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***Abstract*—In this article, motivated by a recent work of R. Brockett [1], a robust global synchronization problem of multistable Brockett oscillators has been studied within an Input-to-State Stability (ISS) framework. Following a recent generalization of the classical ISS theory to multistable systems and its application to the synchronization of multistable systems, two synchronization protocols are designed with respect to compact invariant sets of the unperturbed Brockett oscillator. The conditions obtained in our work are global and applicable to families of non-identical oscillators in contrast to the local analysis of [1]. Numerical simulation examples illustrate our theoretical results.**

***Index Terms*—Input-to-State Stability, synchronization, multistability, Brockett oscillator.**

I. Introduction

Over the past decade, considerable attention has been devoted to the problem of coordinated motion of multiple autonomous agents due to its broad applications in various areas. One critical issue related with multi-agent systems is to develop distributed control policies based on local information that enables all agents to reach an agreement on certain quantities of interest, which is known as the consensus problem. A classic example of distributed coordination/consensus in physics, engineering and biology is the synchronization of arrays of coupled nonlinear oscillators [2]–[7]. Oscillators synchronization has several potential application domains, for instance in power networks [8]–[11], smooth operations of micro-grids [12], real-time distributed control in networked systems [13] and so on.

The problem of synchronization has been addressed by researchers from various technical fields like physics, biology, neuroscience, automatic control, *etc.* To have a better insight on the contribution of automatic control community in this area, interested readers may consult [14]–[21]. In the context of the synchronization of oscillators, R. Brockett has recently introduced the following model [1]:

$$\ddot{x}+\varepsilon\dot{x}\left(\dot{x}^2+x^2-1\right)+x = \varepsilon^2 u, \tag{1}$$

where $x\in\mathbb{R}^n$ is the state, $u\in\mathbb{R}^n$ is the control and $\varepsilon>0$ is a parameter. In [1], a centralized synchronization protocol has been proposed for the model (1), such that the conventional averaging theory does not predict the existence of a periodic (almost periodic) solution for small ε. However, a qualitative synchronization together with a small amplitude irregular motion can be observed through numerical studies. Following [1], for ε sufficiently small, but non-zero, let us introduce the set

$$\begin{aligned} S_\varepsilon \;=\; & \big\{(x,\dot{x})\in\mathbb{R}^2:\dot{x}^2+x^2-1 \\ & +2\varepsilon^2 x\dot{x}\,\mathrm{sign}(\dot{x}^2+x^2-1)=\varepsilon\big\}, \end{aligned}$$

which contains two smooth closed contours: Γ_ε^+ lies outside the unit circle in the $(x,\dot{x})$-space and Γ_ε^- lies inside the unit circle. Both curves approach the unit circle as ε goes to zero. Then the main result of [1] is given below.

Theorem 1. *Let $\Gamma_\varepsilon^\pm$ be as before. Then there exist $\varepsilon_0>0$ such that for all $0<\varepsilon<\varepsilon_0$, the solutions of (1) beginning in the annulus bounded by Γ_ε^+ and Γ_ε^- remain in this annulus for all time, provided that $|u|\le\sqrt{x^2+\dot{x}^2}$.*

Theorem 1 provides a local synchronization result which depends on a small parameter $\varepsilon\neq 0$. Moreover, the result is applicable to the synchronization of identical oscillators only.

The goal of this work is to extend the result of [1] and to develop a protocol of *global* synchronization in the network of (1), for the case of identical and non-identical models of the agents[1]. It is assumed that the oscillators are connected through a N-cycle graph[2] [23]. The proposed solution is based on the framework of ISS for multistable systems [24], [25].

The ISS property provides a natural framework of stability analysis with respect to input perturbations (see [26] and references therein). The classical definition allows the stability properties with respect to arbitrary compact invariant sets (and not simply equilibria) to be formulated and characterized. Nevertheless, the implicit requirement is that these sets should be simultaneously Lyapunov stable and globally attractive, which makes the basic theory not applicable for a global analysis of many dynamical behaviors of interest having multistability [27]–[29], periodic oscillations [30], just to name a few, and only a local analysis remains possible [31]. Some attempts were made to overcome such limitations by introducing the notions of almost global stability [32] and almost input-to-state stability [33], *etc.*

Recently, the authors in [24], [25] have found that a natural way of developing ISS theory for systems with multiple invariant sets consists in relaxing the Lyapunov stability requirement

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The project is partly supported by ANR project WaQMoS (ANR 15 CE 04 0002). H. Ahmed was partly supported by the regional council of Nord-Pas de Calais, France during the period of this work.

The work has been performed while H. Ahmed was part of the NON-A team.

[1]Part of the results has been presented in [22].

[2]A cycle graph C_N is a graph on N nodes containing a single cycle through all nodes, or in other words, N number of vertices connected in a closed chain.

[34] (rather than the global nature of the attractivity property). Using this relatively mild condition, the ISS theory has been generalized in [24], [25], as well as the related literature on time-invariant autonomous dynamical systems on compact spaces [35] for multistable systems. Multistability accounts for the possible coexistence of various oscillatory regimes or equilibria in the state space of the system for the same set of parameters. Any system that exhibits multistability is called a multistable system. Frequently, for a given set of initial conditions and inputs it is very difficult to predict the asymptotic regime that a multistable system will attain asymptotically [36]. Following the results of [24], [25], the authors in [37] have provided conditions for the robust synchronization of multistable systems in the presence of external inputs. Readers can consult [38] for an overview of recent developments in the ISS framework, dealing in particular with the extension of the classical concept to systems with multiple invariant sets and possibly evolving on Riemannian manifolds.

The results presented in [25] and [37] can be applied to provide sufficient conditions for the existence of robust synchronization for identical/non-identical Brockett oscillators in the presence of external inputs. In [22], a global synchronization protocol has been proposed for the case $N = 2$. In this work, this result is extended to the general case $N > 2$ and to this end another synchronization control is proposed, which is not based on the theory of [37] and a special Lyapunov function is designed characterizing synchronization conditions for a family of non-identical Brockett oscillators. In opposite to the local results of [1], the conditions obtained in this work are global. The obtained synchronized system may demonstrate phase or anti-phase synchronization phenomena depending on parameters of the oscillators.

The rest of this paper is organized as follows. Section II introduces some preliminaries about decomposable sets, notions of robustness and conditions of robust synchronization of multistable systems. More details about Brockett oscillators (such as the proof that they possess ISS property) and the synchronization of a family of oscillators (the main results) can be found in sections III and IV, respectively. In Section V, a numerical simulation example is given to illustrate these results. Concluding remarks in Section VI close this article.

II. Preliminaries

This section has been taken from [24], [37].

A. Preliminaries on input-to-stability of multistable systems

Let M be an n-dimensional $\mathcal{C}^2$ connected and orientable Riemannian manifold without a boundary and $x \in M$. Let $f : M \times \mathbb{R}^m \to T_xM$ be a map of class $\mathcal{C}^1$. Throughout this work, we assume that all manifolds are embedded in a Euclidean space of dimension n, so they contain 0. Consider a nonlinear system of the following form:

$$\dot{x}(t) = f\left(x(t), d(t)\right), \tag{2}$$

where the state $x(t) \in M$ and $d(t) \in \mathbb{R}^m$ (the input $d(\cdot)$ is a locally essentially bounded and measurable signal) for $t \geq 0$. We denote by $X\left(t, x; d(\cdot)\right)$ the uniquely defined solution of (2) at time t satisfying $X(0, x; d(\cdot)) = x$. Together with (2), we will analyze its unperturbed version:

$$\dot{x}(t) = f\left(x(t), 0\right). \tag{3}$$

A set $S \subset M$ is invariant for the unperturbed system (3) if $X\left(t, x; 0\right) \in S$, for all $t \in \mathbb{R}$ and for all $x \in S$. For a set $S \subset M$, define the distance to S from a point $x \in M$ by $|x|_S = \inf_{a \in S} \delta(x, a)$ where the $\delta(x_1, x_2)$ denotes the Riemannian distance between x_1 and x_2 in M. We have $|x| = |x|_{\{0\}}$ for $x \in M$, the usual Euclidean norm of a vector $x \in \mathbb{R}^n$. For a signal $d : \mathbb{R} \to \mathbb{R}^m$, the essential supremum norm is defined as $\|d\|_\infty = \operatorname{ess\,sup}_{t \geq 0} |d(t)|$.

