Equilibrium Assignments in Competitive and Cooperative Traffic Flow Routing

Victor Zakharov, Alexander Krylatov

Equilibrium Assignments in Competitive and Cooperative Traffic Flow Routing

Victor Zakharov[1] and Alexander Krylatov[1]

[1] Saint-Petersburg state university, Universitetskaya nab. 7-9, 199034 Saint-Petersburg, Russia
mcvictor@mail.ru, aykrylatov@yandex.ru

Abstract. The goal of the paper is to demonstrate possibilities of collaborative transportation network to minimize total travel time of the network users. Cooperative and competitive traffic flow assignment systems in case of $m \geq 2$ navigation providers (Navigators) are compared. Each Navigator provides travel guidance for its customers (users) on the non-general topology network of parallel links. In both cases the main goals of Navigators are to minimize travel time of their users but the behavioral strategies are different. In competitive case the behavioral strategy of each Navigator is to minimize travel time of traffic flow of its navigation service users while in cooperative case – to minimize travel time of overall traffic flow. Competitive routing is formalized mathematically as a non-zero sum game and cooperative routing is formulated as an optimization problem. It is demonstrated that Nash equilibrium in the navigation game appears to be not Pareto optimal. Eventually it is shown that cooperative routing systems in smart transportation networked environments could give users less value of travel time than competitive one.

Keywords: competitive routing, cooperative routing, traffic flow assignment, Wardrop equilibrium, Nash equilibrium, Pareto optimality.

1 Introduction

One of the important trends during last decades is essential increasing the amount of navigation providers on the roads. The influence of competitive navigation services on a traffic flow assignment on transportation networks is not as good as expected. The problem is that competitive routing proposed by navigation systems to minimize travel time of their customers in case of huge amount of cars driving through the network could leads to decreasing the total travel time of network users. The basic reason for that is the conflict of interests between different navigation providers. From game theory point of view it means that Nash equilibrium traffic assignment leads to less value of total travel time compare to Pareto optimal solution.

Collaboration of navigation provider can be mathematically formalized as centralized traffic assignment which guaranties user equilibrium or system optimum on transportation network. The main principles for collaborative traffic flow assignments were formulated in 1952 by Wardrop who stated two principle of equilibrium traffic flow assignment in transportation networks [1]. In 1956 Beckmann proposed mathematical formalization of Wardrop's principles that is today considered as classical [2, 3]. One of the most disadvantages of this model is computational complexity for using in large networks. For details one can read articles [4, 5]. In the present paper we offer a method for finding in Wardrop's model the traffic flow assignment strategies explicitly in case of linear BPR-delay function.

Considering *competitive routing* we use Nash equilibrium optimality principle. We guess that comparison of cooperative and competitive routing efficiency is highly important to make a dicision either collaborative or competitive principle should be taken as a basis for smart transportation networked environments. Thus relation between Wardrop equilibrium and Nash equilibrium has to be explored. For the first time this issue was raised in [6] where origin-destination areas had been considered as players. Unfortunately such problem formulation as available in [6] is not connected with competitive routing. Another works on this topic mostly have not any analytical representation, for example [7]. In this work we compare Wardrop equilibrium and Nash equilibrium expressed explicitly and show that cooperative routing systems in smart transportation networked environment could give users less value of travel time than competitive one.

Thereby the present work is devoted to the following three interrelated problems: 1) receiving the explicit form of traffic flow assignment strategies; 2) finding relationship between Wardrop and Nash equilibria; 3) comparison of collaborative and competitive smart transportation networked environments.

2 Mathematical Models of Competitive and Cooperative Traffic Flow Routing

In this section we formulate competitive and cooperative traffic flow routing problems mathematically and find corresponding equilibrium strategies of assignments.

First of all, it should be noted that analyzing an arbitrary transportation network we rely on the idea according to which: any transportation network could be decompose to the set of subnets consisting of one origin-destination pair and certain amount of parallel routes [8, 9, 10]. On the one hand, such an idea is reasonable due to the fact that narrowing of the road (using the same link in different routes) leads to congestion [10]. On the other hand, parallel structure of transportation network contributes to avoiding the Braess's paradox [8, 9].

