Immersion and Invariance Observers for Gyro-Free Attitude Control Systems

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I. Introduction

The rigid-body attitude control problem has been extensively studied due to mathematical interest in the study of kinematics, geometric mechanisms, and a wide range of challenging applications such as spacecraft, underwater vehicle and robot manipulator systems [1,2]. Although this problem is now considered to be well understood, it still attracts researchers as control theory develops and applications are diversified. The theoretical difficulties associated with this problem originate not only from the nonlinearity of the dynamics itself, but also from challenging practical constraints additionally imposed to achieve certain objectives – e.g., partial availability of states, system uncertainty, degradation or saturation of actuators [3–9].

In the case where angular velocity measurements are not readily available for control implementation, numerous output feedback controllers have been proposed only utilizing attitude measurements. Recently, it was shown that a set of unit vector measurements can be directly used in a feedback control law without constructing an attitude vector to regulate the body orientation [10, 11]. Such "gyro-free" controllers are mainly based on construction of auxiliary filters or dynamic observers. Thanks to certain passivity properties of the dynamics, output of an attitude

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error filter can be used as a replacement for the rate error feedback [3–5]. Alternatively, angular velocity can be estimated regardless of control input through an observer and the estimates are treated in certainty-equivalence fashion as true rate measurements [12, 13]. Compared to passivity-based controllers, the observer-controller combination requires establishment of a separation property which is not automatically guaranteed for nonlinear systems.

Recently in [14], an exponentially converging angular velocity observer was proposed based on Immersion and Invariance (I&I) design aided by a new dynamic scaling method suggested in [15]. Readers are referred to [16, 17] for further details of the general I&I methodology and to [18, 19] for nonlinear observer applications. Using switching logic, a C^0 continuous angular velocity observer with an exponential convergence was designed in [20] while a C^∞ observer was suggested showing an asymptotic convergence of the estimation errors in [21]. The aforementioned observers are driven by quaternion measurements and combined with proportional-derivative (PD) type controllers establishing an almost global asymptotic stability for the overall closed-loop system.

In this paper, a new lower dimensional I&I observer is proposed based on a novel definition of the angular velocity estimate. Instead of using a filter for the angular velocity estimate to dominate Coriolis effect suggested in [14], the direction cosine matrix associated with the quaternion state is utilized so that cross terms caused by both Coriolis effect and an approximate solution to a partial differential equation (PDE) I&I design inevitably involves are expressed in terms of quaternion filter error in Lyapunov analysis. Consequently, the observer dynamics evolve only in 8-dimensional space instead of the 11-dimensional space in [14] and stability analysis becomes simpler due to the fact that the dynamics of the scaling factor are only driven by the quaternion filter error. Moreover, further simplification is made by utilizing a structural property of the quaternion kinematics. Typically, I&I design with dynamic scaling requires additional term in the time-varying gain associated with the quaternion filter in order to bound the scaling factor [15], but the added term is shown to be zero for this particular application. When this observer is equipped with a PD tracking control, a separation property is established through the use of a partially strictified Lyapunov function candidate which is similar to one in [22].

This note is organized as follows. The rigid-body rotational dynamics and the corresponding quaternion kinematics are described in Section II and some useful lemmas are presented in Section III. Then the I&I observer design is discussed in Section IV. The stability analysis of the PD feedback control is conducted both with true rate measurements in Section V and with its estimates in Section VI, which also establishing the separation property. Section VII provides numerical results showing the performance of the proposed observer and controller. Finally, we ends with some concluding remarks in Section VIII.

II. Model Description

Consider the rigid-body rotational motion. Let $\omega(t) \in \mathbb{R}^3$ and $\mathbf{u}(t) \in \mathbb{R}^3$ be the angular rate and the applied control torque respectively both represented in the body-fixed frame. Then the dynamics are governed by Euler's rotational equations of motion given by

$$J\dot{\omega}(t) = -S(\omega(t))J\omega(t) + \mathbf{u}(t) , \tag{1}$$

where $J \in \mathbb{R}^{3\times3}$ is the positive definite inertia matrix and $S : \mathbb{R}^3 \to \mathbb{R}^{3\times3}$ is the skew-symmetric matrix operator such that it is equivalent to the vector cross product, i.e., $S(\mathbf{x})\mathbf{y} = \mathbf{x} \times \mathbf{y}$ for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^3$.

In this note, the quaternion kinematics are adopted to avoid the singularity:

$$\dot{\mathbf{q}}(t) = \frac{1}{2}E(\mathbf{q}(t))\omega(t) , \tag{2}$$

where the quaternion $\mathbf{q}(t) \in \mathbb{Q} = \{\mathbf{q} = (q_0, \mathbf{q}_v) \in \mathbb{R} \times \mathbb{R}^3 \,|\, \mathbf{q}^T\mathbf{q} = 1\}$ represents the orientation of the body frame $\mathcal{F}_B$ with respect to the inertial frame $\mathcal{F}_I$. The matrix function $E : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^{4\times3}$ in Eq. (2) is defined by

$$E(\mathbf{q}) = \begin{bmatrix} -\mathbf{q}_v^T \\ S(\mathbf{q}_v) + q_0 I_3 \end{bmatrix} , \tag{3}$$

where I_3 is the 3-dimensional identity matrix. Note that, given a quaternion $\mathbf{q} \in \mathbb{Q}$, a direction cosine matrix which also represents the body attitude can be obtained by the following identity:

$$C(\mathbf{q}) = I_3 - 2q_0 S(\mathbf{q}_v) + 2S^2(\mathbf{q}_v) , \tag{4}$$

whose corresponding kinematics are given by

$$\dot{C}(\mathbf{q}(t)) = -S(\omega(t))C(\mathbf{q}(t)) . \tag{5}$$

In our formulation, it is assumed that the inertia matrix J is exactly known and only quaternion measurements are available from attitude sensors. The objective is to design an observer that estimates the states $\mathbf{q}(t)$ and $\omega(t)$ with an exponential convergence using the I&I design method.

