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Distance-preserving orderings in graphs*

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Abstract

For every connected graph G , a subgraph H of G is *isometric* if for every two vertices $x, y \in V(H)$ there exists a shortest xy -path of G in H . A *distance-preserving elimination ordering* of G is a total ordering of its vertex-set $V(G)$, denoted (v_1, v_2, \dots, v_n) , such that any subgraph $G_i = G \setminus (v_1, v_2, \dots, v_i)$ with $1 \leq i < n$ is isometric. This kind of ordering has been introduced by Chepoi in his study on weakly modular graphs. In this note we prove that it is NP-complete to decide whether such ordering exists for a given graph — even if it has diameter at most 2. Then, we describe a heuristic in order to compute a distance-preserving ordering when it exists one that we compare to an exact exponential algorithm and an ILP formulation for the problem. Lastly, we prove on the positive side that the problem of computing a distance-preserving ordering when it exists one is fixed-parameter-tractable in the treewidth.

Keywords: distance-preserving elimination ordering; metric graph theory; NP-complete; exact exponential algorithm; integer linear programming; bounded treewidth.

1 Introduction

Elimination orderings of a graph are total orderings over its vertex-set. Many interesting graph problems can be specified in terms of the existence of an elimination ordering with some given properties. These range from some practical problems in molecular biology and chemistry [8] to the analysis of graph search algorithms [14] and the characterization of some graph classes [10]. On the computational point of view, vertex ordering characterizations of a given graph class often lead to efficient (polynomial-time) recognition algorithms for the graphs in this class [24, 6, 15, 21, 2]. In this work we will consider one specific kind of elimination ordering that is called *distance-preserving elimination ordering* [11]. Precisely, let us remind that a subgraph H of G is *isometric* if the distance between any two vertices in H is the same in H as in G . An elimination ordering (v_1, v_2, \dots, v_n) of G is distance-preserving if it satisfies that each suffix $(v_i, v_{i+1}, \dots, v_n)$ with $i < n$ induces an isometric subgraph of G .

Distance-preserving elimination orderings encompass several other elimination orderings found in the literature [24, 6, 7, 22, 23, 19], all of which can be computed in polynomial time when they exist. In particular, known refinements of distance-preserving elimination orderings comprise the perfect elimination orderings [24], maximum neighbourhood orderings [6], h-extremal orderings [7], semisimplicial elimination orderings [22], dismantlable orderings [23] and more generally *domination elimination orderings* [19]. The latter orderings characterize chordal graphs, dually chordal graphs, homogeneously orderable graphs, cop-win graphs and a subclass of tandem-win graphs [12] respectively, and as above stated they all can be computed in polynomial-time when they exist. However the complexity of deciding whether a distance-preserving elimination ordering exists in a given graph has been left open until this paper. We aim at completing the picture and characterizing the complexity of this problem.

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Related work. In [17] we proved that every graph with a distance-preserving elimination ordering has a *minimum-size cycle basis* with only triangles and quadrangles, that can be easily computed if a distance-preserving elimination ordering is part of the input. This property has been useful in our study on some tree-likeness invariants of graphs (*e.g.*, in comparing treewidth with treelength). However, we left open the complexity of recognizing graphs with a distance-preserving elimination ordering.

Prior work [11, 9] has focused on the existence of distance-preserving elimination orderings in some well-structured graph classes, *i.e.*, the *weakly modular graphs*. In particular, it has been proved recently in [9] that every breadth-first search ordering of a weakly modular graph is distance-preserving, that allows to compute one such ordering in linear time for a given graph in this class.

On the positive side, above stated refinements of distance-preserving elimination orderings [24, 6, 7, 22, 23, 19] can all be computed with greedy algorithms when they exist. Indeed, for all these orderings it can be tested in polynomial-time whether a given vertex can be eliminated first. As example, any dominated vertex can be the starting vertex of some domination elimination ordering. A first hint that computing a distance-preserving elimination ordering can be more difficult is that it is not that simple to choose a starting vertex. For instance, consider the wheel W_5 obtained from a cycle C_5 of length five by adding a universal vertex. Every elimination ordering of W_5 where the universal vertex is the last vertex eliminated is distance-preserving. However, if the universal vertex is eliminated first then the cycle C_5 is an isometric subgraph of W_5 that does not admit a distance-preserving elimination ordering.

Our contributions. We prove on the negative side that it is NP-complete to decide whether a given graph admits a distance-preserving elimination ordering. We find it surprising since as above stated, a broad range of distance-preserving orderings with additional properties can be computed in polynomial time when they exist. Then we show that the problem remains NP-complete even for general graphs with diameter at most two. Note that in a sense our result is optimal w.r.t. the diameter because complete graphs trivially admit a distance-preserving ordering. Our reduction will show how to encode a 3-SAT formula in a graph whose distance-preserving orderings are in many-to-many correspondance with the satisfying assignments for the formula. This line of work resembles to the one in [25] in order to show that it is NP-complete to recognize collapsible complexes. Our work differs from theirs in that we study orderings with very distinct properties and the “simpler” structure of graphs —w.r.t. complexes— further constrains our gadgets to mimic variables and clauses of the formula.

For a given n -vertex graph, finding a distance-preserving ordering by using exhaustive search would require $\mathcal{O}^*(n!)$ -time. Next, we improve on this result by noticing that a meta-theorem on vertex-orderings [3] can be applied to our problem, that leads to a $\mathcal{O}^*(2^n)$ -time complexity. We also propose an Integer Linear Programming formulation which may lead to a better complexity in practice. These two exact algorithms are described and compared to a heuristic in Section 4. Finally, we prove on a more positive side that the problem of computing a distance-preserving ordering when it exists one is fixed-parameter-tractable in the treewidth (Section 5).

Notations. Graphs in this study are finite, simple (hence without loop nor multiple edges) and unweighted. We refer to [5, 20] for standard reference books on graphs (see also [1] for a survey about metric graph theory). Let (v_1, v_2, \dots, v_n) be an elimination ordering of a graph G , we say that vertex v_i , $1 \leq i \leq n$, is the i^{th} vertex to be eliminated, and that vertex v_i is eliminated before vertex v_j , denoted $v_i \prec v_j$, if $i < j$.

2 Local characterization

In what follows, we will avoid considering all the distances in the graph at each time a vertex is eliminated. The following characterization will explain how to do so.

