla u_1\cdot\nabla u_2 \le \frac12\int a^{-1}|\nabla u_1|^2 + \frac12\int a|\nabla u_2|^2$, hence $\int a^{-1}|\nabla u_1|^2 \le \int a|\nabla u_2|^2$.) The Loeper inequality [31, Proposition 3.1] and the bound (5.5) then imply

$$T_1^t \le \|\omega_2^t\|_{\mathrm{L}^\infty}(\|\omega_1^t\|_{\mathrm{L}^\infty}\vee\|\omega_2^t\|_{\mathrm{L}^\infty})W_2(\omega_1^t,\omega_2^t)^2 \le \|(\omega_1^t,\omega_2^t)\|_{\mathrm{L}^\infty}^2 Q^t.$$

We finally turn to T_2. Using the log-Lipschitz property (5.4) and the concavity of the function $f(x) = x(1+\log^- x)^2$, we obtain by Jensen's inequality

$$\begin{aligned}
T_2^t &\lesssim \|\omega_1^t\|_{\mathrm{L}^\infty}^2\int (1+\log^-(|\psi_1^t-\psi_2^t|))^2|\psi_1^t-\psi_2^t|^2\omega^\circ \\
&\le \|\omega_1^t\|_{\mathrm{L}^\infty}^2\Big(1+\log^-\int|\psi_1^t-\psi_2^t|^2\omega^\circ\Big)^2\int|\psi_1^t-\psi_2^t|^2\omega^\circ \\
&\lesssim \|\omega_1^t\|_{\mathrm{L}^\infty}^2(1+\log^- Q^t)^2Q^t.
\end{aligned}$$

We may thus conclude $\partial_t Q \lesssim (1+\|(\omega_1,\omega_2)\|_{\mathrm{L}^\infty})Q(1+\log^- Q)$, and the uniqueness result directly follows from a Grönwall argument. ☐

Acknowledgements

The work of the author is supported by F.R.S.-FNRS (Belgian National Fund for Scientific Research) through a Research Fellowship. The author would also like to thank his PhD advisor Sylvia Serfaty for useful comments on this work.

References

[1] L. Ambrosio, N. Gigli, and G. Savaré. *Gradient flows in metric spaces and in the space of probability measures*. Lectures in Mathematics ETH Zürich. Birkhäuser Verlag, Basel, 2005.

[2] L. Ambrosio, E. Mainini, and S. Serfaty. Gradient flow of the Chapman-Rubinstein-Schatzman model for signed vortices. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 28(2):217–246, 2011.

[3] L. Ambrosio and S. Serfaty. A gradient flow approach to an evolution problem arising in superconductivity. *Comm. Pure Appl. Math.*, 61(11):1495–1539, 2008.

[4] I. S. Aranson and L. Kramer. The world of the complex Ginzburg-Landau equation. *Rev. Mod. Phys.*, 74:99–143, 2002.

[5] H. Bahouri, J.-Y. Chemin, and R. Danchin. *Fourier analysis and nonlinear partial differential equations*, volume 343 of *Grundlehren der Mathematischen Wissenschaften*. Springer, Heidelberg, 2011.

[6] K. Bardos and È. S. Titi. Euler equations for an ideal incompressible fluid. *Uspekhi Mat. Nauk*, 62(3(375)):5–46, 2007.

[7] A. L. Bertozzi, Th. Laurent, and F. Léger. Aggregation and spreading via the Newtonian potential: the dynamics of patch solutions. *Math. Models Methods Appl. Sci.*, 22(suppl. 1):1140005, 39, 2012.

[8] G. Blatter, M. V. Feigel'man, V. B. Geshkenbein, A.I. Larkin, and V.M. Vinokur. Vortices in high-temperature superconductors. *Reviews of Modern Physics*, 66(4), 1994.

[9] H. Brézis and T. Gallouët. Nonlinear Schrödinger evolution equations. *Nonlinear Anal.*, 4(4):677–681, 1980.

[10] J. A. Carrillo, Y.-P. Choi, and M. Hauray. The derivation of swarming models: Mean-field limit and Wasserstein distances. In *Collective Dynamics from Bacteria to Crowds*, CISM International Centre for Mechanical Sciences, pages 1–46. Springer, 2014.

[11] S. J. Chapman and G. Richardson. Vortex pinning by inhomogeneities in type-II superconductors. *Phys. D*, 108:397–407, 1997.

[12] S. J. Chapman, J. Rubinstein, and M. Schatzman. A mean-field model of superconducting vortices. *European J. Appl. Math.*, 7(2):97–111, 1996.

[13] J.-Y. Chemin. *Perfect incompressible fluids*, volume 14 of *Oxford Lecture Series in Mathematics and its Applications*. The Clarendon Press, Oxford University Press, New York, 1998.

[14] R. R. Coifman and Y. Meyer. *Au delà des opérateurs pseudo-différentiels*, volume 57 of *Astérisque*. Société Mathématique de France, Paris, 1978. With an English summary.

[15] R. R. Coifman and Y. Meyer. Nonlinear harmonic analysis, operator theory and P.D.E. In *Beijing lectures in harmonic analysis (Beijing, 1984)*, volume 112 of *Ann. of Math. Stud.*, pages 3–45. Princeton Univ. Press, Princeton, NJ, 1986.