From the subsequent section on, function arguments are often dropped for notational simplicity.

III. Preliminaries

Before presenting our main result, we state some important technical results to make our analysis easier to follow. Detailed proofs are omitted in the interest of brevity.

Lemma 1. *Let $SO(3)$ be the 3-dimensional rotation group defined as*

$$SO(3) = \{R \in \mathbb{R}^{3\times 3} \,|\, R^T R = I_3, \det\{R\} = 1\} . \tag{6}$$

For any $R \in SO(3)$ and $\mathbf{x} \in \mathbb{R}^3$

$$RS(\mathbf{x}) = S(R\mathbf{x})R . \tag{7}$$

In other words, the vector cross product in three-dimension is invariant under proper rotations.

Lemma 2. *For any symmetric matrix $M \in \mathbb{R}^{3\times 3}$ and any vector $\mathbf{x} \in \mathbb{R}^3$, the equality*

$$S(M\mathbf{x}) + MS(\mathbf{x}) + S(\mathbf{x})M = \mathrm{Tr}(M)S(\mathbf{x}) \tag{8}$$

holds, where $\mathrm{Tr}(M)$ is the trace of the matrix M.

Lemma 3. *Consider the function $E(\mathbf{q})$ defined in Eq. (3). For any $\mathbf{q} \in \mathbb{R}^4$, the following equalities are satisfied:*

$$E^T(\mathbf{q})\mathbf{q} = \mathbf{0}\ , \tag{9}$$

$$\|E(\mathbf{q})\|_2 = \|\mathbf{q}\|_2\ . \tag{10}$$

Note Eq. (9) indicates that the unit-norm constraint is invariant under the quaternion kinematics. It is induced by Eqs. (3) and (10) that the function $E(\mathbf{q})$ is globally Lipschitz in $\mathbf{q}$.

IV. I&I Observer Design

A. Observer states

Let $\hat{\mathbf{q}}(t)$ and $\hat{\omega}(t)$ be the observer states that track $\mathbf{q}(t)$ and $\omega(t)$ respectively. Then the errors are defined as

$$\tilde{\mathbf{q}} = \hat{\mathbf{q}} - \mathbf{q}\ , \tag{11}$$

$$\tilde{\omega} = \hat{\omega} - \omega\ . \tag{12}$$

The angular velocity estimate is assumed to be

$$\hat{\omega} = C(\mathbf{q})\bar{\omega} + \boldsymbol{\beta}(\hat{\mathbf{q}}, \mathbf{q})\ , \tag{13}$$

where the mapping $\boldsymbol{\beta} : \mathbb{R}^4 \times \mathbb{Q} \to \mathbb{R}^3$ is to be determined. The objective is to design $\dot{\hat{\mathbf{q}}}(t)$, $\dot{\bar{\omega}}(t)$ and $\boldsymbol{\beta}(t)$ such that

$$\lim_{t\to\infty}(\tilde{\mathbf{q}}(t), \tilde{\omega}(t)) = (\mathbf{0}, \mathbf{0})\ . \tag{14}$$

B. Observer error dynamics

Let us consider the time-derivative of the angular velocity estimate defined in Eq. (13). Using Eq. (5), we obtain

$$\begin{aligned}\dot{\hat{\omega}} &= \dot{C}(\mathbf{q})\bar{\omega} + C(\mathbf{q})\dot{\bar{\omega}} + \dot{\boldsymbol{\beta}}(\hat{\mathbf{q}}, \mathbf{q}) \\ &= -S(\omega)C(\mathbf{q})\bar{\omega} + C(\mathbf{q})\dot{\bar{\omega}} + \dot{\boldsymbol{\beta}}\,. \end{aligned} \tag{15}$$

After arranging terms in the above equation, we have

$$\dot{\hat{\omega}} = S(\tilde{\omega})(\hat{\omega} - \boldsymbol{\beta}) - S(\hat{\omega})C(\mathbf{q})\bar{\omega} + C(\mathbf{q})\dot{\bar{\omega}} + \dot{\boldsymbol{\beta}}\,, \tag{16}$$

which leads to

$$\begin{aligned}J\dot{\tilde{\omega}} &= J\dot{\hat{\omega}} - J\dot{\omega} \\ &= JS(\tilde{\omega})(\hat{\omega} - \boldsymbol{\beta}) - JS(\hat{\omega})C(\mathbf{q})\bar{\omega} + JC(\mathbf{q})\dot{\bar{\omega}} + J\dot{\boldsymbol{\beta}} + S(\hat{\omega} - \tilde{\omega})J(\hat{\omega} - \tilde{\omega}) - \mathbf{u}\,. \end{aligned} \tag{17}$$

Once Lemma 2 is applied and the following relationships are substituted:

$$\bar{J} = \mathrm{Tr}(J)I_3 - 2J\,, \tag{18}$$

$$\boldsymbol{\mu} = C(\mathbf{q})\dot{\bar{\omega}} - S(\hat{\omega})C(\mathbf{q})\bar{\omega} + J^{-1}\left(S(\hat{\omega})J\hat{\omega} - \mathbf{u}\right)\,, \tag{19}$$

the rate error dynamics are obtained as

$$J\dot{\tilde{\omega}} = S(\tilde{\omega})\left(J\tilde{\omega} + \bar{J}\hat{\omega}\right) - JS(\tilde{\omega})\boldsymbol{\beta} + J(\boldsymbol{\mu} + \dot{\boldsymbol{\beta}})\,. \tag{20}$$