Hereby we consider transportation subnet presented by digraph consisted of one origin-destination pair and n parallel links. On this subset m Navigators act. Let us introduce the following notation: $N = \{1,\ldots,n\}$ – set of numbers of routes; $M = \{1,\ldots,m\}$ – set of numbers of Navigators; i – number of the route, $i \in N$; j,q – numbers of Navigators, $j,q \in M$; $F^j > 0$ – traffic flow value of Navigator j; $F = \sum_{j=1}^{m} F^j$ – aggregate traffic flow value of all Navigators; $f_i^j \geq 0$ – traffic flow value of Navigator j through route i; $f_i = \left(f_i^1,\ldots,f_i^m\right)$ – vector of traffic flow values of all

Navigators assigned through route i, wherein $f_i^{-j} = \left(f_i^1,\dots,f_i^{j-1},f_i^{j+1},\dots,f_i^m\right)$; $F_i = \sum_{j=1}^{m} f_i^j$ – traffic flow value through route i; $t_i^0 > 0$ – free travel time through route i; $c_i > 0$ – capacity of i-th route; $d_i(F_i)$ – delay of F_i traffic flow value on route i. Vector of strategies of Navigator j is $f^j = \left(f_1^j,\dots,f_n^j\right)^T$ and $f = \left(f^1,\dots,f^m\right)$.

Now we are ready to formulate competitive and cooperative traffic flow routing problems mathematically. According to [3] cooperative case is expressed by the following optimization problem:

$$\min_f z^{cpr} = \min_f \sum_{i=1}^{n} d_i(F_i)\ F_i\,, \tag{1}$$

with constraints

$$\sum_{i=1}^{n} F_i = F\,, \tag{2}$$

$$f_i^j \geq 0 \quad \forall\, i \in N,\, j \in M, \tag{3}$$

while competitive case can be expressed by the following non-zero sum game between Navigators:

$$\min_{f^j} z_j^{cmp} = \min_{f^j} \sum_{i=1}^{n} d_i(F_i)\ f_i^j \quad \forall\, j \in M, \tag{4}$$

with constraints

$$\sum_{i=1}^{n} f_i^j = F^j \quad \forall\, j \in M, \tag{5}$$

$$f_i^j \geq 0 \quad \forall\, i \in N,\, j \in M, \tag{6}$$

Hereby we can see that in cooperative case Navigators try to achieve system optimum of Wardrop for whole traffic flow [3], whereas in competitive case – each Navigator tries to achieve system optimum only for its own customers. However in competitive case any Navigator j is affected by other Navigators so that mutual influence of Navigators' strategies addresses us to find Nash equilibrium. Further we will use explicit BPR-delay function: $d_i(F_i) = t_i^0\left(1 + \frac{F_i}{c_i}\right)$ [11]. Without loss of generality to find Nash equilibrium we assume

$$t_1^0 < t_2^0 < \ldots < t_n^0. \tag{7}$$

Theorem 1. Subject to (7) Nash equilibrium in the game of Navigators is achieved by the following strategies

$$f_i^{j*} = b_i^j - \frac{1}{m+1}\sum_{q=1}^{m} b_i^q, \tag{8}$$

where

$$b_i^j = \frac{c_i}{t_i^0}\frac{F + \sum_{s=1}^{m} F^s + \sum_{r=1}^{n} c_r}{\sum_{r=1}^{n} \frac{c_r}{t_r^0}} - c_i, \tag{9}$$

$\forall\ i \in N$, when

$$F^j > \frac{1}{m+1}\sum_{i=1}^{n} c_i\left(\frac{t_n^0}{t_i^0} - 1\right), \tag{10}$$

$\forall\, j \in M$.

Theorem 1 provides explicit optimal strategies for Navigators in the case of competitive traffic flow routing. To find optimal strategies in case of cooperative routing it is sufficiently to equate m to 1 in formulas (8)-(10). Indeed (1) shows that in cooperative case collaboration of Navigators is reduced to finding system optimal assignments on all routes for whole traffic flow (unlike optimal assignments on all routes for each Navigator in competitive routing). In other words in cooperative case Navigators try to find F_i^* and it is clear that in such a case f^* is not unique and limited only by condition $F_i^* = \sum_{j=1}^{m} f_i^j\ \ \forall\ i \in N$.

