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***Abstract*— We provide an extension of Razumikhin's theorem for continuous time time-varying systems with time-varying delays. Our result uses a novel 'strictification' technique for converting a nonstrict Lyapunov function into a strict one. Our examples show how our method can sometimes allow broader classes of allowable delays than the results in the literature.**

***Key Words*: delay, Lyapunov, Razumikhin, stability**

I. Introduction

Input delays are ubiquitous in engineering, where they are useful for modeling time consuming information gathering or latencies in engineering processes; see [1], [3], [4], [6], [7], [11], [18], [19], and [20]. However, such systems are usually too complicated to be covered by standard methods for undelayed systems [21]. This note builds on our research (begun in [12], [16], and [17]) on novel methods to prove important stability properties for time delayed systems.

Since the flow map for a nonlinear system usually cannot be written in explicit closed form, it is natural to use indirect methods such as Lyapunov approaches to prove stability for undelayed systems. Lyapunov functions provide a generalized notion of energy in dynamical systems, so the decay of the Lyapunov function in a suitable sense often implies asymptotic convergence of the solutions towards an equilibrium. Classical Lyapunov function approaches require that the time derivative of the Lyapunov function be nonpositive along all solutions of the system, which can sometimes be a demanding condition, especially for time-varying or time delay systems. While classical Lyapunov functions are suited for proving stability of systems without delays, one often replaces Lyapunov functions by Lyapunov-Krasovskii functionals [5] or Ruzumikhin functions to help solve stability problems for delayed systems.

As explained in [16] and [22], time-varying systems with delay are very important, e.g., to model tracking problems; see also [2]. The works [2], [13], [17], and [23] on time delay systems are significant, in part because they use Lyapunov functional ideas to prove stability but allow the time derivatives of the functionals to take positive and negative values along trajectories of the systems. For delayed systems, two important analogs of classical Lyapunov functions are often used. The first are Lyapunov-Krasovskii functionals, which can often be built by adding together a Lyapunov function for the corresponding undelayed system and a second integral term involving the delay; see [15]. A different approach involves Razumikhin's theorem; see [6], [9], and [22, Theorem B.2]. The Razumikhin approach is especially useful for systems with time-varying delay.

This note provides an extension of Razumikhin's theorem for time-varying systems. We extend the strictification technique, developed in particular in [10]. Our first result does not use periodicity. However, due to the importance of periodic systems, we present a slightly simpler result in the periodic case. We obtain less conservative stability conditions than those in [17] and, at least in some cases, those in [23].

II. Definitions and Notation

Throughout this work, all dimensions are arbitrary, unless indicated otherwise. The usual Euclidean norm, and its matrix norm, are denoted by $|\cdot|$, and $|\cdot|_{\mathcal{I}}$ denotes the (essential) supremum over any interval $\mathcal{I} \subseteq \mathbb{R}$. Let C^1 be the set of all continuously differentiable functions, whose domains and ranges will be clear from the context. For each constant delay bound τ, let $C([-\tau, 0], \mathbb{R}^n)$ be the set of all continuous $\mathbb{R}^n$-valued functions defined on $[-\tau, 0]$. Let $C_{\rm in}$ be the set of all absolutely continuous functions $\phi \in C([-\tau, 0], \mathbb{R}^n)$, which we call the set of all *initial functions*. For each continuous function $\varphi : [-\tau, \infty) \to \mathbb{R}^n$ and $t \geq 0$, set $\varphi_t(m) = \varphi(t + m)$ for all $m \in [-\tau, 0]$. A locally bounded function defined on an interval $\mathcal{I} \subseteq \mathbb{R}$ is called *piecewise continuous* provided it is continuous except at finitely many points on each bounded subset of $\mathcal{I}$. For a function ϕ defined on $[0, \infty) \times \mathbb{R}^n$ as being differentiable, we view its partial derivative $\phi_t(0, x)$ with respect to its first argument at 0 as being a right derivative at 0.

Let $\mathcal{K}$ denote the set of all strictly increasing continuous functions $\alpha : [0, \infty) \to [0, \infty)$ such that $\alpha(0) = 0$; if, in addition, α is unbounded, then we say that α is of class $\mathcal{K}_\infty$. A function $F : [0, \infty) \times C_{\rm in} \to \mathbb{R}^n$ is called *uniformly bounded with respect to its first argument* provided there is a function $\alpha \in \mathcal{K}$ and a constant $\bar{c} > 0$ such that $|F(t, \phi))| \leq \bar{c} + \alpha(|\phi|_{[-\tau,0]})$ holds for all $t \geq 0$ and $\phi \in C_{\rm in}$; it is called *Lipschitz continuous* with respect to its second argument provided for each constant $K > 0$, we have $|F(t, \phi) - F(t, \psi)| \leq \alpha(K)|\phi - \psi|_{[-\tau,0]}$ for all $t \geq 0$ and all ϕ and ψ in $C_{\rm in}$ such that $\max\{|\phi|_{[-\tau,0]}, |\psi|_{[-\tau,0]}\} \leq K$.