Lemma 1. *Let $G = (V, E)$ and $u \in V$, the subgraph $G \setminus u$ is isometric if and only if every two non-adjacent neighbours of vertex u have at least two common neighbours in G (including u).*

Proof. If $G \setminus u$ is isometric, then let $x, y \in N_G(u)$ be non-adjacent. Since $d_{G \setminus u}(x, y) = d_G(x, y) = 2$, x and y have another common neighbour than vertex u . Conversely, suppose that every two non-adjacent neighbours of vertex u have at least two common neighbours in G . In particular, every of them have at least one common neighbour in $G \setminus u$. Then, for every two non-adjacent $x, y \in N_G(u)$ the subpath

(x, u, y) can be substituted in any shortest-path of G with the subpath (x, v, y) of $G \setminus u$, where v denotes a common neighbour of x, y . This proves that $G \setminus u$ is an isometric subgraph. \square

Corollary 2. *An elimination ordering \prec of $G = (V, E)$ is distance-preserving if and only if for every $u, v \in V$ at distance $d_G(u, v) = 2$, there is $w \in N_G(u) \cap N_G(v)$ such that $u \prec w$ or $v \prec w$.*

Proof. Let (v_1, v_2, \dots, v_n) be the elimination ordering we consider. For every $0 \leq i < n$, define $G_i = G \setminus \{v_1, \dots, v_{i-1}\}$ (in particular $G_0 = G$). On the one direction, suppose that \prec is distance-preserving. Let $v_i, v_j \in V$ satisfy $d_G(v_i, v_j) = 2$ with $i < j$. Since \prec is distance-preserving, G_i is an isometric subgraph of G . Hence, since $v_i, v_j \in V(G_i)$ and $d_G(v_i, v_j) = 2$, there exists $w \in N_G(v_i) \cap N_G(v_j)$ such that $w \in V(G_i)$, i.e., $v_i \prec w$. On the other direction, suppose that \prec is not distance-preserving. Let $i \geq 0$ be the least index such that G_i is an isometric subgraph of G but $G_{i+1} = G_i \setminus v_i$ is not. By Lemma 1, there exist $x, y \in N_G(v_i) \cap V(G_i)$ nonadjacent such that $N_G(x) \cap N_G(y) \cap V(G_{i+1}) = \emptyset$. In particular, there does not exist any $w \in N_G(x) \cap N_G(y)$ such that $x \prec w$ or $y \prec w$ (else, $w \in V(G_{i+1})$). \square

Sometimes, it may be easier to group vertices into subsets whose vertices can be eliminated in an arbitrary way. On such occasions, we will base on the following consequence of Lemma 1.

Corollary 3. *Let G be a graph, $S \subseteq V(G)$ satisfy that for every $u \in S$, every two non-adjacent neighbours of vertex u have a common neighbour in $G \setminus S$. Then, for any $S' \subseteq S$, the subgraph $G \setminus S'$ is isometric.*

Proof. By contradiction, let $S' \subseteq S$ falsify the corollary with S' being of minimum size w.r.t. this property. Let $u \in S', S'' := S' \setminus u$. The subgraph $G \setminus S''$ is isometric by the minimality of S' . Furthermore, by the hypothesis every two non-adjacent neighbours of u have a common neighbour in $G \setminus S$, hence in $G \setminus S'$ so, $G \setminus S'$ is isometric by Lemma 1. This contradicts the fact that S' falsifies the corollary. \square

3 Hardness results

The purpose of this section is to prove the following result.

Theorem 4. *Deciding whether a given graph G admits a distance-preserving elimination ordering is NP-complete, already if G has diameter at most two.*

Note that since the all-pairs-shortest-paths in a graph can be computed in polynomial-time then it easily follows that the problem is in NP and so, we will only prove the NP-hardness. We will first prove that deciding whether a given graph G admits a distance-preserving elimination ordering is NP-hard, already if G has diameter at most five. This first part of the proof is involved and it is based on a technical reduction from 3-SAT, the standard NP-complete problem [13]. Then, we will show how to lower the diameter to two.

3.1 Main reduction

Given a formula Φ with n variables and m clauses of exactly three literals each, the 3-SAT problem aims to decide whether it exists a boolean assignment of the variables which makes the formula true. In case it does, then the formula Φ is said satisfiable. We will construct a graph G_Φ from an arbitrary formula Φ so that there is a distance-preserving elimination ordering of G_Φ if and only if Φ is satisfiable. This will prove the NP-hardness of our problem. To achieve the result, assume w.l.o.g. that no literal and its negation can be contained in the same clause of Φ (else, any such clause could be removed from Φ), and every variable appears both positively and negatively in the clauses of Φ (else, any clause containing either this variable or its negation could also be removed from Φ). Let us denote by x_1, x_2, \dots, x_n the n variables, and by C_1, C_2, \dots, C_m the m clauses of Φ . The graph G_Φ is defined as follows.

Variable gadget. For every variable x_i , $1 \leq i \leq n$, let us add in G_Φ an induced quadrangle $(x_i, y_i, \bar{x}_i, \bar{y}_i)$. If x_i is in the j^{th} clause of the formula then four more vertices $a_{ij}, b_{ij}, c_{ij}, d_{ij}$ are added and made adjacent to vertex x_i . Similarly if \bar{x}_i is in the j^{th} clause of the formula then four more vertices $a_{ij}, b_{ij}, c_{ij}, d_{ij}$ are added and made adjacent to vertex \bar{x}_i (this is clearly defined because no clause contains both literals x_i, \bar{x}_i by the hypothesis). We refer to Figure 1 for an illustration.

To better understand the role played by the quadrangle $(x_i, y_i, \bar{x}_i, \bar{y}_i)$ in our reduction, we make the following observation that captures well the difficulty of the problem. Indeed, every quadrangle admits

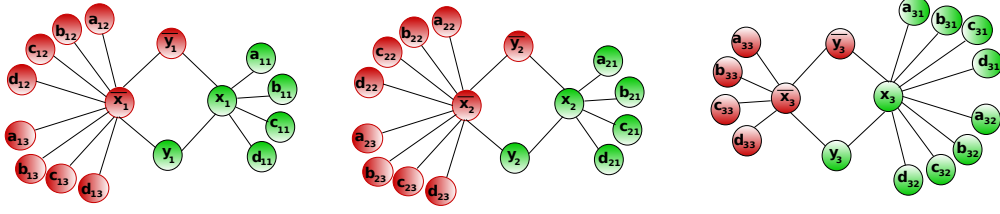


Figure 1: The three variable gadgets for the formula $\Phi = (x_1 \vee x_2 \vee x_3) \wedge (\bar{x}_1 \vee \bar{x}_2 \vee \bar{x}_3) \wedge (\bar{x}_1 \vee \bar{x}_2 \vee x_3)$.

a distance-preserving ordering, furthermore the first vertex to be eliminated can be chosen arbitrarily, but then we are partly constrained for the following as the diametrically opposed vertex mustn't be the second one to be eliminated. We will make use of a similar trick in our reduction so as to mimic a truth table with variable gadgets, ensuring that the second vertex to be eliminated in x_i, \bar{x}_i must be eliminated after one of each pair x_i, \bar{x}_i has already been eliminated.