A function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ is said to belong to class $\mathcal{K}$, *i.e.* $\alpha \in \mathcal{K}$, if it is continuous, strictly increasing and $\alpha(0) = 0$. Furthermore, $\alpha \in \mathcal{K}_\infty$ if $\alpha \in \mathcal{K}$ and it is unbounded, *i.e.* $\lim_{s \to \infty} \alpha(s) = \infty$. For any $x \in M$, the $\alpha-$ and $\omega-$ limit sets for (3) can be defined as follows [39]:

$$\alpha(x) := \left\{ y \in M \mid y = \lim_{n \to -\infty} X(x, t_n) \text{ with } t_n \searrow -\infty \right\},$$
$$\omega(x) := \left\{ y \in M \mid y = \lim_{n \to \infty} X(x, t_n) \text{ with } t_n \nearrow \infty \right\}.$$

For an integer $N \geq 1$ the symbol $\overline{1, N}$ denotes the sequence $1, \ldots, N$.

B. Decomposable sets

Let $\Lambda \subset M$ be a compact invariant set for (3).

Definition 2. [35] A decomposition of Λ is a finite and disjoint family of compact invariant sets $\Lambda_1, \ldots, \Lambda_k$ such that $\Lambda = \bigcup_{i=1}^{k} \Lambda_i$.

For an invariant set Λ, its attracting and repulsing subsets are defined as follows:

$$\begin{aligned} \mathfrak{A}(\Lambda) &= \{x \in M \mid |X(t, x, 0)|_\Lambda \to 0 \text{ as } t \to +\infty\}, \\ \mathfrak{R}(\Lambda) &= \{x \in M \mid |X(t, x, 0)|_\Lambda \to 0 \text{ as } t \to -\infty\}. \end{aligned}$$

Define a relation on invariant sets in M: for $\mathcal{W} \subset M$ and $\mathcal{D} \subset M$, we write $\mathcal{W} \prec \mathcal{D}$ if $\mathfrak{A}(\mathcal{W}) \cap \mathfrak{R}(\mathcal{D}) \neq \emptyset$.

Definition 3. [35] Let $\Lambda_1, \ldots, \Lambda_k$ be a decomposition of Λ, then

1) An r-cycle ($r \geq 2$) is an ordered r-tuple of distinct indices $i_1, \ldots, i_r$ such that $\Lambda_{i_1} \prec \ldots \prec \Lambda_{i_r} \prec \Lambda_{i_1}$.
2) A 1-cycle is an index i such that $(\mathfrak{R}(\Lambda_i) \cap \mathfrak{A}(\Lambda_i)) \backslash \Lambda_i \neq \emptyset$.
3) A filtration ordering is a numbering of the Λ_i so that $\Lambda_i \prec \Lambda_j \Rightarrow i \leq j$.

As we can conclude from Definition 3, the existence of an r-cycle with $r \geq 2$ is equivalent to the existence of a heteroclinic cycle for (3) [40]. Moreover, the existence of a 1-cycle implies the existence of a homoclinic cycle for (3) [40].

Definition 4. Let $\mathcal{W} \subset M$ be a compact set containing all $\alpha-$ and $\omega-$limit sets of (3). We say that $\mathcal{W}$ is decomposable if it admits a finite decomposition without cycles, $\mathcal{W} = \bigcup_{i=1}^{k} \mathcal{W}_i$, for some non-empty disjoint compact sets $\mathcal{W}_i$, forming a filtration ordering of $\mathcal{W}$.

This definition of the compact set $\mathcal{W}$ will be used all through the article.

C. Robustness notions

The following robustness notions for systems in (2) have been introduced in [25].

Definition 5. We say that the system (2) has the practical asymptotic gain (pAG) property if there exist $\eta \in \mathcal{K}_\infty$ and $q \in \mathbb{R}_+$ such that for all $x \in M$ and all measurable essentially bounded inputs $d(\cdot)$, the solutions are defined for all $t \geq 0$ and

$$\limsup_{t\to+\infty} |X(t,x;d)|_{\mathcal{W}} \leq \eta\left(\|d\|_\infty\right) + q. \tag{4}$$

If $q = 0$, then we say that the asymptotic gain (AG) property holds.

Definition 6. We say that the system (2) has the limit property (LIM) with respect to $\mathcal{W}$ if there exists $\mu \in \mathcal{K}_\infty$ such that for all $x \in M$ and all measurable essentially bounded inputs $d(\cdot)$, the solutions are defined for all $t \geq 0$ and the following holds:

$$\inf_{t\geq 0} |X(t,x;d)|_{\mathcal{W}} \leq \mu\left(\|d\|_\infty\right).$$

Definition 7. We say that the system (2) has the practical global stability (pGS) property with respect to $\mathcal{W}$ if there exist $\beta \in \mathcal{K}_\infty$ and $q \in \mathbb{R}_+$ such that for all $x \in M$ and all measurable essentially bounded inputs $d(\cdot)$, the following holds for all $t \geq 0$:

$$|X(t,x;d)|_{\mathcal{W}} \leq q + \beta\left(\max\{|x|_{\mathcal{W}}, \|d\|_\infty\}\right).$$

To characterize (4) in terms of Lyapunov functions, it has been shown in [25] that the following notion is suitable:

Definition 8. We say that a $\mathcal{C}^1$ function $V : M \to \mathbb{R}$ is a practical ISS-Lyapunov function for (2) if there exists $\mathcal{K}_\infty$ functions α_1, α_2, α and γ, and scalars $q, c \in \mathbb{R}_+$ such that

$$\alpha_1(|x|_{\mathcal{W}}) \leq V(x) \leq \alpha_2(|x|_{\mathcal{W}} + c),$$

the function V is constant on each $\mathcal{W}_i$ and the dissipation inequality below holds:

$$DV(x)f(x,d) \leq -\alpha(|x|_{\mathcal{W}}) + \gamma(|d|) + q.$$

If this latter holds for $q = 0$, then V is said to be an ISS-Lyapunov function.

Notice that the existence of α_2 and c follows (without any additional assumptions) by standard continuity arguments.

The main result of [25] connecting these robust stability properties is stated below:

Theorem 9. *Consider a nonlinear system as in (2) and let a compact invariant set $\mathcal{W}$ containing all $\alpha-$ and $\omega-$limit sets of (3) be decomposable (in the sense of Definition 4). Then the following are equivalent:*

1) *The system admits an ISS Lyapunov function;*
2) *The system enjoys the AG property;*
3) *The system admits a practical ISS Lyapunov function;*
4) *The system enjoys the pAG property;*
5) *The system enjoys the LIM property and the pGS.*

A system in (2) that satisfies this list of equivalent properties is called ISS with respect to the set $\mathcal{W}$ [25].

