Residual-based a posteriori estimators for the potential formulations of electrostatic and time-harmonic eddy current problems with voltage or current excitation

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Chao Chen* Emmanuel Creusé † Serge Nicaise‡ Zuqi Tang§

Abstract

In this paper, we present some residual-based a posteriori estimators for the potential formulations of electrostatic and time-harmonic eddy current problems with voltage or current excitation. The reliability of the proposed estimators is proved and some numerical experiments are carried out to illustrate the theoretical analysis.

Key words: A posteriori estimate, Electrostatics, Eddy current, Potential formulation, Voltage and current excitation.

Introduction

Nowadays, the finite element method is widely used for the study of electromagnetic phenomena. One of the main difficulties consists in controlling the accuracy of the numerical solution obtained. In this context, some a posteriori error estimators are needed to give an upper bound of the error and to provide efficient tools allowing an adaptive mesh refinement strategy. Since the pioneering work of Bakuska and Rheinboldt [2, 3], a large number of contributions have been proposed, devoted to different kinds of formulations, equations, numerical methods or approximation spaces.

In this work, we are particularly interested in explicit residual-based error estimators devoted to the approximation of Maxwell's equations in the low frequency regime by harmonic formulations, where the quasi-static approximation occurs [5, 8, 19, 20].

For such kinds of estimators, the isotropic case for standard elliptic boundary value problems is currently well understood [1, 13, 18]. The analysis of residual a posteriori error estimators for the edge elements, in the context of the electric field formulation of Maxwell equations, was successively initiated in [4] with specific assumptions on the coefficients arising in the equations or on the domain regularity. Then, they have been generalized to the case of anisotropic meshes and non regular data [14], as well as to the one of Lipschitz domains [16]. The case of piecewise constant coefficients was also discussed in [14]. Some new results based on interpolation operators on mixed finite element spaces were addressed in [16]. The robustness of the estimations with the data was considered in [7]. Moreover, some papers have also been devoted to design some adaptive mesh loop strategies, in order to ensure the convergent process while the mesh is being refined using error estimators [6, 11, 12].

In our previous work, the potential formulations were considered in the case of one imposed current density generated by a coil [8, 9]. The voltage excitation for another potential formulation has also been

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reported in [5].

In this paper, we focus on the residual based a posteriori estimators derivation in the case of the voltage and of the current excitation (see [10, 15] for details) instead of the imposed current density one. It consequently leads to considering different formulations and to overcoming some specific difficulties in order to derive the reliability of the involved estimators. Our contribution mainly focusses on the main differences compared to our previous works on the topic (e.g. [8, 9]), and we refer to these references to complete the proofs. Let us also note that we do not consider here the question of the efficiency of the proposed estimators, since it is straightforward.

The schedule of the paper is as follows. Section 1 is devoted to the electrostatic problem, where the voltage excitation case and the current excitation one are successively considered. Section 2 deals with the eddy current problem, in both cases. Finally, Section 3 presents some numerical experiments to illustrate some of the theoretical results obtained.

1 Electrostatic problem

Let us consider a simply connected and bounded conductor domain $D_c \subset \mathbb{R}^3$ with a connected boundary $\Gamma = \partial D_c$. We suppose that $\overline{\Gamma} = \overline{\Gamma}_C \cup \overline{\Gamma}_J$ with Γ_J a connected part of Γ and $\Gamma_C \cap \Gamma_J = \emptyset$. Moreover, Γ_C is supposed to be composed of at least two disjoint connected components corresponding to the electric ports: $\Gamma_C = \overset{N}{\underset{i=1}{\cup}} \Gamma_{C_i}$, $N \geq 2$, see Figure 1.

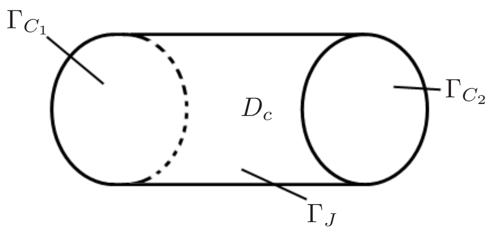


Figure 1: Configuration of the domain, example for two electric ports ($N = 2$).

The equations corresponding to the electrostatic problem are written as:

$$\nabla\times\mathbf{E} = \mathbf{0} \text{ in } D_c, \tag{1.1}$$

$$\nabla\cdot \mathbf{J} = 0 \text{ in } D_c, \tag{1.2}$$

where $\mathbf{E}$ and $\mathbf{J}$ are the electric field and the current flux density respectively. The constitutive law between $\mathbf{J}$ and $\mathbf{E}$ takes the form:

$$\mathbf{J} = \sigma\,\mathbf{E}, \tag{1.3}$$

where σ is defined as the electrical conductivity of the material. The corresponding boundary conditions are given by:

$$\mathbf{J}\cdot\mathbf{n} = 0 \text{ on } \Gamma_J, \tag{1.4}$$

and

$$\mathbf{E}\times\mathbf{n} = \mathbf{0} \text{ on } \Gamma_C, \tag{1.5}$$

where $\mathbf{n}$ denotes the unit outward normal vector on Γ.

Two ways to impose the source term in the model will be considered : the voltage excitation case and the current excitation one.

1.1 Voltage excitation case

In this first case, the scalar potential is imposed on each electric port. This scalar potential is supposed to be equal to 0 on Γ_{C_N}, so that the voltage value on each other electric port is determined by:

$$V\,|_{\Gamma_{C_i}} = V_i - V_N = V_i,\ 1 \leq i \leq N-1,$$

where each V_i is a known constant. Without loss of generality, we consider from now the case $N = 2$ like represented in Figure 1. A voltage tension $V_1 = V$ is imposed between Γ_{C_1} and Γ_{C_2}, and for any oriented path γ from Γ_{C_2} to Γ_{C_1}, we have:

$$\int_\gamma \mathbf{E} \cdot \mathbf{dl} = V.$$

In order to model this problem, we fix one scalar potential $\alpha \in H^1(D_c)$ such that:

$$\alpha = \begin{cases} 1 & \text{on} & \Gamma_{C_1}, \\ 0 & \text{on} & \Gamma_{C_2}. \end{cases} \tag{1.6}$$

Consequently, we get :

$$\int_\gamma -\nabla\alpha \cdot \mathbf{dl} = \alpha\,|_{V_1} - \alpha\,|_{V_2} = 1.$$

From equation (1.1), an electrical scalar potential φ can be introduced so that the electrical field $\mathbf{E}$ takes the form:

$$\mathbf{E} = -\nabla\varphi - V\,\nabla\alpha.$$

Equation (1.2) associated to the constitutive law (1.3) and the boundary conditions (1.4)-(1.5) lead to the following electrical scalar potential formulation:

$$\begin{cases} \nabla\cdot\,(\sigma(\nabla\varphi + V\,\nabla\alpha)) & = & 0 & \text{in} & D_c, \\ \sigma\,(\nabla\varphi + V\,\nabla\alpha)\cdot\mathbf{n} & = & 0 & \text{on} & \Gamma_J, \\ \varphi & = & 0 & \text{on} & \Gamma_{C_1} \cup \Gamma_{C_2}. \end{cases} \tag{1.7}$$

As the scalar potential α is known and the voltage tension V is given, this problem corresponds to the classical Laplace problem with mixed boundary conditions, and our aim is to find φ such that:

$$\begin{cases} \nabla\cdot\,(\sigma\,\nabla\varphi) & = & -\nabla\cdot\,(\sigma\,V\,\nabla\alpha) & \text{in} & D_c, \\ \sigma\,\nabla\varphi\cdot\mathbf{n} & = & -\sigma\,V\,\nabla\alpha\cdot\mathbf{n} & \text{on} & \Gamma_J, \\ \varphi & = & 0 & \text{on} & \Gamma_{C_1} \cup \Gamma_{C_2}. \end{cases}$$

The derivation of residual a posteriori error estimators for this problem is easily available in [1, 18], and will consequently not be detailed here.