Clause tree. Second, a rooted tree of depth two with $8m + 1$ vertices is added in G_Φ . More precisely, the tree is rooted at some newly added vertex r_Φ that has $2m$ children denoted by $s_1, t_1, s_2, t_2, \dots, s_m, t_m$. Informally, for every $1 \leq j \leq m$ both nodes s_j, t_j represent the j^{th} clause of Φ . In particular let $C_j = l_p \vee l_q \vee l_r$ with $p < q < r$ and $l_i \in \{x_i, \bar{x}_i\}$ for every $i \in \{p, q, r\}$. Then, the internal node s_j has three children denoted by $u_j(p, q), u_j(q, r)$ and $u_j(r, p)$, similarly the internal node t_j has three children denoted by $v_j(p, q), v_j(q, r)$ and $v_j(r, p)$. Moreover, any leaf node $u_j(p, q)$ is made adjacent to the pair of vertices a_{pj}, b_{qj} , and in the same way any leaf node $v_j(r, p)$ is made adjacent to the pair of vertices c_{rj}, d_{pj} . We refer to Figure 2 for an illustration.

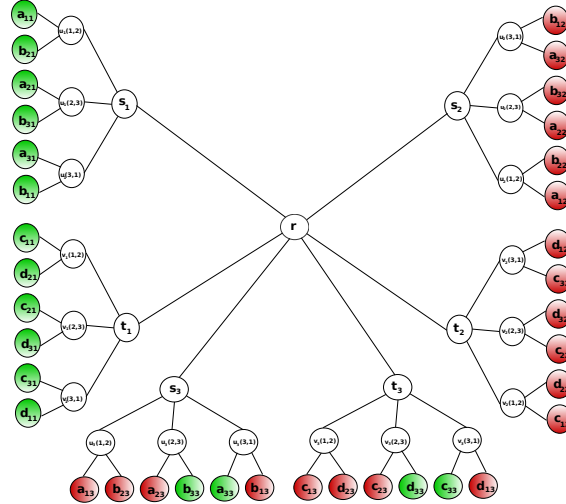
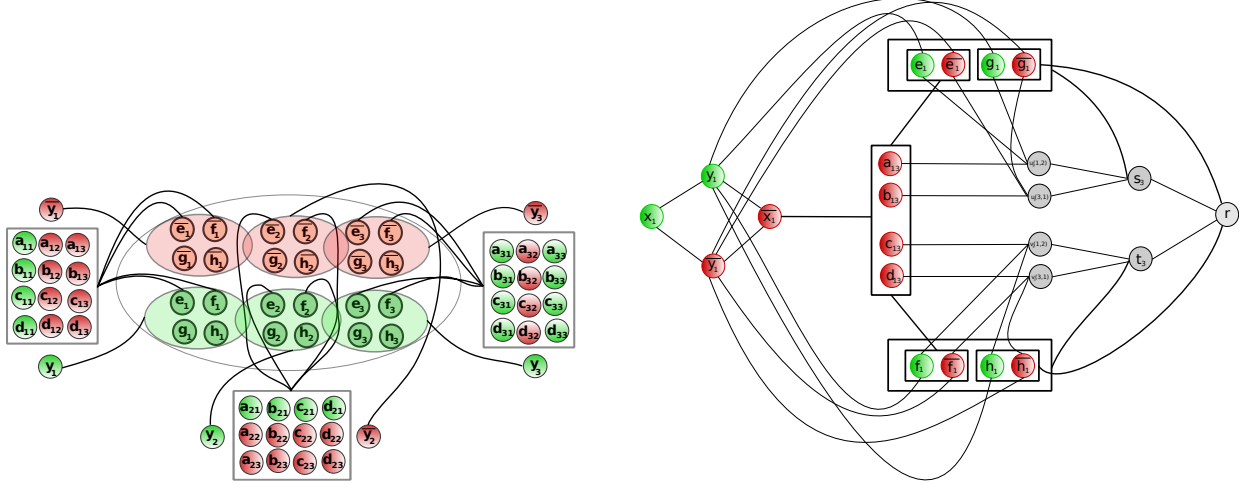


Figure 2: The clause tree for the formula $\Phi = (x_1 \vee x_2 \vee x_3) \wedge (\bar{x}_1 \vee \bar{x}_2 \vee \bar{x}_3) \wedge (\bar{x}_1 \vee \bar{x}_2 \vee x_3)$.

Our reduction will ensure that r_Φ is the unique common neighbour of s_j, t_j in G_Φ . Consequently, in any distance-preserving ordering of G_Φ one of s_j, t_j will need to precede vertex r_Φ , that will imply that the j^{th} clause of Φ is satisfied.

Literal clique. The final and most technical part of our reduction is to construct a clique of G_Φ with $8n$ vertices so as to ensure that a distance-preserving ordering exists if Φ is satisfiable. For every $1 \leq i \leq n$, there are four vertices denoted by e_i, f_i, g_i, h_i (related to variable x_i), in the same way there are four vertices denoted by $\bar{e}_i, \bar{f}_i, \bar{g}_i, \bar{h}_i$ (related to the negated variable \bar{x}_i). Moreover, vertex y_i is made adjacent to every of the four vertices e_i, f_i, g_i, h_i , and in the same way vertex \bar{y}_i is made



(a) Adjacency relations between vertices from the variable gadgets and those from the literal clique.

(b) Adjacency relations w.r.t. literal \bar{x}_1 and clause $C_3 = \bar{x}_1 \vee \bar{x}_2 \vee x_3$.

Figure 3: The literal clique, for the formula $\Phi = (x_1 \vee x_2 \vee x_3) \wedge (\bar{x}_1 \vee \bar{x}_2 \vee \bar{x}_3) \wedge (\bar{x}_1 \vee \bar{x}_2 \vee x_3)$.

adjacent to every of the four vertices $\bar{e}_i, \bar{f}_i, \bar{g}_i, \bar{h}_i$. For every $1 \leq j \leq m$ such that one of x_i, \bar{x}_i is a literal of C_j , the four vertices a_{ij}, b_{ij}, c_{ij} and d_{ij} are made adjacent to every of the four vertices e_i, f_i and \bar{e}_i, \bar{f}_i .

Let $C_j = l_p \vee l_q \vee l_r$ with $p < q < r$ and $l_i \in \{x_i, \bar{x}_i\}$ for every $i \in \{p, q, r\}$, then:

- the three vertices $u_j(p, q), u_j(q, r), u_j(r, p)$ are respectively made adjacent to the quartets of vertices: e_p, g_p and \bar{e}_q, \bar{g}_q ; e_q, g_q and \bar{e}_r, \bar{g}_r ; e_r, g_r and \bar{e}_p, \bar{g}_p ;
- similarly, the three vertices $v_j(p, q), v_j(q, r), v_j(r, p)$ are respectively made adjacent to the quartets of vertices: f_p, h_p and \bar{f}_q, \bar{h}_q ; f_q, h_q and \bar{f}_r, \bar{h}_r ; f_r, h_r and \bar{f}_p, \bar{h}_p ;
- last, vertex s_j is made adjacent to the twelve vertices e_i, g_i and \bar{e}_i, \bar{g}_i with $i \in \{p, q, r\}$; similarly, vertex t_j is made adjacent to the twelve vertices f_i, h_i and \bar{f}_i, \bar{h}_i with $i \in \{p, q, r\}$.