D. Robust synchronization of multistable systems

This section summarizes the result on robust synchronization of multistable systems obtained in [37]. The following family of nonlinear systems is considered in this section:

$$\dot{x}_i(t) = f_i\left(x_i(t), u_i(t), d_i(t)\right),\ i = \overline{1,N},\ N > 1, \tag{5}$$

where the state $x_i(t) \in M_i$ (M_i is n_i-dimensional $\mathcal{C}^2$ connected and orientable Riemannian manifold without a boundary), the control $u_i(t) \in \mathbb{R}^{m_i}$ and the external disturbance $d_i(t) \in \mathbb{R}^{p_i}$ ($u_i(\cdot)$ and $d_i(\cdot)$ are locally essentially bounded and measurable signals) for $t \geq 0$; and the map $f_i : M_i \times \mathbb{R}^{m_i} \times \mathbb{R}^{p_i} \to T_{x_i}M_i$ is $\mathcal{C}^1$, $f_i(0,0,0) = 0$. Denote the common state vector of (5) as $x = [x_1^T, \ldots, x_N^T]^T \in M = \prod_{i=1}^N M_i$, so M is the corresponding Riemannian manifold of dimension $n = \sum_{i=1}^N n_i$ where the family (5) evolves and $d = [d_1^T, \ldots, d_N^T]^T \in \mathbb{R}^p$ with $p = \sum_{i=1}^N p_i$ is the exogenous input.

Assumption 1. *For all $i = \overline{1,N}$, each system in (5) has a compact invariant set $\mathcal{W}_i$ containing all $\alpha-$ and $\omega-$limit sets of $\dot{x}_i(t) = f_i\left(x_i(t), 0, 0\right)$, $\mathcal{W}_i$ is decomposable in the sense of Definition 4, and the system is ISS with respect to the set $\mathcal{W}_i$ and the inputs u_i and d_i.*

This assumption implies that family (5) is composed of robustly stable nonlinear systems.

Let a $\mathcal{C}^1$ function $\psi(x) : M \to \mathbb{R}^q$, $\psi(0) = 0$ be a synchronization measure for (5). We say that the family (5) is synchronized (or reached the consensus) if $\psi\left(x(t)\right) \equiv 0$ for all $t \geq 0$ on the solutions of the network under properly designed control actions

$$u_i(t) = \varphi_i\left(\psi\left(x(t)\right)\right) \tag{6}$$

($\varphi_i : \mathbb{R}^q \to \mathbb{R}^{m_i}$ is a $\mathcal{C}^1$ function, $\varphi_i(0) = 0$) for $d(t) \equiv 0$, $t \geq 0$. In this case, the set $\mathcal{A} = \{x \in \mathcal{W} \mid \psi(x) = 0\}$ contains the synchronous solutions of the unperturbed family in (5) and the problem of synchronization of "natural" trajectories is considered since $\mathcal{A} \subset \mathcal{W}$ (due to $\varphi_i(0) = 0$ in (6), the convergence of ψ (synchronization/consensus) implies that the solutions belong to $\mathcal{W}$).

The main result of [37] states that by selecting the shapes of φ_i, it is possible to guarantee robust synchronization of (5) for any measurable and essentially bounded input d.

Proposition 10. *Let Assumption 1 be satisfied for (5). Then there exist φ_i, $i = \overline{1,N}$ in (6) such that the interconnection (5), (6) has pGS property with respect to the set $\mathcal{W}$.*

For example, the result of Proposition 10 is valid for any bounded functions φ_i, $i = \overline{1,N}$ in (6).

Assumption 2. *The set $\mathcal{A}$ is compact, it contains all $\alpha-$ and $\omega-$limit sets of (5), (6) for $d = 0$, and it is decomposable.*

Therefore, it is assumed that the controls $\varphi_i(\psi)$ ensure the network global synchronization, while the decomposability in general follows from Assumption 1.

Theorem 11. *Let conditions of Proposition 10 be satisfied together with Assumption 2, then the interconnection (5), (6) is ISS with respect to $\mathcal{A}$.*

III. Properties of Brockett oscillator

Let us consider the Brockett oscillator [1]:

$$\ddot{\xi}+b\dot{\xi}\left(\dot{\xi}^2+\xi^2-1\right)+\xi=au, \tag{7}$$

where $\xi\in\mathbb{R}$, $\dot{\xi}\in\mathbb{R}$ are the states variables, $a,b>0$ are parameters and u is the control input. By considering $x_1=\xi$, $\dot{x}_1=x_2=\dot{\xi}$, $x=[x_1,x_2]^T$ and $|x|=\sqrt{x_1^2+x_2^2}$ equation (7) can be written in the state-space form as:

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -x_1+au-bx_2\left(|x|^2-1\right),\end{aligned} \tag{8}$$

where the state of the system (8), *i.e.* x, evolves in the manifold $M=\mathbb{R}^2$. By analyzing equation (8) it can be seen that the unperturbed system admits two invariant sets: namely, the origin $\mathcal{W}_1=\{0\}$ and the limit cycle $\mathcal{W}_2=\Gamma=\left\{x\in M:|x|^2=1\right\}$. So, the invariant set for the trajectories of (8) can be defined as:

$$\mathcal{W}:=\mathcal{W}_1\cup\mathcal{W}_2=\{0\}\cup\Gamma. \tag{9}$$

In order to verify the decomposability of the invariant set $\mathcal{W}$, we need to know the nature of the equilibrium $\mathcal{W}_1$ and the limit cycle $\mathcal{W}_2=\Gamma$. This information can be obtained by analyzing the Lyapunov stability of the unperturbed system (8).

A. Stability of the autonomous Brockett oscillator

Since $\mathcal{W}$ is invariant for the trajectories of (9), then the following proposition provides the stability of the unforced Brockett oscillator ((8) with $u=0$) with respect to $\mathcal{W}$.

Proposition 12. *For (8) with $u=0$, the limit cycle Γ is almost globally asymptotically stable and the origin is unstable.*

Proof: The instability of the origin of the unperturbed system (8) can be verified for a linearized version of the system. The eigenvalues of the linearized system $\lambda_{1,2}=\frac{1}{2}\left(b\pm\sqrt{b^2-4}\right)$ have always positive real parts for any $b>0$. Alternatively, this fact can also be checked through LMI formulation which is given in Remark 13.

To analyze the stability of the limit cycle $\mathcal{W}_2$, let us consider the following Lyapunov function:

$$U(x) = \frac{1}{2}\left(|x|^2-1\right)^2,$$

which is zero on the set $\mathcal{W}_2$ and positive otherwise. Evaluating the total derivative of U along the solutions of (8), we obtain:

$$\begin{aligned}\dot{U} &= \left(|x|^2-1\right)\left\{2aux_2-2bx_2^2\left(|x|^2-1\right)\right\}\\ &= -2bx_2^2\left(|x|^2-1\right)^2+2aux_2\left(|x|^2-1\right)\\ &\leq -2bx_2^2\left(|x|^2-1\right)^2+bx_2^2\left(|x|^2-1\right)^2+\frac{a^2}{b}u^2\\ &\leq -bx_2^2\left(|x|^2-1\right)^2+\frac{a^2}{b}u^2.\end{aligned}$$

Then for $u=0$ we have $\dot{U}\leq 0$ and all trajectories are globally bounded. By LaSalle's invariance principle [41], all trajectories of the system converge to the set where $\dot{U}=0$. Note that $\{x\in M:\dot{U}=0\}=\mathcal{W}_2\cup\{x\in M:x_2=0\}$ and on the line x_2 there is the only invariant solution at the origin (in $\mathcal{W}_1$), therefore $\dot{U}=0$ for all $x\in\mathcal{W}$, which contains all invariant solutions of the system. Since the origin is unstable, it can be concluded that the limit cycle $\mathcal{W}_2$ is almost globally asymptotically stable. ∎