Theorem 2. Total travel time of whole traffic flow in case of cooperative routing is strictly less than in competitive case (Nash equilibrium is not Pareto optimal).

Theorem 2 states that Wardrop equilibrium leads to less travel time than Nash equilibrium. Moreover it indicates that applying cooperative or centralized traffic flow navigation systems in smart networked environments is more preferable than competitive systems in terms of travel time value.

3 Numerical Experiments

In previous section it was shown that employing of cooperative or centralized traffic navigation system in smart transportation networked environments leads to less travel time value than competitive systems. To illustrate this result we have investigated transportation network of one of the central districts in Saint-Petersburg – Vasileostrovsky district.

Consider Fig. 1. Denote two areas: origin area (red bold circle) and destination area (green bold circle). Moreover there are 4 potential routes from origin to destination: red line, orange line, blue line and green line.

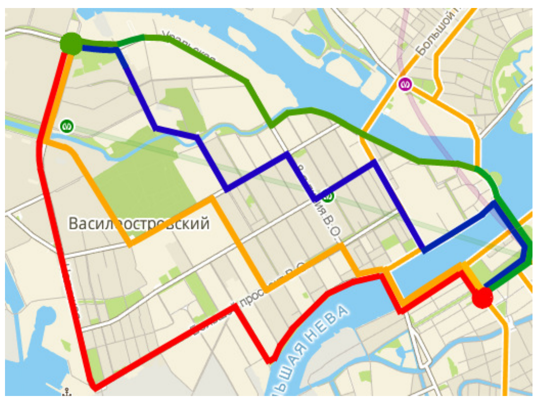

Fig. 1. Transportation network of Vasileostrovsky district of Saint-Petersburg.

Investigated district has the following parameters: $t_1^0 = 7{,}5$ and $c_1 = 300$ (orange line), $t_2^0 = 9$ and $c_2 = 400$ (blue line), $t_3^0 = 12$ and $c_3 = 500$ (green line), $t_4^0 = 13{,}5$ and $c_4 = 600$ (red line). Then we assume that the flow $F = 1\,000$ has to be assigned and we compare final travel time of whole traffic flow when there is 1 (cooperative or centralized system), 2, 4 and 10 competing Navigators acting on the network. It is to be noted that in case of $m > 1$ Navigators we devide whole flow F between Navigators equally. Firstly we propose that there are only 3 routes (orange, blue and green lines). Results are shown in Table 1.

Table 1. Dependence of goal-function value on amount of competing Navigators (3 routes).

Amount of Navigators, m	$\sum_{j=1}^{m} z_j^{cmp}\left(f^*\right)$

1 (cooperation)	17337,06
2	17349,05
4	17375,89
10	17409,27

Thereby, when there is a little network with only three possible routes the more competing Navigators give assignment with the worse travel time of whole traffic flow F but it is not so crucial. Let us increase amount of routes adding just one new route. In such a case results are shown in table 2.

Table 2. Dependence of goal-function value on amount of competing Navigators (4 routes).

Amount of Navigators, m	$\sum_{j=1}^{m} z_j^{cmp}\left(f^*\right)$
1 (cooperation)	16178,624
2	16205,103
4	16264,417
10	16338,157

We can see that the bigger transportation network the worse results given by more competing Navigators.

It has to be mentioned that classical algorithms such as Frank-Wolfe algorithm would demand much more operations for getting the same results. Indeed it needs information about all links in the network and, consequently, dimension of the problem grows extremely when the network becomes larger. Interested one can see [5].

Proposed simple example of only one district of Saint-Petersburg shows that application of collaborative systems in smart transportation networked environments in large cities is more reasonable than cooperative systems.