Let $\mathcal{E} = \{e_i \mid 1 \leq i \leq n\} \cup \{\bar{e}_i \mid 1 \leq i \leq n\}$, $\mathcal{F} = \{f_i \mid 1 \leq i \leq n\} \cup \{\bar{f}_i \mid 1 \leq i \leq n\}$, $\mathcal{G} = \{g_i \mid 1 \leq i \leq n\} \cup \{\bar{g}_i \mid 1 \leq i \leq n\}$ and $\mathcal{H} = \{h_i \mid 1 \leq i \leq n\} \cup \{\bar{h}_i \mid 1 \leq i \leq n\}$ partition the clique. The root vertex r_Φ of the clause tree is made adjacent to every vertex in $\mathcal{G} \cup \mathcal{H}$. We refer to Figure 3 for a partial illustration.

The resulting graph G_Φ has diameter at most five. Indeed, all vertices but the x_i, \bar{x}_i with $1 \leq i \leq n$ are adjacent to the literal clique, therefore it is a 2-distance dominating clique. We will show later how to lower the diameter. Note that several vertices play almost identical roles in the reduction. This redundancy is necessary in order to ensure that most pairs of vertices that are at distance two in G_Φ only have one common neighbour. Indeed, the latter will impose necessary conditions on an elimination ordering of G_Φ to be distance-preserving.

3.2 Proof of correctness

We are now ready to prove that it is NP-hard to decide whether a given graph G admits a distance-preserving elimination ordering. We divide the proof in two propositions, as follows.

Proposition 5. *If Φ is satisfiable, then G_Φ admits a distance-preserving ordering.*

Proof. Let us fix a boolean assignment of the variables x_i satisfying Φ , that exists by the hypothesis. In particular, let $\{l_i, \bar{l}_i\} = \{x_i, \bar{x}_i\}$ be such that l_i is true, let $V_0 = \{\bar{l}_i \mid 1 \leq i \leq n\}$ and let $V_1 = \{l_i \mid 1 \leq i \leq n\}$. Now, consider the following partition of the vertex-set of G_Φ into eleven subsets S_k , with $1 \leq k \leq 11$. Let $G_0 := G$, and let $G_k := G_{k-1} \setminus S_k$ for every $1 \leq k < 11$. We will exhibit from the partition a distance-preserving ordering of G_Φ . To do so, we will prove that for every $1 \leq k \leq 11$, the pair $\langle G_{k-1}, S_k \rangle$ satisfies the sufficient condition of Corollary 3.

- Let $S_1 = V_1$. Let $l_i \in S_1$ be arbitrary. Neighbours of l_i in G_0 are y_i, \bar{y}_i and every of $a_{ij}, b_{ij}, c_{ij}, d_{ij}$ such that $l_i \in C_j$. Let $\alpha, \beta \in N_{G_0}(l_i)$ be non-adjacent. There are four subcases.
 - if $\{\alpha, \beta\} = \{y_i, \bar{y}_i\}$ then \bar{l}_i is a common neighbour of α, β ;
 - if one of α, β is equal to y_i and the other is amongst $a_{ij}, b_{ij}, c_{ij}, d_{ij}$ for some j , then e_i, f_i are common neighbours of α, β ;
 - similarly, if one of α, β is equal to \bar{y}_i and the other is amongst $a_{ij}, b_{ij}, c_{ij}, d_{ij}$ for some j , then \bar{e}_i, \bar{f}_i are common neighbours of α, β ;
 - if α is amongst $a_{ij}, b_{ij}, c_{ij}, d_{ij}$ for some j , and β is amongst $a_{ij'}, b_{ij'}, c_{ij'}, d_{ij'}$ for some j' , then e_i, f_i and \bar{e}_i, \bar{f}_i are common neighbours of α, β .

Therefore, in all cases α, β have a common neighbour in G_1 , and Corollary 3 applies.

- Let S_2 contain every $a_{ij}, b_{ij}, c_{ij}, d_{ij}$ such that clause C_j is satisfied by l_i . Let $w \in S_2$ be arbitrary. There exist j, p, q, r such that neighbours of w in G_1 are composed of $e_p, f_p, \bar{e}_p, \bar{f}_p$ and one of $u_j(p, q), u_j(r, p), v_j(p, q)$ or $v_j(r, p)$. Let $\alpha, \beta \in N_{G_1}(w)$ be non-adjacent. Note that w has only one neighbour in G_1 that is not in the literal clique. Consequently, one of α, β is amongst $u_j(p, q), u_j(r, p), v_j(p, q), v_j(r, p)$. Since the latter four vertices have some neighbour in the literal clique by construction, therefore α, β have a common neighbour in G_2 and Corollary 3 applies.
- Let S_3 contain every $u_j(p, q), v_j(p, q)$ such that one of l_p, l_q satisfies clause C_j . Let $w \in S_3$ be arbitrary. There exist j, p, q such that neighbours of w in G_2 are:
 - either $s_j, e_p, g_p, \bar{e}_q, \bar{g}_q$ and at most one amongst a_{pj}, b_{qj} (if $w = u_j(p, q)$);
 - or $t_j, f_p, h_p, \bar{f}_q, \bar{h}_q$ and at most one amongst c_{pj}, d_{qj} (if $w = v_j(p, q)$).

Let $\alpha, \beta \in N_{G_2}(w)$ be non-adjacent. If $w = u_j(p, q)$ then either $\alpha, \beta \in N_{G_2}[e_p]$ (if $a_{pj} \in N_{G_2}(w)$) or $\alpha, \beta \in N_{G_2}[\bar{e}_q]$. Else, $w = v_j(p, q)$ and so, either $\alpha, \beta \in N_{G_2}[f_p]$ (if $c_{pj} \in N_{G_2}(w)$) or $\alpha, \beta \in N_{G_2}[\bar{f}_q]$. As a result, α, β have a common neighbour in G_3 and so, Corollary 3 applies.

- Let $S_4 = \{s_1, t_1, s_2, t_2, \dots, s_m, t_m\}$ be the vertices representing each clause. Let $w \in S_4$ be arbitrary. Clearly, there exists j such that either $w = s_j$ or $w = t_j$. Furthermore, by the choice of a boolean assignment satisfying Φ , there exists $l_p \in S_1$ satisfying C_j . By construction, this implies $u_j(p, q), u_j(r, p), v_j(p, q), v_j(r, p) \in S_3$ for some q, r . Hence, neighbours of w in G_3 are r_Φ and:
 - either the twelve vertices $e_i, g_i, \bar{e}_i, \bar{g}_i$ with $i \in \{p, q, r\}$, and possibly $u_j(q, r)$ (if $w = s_j$);
 - or the twelve vertices $f_i, h_i, \bar{f}_i, \bar{h}_i$ with $i \in \{p, q, r\}$, and possibly $v_j(q, r)$ (if $w = t_j$).