1.2 Current excitation case

1.2.1 Model description

In this second case, a current intensity I is imposed through each electric port. Namely, we have:

$$\int_{\Gamma_{C_i}} \mathbf{J} \cdot \mathbf{n} = I_i,\ 1 \leq i \leq N.$$

For convenience, we still suppose that $N = 2$. In that case, a current intensity enters the domain through Γ_{C_1} and leaves it through Γ_{C_2}, i.e.:

$$-\int_{\Gamma_{C_1}} \mathbf{J} \cdot \mathbf{n} = \int_{\Gamma_{C_2}} \mathbf{J} \cdot \mathbf{n} = I. \tag{1.8}$$

Compared to the previous voltage excitation case, the main difference lies in the fact that the voltage tension V is in general unknown. Therefore, both the scalar potential function φ and the voltage constant V are now unknowns in the problem. Meanwhile, the corresponding system is coupled in φ and V via (1.7). It should be noted that this system is not sufficient to determine the pair (φ, V). The following lemma yields that the current density is linked by the potential and the voltage by an integral condition.

Lemma 1.1. *The physical condition (1.8) is equivalent to:*

$$\int_{D_c} \sigma\left(\nabla\varphi + V\nabla\alpha\right)\cdot\nabla\alpha = I. \tag{1.9}$$

Proof. As $\nabla\cdot\left(\sigma(\nabla\varphi + V\,\nabla\alpha)\right) = 0$ in D_c, using an integration by parts we get:

$$\int_{D_c} \sigma\left(\nabla\varphi + V\nabla\alpha\right)\cdot\nabla\alpha = \int_{\Gamma} \sigma\left(\nabla\varphi + V\nabla\alpha\right)\cdot\mathbf{n}\,\alpha.$$

Taking into account the boundary conditions (1.6) and (1.7), we obtain:

$$\int_{D_c} \sigma(\nabla\varphi + V\nabla\alpha)\cdot\nabla\alpha = \int_{\Gamma_{C_1}} \sigma(\nabla\varphi + V\nabla\alpha)\cdot\mathbf{n} = -\int_{\Gamma_{C_1}} \mathbf{J}\cdot\mathbf{n},$$

so that (1.8) and (1.9) are equivalent. □

The complete system corresponding to the current excitation case is finally given by:

$$\left\{\begin{array}{rcll} \nabla\cdot\left(\sigma\left(\nabla\varphi + V\,\nabla\alpha\right)\right) & = & 0 & \text{in} \quad D_c, \\ \sigma\left(\nabla\varphi + V\,\nabla\alpha\right)\cdot\mathbf{n} & = & 0 & \text{on} \quad \Gamma_J, \\ \varphi & = & 0 & \text{on} \quad \Gamma_{C_1}\cup\Gamma_{C_2}, \\ \displaystyle\int_{D_c} \sigma\left(\nabla\varphi + V\nabla\alpha\right)\cdot\nabla\alpha & = & I. & \end{array}\right. \tag{1.10}$$

1.2.2 Variational formulation

In the following, for a given domain Ω, the $L^2(\Omega)$ (or $(L^2(\Omega))^3$) norm will be denoted by $\|\cdot\|_\Omega$. The usual norm and semi-norm of $H^1(\Omega)$ (or $(H^1(\Omega))^3$) will be denoted by $\|\cdot\|_{1,\Omega}$ and $|\cdot|_{1,\Omega}$ respectively. We first introduce the following functional space $W = H^1_{\Gamma_C}(D_c)\times\mathbb{R}$, where

$$H^1_{\Gamma_C}(D_c) = \left\{\varphi\in H^1(D_c) : \varphi = 0 \text{ on } \Gamma_C\right\}.$$

This Hilbert space W is equipped with the norm:

$$\|(\varphi, V)\|_W^2 = \|\varphi\|_{1,D_c}^2 + V^2\|\alpha\|_{1,D_c}^2.$$

It follows from (1.10) that for any $(\varphi', V')\in W$

$$\begin{array}{rcl} \displaystyle\int_{D_c} \sigma\left(\nabla\varphi + V\,\nabla\alpha\right)\cdot\nabla\varphi' & = & 0, \\ \displaystyle\int_{D_c} \sigma\left(\nabla\varphi + V\,\nabla\alpha\right)\cdot\nabla\alpha\, V' & = & I\,V', \end{array} \tag{1.11}$$

where (φ, V) is a solution to (1.10). Taking the sum of these two terms, we can consequently derive the corresponding weak formulation which consists to find $(\varphi, V)\in W$ such that, for all $(\varphi', V')\in W$, we have

$$a((\varphi, V), (\varphi', V')) = l((\varphi', V')), \tag{1.12}$$

where a and l are respectively the bilinear and linear forms defined by:

$$\begin{array}{rcl} a((\varphi, V), (\varphi', V')) & = & \displaystyle\int_{D_c} \sigma\left(\nabla\varphi + V\,\nabla\alpha\right)\cdot\left(\nabla\varphi' + V'\,\nabla\alpha\right), \\ l((\varphi', V')) & = & I\,V'. \end{array}$$

The only real difficulty to ensure the well-posedness of this problem by the Lax-Milgram Lemma is to prove that the bilinear form a is coercive on W, which is obtained by the following lemma. Let us note that in the remainder of the paper, C denotes a generic positive constant which is not always the same and depends on the context.

Lemma 1.2. *The bilinear form a is coercive on W, namely there exists $C > 0$ such that:*

$$a((\varphi, V), (\varphi, V)) \geq \ C \, \|(\varphi, V)\|_W^2 \text{ for all } (\varphi, V) \in W.$$

Proof. As we know that $\varphi \in H^1_{\Gamma_C}(D_c)$ and $\alpha \in H^1_{\Gamma_{C_2}}(D_c)$, thanks to the Friedrichs' inequality there exist two constant C_1 and C_2 such that:

$$\|\varphi\|_{D_c} \leq C_1 \, \|\nabla\varphi\|_{D_c},$$

and

$$\|\alpha\|_{D_c} \leq C_2 \, \|\nabla\alpha\|_{D_c}.$$

So we only need to prove that:

$$\int_{D_c} \left|\nabla\varphi + V\nabla\alpha\right|^2 \geq C(\|\nabla\varphi\|_{D_c}^2 + V^2\|\nabla\alpha\|_{D_c}^2) \text{ for all } (\varphi, V) \in W. \tag{1.13}$$

We do it by a contradiction argument.