We will design $\boldsymbol{\mu}(t)$ and then recover $\dot{\bar{\omega}}(t)$ through the following relation:

$$\dot{\bar{\omega}} = C^T(\mathbf{q})\left[\boldsymbol{\mu} + S(\hat{\omega})C(\mathbf{q})\bar{\omega} - J^{-1}\left(S(\hat{\omega})J\hat{\omega} - \mathbf{u}\right)\right]\,. \tag{21}$$

Remark 1. *The key idea of this work compared to the result in [14] is that $\boldsymbol{\beta}$ is a function of only $\hat{\mathbf{q}}$ and $\mathbf{q}$, so the observer does not require a three-dimensional filter for $\hat{\omega}$ such that the filter state converges to $\hat{\omega}$ as time goes to infinity. This is achieved by use of the direction cosine matrix associated with the quaternion state to define the rate estimate in Eq. (13). We prove in the following analysis that the second term $JS(\tilde{\omega})\boldsymbol{\beta}$ in the right-hand side of Eq. (20) will be managed by designing $\boldsymbol{\beta}$ such that $\boldsymbol{\beta} \to \mathbf{0}$ as $\tilde{\mathbf{q}} \to \mathbf{0}$ without introducing extra dynamic extension.*

We also propose the dynamics for the quaternion filter as

$$\dot{\hat{\mathbf{q}}}(t) = -k_q r^2(t)\tilde{\mathbf{q}}(t) + \frac{1}{2}E(\mathbf{q}(t))\hat{\omega}(t)\ , \quad \hat{\mathbf{q}}(0) = \mathbf{q}(0)\ , \tag{22}$$

where k_q is a positive constant and $r(t)$ is the dynamic scaling factor to be determined in the following section. Then the quaternion error dynamics become

$$\dot{\tilde{\mathbf{q}}} = -k_q r^2\tilde{\mathbf{q}} + \frac{1}{2}E(\mathbf{q})\tilde{\omega}\ . \tag{23}$$

C. Dynamic scaling

The original I&I methodology requires the existence of a function $\boldsymbol{\beta}(\mathbf{q})$ such that

$$\frac{\partial\boldsymbol{\beta}}{\partial\mathbf{q}} = k_\omega E^T(\mathbf{q})\ , \tag{24}$$

where k_ω is a positive constant. However, no such solution to the above PDE exists. In order to apply I&I methodology, we rather consider an approximate solution $\boldsymbol{\beta}(\hat{\mathbf{q}}, \mathbf{q})$ such that

$$\frac{\partial\boldsymbol{\beta}}{\partial\mathbf{q}} = k_\omega E^T(\hat{\mathbf{q}}) \tag{25}$$

and $\lim_{t\to\infty} \hat{\mathbf{q}}(t) = \mathbf{q}(t)$. The effect caused by this approximation will be managed by introducing a dynamic scaling factor to be discussed later.

Let

$$\boldsymbol{\beta} = k_\omega E^T(\hat{\mathbf{q}})\mathbf{q} \tag{26}$$

and

$$\boldsymbol{\mu} = k_\omega E^T(\tilde{\mathbf{q}})\left(k_q r^2 \mathbf{q} - \frac{1}{2}E(\mathbf{q})\hat{\omega}\right) . \tag{27}$$

Since

$$\begin{aligned}
\dot{\boldsymbol{\beta}} &= k_\omega E^T(\hat{\mathbf{q}})\dot{\mathbf{q}} - k_\omega E^T(\mathbf{q})\dot{\hat{\mathbf{q}}} \\
&= -\frac{k_\omega}{2}E^T(\hat{\mathbf{q}})E(\mathbf{q})\tilde{\omega} - k_\omega E^T(\tilde{\mathbf{q}})\left(k_q r^2\mathbf{q} - \frac{1}{2}E(\mathbf{q})\hat{\omega}\right) \\
&= -\frac{k_\omega}{2}\tilde{\omega} - \frac{k_\omega}{2}E^T(\tilde{\mathbf{q}})E(\mathbf{q})\tilde{\omega} - k_\omega E^T(\tilde{\mathbf{q}})\left(k_q r^2\mathbf{q} - \frac{1}{2}E(\mathbf{q})\hat{\omega}\right) ,
\end{aligned} \tag{28}$$

the closed-loop dynamics are obtained as

$$J\dot{\tilde{\omega}} = S(\tilde{\omega})\left(J\tilde{\omega} + \bar{J}\hat{\omega}\right) - k_\omega J S(\tilde{\omega})E^T(\hat{\mathbf{q}})\mathbf{q} - \frac{k_\omega}{2}J\tilde{\omega} - \frac{k_\omega}{2}JE^T(\tilde{\mathbf{q}})E(\mathbf{q})\tilde{\omega} . \tag{29}$$

Now consider $V_{\tilde{\omega}} = (1/2)\tilde{\omega}J\tilde{\omega}$ whose time-derivative is given by

$$\dot{V}_{\tilde{\omega}} = -\frac{k_\omega}{2}\|J^{1/2}\tilde{\omega}\|_2^2 - k_\omega\tilde{\omega}^T JS(\tilde{\omega})E^T(\hat{\mathbf{q}})\mathbf{q} - \frac{k_\omega}{2}\tilde{\omega}^T JE^T(\tilde{\mathbf{q}})E(\mathbf{q})\tilde{\omega} . \tag{30}$$