Let $\alpha, \beta \in N_{G_3}(w)$ be non-adjacent. If $w = s_j$ then $\alpha, \beta \in N_{G_3}[g_q]$, else, $w = t_j$ and so, $\alpha, \beta \in N_{G_3}[h_q]$. Therefore, α, β have a common neighbour in G_4 , hence Corollary 3 applies.

- Let $S_5 = \{r_\Phi\}$, r_Φ is simplicial in G_4 , *i.e.*, its neighbourhood $N_{G_4}(r_\Phi) = \mathcal{G} \cup \mathcal{H}$ induces a complete subgraph. It is thus straightforward that Corollary 3 applies.
- Let $S_6 = \mathcal{G} \cup \mathcal{H}$. Let $w \in S_6$ be arbitrary.
 - If $w = g_i$ for some i , then neighbours of g_i in G_5 are those in the literal clique, vertex y_i and every $u_j(i, q) \notin S_3$. Therefore, $N_{G_5}[w] \subseteq N_{G_5}[e_i]$;
 - if $w = \bar{g}_i$ for some i , then neighbours of \bar{g}_i in G_5 are those in the literal clique, vertex \bar{y}_i and every $u_j(p, i) \notin S_3$. Therefore, $N_{G_5}[w] \subseteq N_{G_5}[\bar{e}_i]$;
 - if $w = h_i$ for some i , then neighbours of h_i in G_5 are those in the literal clique, vertex y_i and every $v_j(i, q) \notin S_3$. Therefore, $N_{G_5}[w] \subseteq N_{G_5}[f_i]$;
 - else, $w = \bar{h}_i$ for some i , hence neighbours of \bar{h}_i in G_5 are those in the literal clique, vertex \bar{y}_i and every $v_j(p, i) \notin S_3$. Therefore, $N_{G_5}[w] \subseteq N_{G_5}[\bar{f}_i]$.

Since, $e_i, \bar{e}_i, f_i, \bar{f}_i \in V(G_6)$, therefore Corollary 3 applies.

- Let S_7 contain y_i, \bar{y}_i for every $1 \leq i \leq n$. Let $w \in S_7$ be arbitrary. There is some i such that neighbours of w in G_6 are vertex \bar{l}_i and either e_i, f_i (if $w = y_i$) or \bar{e}_i, \bar{f}_i (if $w = \bar{y}_i$). Moreover, recall that we assume the existence of some $1 \leq j \leq m$ such that \bar{l}_i appears in clause C_j . Indeed, all variables are assumed to appear positively and negatively in the clauses of Φ . In particular, by construction $a_{ij}, b_{ij}, c_{ij}, d_{ij} \notin S_2$ and so, $a_{ij}, b_{ij}, c_{ij}, d_{ij} \in V(G_6)$. The latter four vertices are adjacent to every of \bar{l}_i, e_i, f_i and \bar{e}_i, \bar{f}_i by construction of G_Φ . As a result, for any $\alpha, \beta \in N_{G_6}(w)$ non-adjacent, α, β have a common neighbour in G_7 and so, Corollary 3 applies.
- Let $S_8 = V_0$. Let $\bar{l}_i \in S_8$ be arbitrary. Neighbours of \bar{l}_i in G_7 are those $a_{ij}, b_{ij}, c_{ij}, d_{ij}$ such that \bar{l}_i appears in C_j . Every such neighbour is adjacent to the quartet $e_i, f_i, \bar{e}_i, \bar{f}_i$ of the literal clique, hence Corollary 3 applies.

- Let S_9 contain every $a_{ij}, b_{ij}, c_{ij}, d_{ij}$ such that \bar{l}_i appears in C_j . The proof for this case is similar as for S_2 . Let $w \in S_9$ be arbitrary. There are j, p, q, r such that neighbours of w in G_8 are $e_p, f_p, \bar{e}_p, \bar{f}_p$ and at most one of $u_j(p, q), u_j(r, p), v_j(p, q)$ or $v_j(r, p)$. Let $\alpha, \beta \in N_{G_8}(w)$ be non-adjacent. Necessarily, one of α, β must be one of $u_j(p, q), u_j(r, p), v_j(p, q), v_j(r, p)$ because any other neighbour of w is in the literal clique. Furthermore, $u_j(p, q), u_j(r, p), v_j(p, q), v_j(r, p)$ are respectively adjacent to $e_p, e_r, f_p, f_r \in \mathcal{E} \cup \mathcal{F}$ in the literal clique. Therefore, α, β have a common neighbour in G_9 and so, Corollary 3 applies.
- Let S_{10} contain every $u_j(p, q), v_j(p, q)$ such that \bar{l}_p, \bar{l}_q appear in C_j . Equivalently, those are all of $u_j(p, q), v_j(p, q)$ but the ones already in S_3 . Let $w \in S_{10}$ be arbitrary. There exist j, p, q such that neighbours of w in G_9 are either e_p, \bar{e}_q (if $w = u_j(p, q)$) or f_p, \bar{f}_q (if $w = v_j(p, q)$). As a result, vertex w is simplicial. It thus follows that Corollary 3 trivially applies.
- Finally, let $S_{11} = \mathcal{E} \cup \mathcal{F}$, this is a clique and so, the vertices in S_{11} can be eliminated sequentially while leaving a sequence of isometric subgraphs.

To sum up, one obtains a distance-preserving ordering of G_Φ by sequentially eliminating vertices in S_1 then in S_2 and so on until S_{11} , in an arbitrary way. \square

Proposition 6. *If G_Φ admits a distance-preserving elimination ordering, then Φ is satisfiable.*

Proof. Let \prec be a distance-preserving ordering of G_Φ . For every $1 \leq j \leq m$ we claim that there is $1 \leq i \leq n$ such that some $l_i \in \{x_i, \bar{x}_i\}$ satisfies clause C_j , and $l_i \prec r_\Phi$. Then, we will prove that this implies a boolean assignment of the variables satisfying Φ by showing that $r_\Phi \prec \bar{l}_i$.