III. General Result

A. Statement of Result

We consider a nonlinear time-varying system

$$\dot{x} = F(t, x_t) \tag{1}$$

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whose delay is bounded by some constant τ, having initial conditions in $C([-\tau,0],\mathbb{R}^n)$. We assume:

Assumption 1: The function F is uniformly bounded with respect to its first argument and Lipschitz continuous with respect to its second argument. There is a function $V : [0,\infty)\times\mathbb{R}^n \to [0,\infty)$ that is of class C^1 on $([0,\infty)\times\mathbb{R}^n)\setminus\{0\}$ and functions α_1 and α_2 of class $\mathcal{K}_\infty$ such that

$$\alpha_1(|x|) \le V(t,x) \le \alpha_2(|x|) \tag{2}$$

hold for all $(t,x)\in[0,\infty)\times\mathbb{R}^n$ and such that there are bounded piecewise continuous functions $a:[0,\infty)\to\mathbb{R}$ and $b:[0,\infty)\to[0,\infty)$ such that

$$\dot V(t) \le a(t)V(t,x(t)) + b(t)\sup_{\ell\in[t-\tau,t]} V(\ell,x(\ell)) \tag{3}$$

holds along all trajectories of (1).

Assumption 2: There are positive constants β, ϖ, $\underline{\epsilon}$ and $\overline{\epsilon}$ and a piecewise continuous function $\epsilon:[0,\infty)\to[0,\infty)$ such that $\epsilon(t)\in[\underline{\epsilon},\overline{\epsilon}]$ for all $t\ge 0$ and such that the function

$$\mu(t) = a(t) + b(t) \tag{4}$$

satisfies

$$\left|\int_0^t (\epsilon(\ell)+\mu(\ell))\mathrm{d}\ell\right| \le \beta \tag{5}$$

for all $t\ge 0$. Moreover

$$\kappa(t) = \sup_{\ell\in[t-\tau,t]} \int_\ell^t (-\epsilon(s)-\mu(s))\mathrm{d}s \tag{6}$$

is a real number for each $t\ge 0$, and

$$\sup_{t\ge 0}\left[\left(e^{\kappa(t)}-1\right)b(t) - \epsilon(t)\right] \le -\varpi \tag{7}$$

holds.

See Section V for interesting applications where we can easily check the preceding assumptions. Our main result is:

Theorem 1: If Assumptions 1-2 hold, then the origin of (1) is a globally uniformly asymptotically stable equilibrium.

B. Discussion of Theorem 1

Consider the special case where there is a constant $\underline{\mu}>0$ such that the function (4) satisfies

$$\mu(t) \le -\underline{\mu} \tag{8}$$

for all $t\ge 0$, and $\bar b$ be any constant bound for b. Then, if

$$q \in \left(1, 1+\tfrac{1}{2\bar b}\underline{\mu}\right), \tag{9}$$

then

$$a(t)+qb(t) \;\le\; \mu(t)+(q-1)b(t) \;\le\; -\tfrac{1}{2}\underline{\mu} \tag{10}$$

is satisfied for all $t\ge 0$. Therefore, for all $t\ge 0$ such that

$$qV(t,x(t)) \ge \sup_{\ell\in[t-\tau,t]} V(\ell,x(\ell)), \tag{11}$$

the inequalities

$$\begin{aligned}\dot V(t) &\le a(t)V(t,x(t)) + b(t)qV(t,x(t))\\ &\le -\tfrac{\underline{\mu}}{2}V(t,x(t))\end{aligned} \tag{12}$$

are satisfied. Then Razumikhin's theorem ensures the global uniform asymptotic stability of the origin of (1). However, our objective is precisely to establish stability results in cases where (8) is not satisfied.

Notice that our assumptions include cases where the function a takes positive and negative values. Our discussion of Corollary 1 in the next section will explain the motivation for Assumption 2 and how it compares with the stability conditions obtained in [23].

C. Proof of Theorem 1

Throughout the proof, all inequalities and equalities should be understood to hold for all $t\ge 0$ and along all solutions of (1) unless otherwise indicated. Assumption 1 ensures the existence and the uniqueness of the solutions of (1). Let $\bar a\ge 0$ and $\bar b\ge 0$ be any constant bounds for $|a(t)|$ and b, respectively. We define the following functions

$$\theta(t) = \exp\left(-\int_0^t(\epsilon(s)+\mu(s))\mathrm{d}s\right) \tag{13}$$

and

$$U(t,x) = \theta(t)V(t,x)\ . \tag{14}$$

Then for all $\ell\in[t-\tau,t]$, we have

$$\frac{\theta(t)}{\theta(\ell)} \le e^{\kappa(t)}. \tag{15}$$

Also, (3)-(4) give

$$\begin{aligned}\tfrac{d}{dt}\left(U(t,x(t))\right) &= \dot\theta(t)V(t,x(t)) + \theta(t)\tfrac{d}{dt}\left(V(t,x(t))\right)\\ &\le -(\epsilon(t)+\mu(t))U(t,x(t))\\ &\quad + a(t)U(t,x(t))\\ &\quad + \theta(t)b(t)\sup\nolimits_{\ell\in[t-\tau,t]}V(\ell,x(\ell))\\ &= (-\epsilon(t)-b(t))U(t,x(t))\\ &\quad + b(t)\sup\nolimits_{\ell\in[t-\tau,t]}\theta(t)V(\ell,x(\ell)).\end{aligned}$$