Let us suppose that (1.13) is false. Then, there exists one sequence $\{(\varphi_n, V_n)\}$ such that

$$\|\nabla\varphi_n\|_{D_c}^2 + V_n^2\|\nabla\alpha\|_{D_c}^2 = 1 \tag{1.14}$$

and

$$\int_{D_c} \left|\nabla\varphi_n + V_n\nabla\alpha\right|^2 \to 0 \text{ when } n \to \infty. \tag{1.15}$$

Due to the Friedrichs' inequality, there exists $C > 0$ such that:

$$\|\varphi_n\|_{D_c} \leq C \, \|\nabla\varphi_n\|_{Dc}.$$

Consequently, by using (1.14) we get:

$$\|\varphi_n\|_{H^1(D_c)} \leq \sqrt{1 + C^2}\|\nabla\varphi_n\|_{D_c} \leq \sqrt{1 + C^2}$$

and

$$|V_n| \leq 1/\sqrt{\|\nabla\alpha\|_{D_c}},$$

i.e., $(\varphi_n)_n$ and $(V_n)_n$ are bounded sequences in $H^1(D_c)$ and $\mathbb{R}$ respectively. So by the compact embedding of $H^1(D_c)$ into $L^2(D_c)$, there exists at least one subsequence $((\varphi_{n_k}, V_{n_k}))_k \subset W$ such that:

$$\varphi_{n_k} \to \varphi \text{ in } L^2(D_c) \tag{1.16}$$

and

$$V_{n_k} \to V.$$

Furthermore,

$$\varphi_{n_k} + V_{n_k}\alpha \to \varphi + V\alpha \text{ in } L^2(D_c), \tag{1.17}$$

From (1.15) and (1.17), we obtain

$$\varphi_{n_k} + V_{n_k}\alpha \to \varphi + V\alpha \text{ in } H^1(D_c),$$

and $\nabla(\varphi + V\alpha) = 0$, so that $V = 0$ and $\varphi = 0$ because φ and α are linearly independent in $H^1(D_c)$. On the other hand, we have

$$\begin{aligned} \|\nabla\varphi_{n_k}\|_{D_c}^2 \quad &= \|\nabla\varphi_{n_k} + V_{n_k}\nabla\alpha - V_{n_k}\nabla\alpha\|_{D_c}^2 \\ &\leq 2(\|\nabla\varphi_{n_k} + V_{n_k}\nabla\alpha\|_{D_c}^2 + V_{n_k}^2\|\nabla\alpha\|_{D_c}^2). \end{aligned}$$

Hence, we conclude that

$$\|\nabla\varphi_n\|_{D_c}^2 + V_n^2\|\nabla\alpha\|_{D_c}^2 \to 0,$$

which is in contradiction with (1.14). □

1.2.3 Finite Element approximation

The domain D_c is now discretized by a conforming mesh $\mathcal{T}_h$ made of tetrahedra. The faces of $\mathcal{T}_h$ are denoted by F and its edges by E. Let us note h_T the diameter of T and ρ_T the diameter of its largest inscribed ball. We suppose that for any element T, the ratio h_T/ρ_T is bounded by a constant independant of T and of the mesh size $h = \max_{T\in\mathcal{T}_h} h_T$. The set of faces (resp. edges and nodes) of the triangulation is denoted by $\mathcal{F}$ (resp. $\mathcal{E}$ and $\mathcal{N}$), and we denote h_F the diameter of the face F. The set of internal faces (resp. internal edges and internal nodes) to D_c is denoted by $\mathcal{F}_{int}$ (resp. $\mathcal{E}_{int}$ and $\mathcal{N}_{int}$). The coefficient σ arising in (1.11) is moreover supposed to be constant in each tetrahedron of the mesh, and we will set $\sigma_T = \sigma_{|T}$ for all $T \in \mathcal{T}_h$.

We define the approximation space $W_h = \Theta_h(D_c) \times \mathbb{R}$ with:

$$\Theta_h(D_c) = \Big\{\xi_h \in H^1_{\Gamma_C}(D_c); \xi_{h|T} \in \mathbb{P}_1(T)\ \forall\, T \in \mathcal{T}_h\Big\}.$$

Hence, the corresponding discrete variational formulation is given by:

Find $(\varphi_h, V_h) \in W_h$ such that, for all $(\varphi'_h, V'_h) \in W_h$ we have

$$a((\varphi_h, V_h), (\varphi'_h, V'_h)) = l((\varphi'_h, V'_h)). \tag{1.18}$$

Since $W_h \subset W$, the discrete problem (1.18) is well-posed and admits a unique solution $(\varphi_h, V_h) \in W_h$.

1.2.4 A posteriori residual error analysis

Let us denote $[u]_F$ the jump of the quantity u over a face F of the mesh. For each $T \in \mathcal{T}_h$, the local error indicator η_T is defined by :

$$\eta_T^2 = \eta_{T,1}^2 + \sum_{F\subset\partial T, F\in\mathcal{F}_{int}} \eta_{F,1}^2 + \sum_{F\subset\partial T, F\subset\Gamma_J} \eta_{F,2}^2,$$

with :

$$\begin{aligned}
\eta_{T,1} &= h_T \left\|\nabla\cdot\left(\sigma(\nabla\varphi_h + V_h\nabla\alpha)\right)\right\|_T ,\\
\eta_{F,1} &= h_F^{1/2} \left\|[\sigma\left(\nabla\varphi_h + V_h\nabla\alpha\right)\cdot \mathbf{n}]_F\right\|_F ,\\
\eta_{F,2} &= h_F^{1/2} \left\|\sigma\left(\nabla\varphi_h + V_h\nabla\alpha\right)\cdot \mathbf{n}\right\|_F .
\end{aligned}$$

Then, the global error indicator η is defined by :

$$\eta^2 = \sum_{T\in\mathcal{T}_h} \eta_T^2.$$

Theorem 1.3. *(Reliability of the estimator). There exists a constant $C_{up} \geq 0$, which does not depend on φ, φ_h, V, V_h or h, such that :*

$$\left(\int_{D_c} \sigma\left|\nabla((\varphi + V\alpha) - (\varphi_h + V_h\alpha))\right|^2\right)^{\frac{1}{2}} \leq C_{up}\,\eta. \tag{1.19}$$

Proof. The proof is close to the one devoted to the case where a current is imposed in the domain itself [8]. In particular, the needed Helmholtz decomposition of the error is the same. Consequently, we only recall here the main steps to be followed and underline the main differences.

First, for any $(\varphi', V') \in W$, the residual form associated with (1.12) is defined by :

$$r((\varphi', V')) = l((\varphi', V')) - a((\varphi_h, V_h), (\varphi', V')).$$

As $W_h \subset W$, setting $e_\varphi := \varphi - \varphi_h$ and $e_V := V - V_h$, we obtain :

$$r((e_\varphi, e_V)) = a((e_\varphi, e_V), (e_\varphi, e_V)) = \left\|\sqrt{\sigma}(\nabla e_\varphi + e_V\nabla\alpha)\right\|^2_{D_c} . \tag{1.20}$$

Now, we write

$$r((e_\varphi, e_V)) = r((e_\varphi, 0)) + r((0, e_V)) = r((e_\varphi, 0)), \tag{1.21}$$

since $r((0, e_V)) = 0$ owing to the fact that $(0, e_V)$ belongs to W_h.