To prove the claim, first observe that for every $1 \leq j \leq m$, r_Φ is the unique common neighbour of s_j, t_j in G_Φ . By Corollary 2, it implies $s_j \prec r_\Phi$ or $t_j \prec r_\Phi$. So, assume $s_j \prec r_\Phi$ (the case $t_j \prec r_\Phi$ is symmetrical to this one). Let $u_j(p, q), u_j(q, r), u_j(r, p)$ be the three children of s_j in the clause tree. Note that the latter three vertices pairwise share s_j as their unique common neighbour in G_Φ . Consequently, by Corollary 2 (applied twice) at least two of them must be eliminated before s_j . W.l.o.g., let $u_j(p, q)$ be eliminated before s_j . In such case, note that $u_j(p, q)$ is the unique common neighbour of a_{pj}, b_{qj} by construction of G_Φ . Therefore, by Corollary 2, $a_{pj} \prec u_j(p, q)$ or $b_{qj} \prec u_j(p, q)$. Suppose by symmetry that $a_{pj} \prec u_j(p, q)$. Let $l_p \in \{x_p, \bar{x}_p\}$ appear in C_j . Since l_p and $u_j(p, q)$ share a_{pj} as their unique common neighbour and $a_{pj} \prec u_j(p, q)$, by Corollary 2 $l_p \prec a_{pj} \prec r_\Phi$, that finally proves the claim.

To conclude let us prove for every $1 \leq i \leq n$, there is $l_i \in \{x_i, \bar{x}_i\}$ such that either $l_i \prec r_\Phi \prec \bar{l}_i$ or $r_\Phi \prec l_i \prec \bar{l}_i$. If so, then let us consider any boolean assignment of the variables satisfying for every $1 \leq i \leq n$, l_i is assigned true if $l_i \prec r_\Phi$ (note that if $r_\Phi \prec l_i \prec \bar{l}_i$, then x_i can be valuated in an arbitrary way). Since by the above claim, for every $1 \leq j \leq m$, there is $l_i \prec r_\Phi$ satisfying clause C_j , therefore any such assignment satisfies the formula Φ . By way of contradiction, suppose $l_i \prec \bar{l}_i \prec r_\Phi$ with $\{l_i, \bar{l}_i\} = \{x_i, \bar{x}_i\}$ for some $1 \leq i \leq n$. Since y_i, \bar{y}_i share x_i, \bar{x}_i as their only two common neighbours in G_Φ , by Corollary 2 $y_i \prec \bar{l}_i$ or $\bar{y}_i \prec \bar{l}_i$. Suppose by symmetry $y_i \prec \bar{l}_i$. Then, since y_i is the unique common neighbour between \bar{l}_i and g_i, h_i , we have by Corollary 2 that $g_i \prec y_i$ and $h_i \prec y_i$. However, we claim that the combination of $g_i \prec y_i \prec r_\Phi$ and $h_i \prec y_i \prec r_\Phi$ contradicts the fact that \prec is distance-preserving. Indeed, g_i, h_i are the only two common neighbours of r_Φ and y_i , so, this contradicts Corollary 2. \square

3.3 Reduction to graphs with diameter at most two

As stated before, the graph G_Φ resulting from our reduction in Section 3.1 has diameter at most five. In this section, we improve the result by lowering the diameter to two, thereby proving Theorem 4.

We base on the local view of Corollary 2, which states that in order to obtain a distance-preserving ordering of G it is necessary and sufficient to ensure that vertices at distance two in G still have a common neighbour in the graph at each time a vertex is eliminated. This motivates the following definition of Definition 7 that ensures that any two vertices at distance two in G have the same set of common neighbours in G and G' .

Definition 7. Let G be a connected graph with n vertices, let $\mathcal{H} = \{\{u, v\} \mid d_G(u, v) \geq 3\}$ with $|\mathcal{H}| = p$. The graph G' is obtained from G by adding a clique Z of $n + p$ vertices, defined as follows.

For every vertex $v \in V(G)$, there is $z_v \in Z$ that is adjacent to v in $V(G)$.

For every $u, v \in V(G)$ such that $d_G(u, v) \geq 3$, there is $z_{uv} \in Z$ that is adjacent to u, v in $V(G)$.

Lemma 8. *For any connected graph G , let G' be as in Definition 7, G' has diameter at most two.*

Proof. Let $u, v \in V(G')$ be arbitrary. If $u \in Z$ or $v \in Z$ then $d_{G'}(u, v) \leq 2$ because either $u, v \in Z$ are adjacent or, w.l.o.g., $u \in Z$ and $z_v \in Z$ is a common neighbour of u, v in G' by Definition 7. Else, $u, v \in V(G)$ and so, $d_{G'}(u, v) \leq d_G(u, v)$ because G is an induced subgraph of G' . Moreover, if $d_G(u, v) \geq 3$ then by Definition 7 there is $z_{uv} \in Z$ adjacent to u, v in G' , therefore $d_{G'}(u, v) = 2$. \square

Lemma 9. *For any connected graph G , let G' be as in Definition 7, G admits a distance-preserving ordering if and only if G' admits one.*

Proof. Let (v_1, v_2, \dots, v_n) be a distance-preserving ordering of G . For every $1 \leq i < n$, let $G_i := G \setminus (v_1, \dots, v_i)$ be an isometric subgraph of G , let G'_i be the subgraph of G' induced by $V(G_i) \cup Z$ (by convention, $G_0 := G$, $G'_0 := G'$). We claim that for every $1 \leq i < n$, G'_i is an isometric subgraph of G' . Note that if the claim holds, then (v_1, v_2, \dots, v_n) can be completed into a distance-preserving ordering of G' as follows: vertices v_1, v_2, \dots, v_n are sequentially eliminated, then vertices of the clique Z are eliminated in an arbitrary way¹. To prove the claim, by Lemma 1 it suffices to prove that any two $x, y \in N_{G'_{i-1}}(v_i)$ non-adjacent share a common neighbour in G'_i . If $x, y \in V(G_{i-1})$, then by Lemma 1 they share a common neighbour in G_i , hence in G'_i . Else, one of x, y is in Z , w.l.o.g. say $x \in Z$ and so, $z_y \in Z$ is a common neighbour of x, y in G'_i .

Conversely, let G' admit a distance-preserving ordering. Let \prec be a distance-preserving elimination ordering of G' , and let us consider the restriction (v_1, v_2, \dots, v_n) of the total ordering \prec to the vertices of G . We claim that it is a distance-preserving elimination ordering of G . By contradiction, let i be the least index such that $G_i := G \setminus (v_1, v_2, \dots, v_i)$ is not an isometric subgraph of G (by convention, $G_0 := G$). Let j be such that v_i is the j^{th} vertex to be eliminated in G' w.r.t. \prec , and let G'_j be obtained from G' by removing the j first vertices to be eliminated in G' w.r.t. \prec . Note that G'_j is an isometric subgraph of G' because \prec is distance-preserving by the hypothesis. Moreover, since (v_1, v_2, \dots, v_n) is assumed not to be distance-preserving, then by Lemma 1, there exist $x, y \in N_{G_{i-1}}(v_i)$ non-adjacent whose unique common neighbour in the subgraph G_{i-1} is v_i . In such case, $d_G(x, y) = 2$, therefore x, y have no common neighbour in the clique Z by Definition 7. However, $V(G_i) \subseteq V(G'_j) \subseteq V(G_i) \cup Z$ by construction, therefore x, y have no common neighbour in G'_j , that contradicts the fact that G'_j is an isometric subgraph of G' by Lemma 1. \square

Altogether, we can now prove our main result as follows.