It follows from (15) that

$$\begin{aligned}\tfrac{d}{dt}\left(U(t,x(t))\right) &= -(\epsilon(t)+b(t))U(t,x(t))\\ &\quad + b(t)\sup_{\ell\in[t-\tau,t]}\frac{\theta(t)}{\theta(\ell)}U(\ell,x(\ell))\\ &\le -(\epsilon(t)+b(t))U(t,x(t))\\ &\quad + b(t)\sup_{\ell\in[t-\tau,t]}\frac{\theta(t)}{\theta(\ell)}\\ &\qquad\times\sup_{\ell\in[t-\tau,t]}U(\ell,x(\ell))\\ &\le -(\epsilon(t)+b(t))U(t,x(t))\\ &\quad + e^{\kappa(t)}b(t)\sup_{\ell\in[t-\tau,t]}U(\ell,x(\ell)).\end{aligned} \tag{16}$$

Also, we can use the nonnegativity of b to get

$$\begin{aligned}e^{\kappa(t)}b(t) &\le \exp\left(\int_{t-\tau}^t|\epsilon(s)+\mu(s)|\mathrm{d}s\right)b(t)\\ &\le e^{\tau(\bar\epsilon+\bar a+\bar b)}\bar b.\end{aligned} \tag{17}$$

Consequently, for all $r>0$ such that

$$r \le \frac{\varpi}{2e^{\tau(\bar\epsilon+\bar a+\bar b)}\bar b} \tag{18}$$

the inequality

$$re^{\tau(\bar{\epsilon}+\bar{a}+\bar{b})}\bar{b} \leq \frac{\varpi}{2} \tag{19}$$

is satisfied. This inequality combined with (17) ensures that

$$re^{\kappa(t)}b(t) \leq \frac{\varpi}{2} \ . \tag{20}$$

Also, Assumption 2 ensures that for all $t \geq 0$, we have

$$\left(e^{\kappa(t)} - 1\right) b(t) - \epsilon(t) \leq -\varpi \ . \tag{21}$$

From (20) and (21), it follows that for all $t \geq 0$, we have

$$\left(e^{\kappa(t)} - 1\right) b(t) + re^{\kappa(t)}b(t) - \epsilon(t) \leq -\tfrac{\varpi}{2} \ . \tag{22}$$

Set $q = 1 + r$. Then grouping terms gives

$$\left(qe^{\kappa(t)} - 1\right) b(t) - \epsilon(t) \leq -\tfrac{\varpi}{2} \ . \tag{23}$$

Next note that when $qU(t, x(t)) \geq \sup_{\ell \in [t-\tau,t]} U(\ell, x(\ell))$, the second inequality in (16) and (23) give

$$\begin{array}{rcl} \dot{U}(t) & \leq & -(\epsilon(t) + b(t))U(t, x(t)) \\ & & + e^{\kappa(t)}b(t)qU(t, x(t) \\ & \leq & -\frac{\varpi}{2}U(t, x(t)). \end{array} \tag{24}$$

Also, (5) combined with Assumption 1 gives

$$e^{-\beta}\alpha_1(|x|) \leq U(t, x) \leq e^{\beta}\alpha_2(|x|) \ . \tag{25}$$

Hence, the theorem follows from the classical Razumikhin theorem, by combining the decay estimate (24) with (25).

IV. Periodic Case

A. Main Result for Periodic Case

We now consider key cases where the following holds:

Assumption 3: Assumption 1 holds, and there is a constant $\mathcal{T} > 0$ such that a and b in Assumption 1 are periodic of period $\mathcal{T}$. Also, there exist a piecewise continuous function $\epsilon : \mathbb{R} \to \mathbb{R}$ that is periodic of period $\mathcal{T}$ and constants $\underline{\epsilon} > 0$ and $\bar{\epsilon} > 0$ such that $\epsilon(t) \in [\underline{\epsilon}, \bar{\epsilon}]$ for all $t \geq 0$ and

$$\textstyle\int_0^{\mathcal{T}} [\epsilon(m) + a(m) + b(m)]\mathrm{d}m = 0 \ . \tag{26}$$

Moreover, the function κ defined in (6) is such that

$$\sup_{t\in[0,\mathcal{T}]} \left[\left(e^{\kappa(t)} - 1\right) b(t) - \epsilon(t)\right] < 0 \tag{27}$$

is satisfied.