Since $V_{\tilde{\omega}}$ is not upper bounded by a non-positive function, it cannot be further analyzed. In oder to handle the sign indefinite terms in Eq. (30) we introduce, using the dynamic scaling factor $r(t)$, the scaled rate error defined as

$$\mathbf{z}(t) = \frac{1}{r(t)}\tilde{\omega}(t) , \tag{31}$$

and propose

$$\dot{r} = -k_1 k_\omega (r-1) + k_2 k_\omega \|\tilde{\mathbf{q}}\|_2^2 r , \quad r(0) = 1 , \tag{32}$$

where k_1 and k_2 are positive constants. Note that, since the set $\{r \in \mathbb{R} \,|\, r \geq 1\}$ is positively invariant, the choice of the initial condition ensures $r(t) \geq 0$ for all $t \geq 0$. Along the trajectories of Eqs. (29) and (32), we have

$$J\dot{\mathbf{z}} = S(\mathbf{z})\left(J\tilde{\omega} + \bar{J}\hat{\omega}\right) - \frac{k_\omega}{2}J\mathbf{z} - \frac{\dot{r}}{r}J\mathbf{z} - k_\omega JS(\mathbf{z})E^T(\hat{\mathbf{q}})\mathbf{q} - \frac{k_\omega}{2}JE^T(\tilde{\mathbf{q}})E(\mathbf{q})\mathbf{z} . \tag{33}$$

Then the time-derivative of the Lyapunov-like function defined by

$$V_{\mathbf{z}} = \frac{1}{2k_{\omega}}\mathbf{z}^T J\mathbf{z} \tag{34}$$

is obtained as

$$\dot{V}_{\mathbf{z}} = -\left(\frac{1}{2} - \frac{k_1(r-1)}{r}\right)\mathbf{z}^T J\mathbf{z} - k_2\|\tilde{\mathbf{q}}\|_2^2\mathbf{z}^T J\mathbf{z} - \mathbf{z}^T JS(\mathbf{z})E^T(\hat{\mathbf{q}})\mathbf{q} - \frac{1}{2}\mathbf{z}^T JE^T(\tilde{\mathbf{q}})E(\mathbf{q})\mathbf{z}\,. \tag{35}$$

Let J_M and J_m denote the largest and the smallest eigenvalues of the inertia matrix J so that the inequalities

$$J_m\|\mathbf{z}\|_2^2 \leq \mathbf{z}^T J\mathbf{z} \leq J_M\|\mathbf{z}\|_2^2 \tag{36}$$

and

$$\|S(\mathbf{z})J\mathbf{z}\|_2^2 \leq J_\Delta\|\mathbf{z}\|_2^2\,\mathbf{z}^T J\mathbf{z} \tag{37}$$

hold for any $\mathbf{z} \in \mathbb{R}^3$, where $J_\Delta = J_M - J_m$. Using the above inequalities and the fact that

$$\begin{aligned}\|E^T(\hat{\mathbf{q}})\mathbf{q}\|_2 &= \|E^T(\mathbf{q})\mathbf{q} + E^T(\tilde{\mathbf{q}})\mathbf{q}\|_2\\ &\leq \|\tilde{\mathbf{q}}\|_2\end{aligned} \tag{38}$$

from Lemma 3, the upper bound of $V_{\mathbf{z}}$ is obtained as

$$\dot{V}_{\mathbf{z}} \leq -\begin{bmatrix}\mathbf{z}\\ \|\tilde{\mathbf{q}}\|\mathbf{z}\end{bmatrix}^T (Q_1 \otimes J)\begin{bmatrix}\mathbf{z}\\ \|\tilde{\mathbf{q}}\|\mathbf{z}\end{bmatrix}, \tag{39}$$

where

$$Q_1 = \begin{bmatrix}\frac{1}{2} - k_1 & -\frac{J_M + 2\sqrt{J_m J_\Delta}}{4J_m}\\ -\frac{J_M + 2\sqrt{J_m J_\Delta}}{4J_m} & k_2\end{bmatrix} \tag{40}$$

and the operator $\otimes$ denotes the Kronecker product.

Remark 2. *As mentioned in Remark 1, the choice of $\boldsymbol{\beta}$ in Eq. (26) renders the cross term caused by the Coriolis effect to depend on $\tilde{\mathbf{q}}$ and $\mathbf{z}$. Similarly, the other cross term coming from the* *time derivative of the approximate solution $\boldsymbol{\beta}(\hat{\mathbf{q}}, \mathbf{q})$ is a function of $\tilde{\mathbf{q}}$ and $\mathbf{z}$. Together they give r-dynamics driven by $\|\tilde{\mathbf{q}}\|_2^2$ only.*

If k_1 and k_2 are additionally constrained as

$$k_1 < \frac{1}{2} \,, \tag{41}$$

$$k_2 > \frac{\left(J_M + 2\sqrt{J_m J_\Delta}\right)^2}{8J_m^2\,(1 - 2k_1)} \,, \tag{42}$$

the matrix Q_1 becomes positive definite, and thus there exists a finite constant $c_1 > 0$ such that

$$\dot{V}_{\mathbf{z}} \leq -c_1 \|\mathbf{z}\|_2^2 \,. \tag{43}$$

Moreover, since

$$\dot{V}_{\mathbf{z}} \leq -\frac{2k_\omega c_1}{J_M} V_{\mathbf{z}} \,, \tag{44}$$

the norm of $\mathbf{z}(t)$ is bounded by

$$\|\mathbf{z}(t)\|_2 \leq \|\mathbf{z}(0)\|_2 \sqrt{\frac{J_M}{J_m}} \exp\left\{-\frac{k_\omega c_1}{J_M} t\right\} \,, \tag{45}$$

which implies that $\mathbf{z}(t)$ converges to zero exponentially fast.