Remark 13. To check the instability of the origin in an alternative way, let us consider a small closed ball with the radius $\rho>0$ around the origin $B(\rho)=\left\{x\in\mathbb{R}^2:|x|^2\leq\rho\right\}$. Inside this ball, by imposing the parameter $b=1$ without loosing generality, the unperturbed system of (8) can be written as the following uncertain linear system:

$$\dot{x}=Ax, A=\begin{bmatrix}0 & 1\\ -1 & -(\tilde{\rho}-1)\end{bmatrix}, \tilde{\rho}\in[0,\rho], \tag{10}$$

where the matrix $A\in\mathbb{R}^{2\times2}$ belongs to the domain $\mathcal{D}_A$ defined as:

$$\mathcal{D}_A\triangleq\left\{A:A=\beta_1A_1+\beta_2A_2,\beta_1,\beta_2>0,\sum_{i=1}^{2}\beta_i=1\right\} \tag{11}$$

with $A_1=\begin{bmatrix}0 & 1\\ -1 & 1\end{bmatrix}$ and $A_2=\begin{bmatrix}0 & 1\\ -1 & -(\rho-1)\end{bmatrix}$. Then, by applying Chetaev instability theorem [42], it can be concluded that the origin is unstable if there exist $P>0,Q>0$ such that for $i=1,2$

$$A_i^TP+PA_i\succeq Q. \tag{12}$$

The LMI (12) can be easily verified by using any standard solvers like Yalmip [43]. For example, let us select $\rho=0.2$, then the following values are obtained satisfying LMI (12):

$$P=\begin{bmatrix}21.4643 & -6.8278\\ \star & 17.8390\end{bmatrix}, Q=\begin{bmatrix}6.1040 & -1.3080\\ \star & 7.8838\end{bmatrix}.$$

As a result, it can be concluded that the origin is unstable.

B. Stability of the non-autonomous Brockett oscillator

In the previous section, we have proved the stability of the unperturbed system with $u=0$. In this section, we will analyze the stability of the Brockett oscillator in the presence of input u. As it was shown in the previous section, $\mathcal{W}$ contains all $\alpha-$ and $\omega-$limit sets of the unperturbed system in (8), and it admits a decomposition without cycles. Consequently the result of [24] can be applied to show the robust stability of the Brockett oscillator in (8) with respect to $\mathcal{W}$:

Proposition 14. *The system (8) is ISS with respect to the set $\mathcal{W}$.*

Proof: To prove the ISS property, let us introduce two new variables y and h as,

$$\begin{aligned}&y(x)=|x|^2-1,\ \dot{y}=-2bx_2^2y+2ax_2u;\\ &h(x)=(x_1+x_2)y,\ \dot{h}=a[y+2x_2(x_1+x_2)]u\\ &\qquad-(h-2x_2y+bx_2y^2+2bx_2^2h).\end{aligned}$$

Next, let us consider the following Lyapunov function for (8) with some $c,d>0$:

$$W(x) = \frac{1}{2}\left(h^2(x)+cy^2(x)+\frac{1}{2}dy^4(x)\right). \tag{13}$$

Notice that $W(x)=0$ for all $x\in\mathcal{W}_2$ and positive otherwise. Therefore, there exist α_1,$\alpha_2\in\mathcal{K}_\infty$ such that the first condition of Definition 8 is satisfied for all $x\in M$ for the above function $W(x)$. Evaluating the total derivative of W, along the solutions of (8), we obtain

$$\begin{aligned}\dot{W} &= cy\dot{y}+h\dot{h}+dy^3\dot{y}\\ &= 2au[h(x_2^2+x_1x_2+0.5y)+x_2y(c+dy^2)]\\ &\quad -h^2-2bx_2^2[h^2+y^2(c+dy^2)]+hx_2y(2-by).\end{aligned}$$

Next by applying Young's inequality, we can derive the series of relations:

$$2hx_2y\le\frac{1}{2}h^2+2x_2^2y^2,\ hx_2y^2\le\frac{h^2}{4b}+bx_2^2y^4,$$
$$x_2^2hu\le x_2^2\left(\frac{b}{4a}h^2+\frac{a}{b}u^2\right),$$
$$hux_1x_2\le\frac{b}{4a}h^2x_2^2+\frac{a}{b}x_1^2u^2,\ huy\le\frac{h^2}{16a}+4ay^2u^2,$$
$$(c+dy^2)ux_2y\le(c+dy^2)\left(\frac{b}{2a}x_2^2y^2+\frac{a}{2b}u^2\right).$$

By substituting these inequalities for $c=\frac{3}{b}$, $d=2b$ and after simplification, we obtain

$$\begin{aligned}\dot{W} &\le -h^2\left(\frac{1}{8}+bx_2^2\right)-x_2^2y^2(1+b^2y^2)\\ &\quad +\frac{a^2}{b^2}[2b|x|^2+10b^2y^2+3]u^2.\end{aligned}$$

From the properties of the functions h and y we can substantiate that

$$\frac{1}{8}h^2+x_2^2y^2\ge\frac{5-\sqrt{17}}{8}|x|^2y^2,$$

and W is a practical ISS Lyapunov function for (8) since

$$|x|^2\ge\max\{q,16\frac{2bq+10b^2(q-1)^2+3}{(5-\sqrt{17})(q-1)^2}\frac{a^2}{b^2}u^2\}$$
$$\Rightarrow\dot{W}\le-\frac{5-\sqrt{17}}{16}|x|^2y^2$$

for any $q>1$. Consequently, using Theorem 9 it can be concluded that the system (8) is ISS with respect to the set $\mathcal{W}$ from the input u. ∎

Remark 15. It is straightforward to check that there exists a function $\alpha\in\mathcal{K}_\infty$ such that for all $x\in M$ and $u=0$ we have $\dot{W}\le-\alpha(|x|_{\mathcal{W}})$. Thus W is a global Lyapunov function establishing multistability of (8) with respect to $\mathcal{W}$ for $u=0$.

IV. Synchronization of Brockett oscillators

The following family of Brockett oscillators is considered in this section for some $N>1$:

$$\begin{aligned}&\dot{x}_{1i}=x_{2i},\\ &\dot{x}_{2i}=a_iu_i-x_{1i}-b_ix_{2i}\left(|x_i|^2-1\right),\ i=\overline{1,N},\end{aligned}\tag{14}$$

where $a_i,b_i>0$ are the parameters of an individual oscillator, the state $x_i=[x_{1i}\ x_{2i}]^T\in M_i=\mathbb{R}^2$, the control $u_i\in\mathbb{R}$ ($u_i:\mathbb{R}_+\to\mathbb{R}$ is locally essentially bounded and measurable signal). Denote the common state vector of (5) as $x=[x_1^T,\dots,x_N^T]^T\in M=\prod_{i=1}^N M_i$, so M is the corresponding Riemannian manifold of dimension $n=2N$ where the family (5) behaves and $u=[u_1,\dots,u_N]^T\in\mathbb{R}^N$ is the common input. Through propositions 12 and 14, it has been shown that each member of family (14) is robustly stable with respect to the set $\mathcal{W}_i=\mathcal{W}_{1i}\cup\mathcal{W}_{2i}$, where $\mathcal{W}_{1i}=\{0\}$ and $\mathcal{W}_{2i}=\{x_i\in M_i:|x_i|^2=1\}$. Consequently, the family (14) is a robustly stable nonlinear system. As a result, Assumption 1 is satisfied for the case of (14).

There are several works devoted to synchronization and design of consensus protocols for such a family or oscillatory network [44]–[46].