We are ready to state and prove the following result:

Corollary 1: Let (1) satisfy Assumption 3. Then the origin of (1) is globally uniformly asymptotically stable.

B. Discussion of Corollary 1

A crucial question is whether Assumption 2 or Assumption 3 is restrictive with respect to the conditions obtained in [23]. When the delays are small, our conditions are less restrictive than those imposed in [23] in the general nonlinear case.

To see why, consider the case where a and b have some period $\mathcal{T}$ and choose the constant function $\epsilon(t) = \epsilon_*$, where

$$\epsilon_* = -\tfrac{1}{\mathcal{T}} \textstyle\int_0^{\mathcal{T}} \mu(\ell)\mathrm{d}\ell \tag{28}$$

and $\mu = a + b$ as before. Then the function (6) satisfies

$$\kappa(t) = \sup_{\ell\in[t-\tau,t]} \textstyle\int_\ell^t (-\epsilon_* - \mu(s))\mathrm{d}s \leq \tau s_\mu \tag{29}$$

for all $t \geq 0$, where $s_\mu = \sup_{s\in[0,\mathcal{T}]}(-\epsilon_* - \mu(s))$. Then Assumption 3 holds if (a) $\epsilon_* > 0$ and (b) the inequality

$$\sup_{t\in[0,\mathcal{T}]} \left[\left(e^{\tau s_\mu} - 1\right) b(t)\right] < \epsilon_* \tag{30}$$

is satisfied. This is the case if $\tau \in (0, \tau_*)$, where

$$\tau_* = \tfrac{1}{s_\mu} \ln\left(1 + \tfrac{\epsilon_*}{\bar{b}}\right) . \tag{31}$$

Another sufficient condition can be obtained when μ is Lipschitz. Let us consider this case, and let $\lambda_\mu > 0$ denote a Lipschitz constant of μ. Then, since

$$\epsilon_* = -\tfrac{1}{\mathcal{T}} \textstyle\int_{s-\mathcal{T}}^s \mu(\ell)\mathrm{d}\ell, \tag{32}$$

for all $s \geq 0$, it follows that

$$\kappa(t) = \sup_{\ell\in[t-\tau,t]} \int_\ell^t \left(\frac{1}{\mathcal{T}} \int_{s-\mathcal{T}}^s \mu(m)\mathrm{d}m - \mu(s)\right) \mathrm{d}s \ . \tag{33}$$

Since the function

$$\varphi(t) = \tfrac{1}{\mathcal{T}} \textstyle\int_{t-\mathcal{T}}^t \int_s^t \mu(m)\mathrm{d}m\mathrm{d}s \tag{34}$$

satisfies

$$\dot{\varphi}(t) = \mu(t) - \tfrac{1}{\mathcal{T}} \textstyle\int_{t-\mathcal{T}}^t \mu(m)\mathrm{d}m, \tag{35}$$

we can integrate $\dot{\varphi}$ over $[\ell, t]$ to get

$$\kappa(t) = \sup_{\ell\in[t-\tau,t]} (-\varphi(t) + \varphi(\ell)) \,. \tag{36}$$

Since we can change variables to get

$$\varphi(\ell) = \tfrac{1}{\mathcal{T}} \textstyle\int_{t-\mathcal{T}}^t \int_s^t \mu(m - t + \ell)\mathrm{d}m\mathrm{d}s, \tag{37}$$

our choice $\lambda_\mu > 0$ of the Lipschitz constant and (36) give

$$\begin{array}{l} \kappa(t) = \\ \displaystyle\sup_{\ell\in[t-\tau,t]} \frac{1}{\mathcal{T}} \int_{t-\mathcal{T}}^t\int_s^t [-\mu(m) + \mu(m - t + \ell)]\mathrm{d}m\mathrm{d}s \\ \displaystyle\leq \sup_{\ell\in[t-\tau,t]} \frac{\lambda_\mu}{\mathcal{T}} \int_{t-\mathcal{T}}^t \int_s^t (t - \ell)\mathrm{d}\ell\mathrm{d}s \leq \frac{\mathcal{T}\lambda_\mu\tau}{2} \ , \end{array} \tag{38}$$

by upper bounding the last integrand in (38) by τ. Hence, Assumption 3 is satisfied if $\epsilon_* > 0$ and the inequality

$$\left(e^{\frac{\mathcal{T}\lambda_\mu\tau}{2}} - 1\right) \bar{b} < \epsilon_* \tag{39}$$

are satisfied. This leads to the delay dependent condition

$$\lambda_\mu\tau\mathcal{T} < 2\ln\left(1 + \tfrac{\epsilon_*}{\bar{b}}\right) \ . \tag{40}$$

Similar conditions cannot be derived from [23], because the conditions in [23] do not depend on a Lipschitz constant and are delay independent, except in the particular case of the linear systems. In this particular case, our conditions can be less restrictive than those of [23] too. To understand why, consider the simple one-dimensional time-invariant system

$$\dot{x}(t) = -qx(t) + (q - 1)x(t - \tau) \tag{41}$$

where $q > 1$ and $\tau \geq 0$ and constants. We consider (41) because it is easier to compare techniques in the time-invariant case than in the time-varying case. It is well known that for any $\tau \geq 0$, the origin of (41) is globally exponentially stable. We show that our method makes it possible to prove this result, while the results in [23] lead to an extra constraint.