To show the convergence of $\tilde{\omega}$, consider

$$V_o = \frac{1}{2} \tilde{\mathbf{q}}^T \tilde{\mathbf{q}} + \zeta_1 V_{\mathbf{z}} \,, \tag{46}$$

where $\zeta_1 > 1/(8kc_1)$. Along the trajectories of the closed-loop dynamics the time-derivative of V_o yields

$$\dot{V}_o = -k_q r^2 \|\tilde{\mathbf{q}}\|_2^2 + \frac{r}{2} \tilde{\mathbf{q}}^T E(\mathbf{q}) \mathbf{z} + \zeta_1 \dot{V}_{\mathbf{z}} \,. \tag{47}$$

This is upper bounded by

$$\dot{V}_o \leq -\frac{k_q}{2} r^2 \|\tilde{\mathbf{q}}\|_2^2 - \begin{bmatrix} r\|\tilde{\mathbf{q}}\|_2 \\ \|\mathbf{z}\|_2 \end{bmatrix}^T Q_2 \begin{bmatrix} r\|\tilde{\mathbf{q}}\|_2 \\ \|\mathbf{z}\|_2 \end{bmatrix} , \tag{48}$$

where

$$Q_2 = \begin{bmatrix} \frac{k}{2} & -\frac{1}{4} \\ -\frac{1}{4} & \zeta_1 c_1 \end{bmatrix} \tag{49}$$

is positive definite.

D. Boundedness of scaling factor

It is essential to establish the boundedness of $r(t)$ to show the convergence of $\tilde{\mathbf{q}}(t)$ since Eqs. (46) and (48) are only valid as long as $r(t)$ is finite. To prove $r \in L_\infty$, we construct another Lyapunov function candidate

$$W = V_o + \frac{k_q}{8k_2 k_\omega} r^2 \tag{50}$$

whose time derivative is given by

$$\begin{aligned} \dot{W} &= \dot{V}_o - \frac{k_1 k_q}{4k_2 k_\omega} r(r-1) + \frac{k_q}{4} r^2 \|\tilde{\mathbf{q}}\|_2^2 \\ &\leq -\frac{k_q}{4} r^2 \|\tilde{\mathbf{q}}\|_2^2 \, . \end{aligned} \tag{51}$$

Since $W(t)$ is non-increasing, the scaling factor is bounded for all $t \in [0, \infty)$.

Remark 3. *To ensure $r \in L_\infty$, the quaternion estimate dynamics typically have the form*

$$\dot{\hat{\mathbf{q}}}(t) = -K(t)\tilde{\mathbf{q}}(t) + \frac{1}{2} E(\mathbf{q}(t))\hat{\omega}(t) \tag{52}$$

with

$$K(t) = k_q r^2(t) + \epsilon r^2(t) \boldsymbol{\psi}(\mathbf{q}(t), \tilde{\mathbf{q}}(t)) \, , \tag{53}$$

where ϵ is a positive constant and $\boldsymbol{\psi}$ is a function that cancels the positive term in $\dot{W}$ [15]. However, *owing to the fact that the function $E(\mathbf{q})$ is globally Lipschitz, the increasing term in the $\dot{r}$ equation becomes proportional to $||\tilde{\mathbf{q}}||_2^2$ instead of $||E(\tilde{\mathbf{q}})||_2^2$, and thus the additional term $\boldsymbol{\psi}$ can be set to be zero so that the first term in the dynamic gain $K(t)$ absorbs the other.*

E. Convergence properties

As $r(t)$ is finite for all $t \geq 0$, standard Lyapunov analysis is now applied. The positive definite matrix Q_2 in Eq. (48) allows us to find some $c_2 > 0$ such that

$$\dot{V}_o \leq -\frac{k_q}{2}||\tilde{\mathbf{q}}||_2^2 - c_2||\mathbf{z}||_2^2 \,, \tag{54}$$

which implies that there exists $c_3 > 0$ such that

$$\dot{V}_o \leq -c_3 V_o \,. \tag{55}$$

Therefore, it is concluded that both $\tilde{\mathbf{q}}(t)$ and $\mathbf{z}(t)$ converge to zero exponentially fast. Finally, since

$$\begin{aligned} ||\tilde{\omega}(t)||_2 &\leq r(t)||\tilde{\omega}(0)||_2\sqrt{\frac{J_M}{J_m}}\exp\left\{-\frac{k_\omega c_1}{J_M}t\right\} \\ &\leq \sup_{t\geq 0}\{r(t)\}||\tilde{\omega}(0)||_2\sqrt{\frac{J_M}{J_m}}\exp\left\{-\frac{k_\omega c_1}{J_M}t\right\} \end{aligned} \tag{56}$$

from Eq. (45), we conclude that

$$\lim_{t\to\infty}\tilde{\omega}(t) = \mathbf{0} \tag{57}$$

for any $\omega(0) \in \mathbb{R}^3$and that the convergence rate is exponential.