A. Problem statement

Let a $\mathcal{C}^1$ function $\psi:M\to\mathbb{R}^q$, $\psi(0)=0$ be a synchronization measure for (14). We say that the family (14) is synchronized (or reached the consensus) if $\psi(x(t))\equiv0$ for all $t\ge0$ on the solutions of the network under properly designed control actions

$$u_i(t)=\varphi_i\left[\psi(x(t))\right],\tag{15}$$

where $\varphi_i:\mathbb{R}^q\to\mathbb{R}$ is a $\mathcal{C}^1$ function, $\varphi_i(0)=0$. Due to the condition $\varphi_i(0)=0$, the convergence of ψ (synchronization/consensus) implies that the solutions of the interconnection belong to $\mathcal{W}=\prod_{i=1}^N\mathcal{W}_i$. In this case the set $\mathcal{A}=\{x\in\mathcal{W}\mid\psi(x)=0\}$ contains the synchronous solutions of the family in (15) and the problem of synchronization of "natural" trajectories is considered since $\mathcal{A}\subset\mathcal{W}$.

In this work we deal with the following synchronization measure:

$$\begin{aligned}\psi &= [\psi_1,\dots,\psi_N]^T,\\ \psi_i &= \begin{cases}x_{2(i+1)}-x_{2i}, & i=\overline{1,N-1}\\ x_{21}-x_{2N}, & i=N\end{cases}.\end{aligned}$$

From a graph theory point of view, the oscillators are connected through a N-cycle graph [23] (each oscillator needs only the information of its next neighbor), *i.e.*

$$\psi=S\begin{bmatrix}x_{21}\\ \vdots\\ x_{2N}\end{bmatrix},\ S=\begin{bmatrix}-1&1&0&\cdots&0\\ 0&-1&1&\ddots&\\ &\ddots&\ddots&\ddots&\\ &&&&1\\ 1&&&&-1\end{bmatrix},$$

and any other connection type can be studied similarly. Moreover, the interconnection matrix S has Metzler form since all off-diagonal elements are positive. Next, let us define the synchronization error among the various states of the oscillators as follows for $i=\overline{1,N-1}$:

$$e_{2i-1}=x_{1i}-x_{1(i+1)},\ \dot{e}_{2i-1}=x_{2i}-x_{2(i+1)}=e_{2i}$$

and $e_{2N-1}=x_{1N}-x_{11}$, $\dot{e}_{2N-1}=x_{2N}-x_{21}=e_{2N}$. Thus,

$$\begin{aligned}\psi_i &= -e_{2i}\quad i=\overline{1,N},\\ \psi_N &= -\sum_{i=1}^{N-1}e_{2i}\end{aligned}$$

and the quantity $e = [e_1, e_2, \ldots, e_{2N}] = 0$ implies that $\psi = 0$ (the synchronization state is reached). For $y_i = |x_i|^2 - 1$ the error dynamics can be written in the form:

$$\begin{aligned}
\dot{e}_{2i-1} &= e_{2i},\ i = \overline{1, N}, \\
\dot{e}_{2i} &= -e_{2i-1} + a_i u_i - a_{i+1} u_{i+1} - b_i x_{2i} y_i \\
&\quad + b_{i+1} x_{2(i+1)} y_{i+1},\ i = \overline{1, N-1}, \\
\dot{e}_{2N} &= -e_{2N-1} + a_N u_N - a_1 u_1 - b_N x_{2N} y_N \\
&\quad + b_1 x_{21} y_1.
\end{aligned} \tag{16}$$

Since $e_{2N-j} = \sum_{i=1}^{N-1} e_{2i-j}$ for $j = 0, 1$, then formally only $N - 1$ errors can be considered in (16).

In order to design the controls we will consider in this work the following Lyapunov function

$$V(x) \ = \ \sum_{i=1}^{N} \frac{\alpha_i}{4} y_i^2 + \frac{1}{2} \sum_{i=1}^{2N} e_i^2, \tag{17}$$

where $\alpha_i \geq 0$ are weighting parameters. Notice that $V(x) = 0$ for all $x \in \mathcal{A} \cap \prod_{i=1}^{N} \mathcal{W}_{2i}$ and positive otherwise. Such a choice of Lyapunov function is very natural for our goal since it has two items: the former one characterizes stability of each oscillator, while the latter item evaluates synchronicity of the network.

B. Preliminary results

In [22] for $N = 2$ and

$$u = k\psi,\ k > 0, \tag{18}$$

e.g. $\varphi(\psi) = k\psi$ in (15), the following result has been proven using $V(x)$:

Theorem 16. *[22] The family of Brockett oscillators (14) with $N = 2$ is synchronized by (18),* i.e. *in (14),(18) all solutions stay bounded for all $t \geq 0$ and the set $\mathcal{A}$ is globally asymptotically attractive.*

The result of this theorem is a particular case of Proposition 19 given below for $N > 2$. It has been observed in numerical experiments that for $N > 2$ and (18) the synchronization persists, but the proof cannot be extended to the case $N > 2$ since (17) is not a Lyapunov function in such a case.

Remark 17. To overcome this problem, based on the idea presented in [47], the following modification to the control law (18) can be proposed:

$$u_i = k\psi_i + b_i x_{2i} y_i. \tag{19}$$

Since the modified control law (19) compensates the nonlinear part of (14), as a result the closed loop system becomes linear. In this case, it is trivial to show that the closed loop system (14) and (19) is globally asymptotically synchronized.

Theorem 16 guarantees global asymptotic stability of the synchronized behavior, but not the robustness. Note that the controls (18) and (19) are not bounded, then it is impossible to apply the result of Proposition 14 to prove robust stability of $\mathcal{W}$. Moreover, in many application areas, the control is bounded due to actuator limitations. With such a motivation, take a bounded version of (15), then from propositions 10 and 14 convergence of all trajectories in a vicinity of $\mathcal{W}$ immediately follows. If (15) is properly bounded then any accuracy of approaching $\mathcal{W}$ can be guaranteed, and the next result summarizes the conditions of synchronization:

Corollary 18. *Let the set $\mathcal{A}$ contain all $\alpha-$ and $\omega-$limit sets of (14), (15) and it is decomposable for given bounded $\varphi_i, i = \overline{1, N}$, then the interconnection (14), (15) is synchronized,* i.e. *in (14), (15) all solutions stay bounded for all $t \geq 0$ and the set $\mathcal{A}$ is globally asymptotically attractive.*

Proof: In the conditions of the corollary Assumption 2 is satisfied for (14), (15). The proof follows from the result of Theorem 11 since Assumption 1 is satisfied due to Proposition 14. ■

If we assume that (15) contains an additional perturbation $d \in \mathbb{R}^N$:

$$u_i(t) = \varphi_i \left[\psi(x(t)) + d_i(t) \right],\ i = \overline{1, N},$$

which models the connection errors and coupling imperfections, then ISS property with respect to the set $\mathcal{A}$ can be proven in the conditions of Corollary 18 (the result of Theorem 11).

C. Global synchronization control

Consider a variant of synchronization control in the following form:

$$\begin{aligned}
u &= k \begin{bmatrix} -2 & 1 & 0 & \cdots & 1 \\ 1 & -2 & 1 & \cdots & 0 \\ 0 & 1 & -2 & \cdots & 0 \\ \vdots & & \vdots & \ddots & \\ 1 & \cdots & 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_{21} \\ x_{22} \\ \vdots \\ x_{2(N-1)} \\ x_{2N} \end{bmatrix} \\
&= kS^T \begin{bmatrix} e_2 \\ e_4 \\ \vdots \\ e_{2N-2} \\ e_{2N} \end{bmatrix},
\end{aligned} \tag{20}$$

where $k > 0$ is the coupling strength. Obviously, the control (20) can be rewritten as (15):

$$u = -kS^T \psi.$$

With such a control each i^{th} oscillator is connected with its neighbors $(i-1)^{\text{th}}$ and $(i+1)^{\text{th}}$ oscillators, and the closed loop network (14), (20) is organized again in the form of N-cycle graph [23]. Note that for $N = 2$ the control (20) takes the form of (18).