Taking $V(x) = |x|$, Assumption 3 is satisfied with $a = -q$, $b = q - 1$, $\kappa = 0$, and $\epsilon = q - (q - 1) = 1$. Hence, Corollary 1 ensures that the origin of (41) is exponentially stable. This cannot be deduced from the Razumikhin result for linear systems in [23], since this earlier result only proves stability of (41) for values of τ smaller than a function of q.

C. Proof of Corollary 1

We show that Assumption 3 implies that Assumption 2 is satisfied, as follows. Let $t > 0$ be given, and let k be the integer such that $t \in [k\mathcal{T}, (k+1)\mathcal{T})$. Then

$$\begin{array}{rcl} \left|\int_0^t(\epsilon(\ell) + \mu(\ell))\mathrm{d}\ell\right| & \leq & \left|\int_0^{k\mathcal{T}}(\epsilon(\ell) + \mu(\ell))\mathrm{d}\ell\right| \\ & & + \left|\int_{k\mathcal{T}}^t(\epsilon(\ell) + \mu(\ell))\mathrm{d}\ell\right| . \end{array} \tag{42}$$

Since (26) implies that

$$\int_0^{k\mathcal{T}}(\epsilon(\ell) + \mu(\ell))\mathrm{d}\ell = 0, \tag{43}$$

it follows that

$$\begin{array}{rcl} \left|\int_0^t(\epsilon(\ell) + \mu(\ell))\mathrm{d}\ell\right| & \leq & \int_{k\mathcal{T}}^t(\overline{\epsilon} + \overline{a} + \overline{b})\mathrm{d}\ell \\ & \leq & \mathcal{T}(\overline{\epsilon} + \overline{a} + \overline{b}) . \end{array} \tag{44}$$

Next, we prove that κ is periodic of period $\mathcal{T}$. We have

$$\begin{array}{rcl} \kappa(t+\mathcal{T}) & = & \sup_{\ell\in[t+\mathcal{T}-\tau,t+\mathcal{T}]} \int_\ell^{t+\mathcal{T}}(-\epsilon(m) - \mu(m))\mathrm{d}m \\ & = & \sup_{\ell\in[t-\tau,t]} \int_{\ell+\mathcal{T}}^{t+\mathcal{T}}(-\epsilon(m) - \mu(m))\mathrm{d}m \\ & = & \sup_{\ell\in[t-\tau,t]} \int_\ell^t(-\epsilon(m+\mathcal{T}) - \mu(m+\mathcal{T}))\mathrm{d}m. \end{array}$$

Since both μ and ϵ are periodic of period $\mathcal{T}$, it follows that

$$\begin{array}{l} \kappa(t+\mathcal{T}) = \\ \sup_{\ell\in[t-\tau,t]} \displaystyle\int_\ell^t(-\epsilon(m) - \mu(m))\mathrm{d}m = \kappa(t) . \end{array} \tag{45}$$

From the fact that κ, ϵ and μ are all periodic of period $\mathcal{T}$, we easily deduce that for all $t \geq 0$, we have

$$\sup_{t\geq0}\left[\left(e^{\kappa(t)} - 1\right)b(t) - \epsilon(t)\right] \leq -\varpi \tag{46}$$

where

$$\varpi = -\sup_{t\in[0,\mathcal{T}]}\left[\left(e^{\kappa(t)} - 1\right)b(t) - \epsilon(t)\right] . \tag{47}$$

The corollary now follows from Theorem 1.

V. Examples

A. First Example

First consider the one dimensional system

$$\dot{x}(t) = -x(t) + b(t)x(t - \tau(t)) \tag{48}$$

where $b : \mathbb{R} \to \mathbb{R}$ is a periodic function of period 1 such that there are constants $c \in (0, 1)$ and $d > 0$ satisfying (i) $b(t) = 0$ for all $t \in [0, c)$ and (ii) $b(t) = d$ for all $t \in [c, 1)$. Assume that $\tau : [0, +\infty) \to [0, +\infty)$ is continuous and bounded by a constant $\overline{\tau} > 0$. This system is slightly more general than the one used in [17] and [23], since it includes a time-varying delay. See below for higher dimensional examples.

Since (48) is periodic, we apply Corollary 1. With $V(x) = |x|$ and $a(t) = -1$, Assumption 1 is satisfied. Let

$$\epsilon_* = -\int_0^1[-1 + b(m)]dm = 1 - d(1 - c) \tag{49}$$

We consider the case where $\overline{\tau} < \frac{1}{4}$ and give conditions ensuring that Assumption 3 is satisfied.