From the classic residual based estimator method for the Laplace problem [1, 18], there exists a constant $C > 0$ such that:

$$r((e_\varphi, 0)) \leq C \cdot \eta \cdot \left\| \sqrt{\sigma}\, \nabla e_\varphi \right\|_{D_c} \tag{1.22}$$

Since $(e_\varphi, e_V) \in W$, from (1.13), we obtain

$$\|\sqrt{\sigma}\nabla e_\varphi\|_{D_c} \leq \max_K\{\sqrt{\sigma_K}\} \left\|\nabla e_\varphi + e_V \nabla\alpha\right\|_{D_c} \leq \frac{\max_K\{\sqrt{\sigma_K}\}}{\min_{\{}\sqrt{\sigma_K}\}} \left\|\sqrt{\sigma}(\nabla e_\varphi + e_V\nabla\alpha)\right\|_{D_c}. \tag{1.23}$$

Finally, the relations (1.20), (1.21), (1.22) and (1.23) yield (1.19). □

2 Eddy current problem

Let us consider now an open simply connected domain $D \subset \mathbb{R}^3$, with a Lipschitz boundary ∂D, as shown in Figure 2. Here the conductor domain D_c is included in the domain D. We denote D_{nc} the non-conductor domain such that $D = D_c \cup D_{nc}$.

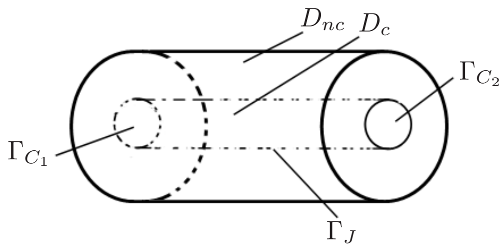


Figure 2: Configuration of the domain, example for two electric ports ($N = 2$).

Let us denote by $\Gamma_C = \partial D_c \cap \partial D$ and $\Gamma_J = \partial D_c \backslash \Gamma_C$, where Γ_C is supposed to be composed of at least two disjoint connected components like in the electrostatic case : $\Gamma_C = \bigcup_{i=1}^{N} \Gamma_{C_i}$, $N \geq 2$. The eddy current problem corresponds to the time harmonic approximation of Maxwell equations, given by:

$$\begin{aligned} \nabla\times \mathbf{E} &= -j\omega\mathbf{B} \text{ in } D_c, \\ \nabla\times \mathbf{H} &= \mathbf{J} \text{ in } D_c, \\ \nabla\cdot \mathbf{B} &= 0 \text{ in } D, \end{aligned}$$

where $\mathbf{E}$ and $\mathbf{H}$ are respectively the electric and the magnetic fields, $\mathbf{B}$ is the magnetic flux density and $\mathbf{J}$ is the current flux density. This current flux density can be decomposed into two parts:

$$\mathbf{J} = \mathbf{J}_{ind} + \mathbf{J}_s,$$

where $\mathbf{J}_s$ is a known distribution current density generally generated by a coil and $\mathbf{J}_{ind}$ represents the eddy current.

The material constitutive laws are given by:

$$\mathbf{B} = \mu\mathbf{H},$$

$$\mathbf{J} = \sigma\mathbf{E},$$

where μ represents the magnetic permeability of the material.

The associated boundary conditions are $\mathbf{B}\cdot\mathbf{n}=0$ on $\partial D\backslash\Gamma_C$, $\mathbf{E}\times\mathbf{n}=\mathbf{0}$ on Γ_C and $\mathbf{J}\cdot\mathbf{n}=0$ on Γ_J.

In [8], we solved the problem with $\mathbf{J}_s\neq\mathbf{0}$, corresponding to a known distribution current density generated by a coil in the non-conductor domain. As discussed in the electrostatic problem, we consider here $\mathbf{J}_s=\mathbf{0}$, with the similar voltage and current excitation cases.

2.1 Voltage excitation case

From $\nabla\cdot\mathbf{B}=0$, a vector potential $\mathbf{A}$ is introduced such that $\mathbf{B}=\nabla\times\mathbf{A}$ in D, with $\mathbf{A}\times\mathbf{n}=\mathbf{0}$ on ∂D. This boundary condition for $\mathbf{A}$ guarantees $\mathbf{B}\cdot\mathbf{n}=0$ on ∂D. With similar arguments as for the electrostatic case, the electrical field $\mathbf{E}$ in the conductor domain can been written as:

$$\mathbf{E}=-j\omega\mathbf{A}-\nabla\varphi-V\nabla\alpha.$$

So the system to solve is given by:

$$\left\{\begin{array}{rcll}\nabla\times\left(\dfrac{1}{\mu}\nabla\times\mathbf{A}\right)+\sigma(j\omega\mathbf{A}+\nabla\varphi+V\,\nabla\alpha) & = & \mathbf{0} & \text{in}\quad D,\\[2ex] \nabla\cdot(\sigma(j\omega\mathbf{A}+\nabla\varphi+V\,\nabla\alpha)) & = & 0 & \text{in}\quad D_c,\end{array}\right.$$

with *ad hoc* boundary conditions on $\mathbf{A}$ and φ. As V is a known (scalar) constant, the system can be written as:

$$\left\{\begin{array}{rcll}\nabla\times\left(\dfrac{1}{\mu}\nabla\times\mathbf{A}\right)+\sigma(j\omega\mathbf{A}+\nabla\,\varphi) & = & -\sigma V\,\nabla\alpha & \text{in}\quad D,\\[2ex] \nabla\cdot(\sigma(j\omega\mathbf{A}+\nabla\varphi)) & = & -\nabla\cdot(\sigma V\,\nabla\alpha) & \text{in}\quad D_c.\end{array}\right.$$

This system is similar to the $\mathbf{A}/\varphi$ formulation discussed in [8]. The only difference consists in the boundary condition on φ, but it does not induce any significantly different analysis. Consequently, we do not recall the corresponding results here.