Proof of Theorem 4. The problem is in NP. In order to prove the NP-hardness, let Φ be any instance for 3-SAT. The graph G_Φ , described in Section 3.1, can be constructed from Φ in polynomial time. Furthermore, by the combination of Propositions 6 and 5, G_Φ admits a distance-preserving ordering if and only if Φ is satisfiable. Finally, let G'_Φ be obtained from G_Φ as defined in Definition 7. By Lemma 8, G'_Φ has diameter at most two, furthermore by Lemma 9, G'_Φ admits a distance-preserving ordering if and only if G_Φ admits one, that is if and only if Φ is satisfiable. Since 3-SAT is NP-complete [13], this proves the hardness and so, the result. \square

4 Exact algorithms and heuristics

The purpose of the section is to describe algorithms in order to compute a distance-preserving ordering for a given graph G when it exists. Exhaustive-search on all possible vertex-orderings of the graph would require $\mathcal{O}^*(n!) = 2^{\mathcal{O}(n \log n)}$ -time². Furthermore, since we proved in prior Section 3 that the problem is NP-hard, a polynomial-time algorithm is unlikely to exist. In this section, we describe algorithms – most of them being exact – that are faster than exhaustive search theoretically or in practice.

¹In fact, if vertices $z_{v_1}, z_{v_2}, \dots, z_{v_n}$ are the last removed in Z then one obtains a breadth-first search ordering rooted at z_n . This proves that the problem of deciding whether there exists a breadth-first search ordering that is distance-preserving is NP-complete.

²The notation $\mathcal{O}^*(f(n))$ is for a complexity $f(n) \cdot n^{\mathcal{O}(1)}$

4.1 Exact exponential algorithm

A meta-theorem for computing vertex-orderings in graphs with given properties was proved in [3]. It bases on dynamic programming. We first prove the theorem applies to distance-preserving orderings.

Theorem 10. *The problem of deciding whether a given graph admits a distance-preserving elimination ordering can be solved in $\mathcal{O}^*(2^n)$ -time and space, or in $\mathcal{O}^*(4^n)$ -time and polynomial-space.*

Proof. Let the function f_G map every subset $S \subseteq V(G)$ to its number of pairs $x, y \in S$ of nonadjacent vertices with no common neighbour in S . Our aim is to compute an elimination ordering (v_1, v_2, \dots, v_n) of G that minimizes $\max_{1 \leq i < n} f_G(V_{i+1})$, with $V_{i+1} = \{v_{i+1}, v_{i+2}, \dots, v_n\}$. Indeed, by Corollary 2, G admits a distance-preserving elimination ordering if and only if there is one such ordering such that $\max_{1 \leq i < n} f_G(V_{i+1}) = 0$. By a meta-theorem from [3], since $f_G(S)$ can be computed in polynomial-time for any choice of S , an ordering that minimizes $\max_{1 \leq i < n} f_G(V_{i+1})$ can be computed in $\mathcal{O}^*(2^n)$ -time and space, or in $\mathcal{O}^*(4^n)$ -time and polynomial-space. \square

4.2 Integer linear programming

Integer linear programming (ILP) formulations have been proved useful in practical computation of vertex orderings [8, 16]. For completeness, we hence propose an ILP formulation that fits to our problem. Like in [16], total ordering on the vertices is expressed through n^2 binary variables $x_{v,i}$, each denoting whether vertex $v \in V$ is amongst the i first vertices to be eliminated.

$$\begin{aligned} \sum_{v \in V} x_{v,i} &= i, \quad \forall 1 \leq i \leq n \\ x_{v,i} &\leq x_{v,i+1}, \quad \forall v \in V, \forall 1 \leq i \leq n \end{aligned} \tag{1}$$

In order to ensure that the total ordering is distance-preserving, we impose that for all pairs of vertices $u, v \in V$ at distance two in G , at least one of u or v must be eliminated before some of their common neighbours w . It can be expressed as follows:

$$\sum_{w \in N_G(u) \cap N_G(v)} x_{w,i} \leq x_{u,i} + x_{v,i} + (|N_G(u) \cap N_G(v)| - 1), \quad \forall u, v \text{ s.t. } d_G(u, v) = 2, \forall 1 \leq i \leq n. \tag{2}$$

The correctness of our formulation follows from Corollary 2 directly.

5 A polynomial case

A *tree-decomposition* (T, \mathcal{X}) of G is a pair consisting of a tree T and of a family $\mathcal{X} = (X_t)_{t \in V(T)}$ of subsets of V indexed by the nodes of T and satisfying:

- $\bigcup_{t \in V(T)} X_t = V$;
- for any edge $e = \{u, v\} \in E$, there exists $t \in V(T)$ such that $u, v \in X_t$;
- for any $v \in V$, $\{t \in V(T) \mid v \in X_t\}$ induces a subtree, denoted by T_v , of T .

The sets X_t are called *the bags* of the decomposition. Furthermore, the *width* of (T, \mathcal{X}) is equal to $\max_{t \in V(T)} |X_t| - 1$, and the *treewidth* of G is the minimum possible width of its tree-decompositions.

It is well-known that many NP-hard problems are fixed-parameter tractable (FPT) in the treewidth [18]. Furthermore, the existence of distance-preserving orderings has been proved useful in our comparative study of treewidth with some other properties of the tree-decompositions of graphs [17]. We prove that the problem of computing a distance-preserving ordering when it exists is polynomial-time solvable on graphs with bounded treewidth.

Theorem 11. *For every $G = (V, E)$ with treewidth at most k , it can be computed a distance-preserving ordering when it exists in time $2^{2^{\mathcal{O}(k)}} \cdot n^{\mathcal{O}(1)}$.*

Proof. For simplicity, we will work on a specific kind of tree-decompositions, called nice tree-decompositions. A tree-decomposition (T, \mathcal{X}) is *nice* if T is rooted in some node $r \in V(T)$, any node of T has at most two children and, for any $t \in V(T)$,

- either t is a leaf of T and $|X_t| = 1$ (*Leaf Node*);
- or t has one child u and there exists $v \in V$ such that $X_u = X_t \cup \{v\}$ (*Forget Node*);
- or t has one child u and there exists $v \in V$ such that $X_t = X_u \cup \{v\}$ (*Introduced Node*);
- or t has two children u and w and $X_u = X_w = X_t$ (*Join Node*).