To get less restrictive conditions than those obtained with $\epsilon(t) = \epsilon_*$, pick any constant $\nu \in (0, \min\{1, \epsilon_*/c\})$, and let ϵ be the periodic 1 function such that (a) $\epsilon(t) = \nu$ for all $t \in [0, c)$ and (b) $\epsilon(t) = \frac{\epsilon_* - c\nu}{1-c}$ for all $t \in [c, 1]$. Then

$$\epsilon(t) = \frac{1 - c\nu}{1 - c} - d \text{ for all } t \in [c, 1]. \tag{50}$$

Also, for all $t \geq 1$, we have $\epsilon(t) \in [\underline{\epsilon}, \overline{\epsilon}]$, where

$$\begin{array}{l} \underline{\epsilon} = \min\left\{\nu, \frac{1-c\nu}{1-c} - d\right\} > 0 \\ \text{and } \overline{\epsilon} = \max\left\{\nu, \frac{1-c\nu}{1-c} - d\right\} . \end{array} \tag{51}$$

We have

$$\kappa(t) = \sup_{\ell\in[t-\overline{\tau},t]} \int_\ell^t [-\epsilon(m) + 1 - b(m)]dm \tag{52}$$

and

$$\int_0^t (\epsilon(t) - 1 + b(t))\,\mathrm{d}t = 0. \tag{53}$$

Assumption 3 is satisfied if

$$\sup_{t\in[0,1]}\left[\left(e^{\kappa(t)} - 1\right)b(t) - \epsilon(t)\right] < 0 . \tag{54}$$

From the definition of b, it follows that (54) is equivalent to

$$\exp\left(\sup_{t\in[c,1]}\kappa(t)\right) < \frac{1 - c\nu}{d(1 - c)} . \tag{55}$$

Now observe that

$$\begin{array}{rcl} -\epsilon(m) + 1 - b(m) & = & -\left[\frac{1-c\nu}{1-c} - d\right] + 1 - d \\ & = & c\frac{\nu-1}{1-c} \end{array} \tag{56}$$

for all $m \in [c, 1]$, while $-\epsilon(m) + 1 - b(m) = -\nu + 1$ for all $m \in [0, c]$. We deduce that

$$\sup_{t\in[c,1]}\kappa(t) \leq \overline{\tau}(1 - \nu). \tag{57}$$

Then (55) is satisfied if

$$\overline{\tau} < \frac{1}{1-\nu}\ln\left(\frac{1-c\nu}{d(1-c)}\right) . \tag{58}$$

Recall that

$$\nu < \frac{\epsilon_*}{c} = \frac{1-d(1-c)}{c} \tag{59}$$

and $\nu < 1$. Taking the limit when ν goes to zero, we get the inequality $\overline{\tau} < \tau_*$, where $\tau_* = -\ln(d(1 - c))$.

The work [23] uses a Razumikhin approach as well, but it does not provide a delay dependent result. However, [23] uses a Krasovskii aproach to provide a delay dependent result, which only applies to (48) when τ is constant.

B. Second Example

Consider the system

$$\dot{x} = -m(t)^\top m(t)u(t-\tau(t)) \tag{60}$$

where x is valued in $\mathbb{R}^n$ for any n, the function $m : \mathbb{R} \to \mathbb{R}^n$ is a continuous and has some period $\omega > 0$, $u \in \mathbb{R}^n$ is the input, and the delay τ is a time-varying piecewise continuous function that bounded by a constant $\overline{\tau} > 0$.

We studied (60) in [15], but only in the case where τ is constant. The approach in [15] is based in the construction of a Lyapunov-Krasovskii functional, which is written as the sum of a strict Lyapunov function for the corresponding undelayed system plus a double integral term. In this section, we use Corollary 1 to provide stabilizability conditions in the case where the delay is time-varying. Notice also that the result of [14] does not apply to systems with a time-varying delay. We first introduce this assumption:

Assumption 4: The matrix $M \in \mathbb{R}^{n\times n}$ defined by

$$M = \frac{1}{\omega}\int_0^\omega m(s)^\top m(s)\mathrm{d}s \tag{61}$$

is positive definite.

Let us define the constants

$$k_m = \frac{1}{\omega}\int_0^\omega |m(s)|^2\mathrm{d}s \;\text{ and }\; s_m = \sup_{t\in[0,\omega]} |m(t)| \tag{62}$$

and let λ_M be the smallest eigenvalue of M. Then $k_m > 0$, $s_m > 0$ and $\lambda_M > 0$. We now add these three assumptions:

Assumption 5: The inequality

$$\frac{2\lambda_M}{1+2\omega s_m^2} > 2\left(s_m^4\overline{\tau}^2 + 2\omega k_m\right)k_m \tag{63}$$

is satisfied.