2.2 Current excitation case

2.2.1 Model description

As mentioned in the electrostatic problem, in the current excitation case, the voltage potential scalar V is unknown. With one imposed current density I through the electric port as defined in (1.8), we get the additional integral equation:

$$\int_{D_c}\sigma(j\omega\mathbf{A}+\nabla\varphi+V\,\nabla\alpha)\cdot\nabla\alpha=I.$$

The system to solve is consequently given by:

$$\left\{\begin{array}{rcll}\nabla\times\left(\dfrac{1}{\mu}\nabla\times\mathbf{A}\right)+\sigma(j\omega\mathbf{A}+\nabla\varphi+V\,\nabla\alpha) & = & \mathbf{0} & \text{in}\quad D,\\[2ex] \nabla\cdot(\sigma(j\omega\mathbf{A}+\nabla\varphi+V\,\nabla\alpha)) & = & 0 & \text{in}\quad D_c.\\[2ex] \displaystyle\int_{D_c}\sigma(j\omega\mathbf{A}+\nabla\varphi+V\,\nabla\alpha)\cdot\nabla\alpha & = & I, & \end{array}\right. \tag{2.1}$$

with the boundary conditions on $\mathbf{A}$ and φ given by:

$$\left\{\begin{array}{rll}\mathbf{A}\times\mathbf{n}=\mathbf{0} & \text{on} & \partial D,\\[2ex] \varphi=0 & \text{on} & \Gamma_{C_1}\cup\Gamma_{C_2},\\[2ex] \sigma\,(j\omega\mathbf{A}+\nabla\varphi+V\,\nabla\alpha)\cdot\mathbf{n}=0 & \text{on} & \Gamma_J.\end{array}\right. \tag{2.2}$$

2.2.2 Variational formulation

We suppose in the following that $\sigma \in L^\infty(D)$, $\sigma = 0$ in D_{nc}, and that there exists a positive constant $\sigma_0 \in \mathbb{R}_+^*$ such that $\sigma > \sigma_0$ in D_c. The variational formulation of (2.1)-(2.2) is:

Find $(\mathbf{A}, \varphi, V) \in Q$, such that for all $(\mathbf{A}', \varphi', V') \in Q$, we have

$$\int_D \frac{1}{\mu} \nabla\times\mathbf{A} \cdot (\nabla\times\overline{\mathbf{A}'}) + \int_{D_c} \sigma(j\omega\mathbf{A} + \nabla\varphi + V\nabla\alpha) \cdot \overline{\mathbf{A}'} = \mathbf{0}, \tag{2.3}$$

$$\int_{D_c} \sigma(j\omega\mathbf{A} + \nabla\varphi + V\nabla\alpha) \cdot \nabla\overline{\varphi'} = 0, \tag{2.4}$$

$$\int_{D_c} \sigma(j\omega\mathbf{A} + \nabla\varphi + V\nabla\alpha) \cdot \nabla\alpha V' = I\, V', \tag{2.5}$$

where $Q = X^0(D) \times W$ and

$$\begin{aligned} X(D) &= H_0(\mathrm{curl}, \mathrm{D}) = \left\{\mathbf{A} \in \mathrm{L}^2(\mathrm{D}); \nabla\times\mathbf{A} \in \mathrm{L}^2(\mathrm{D}) \text{ and } \mathbf{A}\times\mathbf{n} = \mathbf{0} \text{ on } \partial\mathrm{D}\right\}, \\ X^0(D) &= \left\{\mathbf{A} \in X(D), (\mathbf{A}, \nabla\psi) = 0, \forall\psi \in H_0^1(D)\right\}. \end{aligned}$$

It can be seen that the gauge condition (the coulomb one $\nabla\cdot\mathbf{A} = 0$) is taken into account in the definition of the space $X^0(D)$ itself. Now the space $X(D)$ is equipped with its usual norm:

$$\|\mathbf{A}\|^2_{X(D)} = \|\mathbf{A}\|^2_D + \|\nabla\times\mathbf{A}\|^2_D.$$

And the natural norm $\|\cdot\|_Q$ associated with the Hilbert space Q is given by:

$$\|(\mathbf{A}, \varphi, V)\|^2_Q = \|\mathbf{A}\|^2_{X(D)} + \|\varphi\|^2_{1,D_c} + V^2\|\alpha\|^2_{1,D_c}.$$

From (2.3),(2.4),(2.5), we propose an equivalent variational formulation consisting in finding $(\mathbf{A}, \varphi, V) \in Q$ such that:

$$a((\mathbf{A}, \varphi, V), (\mathbf{A}', \varphi', V')) = l((\mathbf{A}', \varphi', V')),\ \forall(\mathbf{A}', \varphi', V') \in Q, \tag{2.6}$$

where a and l are respectively the bilinear and linear forms defined by:

$$\begin{aligned} a((\mathbf{A}, \varphi, V), (\mathbf{A}', \varphi', V')) = &\int_D \frac{1}{\mu} \nabla\times\mathbf{A} \cdot \nabla\times\overline{\mathbf{A}'} \\ &-\frac{j}{\omega} \int_{D_c} \sigma(j\omega\mathbf{A} + \nabla\varphi + V\nabla\alpha) \cdot (\overline{j\omega\mathbf{A}' + \nabla\varphi' + V'\nabla\alpha}), \\ l((\mathbf{A}', \varphi', V')) = &-\frac{j}{\omega} I\, V'. \end{aligned}$$

The next lemma implies the coercivity of the bilinear form a.

Lemma 2.1. *The bilinear form $\sqrt{2}e^{j\frac{\pi}{4}} a$ is coercive on Q. More precisely, there exists $C > 0$ such that:*

$$|\sqrt{2}e^{j\frac{\pi}{4}} a\left((\mathbf{A}, \varphi, V), (\mathbf{A}, \varphi, V)\right)| \geq C\|(\mathbf{A}, \varphi, V)\|^2_Q, \qquad \forall\, (\mathbf{A}, \varphi, V) \in Q.$$

Proof. First, let us notice that :

$$\Re\left[\sqrt{2}e^{j\frac{\pi}{4}} a\left((\mathbf{A}, \varphi, V), (\mathbf{A}, \varphi, V)\right)\right] = \int_D \frac{1}{\mu} |\nabla\times\mathbf{A}|^2 + \int_{D_c} \frac{\sigma}{\omega} |j\omega\mathbf{A} + \nabla\varphi + V\nabla\alpha|^2.$$

Our aim is to prove that there exists $C > 0$ such that for all $(\mathbf{A}, \varphi, V) \in Q$,

$$\int_D \frac{1}{\mu} |\nabla \times \mathbf{A}|^2 + \int_{D_c} \frac{\sigma}{\omega} |j\omega \mathbf{A} + \nabla \varphi + V \nabla \alpha|^2 \geq C \, \|(\mathbf{A}, \varphi, V)\|_Q^2. \tag{2.7}$$

For any $\eta > 1$ we have:

$$\|j\omega \mathbf{A} + \nabla \varphi + V \nabla \alpha\|_{D_c}^2 + \|\nabla \times \mathbf{A}\|_D^2 \quad \geq \quad (1-\eta)\|\omega \mathbf{A}\|_{D_c}^2 + \frac{(\eta-1)}{\eta}\|\nabla \varphi + V \nabla \alpha\|_{D_c}^2 + \|\nabla \times \mathbf{A}\|_D^2.$$

Consequently, from (1.13) we derive:

$$\|j\omega \mathbf{A} + \nabla \varphi + V \nabla \alpha\|_{D_c}^2 + \|\nabla \times \mathbf{A}\|_D^2 \geq (1-\eta)\|\omega \mathbf{A}\|_{D_c}^2 + C\,\frac{(\eta-1)}{\eta}\Big(\|\nabla \varphi\|_{D_c}^2 + V^2\|\nabla \alpha\|_{D_c}^2\Big) + \|\nabla \times \mathbf{A}\|_D^2.$$