Remark 4. *As $\tilde{\mathbf{q}}$ converges to zero exponentially fast, so does $\boldsymbol{\beta}$ by Eq. (38). Therefore, we can also conclude that*

$$\lim_{t\to\infty}C(\mathbf{q}(t))\bar{\omega}(t) = \omega(t) \tag{58}$$

with an exponential convergence rate. Then we can interpret $\bar{\boldsymbol{\omega}}$ as the estimate for the angular velocity expressed in the inertial frame $\mathcal{F}_I$.

V. Full State Feedback Tracking Control

It is known that PD controllers stabilize the attitude tracking error when all states are available for the controller implementation. Let $\mathbf{q}_r(t) \in \mathbb{Q}$ be the quaternion representing the reference frame $\mathcal{F}_R$ with respect to the inertial frame $\mathcal{F}_I$, and let $\boldsymbol{\omega}_r(t) \in \mathbb{R}^3$ be the reference angular velocity in the reference frame. Define the quaternion tracking error $\mathbf{e}_q = (e_0, \mathbf{e}_v) \in \mathbb{Q}$ such that

$$C(\mathbf{e}_q) = C(\mathbf{q})C^T(\mathbf{q}_r) \tag{59}$$

and the rate error as

$$\mathbf{e}_\omega = \boldsymbol{\omega} - C(\mathbf{e}_q)\boldsymbol{\omega}_r \,. \tag{60}$$

If the control torque is given by

$$\mathbf{u} = -k_p\mathbf{e}_v - k_v\mathbf{e}_\omega + JC(\mathbf{e}_q)\dot{\boldsymbol{\omega}}_r + S(\boldsymbol{\omega}_r^B)J\boldsymbol{\omega}_r^B \,, \tag{61}$$

where $k_p, k_v > 0$ are the control gains and $\boldsymbol{\omega}_r^B = C(\mathbf{e}_q)\boldsymbol{\omega}_r$ is the reference angular velocity in the body frame, then the tracking error dynamics are obtained as

$$\dot{\mathbf{e}}_q = \frac{1}{2}E(\mathbf{e}_q)\mathbf{e}_\omega \tag{62}$$

$$J\dot{\mathbf{e}}_\omega = -S(\mathbf{e}_\omega)(J\mathbf{e}_\omega - \bar{J}\boldsymbol{\omega}_r^B) - k_p\mathbf{e}_v - k_v\mathbf{e}_\omega \,. \tag{63}$$

To prove $(\mathbf{e}_v, \mathbf{e}_\omega) \to (\mathbf{0}, \mathbf{0})$ as $t \to \infty$, consider the partially strictified Lyapunov function candidate taken from [22]

$$V_c = \zeta_2 k_p\left[\mathbf{e}_v^T\mathbf{e}_v + (e_0 - 1)^2\right] + \frac{\zeta_2}{2}\mathbf{e}_\omega^T J\mathbf{e}_\omega + \mathbf{e}_v^T J\mathbf{e}_\omega \,, \tag{64}$$

where $\zeta_2 > \frac{k_v}{4k_p} + \frac{J_M}{2k_v}$. Since

$$V_c \geq \zeta_2 k_p(e_0 - 1)^2 + \begin{bmatrix} \mathbf{e}_v \\ \mathbf{e}_\omega \end{bmatrix}^T (Q_3 \otimes J)\begin{bmatrix} \mathbf{e}_v \\ \mathbf{e}_\omega \end{bmatrix} \tag{65}$$

and

$$\dot{V}_c \leq - \begin{bmatrix} \mathbf{e}_v \\ \mathbf{e}_\omega \end{bmatrix}^T (Q_4 \otimes I_3) \begin{bmatrix} \mathbf{e}_v \\ \mathbf{e}_\omega \end{bmatrix} , \tag{66}$$

where the positive definite matrices Q_3 and Q_4 are defined by

$$Q_3 = \begin{bmatrix} \frac{\zeta_2 k_p}{J_M} & \frac{1}{2} \\ \frac{1}{2} & \frac{\zeta_2}{2} \end{bmatrix} \tag{67}$$

and

$$Q_4 = \begin{bmatrix} k_p & \frac{k_v}{2} \\ \frac{k_v}{2} & \zeta_2 k_v - \frac{J_M}{2} \end{bmatrix} , \tag{68}$$

we can conclude, by invoking Barbalat's lemma, that

$$\lim_{t \to \infty} (\mathbf{e}_v(t), \mathbf{e}_\omega(t)) = (\mathbf{0}, \mathbf{0}) . \tag{69}$$

VI. Separation Property

As the angular velocity measurements are not available, the PD control is modified as

$$\mathbf{u} = -k_p \mathbf{e}_v - k_v \hat{\mathbf{e}}_\omega + JC(\mathbf{e}_q)\dot{\boldsymbol{\omega}}_r + S(\boldsymbol{\omega}_r^B) J \boldsymbol{\omega}_r^B , \tag{70}$$

where

$$\hat{\mathbf{e}}_\omega = \hat{\boldsymbol{\omega}} - C(\mathbf{e}_q)\boldsymbol{\omega}_r . \tag{71}$$

Consequently, the rate error dynamics are changed to

$$J\dot{\mathbf{e}}_\omega = -S(\mathbf{e}_\omega)(J\mathbf{e}_\omega - \bar{J}\boldsymbol{\omega}_r^B) - k_p \mathbf{e}_v - k_v \mathbf{e}_\omega - k_v \tilde{\boldsymbol{\omega}} , \tag{72}$$

which leads to

$$\dot{V}_c \leq -\begin{bmatrix} \mathbf{e}_v \\ \mathbf{e}_\omega \end{bmatrix}^T (Q_4 \otimes I_3) \begin{bmatrix} \mathbf{e}_v \\ \mathbf{e}_\omega \end{bmatrix} - k_v \mathbf{e}_v^T \tilde{\omega} - \zeta_2 k_v \mathbf{e}_\omega^T \tilde{\omega} \ . \tag{73}$$