Assumption 6: The function $|m(t)|$ is globally Lipschitz, with Lipschitz constant $l_m > 0$.

Assumption 7: The inequality

$$\begin{aligned} &2\left(s_m^4\overline{\tau}^2 + 2\omega k_m\right)s_m\overline{\tau}\omega < \\ &\ln\left(1 + \frac{\frac{\lambda_M}{1+2\omega s_m^2} - 2\left(s_m^4\overline{\tau}^2+2\omega k_m\right)k_m}{\bar{b}}\right) \end{aligned} \tag{64}$$

is satisfied, where $\bar{b} = 2(s_m^4\overline{\tau}^2 + 2\omega k_m)s_m^2$.

One can easily determine values of $\overline{\tau}$ such that (64) is satisfied.

Corollary 2: Let the system (60) satisfy Assumptions 4-7. Then (60) in closed loop with the feedback

$$u(x(t)) = x(t) \tag{65}$$

is globally uniformly exponentially stabilized to the origin.

Proof: The feedback (65) produces the system

$$\dot{x} = -m(t)^\top m(t)x(t-\tau(t)). \tag{66}$$

Our objective is now to prove that the origin of (66) is globally uniformly exponentially stable. This system is a linear time-varying system and therefore we can restrict our stability analysis to the time interval $[2\overline{\tau}, +\infty)$.

For all $t \geq 2\overline{\tau}$, we can rewrite the system as

$$\begin{aligned} &\dot{x} = -m(t)^\top m(t)x(t) - \\ &m(t)^\top m(t)\textstyle\int_{t-\tau(t)}^t m(s)^\top m(s)x(s-\tau(s))\mathrm{d}s. \end{aligned} \tag{67}$$

Let us consider $\nu_1(x) = \frac{1}{2}|x|^2$ and

$$\nu_2(t,x) = \tfrac{1}{\omega}x^\top \textstyle\int_{t-\omega}^t\int_\ell^t m(s)^\top m(s)\mathrm{d}s\mathrm{d}\ell x \tag{68}$$

By (67), we deduce that along the trajectories of (66),

$$\begin{aligned} &\dot{\nu}_1(t) = -|m(t)x(t)|^2 - \\ &x(t)^\top m(t)^\top m(t)\textstyle\int_{t-\tau(t)}^t m(s)^\top m(s)x(s-\tau(s))ds \end{aligned} \tag{69}$$

and

$$\begin{aligned} &\dot{\nu}_2(t) = |m(t)x(t)|^2 - x(t)^\top Mx(t) - \\ &\tfrac{2}{\omega}x(t)^\top \textstyle\int_{t-\omega}^t\int_\ell^t m(s)^\top m(s)\mathrm{d}s\mathrm{d}\ell m(t)^\top m(t)x(t-\tau(t)), \end{aligned}$$

where M is the matrix defined in (61).

Let us introduce the function

$$V(t,x) = 2\nu_1(x) + \nu_2(t,x). \tag{70}$$

Then for all $t \geq 0$ and $x \in \mathbb{R}^n$, the inequalities

$$\tfrac{1}{2}|x|^2 \leq V(t,x) \leq \left(1 + \tfrac{1}{2}\omega s_m^2\right)|x|^2 \tag{71}$$

are satisfied. For all $t \geq 2\overline{\tau}$,

$$\begin{aligned} &\dot{V}(t) = -x(t)^\top Mx(t) - |m(t)x(t)|^2 \\ &-2x(t)^\top m(t)^\top m(t)\textstyle\int_{t-\tau(t)}^t m(s)^\top m(s)x(s-\tau(s))ds \\ &-\tfrac{2}{\omega}x(t)^\top \textstyle\int_{t-\omega}^t\int_\ell^t m(s)^\top m(s)\mathrm{d}s\mathrm{d}\ell m(t)^\top m(t)x(t-\tau(t)). \end{aligned}$$

Using Young's inequality, we obtain

$$\begin{aligned} &\dot{V}(t) \leq -x(t)^\top Mx(t) + \\ &\left|m(t)\textstyle\int_{t-\tau(t)}^t m(s)^\top m(s)x(s-\tau(s))ds\right|^2 \\ &-\tfrac{2}{\omega}x(t)^\top \textstyle\int_{t-\omega}^t\int_\ell^t m(s)^\top m(s)\mathrm{d}s\mathrm{d}\ell \\ &\times m(t)^\top m(t)x(t-\tau(t)). \end{aligned} \tag{72}$$

Then Jensen's inequality gives

$$\begin{aligned} &\dot{V}(t) \leq -x(t)^\top Mx(t) \\ &+\tau(t)|m(t)|^2\textstyle\int_{t-\tau(t)}^t\left|m(s)^\top m(s)\right|^2|x(s-\tau(s))|^2\mathrm{d}s \\ &+\tfrac{2|m(t)|^2}{\omega}\textstyle\int_{t-\omega}^t\int_\ell^t|m(s)|^2\mathrm{d}s\mathrm{d}\ell|x(t)||x(t-\tau(t))|. \end{aligned} \tag{73}$$