As $\mathbf{A} \in X^0(D)$ and $1 - \eta < 0$, the Friedrichs-Poincaré inequality $\|\mathbf{A}\|_D \leq \tilde{C}\, \|\nabla \times \mathbf{A}\|_D$ yields:

$$\begin{aligned}
\|j\omega \mathbf{A} + \nabla \varphi + V \nabla \alpha\|_{D_c}^2 + \|\nabla \times \mathbf{A}\|_D^2 \quad &\geq \quad (1-\eta)\,\omega^2\, \tilde{C}^2\|\nabla \times \mathbf{A}\|_D + C\,\frac{(\eta-1)}{\eta}\Big(\|\nabla \varphi\|_{D_c}^2 + V^2\|\nabla \alpha\|_{D_c}^2\Big) + \|\nabla \times \mathbf{A}\|_D^2 \\
&= \quad ((1-\eta)\,\omega^2\tilde{C}^2 + 1)\|\nabla \times \mathbf{A}\|_D^2 + C\,\frac{(\eta-1)}{\eta}\Big(\|\nabla \varphi\|_{D_c}^2 + V^2\|\nabla \alpha\|_{D_c}^2\Big)
\end{aligned}$$

Choosing now η such that $1 < \eta < 1 + \dfrac{1}{\tilde{C}^2\omega^2}$ and using once again the Friedrichs-Poincaré inequality $\|\mathbf{A}\|_D \leq \tilde{C}\, \|\nabla \times \mathbf{A}\|_D$ lead to the coercivity of the bilinear form. $\square$

Theorem 2.2. The weak formulation (2.6) admits a unique solution $(\mathbf{A}, \varphi, V) \in Q$.

Proof. The sesquilinear form $\sqrt{2}e^{j\frac{\pi}{4}}a$ is continuous on $Q \times Q$ and coercive on Q by Lemma 2.1. Hence, Lax-Milgram's lemma ensures existence and uniqueness of a solution $(\mathbf{A}, \varphi, V) \in Q$ to (2.6). $\square$

Now, we derive a very important result, coming from the Helmholtz decomposition of $X(D)$.

Lemma 2.3. *Let $(\mathbf{A}, \varphi, V) \in Q$ be the unique solution of (2.6). Then for all $(\mathbf{A}', \varphi', V') \in X(D) \times W$, we have:*

$$a((\mathbf{A}, \varphi, V), (\mathbf{A}', \varphi', V')) = l((\mathbf{A}', \varphi', V')).$$

Proof. The key point is the Helmholtz decomposition on the space $X(D)$. For any $\mathbf{A}' \in X(D)$, we have

$$\mathbf{A}' = \Psi + \nabla \tau,$$

with $\Psi \in X^0(D)$ and $\tau \in H_0^1(D)$. The conclusion then follows by noticing that $a((\mathbf{A}, \varphi, V), (\nabla \tau, 0, 0)) = l((\nabla \tau, 0, 0)) = 0$, for all $\tau \in H_0^1(D)$. $\square$

2.2.3 Finite Element approximation

The approximation space Q_h is defined by $Q_h = X_h^0(D) \times W_h$, where:

$$X_h(D) = X(D) \cap \mathcal{ND}_1(D, \mathscr{T}_h) = \Big\{\mathbf{A}_h \in X(D); \mathbf{A}_{h|T} \in \mathcal{ND}_1(T),\ \forall\, T \in \mathscr{T}_h\Big\},$$

$$\mathcal{ND}_1(T) = \left\{\mathbf{A}_h : \begin{array}{ccc} T & \longrightarrow & \mathbb{C}^3 \\ \mathbf{x} & \longrightarrow & \mathbf{a} + \mathbf{b} \times \mathbf{x} \end{array}, \mathbf{a}, \mathbf{b} \in \mathbb{C}^3\right\},$$

$$\Theta_h^0(D) = \Big\{\xi_h \in H_0^1(D); \xi_{h|T} \in \mathbb{P}_1(T)\ \forall\, T \in \mathscr{T}_h\Big\},$$

$$X_h^0(D) = \Big\{\mathbf{A}_h \in X_h; (\mathbf{A}_h, \nabla \xi_h) = 0\ \forall\, \xi_h \in \Theta_h^0\Big\},$$

and the discretized weak formulation is given by :

Find $(\mathbf{A}_h, \varphi_h, V_h) \in Q_h$ such that:

$$a\left((\mathbf{A}_h, \varphi_h, V_h), (\mathbf{A}_h', \varphi_h', V_h)\right) = l\left((\mathbf{A}_h', \varphi_h', V_h)\right),\ \forall (\mathbf{A}_h', \varphi_h', V_h') \in Q_h.$$

The uniqueness of the solution is obtained by the similar arguments discussed in the continuous case using the discrete Friedrichs-Poincaré inequality.

2.2.4 A posteriori residual error analysis

We aim here to obtain some a posteriori error estimators in the current excitation case.

For each $T \in \mathscr{T}_h$, the local error indicator η_T is defined by :

$$\eta_T^2 = \eta_{T,1}^2 + \eta_{T,2}^2 + \sum_{F \subset \partial T, F \in \mathcal{F}_{int}} (\eta_{F,1}^2 + \eta_{F,2}^2), \tag{2.8}$$

with :

$$\begin{aligned}
\eta_{T,1} &= h_T \left\| \nabla\times \left(\frac{1}{\mu} \nabla\times \mathbf{A}_h \right) + \sigma(j\omega \mathbf{A}_h + \nabla\varphi_h + V_h \nabla\alpha) \right\|_T , \\
\eta_{T,2} &= h_T \left\| \nabla\cdot(\sigma(j\omega \mathbf{A}_h + \nabla\varphi_h + V_h \nabla\alpha)) \right\|_T , \\
\eta_{F,1} &= h_F^{1/2} \left\| \left[\frac{1}{\mu} \nabla\times \mathbf{A}_h \times \mathbf{n} \right]_F \right\|_F , \\
\eta_{F,2} &= h_F^{1/2} \left\| [\sigma(j\omega \mathbf{A}_h + \nabla\varphi_h + V_h \nabla\alpha) \cdot \mathbf{n}]_F \right\|_F ,
\end{aligned}$$

Then, the global error indicator η is defined by :

$$\eta^2 = \sum_{T \in \mathscr{T}_h} \eta_T^2. \tag{2.9}$$

Before giving the estimates of the global error, we need the following lemma.

Lemma 2.4. *Let us define the errors respectively on $\mathbf{A}$, φ and V by:*

$$\mathbf{e_A} = \mathbf{A} - \mathbf{A}_h, \tag{2.10}$$

$$\mathbf{e}_\varphi = \varphi - \varphi_h, \tag{2.11}$$

$$\mathbf{e}_V = V - V_h. \tag{2.12}$$

Then, there exists a constant $C > 0$ such that

$$\int_D \frac{1}{\mu} |\nabla\times \mathbf{e_A}|^2 + \int_{D_c} \frac{\sigma}{\omega} |j\omega \mathbf{e_A} + \nabla e_\varphi + e_V \nabla\alpha|^2 \geq C \left(\|j\omega \mathbf{e_A} + \nabla e_\varphi\|_{D_c}^2 + \|\nabla\times \mathbf{e_A}\|_D^2 \right) \tag{2.13}$$