To dominate the cross terms in the right hand side of Eq. (73), consider the Lyapunov function candidate

$$V = V_c + \zeta_3 V_z \ , \tag{74}$$

where ζ_3 is a positive constant to be determined. From Eqs. (43) and (73), the upper bound of its time derivative along the closed-loop trajectories is obtained as

$$\dot{V} \leq -\begin{bmatrix} \mathbf{e}_v \\ \mathbf{e}_\omega \\ \mathbf{z} \end{bmatrix}^T (Q_5 \otimes I_3) \begin{bmatrix} \mathbf{e}_v \\ \mathbf{e}_\omega \\ \mathbf{z} \end{bmatrix} , \tag{75}$$

where the matrix $Q_5(t) \in \mathbb{R}^{3\times 3}$ is given by

$$Q_5(t) = \begin{bmatrix} Q_4 & R(t) \\ R^T(t) & \zeta_3 c_1 \end{bmatrix} \tag{76}$$

with

$$R(t) = \frac{k_v r(t)}{2} \begin{bmatrix} 1 \\ \zeta_2 \end{bmatrix} . \tag{77}$$

Since Q_4 is positive definite, there exists $c_4 > 0$ such that $Q_4 \geq c_4 I_2$, discussed earlier. Let $T(t) \in \mathbb{R}^{2\times 2}$ be the Schur complement of Q_4 in $Q_5(t)$ defined by

$$T(t) = Q_4 - \frac{1}{\zeta_3 c_1} R(t) R^T(t) \ . \tag{78}$$

Then the choice

$$\zeta_3 > \sup_{t\geq 0}\{r^2(t)\} \frac{k_v^2(1+\zeta_2^2)}{4 c_1 c_4} \tag{79}$$

makes $T(t)$, and thus, $Q_5(t)$ positive definite for all $t \in [0, \infty)$. Applying Barbalat's lemma, we finally have

$$\lim_{t\to\infty} (\mathbf{e}_v(t), \mathbf{e}_\omega(t), \mathbf{z}(t)) = (\mathbf{0}, \mathbf{0}, \mathbf{0}) . \tag{80}$$

VII. Numerical Simulations

In this section, numerical simulation results are provided to demonstrate the performance of the proposed observer and observer-based PD type tracking controller. The inertia matrix of the rigid spacecraft is taken from [21].

$$J = \begin{bmatrix} 10 & 1.2 & 0.5 \\ 1.2 & 19 & 1.5 \\ 0.5 & 1.5 & 25 \end{bmatrix} \text{kg m}^2 \tag{81}$$

The initial conditions are set to

$$\mathbf{q}_v(0) = \begin{bmatrix} -0.1 \\ 0.1 \\ -0.1 \end{bmatrix}, \quad \omega(0) = \begin{bmatrix} 0.005 \\ 0.006 \\ 0.004 \end{bmatrix} \text{rad/s}$$

with $q_0 = \sqrt{1 - \|\mathbf{q}_v(0)\|_2^2}$. The reference quaternion is set to $\mathbf{q}_r(0) = (0.9487, 0.1826, 0.1826, 0.1826)$ initially and its dynamics are assumed to be driven by the reference angular velocity profile $\boldsymbol{\omega}_r(t) = (\omega_r(t), \omega_r(t), \omega_r(t))$ rad/s, where

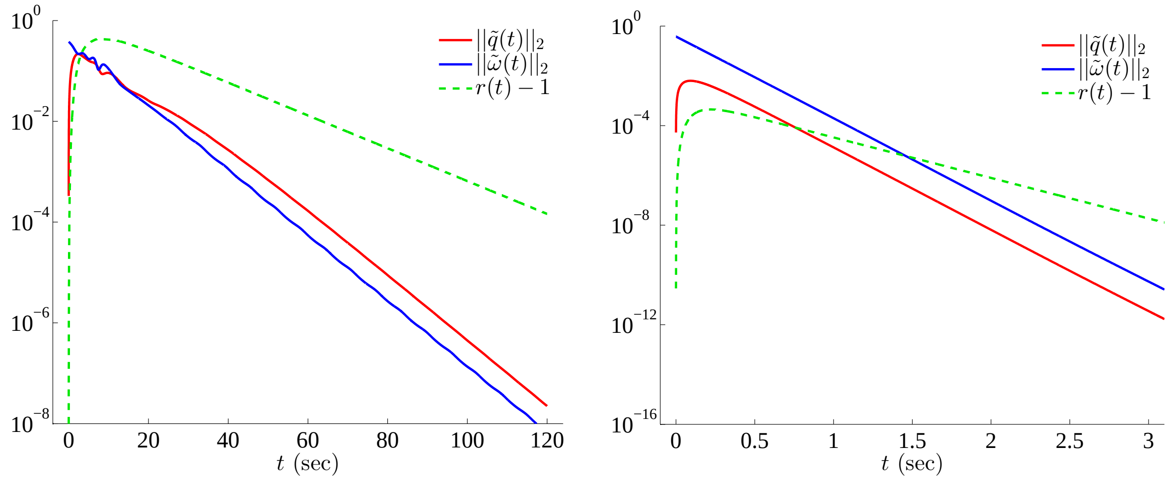
$$\omega_r(t) = 0.3\cos(t)\left(1 - e^{-0.01t^2}\right) + (0.08\pi + 0.006\sin(t))\, te^{-0.01t^2} . \tag{82}$$