From the definition of k_m in (62), it follows that

$$\begin{aligned} &\dot{V}(t) \leq -x(t)^\top Mx(t) + \\ &\tau(t)|m(t)|^2\textstyle\int_{t-\tau(t)}^t\left|m(s)^\top m(s)\right|^2|x(s-\tau(s))|^2\mathrm{d}s \\ &+2|m(t)|^2\omega k_m|x(t)||x(t-\tau(t))|. \end{aligned} \tag{74}$$

Consequently, the bounds (71) give

$$\begin{aligned} &\dot{V}(t) \leq -\lambda_M|x(t)|^2 + 2\omega k_m|m(t)|^2|x(t)||x(t-\tau(t))| \\ &+\tau(t)s_m^4|m(t)|^2\textstyle\int_{t-\tau(t)}^t|x(s-\tau(s))|^2\mathrm{d}s, \end{aligned}$$

and therefore also

$$\begin{array}{l}\dot{V}(t) \leq -\frac{\lambda_M}{1+\omega s_m^2} V(t,x(t)) \\ +2\tau(t)s_m^4|m(t)|^2 \int_{t-\tau(t)}^t V(s-\tau(s), x(s-\tau(s)))\mathrm{d}s \\ +4\omega k_m|m(t)|^2\sqrt{V(t,x(t))V(t-\tau(t),x(t-\tau(t)))}.\end{array} \tag{75}$$

It follows that

$$\begin{array}{rcl}\dot{V}(t) & \leq & -\frac{\lambda_M}{1+\omega s_m^2} V(t,x(t)) \\ & & +2\tau(t)^2 s_m^4|m(t)|^2 \sup_{s\in[t-2\overline{\tau},t]} V(s,x(s)) \\ & & +4\omega k_m|m(t)|^2 \sup_{s\in[t-2\overline{\tau},t]} V(s,x(s)) \\ & = & -\frac{\lambda_M}{1+\omega s_m^2} V(t,x(t)) \\ & & +2\left(\overline{\tau}^2 s_m^4 + 2\omega k_m\right)|m(t)|^2 \\ & & \times \sup_{s\in[t-2\overline{\tau},t]} V(s,x(s))\end{array} \tag{76}$$

Now, we apply Corollary 1 with

$$a(t) = -\frac{\lambda_M}{1+\omega s_m^2} \text{ and } b(t) = 2\left(s_m^4\overline{\tau}^2 + 2\omega k_m\right)|m(t)|^2.$$

Then we have

$$\begin{array}{rcl}\mu(t) & = & a(t)+b(t) \\ & = & -\frac{\lambda_M}{1+\omega s_m^2} + 2\left(s_m^4\overline{\tau}^2 + 2\omega k_m\right)|m(t)|^2\end{array} \tag{77}$$

and

$$\begin{array}{l}\epsilon_* = -\frac{1}{\omega}\int_0^\omega \mu(\ell)\mathrm{d}\ell \\ = -\frac{1}{\omega}\int_0^\omega \left[-\frac{\lambda_M}{1+\omega s_m^2} + 2\left(s_m^4\overline{\tau}^2 + 2\omega k_m\right)|m(\ell)|^2\right]\mathrm{d}\ell.\end{array}$$

Therefore,

$$\epsilon_* = \frac{\lambda_M}{1+\omega s_m^2} - 2\left(s_m^4\overline{\tau}^2 + 2\omega k_m\right)k_m.$$

From (63), it follows that $\epsilon_* > 0$. Moreover, μ is globally Lipschitz with $l_\mu = 4\left(s_m^4\overline{\tau}^2 + 2\omega k_m\right)s_m l_m$ as Lipschitz constant. We deduce from our discussion of Corollary 1 that

$$l_\mu\overline{\tau}\omega < 2\ln\left(1 + \frac{\epsilon_*}{\sup_{t\in[0,\omega]} b(t)}\right) \tag{78}$$

ensures global asymptotic stability; see (40). It is equivalent to (64). This concludes the proof. ■

VI. Conclusions

Time delay systems with time-varying delays play a central role in controls, but their stability analysis is often beyond the scope of Lyapunov-Krasovskii functional or other more traditional techniques. To help overcome these significant challenges, we applied a strictification approach to extend Razumikhin's theorem for time-varying systems. The approach entails converting a nonstrict Lyapunov-like function into a strict one. A key feature of our analysis is that we do not require the Lyapunov functional to decay along trajectories when the delay is set to 0, which puts our strictification analysis outside the scope of existing results on the Razumikhin approach. We illustrated how our method produces a larger value for the allowable time delay than what was obtained in earlier literature. In future work, we plan to extend our analysis to systems with perturbations, as well as adaptive cases where the objectives include both tracking and parameter identification.

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