Let us calculate the derivative of the Lyapunov function $V(x)$ for (14), (20) (in the calculations below we will use convention for indexes that $N+1=1$):

$$\begin{aligned}\dot{V} &= \sum_{i=1}^{N}[\alpha_i(-b_i x_{2i}^2 y_i^2 + a_i x_{2i} y_i u_i) \\ &\quad + e_{2i}(a_i u_i - a_{i+1}u_{i+1} - b_i x_{2i} y_i + b_{i+1} x_{2(i+1)} y_{i+1})] \\ &= \sum_{i=1}^{N}[a_i(\alpha_i x_{2i} y_i + e_{2i} - e_{2i-2})u_i \\ &\quad + b_i(e_{2i-2} - e_{2i})x_{2i}y_i - \alpha_i b_i x_{2i}^2 y_i^2] \\ &= \sum_{i=1}^{N}[a_i(\alpha_i x_{2i} y_i + e_{2i} - e_{2i-2})k(e_{2i-2} - e_{2i}) \\ &\quad + b_i(e_{2i-2} - e_{2i})x_{2i}y_i - \alpha_i b_i x_{2i}^2 y_i^2] \\ &= \sum_{i=1}^{N}[\{a_i\alpha_i k + b_i\}(e_{2i-2} - e_{2i})x_{2i}y_i \\ &\quad - k a_i(e_{2i-2} - e_{2i})^2 - \alpha_i b_i x_{2i}^2 y_i^2].\end{aligned}$$

Select $\alpha_i = \frac{b_i}{k a_i}$, then

$$\begin{aligned}\dot{V} &= \sum_{i=1}^{N} b_i[2(e_{2i-2} - e_{2i})x_{2i}y_i \\ &\quad - \alpha_i^{-1}(e_{2i-2} - e_{2i})^2 - \alpha_i x_{2i}^2 y_i^2] \\ &= -\sum_{i=1}^{N} b_i[\alpha_i^{-0.5}(e_{2i-2} - e_{2i}) - \alpha_i^{0.5} x_{2i} y_i]^2 \\ &\leq 0\end{aligned}$$

Since V is positive definite with respect to the set $\mathcal{A} \cap \prod_{i=1}^{N} \mathcal{W}_{2i}$, which is compact, then all trajectories in the system are globally bounded. By LaSalle's invariance principle all trajectories of the system converge to the largest invariant set in

$$\begin{aligned}\Omega &= \{x \in M : \dot{V}(x) = 0\} \\ &= \{x \in M : e_{2i-2} - e_{2i} = \alpha_i x_{2i} y_i,\ i = \overline{1,N}\}.\end{aligned}$$

Note that $u_i = k(e_{2i-2} - e_{2i}) = k\alpha_i x_{2i} y_i = \frac{b_i}{a_i} x_{2i} y_i$ in the set Ω, then on that set the control performs compensation of nonlinearity as (19) and asymptotically the dynamics of synchronization errors take the form for $i = \overline{1,N}$:

$$\begin{aligned}\dot{e}_{2i-1} &= e_{2i}, \\ \dot{e}_{2i} &= -e_{2i-1} + a_i u_i - a_{i+1} u_{i+1} - b_i x_{2i} y_i \\ &\quad + b_{i+1} x_{2(i+1)} y_{i+1} \\ &= -e_{2i-1}\end{aligned}$$

and

$$\begin{aligned}\dot{y}_i &= -2b_i x_{2i}^2 y_i + 2a_i x_{2i} u_i = 0, \\ \dot{x}_{1i} &= x_{2i}, \\ \dot{x}_{2i} &= -x_{1i},\end{aligned}$$

i.e. the norms $|x_i|$ and $|(e_{2i-1}, e_{2i})|$ for all $i = \overline{1,N}$ become constant on Ω. Therefore, the following result has been proven:

Proposition 19. *For any $k > 0$ in the system (14), (20) all trajectories are bounded for all $t \geq 0$ and asymptotically converge to the largest invariant set in*

$$\begin{aligned}\Omega_\infty &= \{x \in M : |x_i| = const,\ e_{2i-1}^2 + e_{2i}^2 = const, \\ &\quad x_{2(i-1)} + x_{2(i+1)} = (2 + \alpha_i(|x_i|^2 - 1))x_{2i}, \\ &\quad i = \overline{1,N}\}.\end{aligned}$$

As we can conclude, the set Ω_∞ includes the dynamics of interest with synchronization at the unit circle (when $|x_i| = 1$ for all $i = \overline{1,N}$) or on a circle (when $|x_i| \neq 0$ for all $i = \overline{1,N}$). Indeed, the relations

$$\beta_i x_{2i} = x_{2(i-1)} + x_{2(i+1)} \tag{21}$$

with constant $\beta_i = 2 + \alpha_i(|x_i|^2 - 1)$, which satisfy in the set Ω_∞ for all $i = \overline{1,N}$, can be interpreted as a kind of synchronization, with another synchronization measure (the previously introduced $\psi(x(t))$ may be non zero in general case). Note that different, phase or anti-phase, patterns can be obtained in (14), (20) depending on values of parameters. The case when $|x_i| = 0$ for all $i = \overline{1,N}$ corresponds also to synchronization, but it is not interesting from application point of view since there is no oscillating solution in this case.

Theorem 20. *For any $k > 0$, if there is an index $1 \leq i \leq N$ such that $2a_i k < b_i$, then in the system (14), (20) all trajectories are bounded and almost all of them converge to the largest invariant set in*

$$\begin{aligned}\Omega'_\infty &= \{x \in M : |x_i| = const \neq 0,\ e_{2i-1}^2 + e_{2i}^2 = const, \\ &\quad x_{2(i-1)} + x_{2(i+1)} = (2 + \alpha_i(|x_i|^2 - 1))x_{2i}, \\ &\quad i = \overline{1,N}\}.\end{aligned}$$

Proof: Since all conditions of Proposition 19 are satisfied, then all trajectories converge to the set Ω_∞. By substitution of the control (20) in the equations of (14) we obtain:

$$\begin{aligned}\dot{x}_{1i} &= x_{2i}, \\ \dot{x}_{2i} &= a_i u_i - x_{1i} - b_i x_{2i}\left(|x_i|^2 - 1\right) \\ &= a_i k(x_{2(i-1)} - 2x_{2i} + x_{2(i+1)}) - x_{1i} \\ &\quad - b_i x_{2i}\left(|x_i|^2 - 1\right) \\ &= -x_{1i} - (2a_i k - b_i)x_{2i} + a_i k(x_{2(i-1)} \\ &\quad + x_{2(i+1)}) - b_i x_{2i}|x_i|^2.\end{aligned}$$

Linearizing this system around the origin ($|x_i| = 0$ for all $i = \overline{1,N}$) we conclude that this equilibrium is unstable if there exists at least one index $1 \leq i \leq N$ with $2a_i k < b_i$. Thus, for almost all initial conditions trajectories converge to a subset of Ω_∞ where $|x_i| \neq 0$, *i.e.* to the set Ω'_∞ (see Proposition 11 in [48]). ■