Proof. In order to prove (2.13), we are going to show that:

$$\int_D \frac{1}{\mu} |\nabla\times \mathbf{e_A}|^2 + \int_{D_c} \frac{\sigma}{\omega} |j\omega \mathbf{e_A} + \nabla e_\varphi + e_V \nabla\alpha|^2 \geq C \left(\|j\omega \mathbf{e_A} + \nabla e_\varphi\|_{D_c}^2 + \|\nabla\times \mathbf{e_A}\|_D^2 + e_V^2 \|\nabla\alpha\|_{D_c}^2 \right).$$

Since $\mathbf{e_A} \in X(D)$, using the Helmholtz decomposition we have:

$$\mathbf{e_A} = \mathbf{w} + \nabla\varphi_0,$$

where $\mathbf{w} \in X^0(D)$ and $\varphi_0 \in H_0^1(D)$. As $(\mathbf{w}, (e_\varphi + j\omega\varphi_0), e_V) \in Q$, from (2.7) we get:

$$\int_D \frac{1}{\mu} |\nabla\times \mathbf{w}|^2 + \int_{D_c} \frac{\sigma}{\omega} |j\omega \mathbf{w} + \nabla(e_\varphi + j\omega\varphi_0) + e_V \nabla\alpha|^2 \geq C(\|\nabla\times \mathbf{w}\|_D^2 + \|\nabla(e_\varphi + j\omega\varphi_0)\|_{D_c}^2 + e_V^2 \|\nabla\alpha\|_{D_c}^2).$$

Then, the standard triangular inequality and the Poincaré Friedrichs inequality $\|\mathbf{w}\|_D \leq C\, \|\nabla\times \mathbf{w}\|_D$ yields:

$$\begin{aligned}
\|\nabla\times \mathbf{w}\|_D^2 + \|\nabla(e_\varphi + j\omega\varphi_0)\|_{D_c}^2 + e_V^2 \|\nabla\alpha\|_{D_c}^2 &\geq C_2(\|j\omega \mathbf{w} + j\omega\nabla\varphi_0 + \nabla e_\varphi\|_{D_c}^2 + \|\nabla\times \mathbf{w}\|_D^2 + e_V^2 \|\nabla\alpha\|_{D_c}^2) \qquad (2.14) \\
&= C_2 \left(\|j\omega \mathbf{e_A} + \nabla e_\varphi\|_{D_c}^2 + \|\nabla\times \mathbf{e_A}\|_D^2 + e_V^2 \|\nabla\alpha\|_{D_c}^2 \right).
\end{aligned}$$

$\square$

Theorem 2.5. *(Reliability of the estimator). There exists $C_{up} \geq 0$ such that:*

$$\left(\int_D \frac{1}{\mu} \mid \nabla\times\mathbf{e_A} \mid^2 + \int_{D_c} \frac{\sigma}{\omega} \mid j\omega\mathbf{e_A} + \nabla e_\varphi + e_V \nabla\alpha \mid^2\right)^{1/2} \leq C_{up}\,\eta. \tag{2.15}$$

Proof. First, for any $(\mathbf{A}', \varphi', V') \in Q$, the residual form associated with (2.6) is defined by :

$$r((\mathbf{A}', \varphi', V')) = l((\mathbf{A}', \varphi', V')) - a((\mathbf{A}_h, \varphi_h, V_h), (\mathbf{A}', \varphi', V')).$$

Using the definition of the errors (2.10), (2.11) and (2.12), we obtain :

$$\mathcal{R}(\sqrt{2}e^{j\frac{\pi}{4}} r((\mathbf{e_A}, e_\varphi, e_V))) = \int_D \frac{1}{\mu} \mid \nabla\times\mathbf{e_A} \mid^2 + \int_{D_c} \frac{\sigma}{\omega} \mid j\omega\mathbf{e_A} + \nabla e_\varphi + e_V \nabla\alpha \mid^2 .$$

Now, we write

$$r((\mathbf{e_A}, e_\varphi, e_V)) = r((\mathbf{e_A}, e_\varphi, 0)) + r((\mathbf{0}, 0, e_V)) = r((\mathbf{e_A}, e_\varphi, 0)),$$

since $r((\mathbf{0}, 0, e_V)) = 0$ for $(\mathbf{0}, 0, e_V) \in Q_h$. As shown in Theorem 4.1 of [8], we get

$$r((\mathbf{e_A}, e_\varphi, 0)) \leq C \cdot \eta \cdot ((\|j\omega\mathbf{e_A} + \nabla e_\varphi\|^2_{D_c} + \|\nabla\times\mathbf{e_A}\|^2_D))^{1/2}.$$

From Lemma 2.4, we derive:

$$(\|j\omega\mathbf{e_A} + \nabla e_\varphi\|^2_{D_c} + \|\nabla\times\mathbf{e_A}\|^2_D) \leq C \left(\int_D \frac{1}{\mu}|\nabla\times\mathbf{e_A}|^2 + \int_{D_c} \frac{\sigma}{\omega}|j\omega\mathbf{e_A} + \nabla e_\varphi + e_V\nabla\alpha|^2\right).$$

This completes the proof of the lemma.

□

3 Numerical experiments

To illustrate our theoretical developments, we consider in this section the example of an eddy current problem in the current excitation case (see section 2.2), for which the analytical solution is known. The computations are performed with the use of the software Code_Carmel3D of EDF R& D.

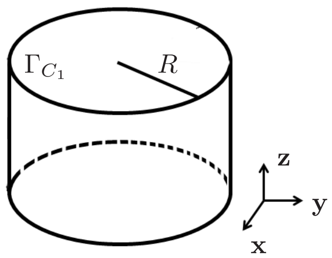


Figure 3: Cylindrical conductor.

We consider a cylindrical conductor of radius R and height R (see Figure 3), where $R = 1\,m$. The conductivity of the conductor is equal to $1\,MS/m$ and the relative magnetic permeability is equal to $1\,H/m$. We impose a total current $I = 1\,A$ on each cross section of the conductor, like explained in subsection 1.2.1. The frequency $f = \omega/2\pi$ of the problem is equal to $f = 50Hz$.

Let us define r the distance from any point (x, y, z) to the axis of the cylinder (i.e. $r = \sqrt{x^2 + y^2}$). The analytical solution to (2.1) for the eddy current $\mathbf{J}_{ind} = \sigma(j\omega\mathbf{A} + \nabla\varphi + V\,\nabla\alpha)$ and the magnetic flux density $\mathbf{B} = \nabla\times\mathbf{A}$ is given by [17]:

$$\mathbf{J}_{ind} = (0, 0, J(r)) \text{ and } \mathbf{B} = \left(-\frac{y}{r}, \frac{x}{r}, 0\right) B(r)$$