4 Conclusion

In this work we conducted a comparative analysis of two smart transportation networked environments: one based on cooperative routing and another based on competitive routing. Mathematical expression of cooperative and competitive cases led to the conclusion that cooperative navigation systems could provide aggregate traffic flow of all transportation network users with less total travel time than competitive one. It is clear that government of any large city is interested in improving the transportation situation uniformly on the entire network. Thereby in terms of governance collaborative traffic flow navigation systems based on explicit assignment strategies in smart transportation networked environments are highly accurate and operative equipment for decision making in transportation area.

In our further works we are going to generalize obtained results for the large transportation networks. For this purpose we are investigating whole transportation network of Saint-Petersburg city using developed technique.

References

1. Wardrop, J.G.: Some Theoretical Aspects of Road Traffic Research. Proc. Inst. Civ. Eng. 2(1), 325--378 (1952)
2. Beckmann, M.J., McGuire, C.B., Winsten, C.B.: Studies in the Economics of Transportation. CT: Yale University Press, New Haven (1956)
3. Sheffi, Y.: Urban Transportation Networks: Equilibrium Analysis with Mathematical Programming Methods. Prentice-Hall Inc., New Jersey (1985)
4. Frank, M., Wolfe, P.: An Algorithm for Quadratic Programming. Naval Research Logistics Quarterly. 3, 95--110 (1956)
5. Shvetsov, V.I.: Algorithms for Distributing Traffic Flows. Automation and Remote Control. 10, 148--157 (2009)
6. Haurie, A., Marcotte, P.: On the Relationship between Nash-Cournot and Wardrop Equilibria. Networks. 15, 295--308 (1985)
7. Altman, E., Wynter, L.: Equilibrium, Games, and Pricing in Transportation and Telecommunication Networks. Networks and Spatial Economics. 4, 7--21 (2004)
8. Korilis, Y.A., Lazar, A.A., Orda, A.: Architecting Noncooperative Networks. IEEE Journal on selected areas in communications. 13(7), 1241--1251 (1995)
9. Korilis, Y.A., Lazar, A.A., Orda, A.: Avoiding the Braess Paradox in Non-Cooperative Networks. J. Appl. Prob. 36, 211--222 (1999)
10. Daganzo, C.F.: The Cell Transmission Model: A Dynamic Representation of Highway Traffic Consistent with the Hydrodynamic Theory. Transpn. Res. B. 28, 269--287 (1994)
11. Traffic Assignment Manual. In: U.S. Bureau of Public Roads (eds.) U.S. Department of Commerce. Washington, D.C. (1964)
12. Zakharov, V., Krylatov, A., Ivanov, D.: Equilibrium Traffic Flow Assignment in Case of Two Navigation Providers. In: Camarinha-Matos, L., Scherer, R. (eds.) Collaborative Systems for Reindustrialization, Proceedings of the 14th IFIP Conference on Virtual Enterprises PRO-VE 2013, pp. 156--163. Springer, Berlin (2013)

Appendix

Proof of Theorem 1. Using Theorem 1 from [12] we can get the following expression

$$f_i^{1*} + \ldots + 2 f_i^{j*} + \ldots + f_i^{m*} = \left(\frac{\omega_j}{t_i^0} - 1 \right) c_i, \quad \forall j \in M. \qquad \textbf{(11)}$$

Summing such expression by i and j we get

$$\sum_{s=1}^{m} F^s + F^j = \sum_{i=1}^{n} \left(\frac{\omega_j}{t_i^0} - 1 \right) c_i, \qquad \textbf{(12)}$$

and, consequently,

$$\omega_j = \frac{F^j + \sum_{s=1}^{m} F^s + \sum_{i=1}^{n} c_i}{\sum_{i=1}^{n} \frac{c_i}{t_i^0}}.$$

Summing (11) by i and expressing F^j we eventually obtain the condition (10) subject to (7). ■

Proof of Theorem 2. Union of domains of functions $z_j^{cmp}(f)$ are wider than domain of function $z^{cpr}(f)$, thereby $\sum_{j=1}^{m} z_j^{cmp}(f^*) \geq z^{cmp}(f^*)$. Moreover equality is possible if and only if $m = 1$. Consequently, $\sum_{j=1}^{m} z_j^{cmp}(f^*) > z^{cpr}(f^*)$. ■