[16] J.-M. Delort. Existence de nappes de tourbillon en dimension deux. *J. Amer. Math. Soc.*, 4(3):553–586, 1991.

[17] Q. Du and P. Zhang. Existence of weak solutions to some vortex density models. *SIAM J. Math. Anal.*, 34(6):1279–1299, 2003.

[18] M. Duerinckx. Mean-field limits for some Riesz interaction gradient flows. Preprint, arXiv:1508.03377, 2015.

[19] M. Duerinckx and S. Serfaty. Mean-field dynamics for Ginzburg-Landau vortices with pinning and applied force. In preparation, 2016.

[20] W. E. Dynamics of vortex liquids in Ginzburg-Landau theories with applications to superconductivity. *Phys. Rev. B*, 50:1126–1135, 1994.

[21] A. Gulisashvili and M. A. Kon. Exact smoothing properties of Schrödinger semigroups. *Amer. J. Math.*, 118(6):1215–1248, 1996.

[22] Q. Han and F. Lin. *Elliptic partial differential equations*, volume 1 of *Courant Lecture Notes in Mathematics*. New York University, Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI, 1997.

[23] R. L. Jerrard and D. Smets. Vortex dynamics for the two dimensional non homogeneous Gross-Pitaevskii equation. *Annali Scuola Norm. Sup. Pisa*, 14(3):729–766, 2015.

[24] R. L. Jerrard and D. Spirn. Hydrodynamic limit of the Gross-Pitaevskii equation. *Comm. Partial Differential Equations*, 40(2):135–190, 2015.

[25] H.-Y. Jian and B.-H. Song. Vortex dynamics of Ginzburg-Landau equations in inhomogeneous superconductors. *J. Differential Equations*, 170(1):123–141, 2001.

[26] T. Kato and G. Ponce. Commutator estimates and the Euler and Navier-Stokes equations. *Comm. Pure Appl. Math.*, 41(7):891–907, 1988.

[27] M. Kurzke, J. L. Marzuola, and D. Spirn. Gross-Pitaevskii vortex motion with critically-scaled inhomogeneities. Preprint, arXiv:1510.08093v1, 2015.

[28] M. Kurzke and D. Spirn. Vortex liquids and the Ginzburg-Landau equation. *Forum Math. Sigma*, 2:e11, 63, 2014.

[29] F. Lin and P. Zhang. On the hydrodynamic limit of Ginzburg-Landau vortices. *Discrete Contin. Dynam. Systems*, 6(1):121–142, 2000.

[30] P.-L. Lions. *Mathematical topics in fluid mechanics. Vol. 2*, volume 10 of *Oxford Lecture Series in Mathematics and its Applications*. The Clarendon Press, Oxford University Press, New York, 1998. Compressible models, Oxford Science Publications.

[31] G. Loeper. Uniqueness of the solution to the Vlasov-Poisson system with bounded density. *J. Math. Pures Appl. (9)*, 86(1):68–79, 2006.

[32] N. Masmoudi and P. Zhang. Global solutions to vortex density equations arising from sup-conductivity. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 22(4):441–458, 2005.

[33] N. G. Meyers. An L^p-estimate for the gradient of solutions of second order elliptic divergence equations. *Ann. Scuola Norm. Sup. Pisa (3)*, 17:189–206, 1963.

[34] F. Otto. The geometry of dissipative evolution equations: the porous medium equation. *Comm. Partial Differential Equations*, 26(1-2):101–174, 2001.

[35] E. Sandier and S. Serfaty. *Vortices in the magnetic Ginzburg-Landau model.* Progress in Nonlinear Differential Equations and their Applications, 70. Birkhäuser Boston Inc., Boston MA, 2007.

[36] S. Serfaty. Mean-field limits of the Gross-Pitaevskii and parabolic Ginzburg-Landau equations. Preprint, arXiv:1507.03821, 2015.

[37] S. Serfaty and I. Tice. Ginzburg-Landau vortex dynamics with pinning and strong applied currents. *Arch. Ration. Mech. Anal.*, 201(2):413–464, 2011.

[38] S. Serfaty and J. L. Vázquez. A mean field equation as limit of nonlinear diffusions with fractional Laplacian operators. *Calc. Var. Partial Differential Equations*, 49(3-4):1091–1120, 2014.

[39] I. Tice. Ginzburg-Landau vortex dynamics driven by an applied boundary current. *Comm. Pure Appl. Math.*, 63(12):1622–1676, 2010.

[40] D. Tilley and J. Tilley. *Superfluidity and superconductivity.* Adam Hilger, second edition, 1986.

[41] M. Tinkham. *Introduction to superconductivity.* McGraw-Hill Inc., second edition, 1996.

[42] V. I. Yudovich. Non-stationary flows of an ideal incompressible fluid. *Ž. Vyčisl. Mat. i Mat. Fiz.*, 3:1032–1066, 1963.

(Mitia Duerinckx) Université Libre de Bruxelles (ULB), Brussels, Belgium, & MEPHYSTO team, Inria Lille–Nord Europe, Villeneuve d'Ascq, France, & Laboratoire Jacques-Louis-Lions, Université Pierre et Marie Curie (UPMC), Paris, France

E-mail address: mduerinc@ulb.ac.be