with

$$J(r) = C_0\ J_0(ikr), \tag{3.1}$$

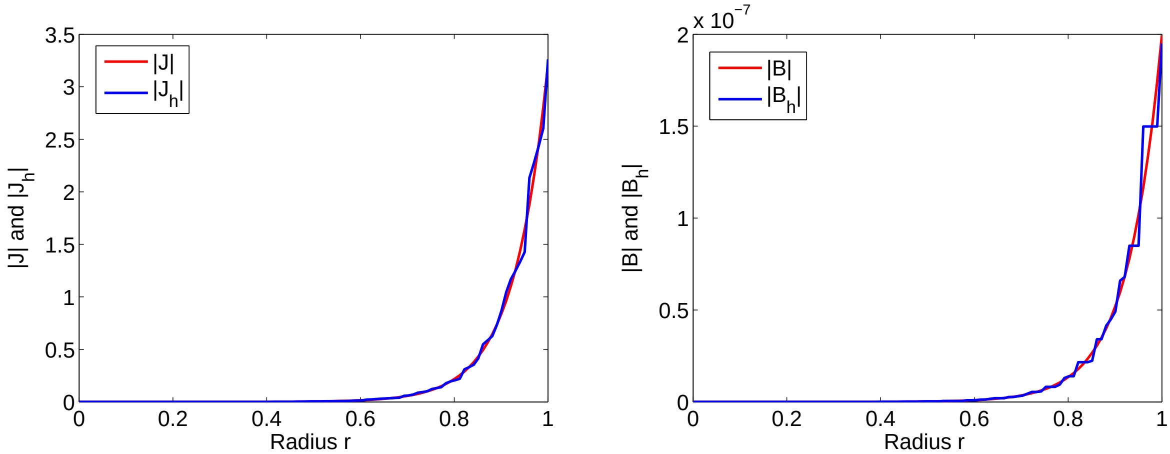
$$B(r) = \frac{ik}{\omega\sigma} C_0\ J_1(ikr), \tag{3.2}$$

where $k^2 = i\omega\sigma\mu$, and J_0, J_1 are the well-known Bessel functions.

The constant C_0 arising in (3.1) and (3.2) can be determined by:

$$C_0 = \frac{I}{\displaystyle\int_0^R 2\pi r J_0(ikr)dr}.$$

First, the modulus of the analytical solution (namely $|\mathbf{J}_{ind}|$ and $|\mathbf{B}|$) and of the numerical one obtained by code_Carmel3D (namely $|\mathbf{J}_{indh}|$ and $|\mathbf{B}_h|$) for different values of r are compared in Figure 4, which shows a very good correspondance.

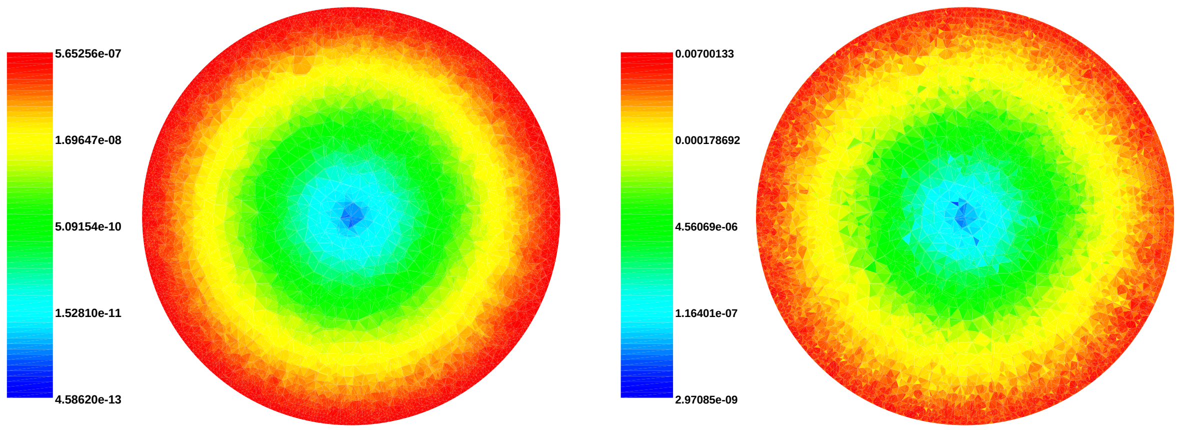


(a) Eddy current

(b) Magnetic flux density

Figure 4: Comparison of the norms of the analytical and the numerical solutions as a function of r .

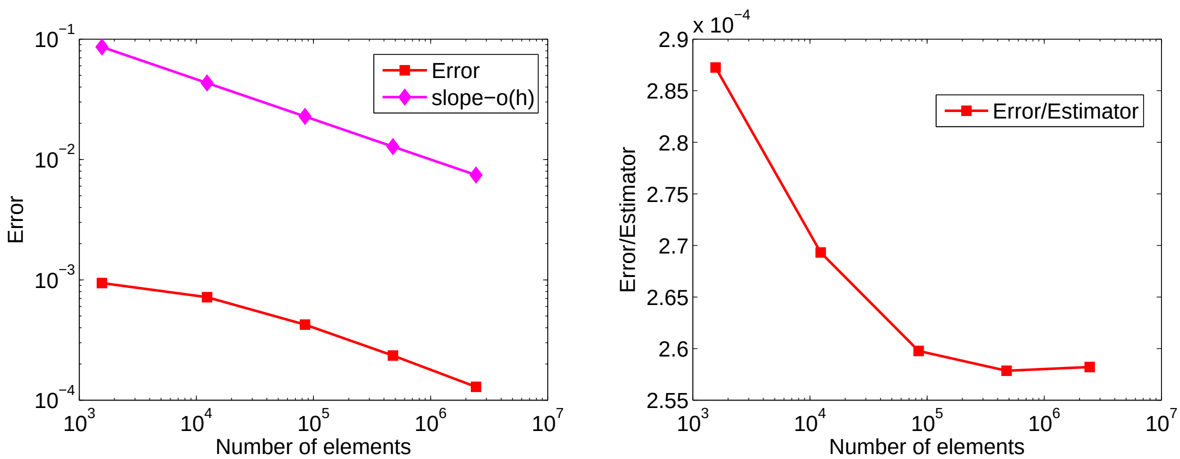
Secondly, the cylinder is cut by the plane $z = R/2$ and we plot in this plane the distributions of the error and of the estimator defined in (2.8), whose maps are given in Figure 5. We can observe that the estimator gives us a good relative distribution of the error.



(a) Error (b) Estimator

Figure 5: Error and Estimator maps in the plane $z = R/2$.

Finally, we consider five meshes with successive refinements. In Figure 6(a), we plot the exact error defined by the left-hand-side of (2.15) as a function of the total number of elements of the mesh. It indicates for the more refined meshes that the rate of convergence is equal to $o(h)$, as theoretically expected. The effectivity index, defined as the ratio between this error and the global estimator defined by (2.9), is plotted in Figure 6(b) for each mesh. It converges towards a constant when the size of the mesh goes to zero, which illustrates the reliability of the estimator as expected by Theorem 2.5.



(a) Error convergence. (b) Effectively index.

Figure 6: Error convergence and Effectivity index.

Conclusion

In this work, we have recalled the modelling of electrostatic and eddy current problems with potential formulations, both for the voltage excitation and the current excitation cases. We proved that in the voltage excitation case, both for electrostatic and eddy current problems, the derivation of the corresponding residual a posteriori error estimators was straightforward, considering the already known results available in the literature. On the contrary, for the current excitation case and for both the electrostatic and the eddy current problems, we gave the necessary additional theoretical results that allow to derive the corresponding estimators, for which the reliability was proved. Finally, a numerical test devoted to the eddy current problem was performed on the current excitation case to illustrate the obtained theoretical results.

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