To show the tracking performance, we set the control gains to $k_p = 1.5$ and $k_v = 5$. For the observer, we use

$$k_1 = \frac{1}{4}, \quad k_2 = \frac{\left(J_M + 2\sqrt{J_m J_\Delta}\right)^2}{4J_m^2(1 - 2k_1)} . \tag{83}$$

The gains k_q and k_ω are closely related to the convergence rate of the observer state errors. In the simulation, they are selected such that the convergence rates for the quaternion and angular velocity errors are approximately the same. For comparison, two sets of these gains are chosen:

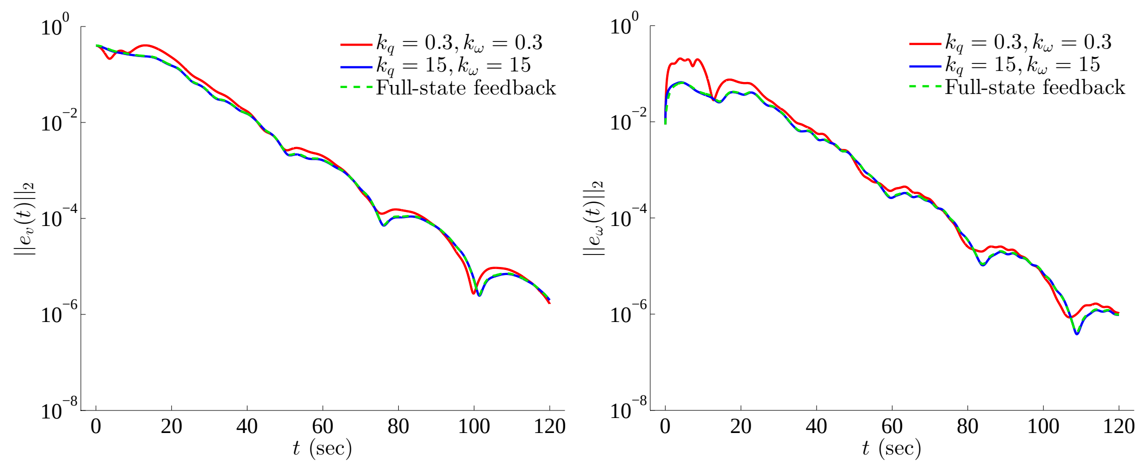
$(k_q, k_\omega) = (0.3, 0.3)$ and $(15, 15)$. The initial guess for the angular velocity is selected as $\hat{\boldsymbol{\omega}}(0) = (-0.3, 0.1, -0.2)$ rad/s.



a) $k_q = 0.3, k_\omega = 0.3$ **b)** $k_q = 15, k_\omega = 15$

Figure 1. Time histories of observer errors

The performance of the proposed observer is shown in Fig. 1. The high gains ensure a fast convergence rate as expected. Given fixed gains k_q and k_ω, the convergence rate of the scaling factor is further restricted by the other observer gains. So the scaling factor converges faster to unity if k_1 and k_2 are close to their constraint boundaries described in Eqs. (41) and (42).



a) Quaternion errors **b) Angular velocity errors**

Figure 2. Time histories of tracking errors

The tracking performance of the observer-based PD controller is demonstrated and compared with the full-state feedback control in Fig. 2. When the small gain set is selected for the observer, the tracking error norms are deviated from those of the full-state feedback control, but the convergence rate is not very different because the convergence rate of the observer errors is still faster than that of the tracking errors in the ideal PD control. When the observer gains are increased to the values in the second set, the trajectories are almost identical to the full-state feedback control. Figure 3 also shows the same tendency in the control effort. For the high gain observer, the trajectory of the control torque norm catches up the full-state feedback control within a second.

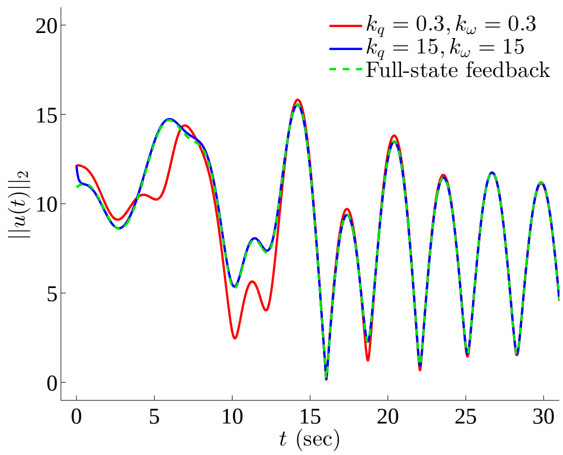


Figure 3. Time histories of control torques

VIII. Conclusion

In this note, a novel observer-based attitude tracking controller is proposed in the absence of rate measurements. Based on the I&I design method in conjunction with dynamic scaling, an angular velocity observer employing the attitude quaternion measurements is designed. It is shown that the angular velocity estimate globally converges to the true value exponentially fast independent of the choice of control torque. By defining the rate estimate using a direction cosine matrix, the observer structure is far simplified and requires less state variables compared to the existing I&I design. A separation property is then established for the PD tracking controller while asymptotic convergence of the tracking errors is shown. To demonstrate the observer performance, numerical simulations are conducted. Tracking results of two observers with relatively high and low gains are provided and compared with the ideal case where true angular velocity is available for feedback control.

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