In the set Ω_∞ we have for all $i = \overline{1,N}$:

$$x_{1i}^2 + x_{2i}^2 = r_i^2,$$
$$\rho_i^2 = e_{2i-1}^2 + e_{2i}^2 = r_i^2 + r_{i+1}^2 - 2(x_{1i}x_{1(i+1)} + x_{2i}x_{2(i+1)})$$

for some $r_i \in \mathbb{R}_+$ and $\rho_i \in \mathbb{R}_+$, and

$$x_{2(i-1)} + x_{2(i+1)} = \beta_i x_{2i},\ x_{1(i-1)} + x_{1(i+1)} = \beta_i x_{1i} + c_i \tag{22}$$

for $\beta_i = 2 + \alpha_i(r_i^2 - 1)$, $\alpha_i = \frac{b_i}{k a_i}$ and some $c_i \in \mathbb{R}$. Note that if $b_i > 2a_i k$ then β_i can take non-positive values. If

$\beta_i = 0$, then from the above equations $x_{2(i-1)} = -x_{2(i+1)}$ and $x_{1(i-1)} = c_i - x_{1(i+1)}$; using these relations and taking sum of ρ_i^2 and ρ_{i-1}^2 we obtain

$$\rho_i^2 + \rho_{i-1}^2 = 2r_i^2 + r_{i+1}^2 + r_{i-1}^2 - 2x_{1i}c_i,$$

consequently, for $c_i \neq 0$ the variable x_{1i} has to be constant, which is impossible in Ω'_∞, then $c_i = 0$ leading to an equality

$$\rho_i^2 + \rho_{i-1}^2 = 2r_i^2 + r_{i+1}^2 + r_{i-1}^2,\ r_i = \sqrt{1 - 2\alpha_i^{-1}}. \quad (23)$$

Assume that $\beta_i \neq 0$, then finding from equations (22) the expressions for x_{1i} and x_{2i} and substituting them into the equation for ρ_i we obtain:

$$\begin{aligned}\rho_i^2 &= r_i^2 + (1 - 2\beta_i^{-1})r_{i+1}^2 - 2\beta_i^{-1}(x_{1(i-1)}x_{1(i+1)} \\ &\quad + x_{2(i-1)}x_{2(i+1)} - c_i x_{1(i+1)}).\end{aligned}$$

Taking square of both sides in (22) and adding them we get:

$$\begin{aligned}\beta_i^2 r_i^2 + c_i(2\beta_i x_{1i} + c_i) &= r_{i-1}^2 + r_{i+1}^2 + 2(x_{1(i-1)}x_{1(i+1)} \\ &\quad + x_{2(i-1)}x_{2(i+1)}),\end{aligned}$$

from which the expression for $x_{1(i-1)}x_{1(i+1)} + x_{2(i-1)}x_{2(i+1)}$ can be derived and substituted in the expression for ρ_i^2:

$$\begin{aligned}2c_i(\beta_i^{-1}x_{1(i+1)} - x_{1i}) &= \rho_i^2 - r_i^2 - (1 - 2\beta_i^{-1})r_{i+1}^2 \\ &\quad + \beta_i^{-1}(\beta_i^2 r_i^2 - r_{i-1}^2 - r_{i+1}^2 + c_i^2),\end{aligned}$$

where the right-hand side is a constant. Differentiating this equation we conclude that either $c_i = 0$ or

$$x_{2(i+1)} = \beta_i x_{2i}$$

that from (22) implies $x_{2(i-1)} = 0$ for all $i = \overline{1, N}$. Thus, if we are interested in the solution into Ω'_∞, then we have to select the option $c_i = 0$, which leads to the set of equations

$$\rho_i^2 = (1 - \beta_i)r_i^2 + (1 - \beta_i^{-1})r_{i+1}^2 + \beta_i^{-1}r_{i-1}^2$$

or, equivalently,

$$\begin{aligned}\rho_i^2 &= \frac{1 + \alpha_i(r_i^2 - 1)}{2 + \alpha_i(r_i^2 - 1)}r_{i+1}^2 - (1 + \alpha_i(r_i^2 - 1))r_i^2 \\ &\quad + \frac{1}{2 + \alpha_i(r_i^2 - 1)}r_{i-1}^2 \qquad (24)\end{aligned}$$

for all $i = \overline{1, N}$. For the subsystems with $b_i > 2a_i k$ (the solution $2 + \alpha_i(r_i^2 - 1) = 0$ is admissible), the corresponding equation in (24) has to be replaced with (23).

Note that the system of equations (24) for $r_i = 1$, $i = \overline{1, N}$ admits the only solution $\rho_i = 0$, $i = \overline{1, N}$. If we assume that $\rho_i = 0$, $i = \overline{1, N}$, then by definition $r_i^2 = r_{i+1}^2$ and (24) can be reduced to

$$0 = \alpha_i(r_i^2 - 1)r_i^2,\ i = \overline{1, N},$$

which in Ω'_∞ has the only admissible solution $r_i = 1$, $i = \overline{1, N}$, as we need. Unfortunately, the equation (24) (as well as (23)) admits also other solutions with $r_i \in (0, 1)$ and $\rho_i \neq 0$.

In order to exclude other solutions with $\rho_i \neq 0$ let us consider a Lyapunov function

$$W = \frac{1}{2}\sum_{i=1}^N |x_i|^2 = \frac{1}{2}\sum_{i=1}^N x_{1i}^2 + x_{2i}^2,$$

whose time derivative has the form:

$$\begin{aligned}\dot{W} &= -\sum_{i=1}^N b_i x_{2i}^2 \left(|x_i|^2 - k\frac{a_{i+1} + a_{i-1} - 2a_i}{2b_i} - 1\right) \\ &\quad - \sum_{i=1}^N k\frac{a_i + a_{i+1}}{2}e_{2i}^2.\end{aligned}$$

According to Theorem 20, asymptotically $\dot{W} = 0$ in the set Ω'_∞, then

$$\sum_{i=1}^N b_i x_{2i}^2 \left(|x_i|^2 - \frac{a_{i+1} + a_{i-1} - 2a_i}{2b_i k^{-1}} - 1\right) + \frac{a_i + a_{i+1}}{2k^{-1}}e_{2i}^2 = 0.$$

Note that in the set Ω'_∞ we have $x_{2i} = r_i \sin(\phi_i - t)$ for all $i = \overline{1, N}$, where $r_i = |x_i|$ and $\phi_i \in [0, 2\pi)$ are some constants depending on the system parameters and initial conditions, then the equation above can be rewritten as follows:

$$\begin{aligned}0 &= \sum_{i=1}^N k\frac{a_i + a_{i+1}}{2}r_i r_{i+1}\left(\cos(\phi_i - \phi_{i+1}) - 1\right) \qquad (25) \\ &\quad + r_i\left[k(a_i + a_{i+1})r_{i+1}\sin^2\left(\frac{\phi_i + \phi_{i+1}}{2} - t\right)\right. \\ &\quad \left. - r_i\left(b_i(r_i^2 - 1) + 2ka_i\right)\sin^2(\phi_i - t)\right].\end{aligned}$$

This equation has a trivial solution $\phi_i = \phi_{i+1}$ and $r_i = 1$ for all $i = \overline{1, N}$ (the case of synchronization). Differentiating this equality, we obtain:

$$\begin{aligned}0 &= \sum_{i=1}^N k(a_i + a_{i+1})r_i r_{i+1}\sin(\phi_i + \phi_{i+1} - 2t) \\ &\quad - r_i^2\left(b_i(r_i^2 - 1) + 2ka_i\right)\sin(2\phi_i - 2t),\end{aligned}$$

and differentiating once more:

$$\begin{aligned}0 &= \sum_{i=1}^N k(a_i + a_{i+1})r_i r_{i+1}\left(1 - 2\sin^2\left(\frac{\phi_i + \phi_{i+1}}{2} - t\right)\right) \qquad (26) \\ &\quad - r_i^2\left(b_i(r_i^2 - 1) + 2ka_i\right)\left(1 - 2\sin^2(\phi_i - t)\right).\end{aligned}$$