In what follows, let (T, \mathcal{X}) be a nice tree-decomposition of width $\mathcal{O}(k)$. It can be computed in time $2^{\mathcal{O}(k)}n$ [4]. For every $t \in V(T)$, let T_t be the subtree rooted at node t and let $V_t = \bigcup_{u \in T_t} X_u$. We aim at computing all the orderings on V_t that can be extended to a distance-preserving ordering of G . In order to do so, we will represent such an ordering as follows:

- its subordering \prec_t on X_t ;
- the collection \mathcal{C}_t of pairs $(N(v) \cap X_t, pos_v)$ for every $v \in V_t \setminus X_t$, where pos_v is the number of neighbours in $N(v) \cap X_t$ preceding vertex v ;
- finally, a set \mathcal{P}_t of pairs $x, y \in X_t$ at distance two in G that are preceded by any of their common neighbours in V_t .

Since there are $2^{\mathcal{O}(k)}$ possibilities for $N(v) \cap X_t$ and there are $\mathcal{O}(k)$ possibilities for pos_v , there are $k! \cdot 2^{\mathcal{O}(k)2^{\mathcal{O}(k)}} \cdot \mathcal{O}(k^2) = 2^{2^{\mathcal{O}(k)}}$ possible representations. Furthermore, we recall that by Corollary 2, an ordering \prec is distance-preserving if and only if for every $x, y \in V$ at distance two there exists a common neighbour $z \in N_G(x) \cap N_G(y)$ such that $x \prec z$ or $y \prec z$. Hence, we will consider one of the above representations to be valid if it represents an ordering \prec'_t of V_t with the following property: for every $x, y \in V_t$ at distance two, there exists a common neighbour $z \in N_G(x) \cap N_G(y)$ such that either $\{x, y\} \in \mathcal{P}_t$ and $z \in V \setminus V_t$, or $z \in V_t$ and one of x or y precedes z w.r.t. \prec'_t . Note that by Corollary 2, a valid representation at the root is equivalent to the existence of a distance-preserving ordering of G .

- **Case of a Leaf Node.** In this situation, $V_t = X_t = \{v\}$ for some $v \in V$. So, there is a unique valid representation $\langle \prec_t = (v), \mathcal{C}_t = \emptyset, \mathcal{P}_t = \emptyset \rangle$.
- **Case of a Forget Node.** Let $u \in V(T)$ be the unique child of node t and let $v \in V$ be such that $X_t = X_u \setminus \{v\}$. Consider any valid representation at node u . If there is a pair containing v in \mathcal{P}_u then it cannot be extended to a valid representation at node t . Else, it can be transformed into a valid representation at node t by taking the restriction of \prec_u to X_t and by constructing \mathcal{C}_t as follows. We put the pair $(N(v) \cap X_t, pos_v)$ in \mathcal{C}_t (that can be easily computed from \prec_u) and for every pair $(N(v') \cap X_u, pos_{v'}) \in \mathcal{C}_u$, either v is among the $pos_{v'}$ first neighbours in $N(v') \cap X_u$ w.r.t. \prec_u , in which case we put $(N(v') \cap X_t, pos_{v'} - 1)$ in \mathcal{C}_t , or we put $(N(v') \cap X_t, pos_{v'})$ in \mathcal{C}_t .
- **Case of an Introduced Node.** Let $u \in V(T)$ be the unique child of node t and let $v \in V$ be such that $X_t = X_u \cup \{v\}$. Consider any valid representation at node u . We consider the $\mathcal{O}(k)$ possible ways to insert v w.r.t. \prec_u , in order to obtain the subordering \prec_t . For every \prec_t , we need to consider all vertices $x \in V_t$ that are at distance two from v . We distinguish between two subcases.
 - Suppose that $x \in X_u$. We check whether there exists a common neighbour z such that either $z \in X_u$ and it is preceded by one of x or v , or $z \notin V_t$. If we cannot find such vertex z then it is not possible to extend to a valid representation at node t . Furthermore, in the latter case when $z \notin V_t$, we put the pair x, v in an intermediate set \mathcal{Q}_t (used later in order to define \mathcal{P}_t).
 - Else, $x \notin X_u$. In particular, there is a pair $(N(x) \cap X_u, pos_x) \in \mathcal{C}_u$ such that $N(x) \cap N(v) \neq \emptyset$ (note that $N(x) \cap N(v) \subseteq X_u$). In this situation, there must exist a common neighbour $z \in X_u$ that is preceded by at least one of v or x (else, we cannot extend to a valid representation at node t). So, let $N^+(v)$ be the vertices of $N(v) \cap X_u$ that are preceded by v w.r.t. \prec_t , and let $N^+(x)$ be obtained from $N(x) \cap X_u$ by removing its pos_x first neighbours in X_u w.r.t. \prec_t . We are left to test for $N^+(v) \cap N(x) \neq \emptyset$ (v precedes z) or $N^+(x) \cap N(v) \neq \emptyset$ (x precedes z).

Finally, let \mathcal{P}_t be obtained from $\mathcal{Q}_t \cup \mathcal{P}_u$ by removing the pairs $\{x, y\} \in \mathcal{P}_u$ such that $v \in N_G(x) \cap N_G(y)$ and $x \prec_t v$ or $y \prec_t v$. The resulting representation at node t is valid if and only if there is no pair remaining in \mathcal{P}_t whose vertex v is the unique common neighbour in $V \setminus V_u$. Note that we needn't modify the collection $\mathcal{C}_u = \mathcal{C}_t$.

- **Case of a Join Node.** Let u, w be the two children nodes of t . Recall that $X_u = X_w = X_t$. Consider any valid representation at node u , and any valid representation at node w . They can be merged into a valid representation at node t only if $\prec_u = \prec_w$. If so, let $\prec_t = \prec_u$, let $\mathcal{P}_t = \mathcal{P}_u \cap \mathcal{P}_w$ and let $\mathcal{C}_t = \mathcal{C}_u \cup \mathcal{C}_w$. In order to decide whether this can be transformed into a valid representation at node t , we need to consider all the pairs $v_u \in V_u \setminus X_u$, $v_w \in V_w \setminus X_w$ at distance two in G , i.e., all the pairs $(N(v_u) \cap X_u, pos_{v_u}) \in \mathcal{C}_u$, $(N(v_w) \cap X_w, pos_{v_w}) \in \mathcal{C}_w$ such that $N(v_u) \cap N(v_w) \neq \emptyset$

(note that $N(v_u) \cap N(v_w) \subseteq X_t$). For all such pairs, there must exist a common neighbour $z \in X_t$ that is preceded by at least one of v_u or v_w (else, we cannot extend to a valid representation at node t). So, let $N^+(v_u)$ be obtained from $N(v_u) \cap X_t$ by removing its pos_{v_u} first neighbours in X_t w.r.t. \prec_t and let $N^+(v_w)$ be defined in a similar fashion. We are left to test for $N^+(v_u) \cap N(v_w) \neq \emptyset$ (v_u precedes z) or $N^+(v_w) \cap N(v_u) \neq \emptyset$ (v_w precedes z).

□

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