Finally combining (25) and (26), we derive a time-invariant equation

$$\begin{aligned}0 &= \sum_{i=1}^N k(a_i + a_{i+1})r_i r_{i+1}\cos(\phi_i - \phi_{i+1}) \\ &\quad - r_i^2\left(b_i(r_i^2 - 1) + 2ka_i\right),\end{aligned}$$

which describes all possible relations between ϕ_i and r_i for $i = \overline{1, N}$ such that the corresponding trajectories are in Ω'_∞. Note that by definition:

$$\rho_i^2 = r_i^2 + r_{i+1}^2 - 2r_i r_{i+1}\cos(\phi_i - \phi_{i+1}),$$

then we obtain

$$\begin{aligned}0 &= \sum_{i=1}^N (\rho_i^2 - r_i^2 - r_{i+1}^2)k(a_i + a_{i+1}) \\ &\quad + 2r_i^2\left(b_i(r_i^2 - 1) + 2ka_i\right), \qquad (27)\end{aligned}$$

which together with N equations in (24) form the system of $N + 1$ nonlinear algebraic equations for $2N$ unknowns (r_i and

ρ_i) describing the kind of synchronization in (14), (20) that is admissible in Ω'_∞ ((27) is not a linear combination of (24)).

Corollary 21. *Let all conditions of Theorem 20 be satisfied, and all solutions of* (24)*,* (27) *with* $r_i \neq 1$ *admit the restriction:*

$$r_i^2 < \frac{1}{3}\left(1 - 2\frac{ka_i}{b_i}\right) \tag{28}$$

for some $1 \leq i \leq N$*. Then for almost all initial conditions the system (14), (20) is synchronized.*

As we can note, $b_i > 2a_i k$ is a necessary condition for Corollary 21 to satisfy.

Proof: Let us consider the dynamics of the variables r_i:

$$\dot{r}_i = x_{2i}\frac{a_i k(x_{2(i-1)} + x_{2(i+1)}) - x_{2i}(b_i[|x_i|^2 - 1] + 2a_i k)}{r_i},$$

then considering only trajectories in Ω'_∞ and substituting $x_{2i} = r_i \sin(\phi_i - t)$ we obtain (the same equation can be derived considering (14), (20) in polar coordinates r_i and θ_i (the amplitude $r_i = \sqrt{x_{1i}^2 + x_{2i}^2}$ and phase $\theta_i = \arctan\left(\frac{x_{2i}}{x_{1i}}\right)$ of an oscillator) and selecting $\theta_i = \phi_i - t$):

$$\begin{aligned}\dot{r}_i = a_i k \sin(\phi_i - t)(r_{i-1}\sin(\phi_{i-1} - t) + r_{i+1}\sin(\phi_{i+1} - t))\\ - r_i \sin^2(\phi_i - t)(b_i[r_i^2 - 1] + 2a_i k).\end{aligned}$$

For any constant values r_i and ϕ_i, which constitute a solution of the system of equations (24) and (27), introduce linearization of the dynamics of r_i taking ϕ_i as constants:

$$\begin{aligned}\dot{\delta r_i} = -\delta r_i \sin^2(\phi_i - t)(b_i[3r_i^2 - 1] + 2a_i k)\\ + a_i k \sin(\phi_i - t)(\delta r_{i-1}\sin(\phi_{i-1} - t) + \delta r_{i+1}\sin(\phi_{i+1} - t)),\end{aligned}$$

where δr_i represents the deviation with respect to r_i for i^{th} oscillator in the linearized dynamics. Let us investigate a Lyapunov function showing instability of this time-varying system in the given equilibrium:

$$U(\delta r_1, \ldots, \delta r_N) = \frac{1}{2}\sum_{i=1}^{N}\delta r_i^2,$$

then

$$\begin{aligned}\dot{U} &= \sum_{i=1}^{N} a_i k \delta r_i \sin(\phi_i - t)\{\delta r_{i-1}\sin(\phi_{i-1} - t)\\ &\quad + \delta r_{i+1}\sin(\phi_{i+1} - t)\}\\ &\quad - \delta r_i^2 \sin^2(\phi_i - t)(b_i[3r_i^2 - 1] + 2a_i k)\\ &= \sum_{i=1}^{N} k[a_i + a_{i+1}]\delta r_i \sin(\phi_i - t)\delta r_{i+1}\sin(\phi_{i+1} - t)\\ &\quad - \delta r_i^2 \sin^2(\phi_i - t)(b_i[3r_i^2 - 1] + 2a_i k).\end{aligned}$$

It is easy to check that if the condition (28) is satisfied for some $1 \leq i \leq N$ and $\delta r_j = 0$ for all $1 \leq j \neq i \leq N$, then $\dot{U}(t) > 0$ for almost all instants of time $t \geq t_0 \geq 0$, which implies instability of the linearized dynamics. Applying the same Lyapunov function U to the original nonlinear system it is possible to prove its local instability at that equilibrium point. Finally, if $b_i[3r_i^2 - 1] + 2a_i k < 0$ (under the condition (28)), then $b_i[r_i^2 - 1] + 2a_i k < 0$ and the result follows Proposition 11 in [48]. ■

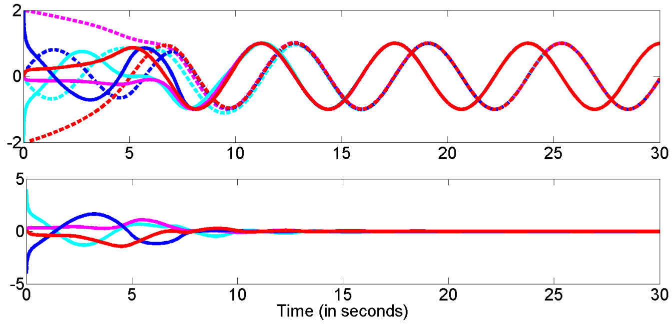

Figure 1. The results of simulation of (14), (20) for $N = 4$. Top) Oscillator states, solid lines - x_{2i}, broken lines - x_{1i}; Bottom) Control signals.

V. Examples and Simulations

To illustrate the theoretical results, we will consider $N = 4$ non-identical Brockett oscillators in (14) with parameters

$$a = [1.25\ 1\ 0.75\ 0.5]^T,\ b = [5\ 4.5\ 3.5\ 3]^T,\ k = 1.$$

The chosen parameters respect the necessary condition $b_i > 2a_i k$ of Corollary 21 for all $i = \overline{1, N}$. In order to check (28) the system of equations (24), (27) was solved using a Newton iterative method for 1000 random initial conditions. If the norm of the error in the equations (24), (27) on the last step was less than 0.1, then it was assumed that a solution to (24), (27) has been found and (28) was tested for the found values of r_i, and (28) was always verified. Then, according to Corollary 21, the system (14), (20) is synchronized and it converges to the unit circle.

The simulation results are given in Fig. 1. Experimental study of the results presented in this work can be found in [49].

VI. Conclusions

This paper studied the problem of global robust synchronization of non-identical Brockett oscillators. To this end, global stability and ISS analysis were done for an individual oscillator (with respect to the set $\mathcal{W}$ composed by the equilibrium at the origin and the limit cycle at the unit sphere). These results make Brockett oscillator a promising benchmark model for the investigation of synchronization and consensus phenomena. Next, two synchronization control strategies were proposed. The first one imposes restriction on the synchronization control amplitude and uses generic ISS arguments. The second synchronization control design is based on a special Lyapunov function proposed in this work, and it allows the kind of synchronous motions to be evaluated. Numerical simulations demonstrated the effectiveness of our method by applying it to networks of non-identical and identical Brockett oscillators.

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