Fast Mixing with Quantum Walks vs. Classical Processes

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Abstract

Quantum walks have been linked to acceleration in various information processing tasks, and proposed as a possible model for quantum-enhanced behavior in biological systems. These links and acceleration claims have been made with various levels of detail. Here we consider discrete-time quantum walks, and focus on the task of mixing, i.e., distributing the state over a graph. Previous papers have observed that the so-called coined quantum walks can accelerate mixing on certain graphs with respect to the optimal classical Markov chain. We here show that the same speedup can be attained with a classical process, if a similar classical coin is added. We establish a precise correspondence between the mixing performance of quantum walks and such "lifted walks" for all (finite) graphs, and thereby improve known bounds on quantum walk mixing time. We conclude that the advantage of quantum walks with respect to classical processes is not in the mixing speed of the *optimal* design. However, a notable quantum advantage might reside in the fact that the mixing speed obtained with suboptimal designs, due to for instance limited graph knowledge, appears to be generically faster.

Motivation

Random walks on graphs are an important tool for algorithmic applications, see for instance Markov chain Monte-Carlo [1], and as physical models of (information or energy) diffusion. Different variations of this concept have been proposed in the quantum setting, both in continuous-time [2] and in discrete-time [3, 4], where the graph nodes index a set of orthonormal vectors in the Hilbert space and the associated quantum dynamics cannot couple vectors whose indices are not joined by a graph edge. A major motivation for the proposal and study of these quantum models concerns their potential speed-up in carrying information through the graph in various senses. A particular hitting problem was shown to be solved exponentially faster with a continuous-time quantum walk – associated to a distributed graph-relying quantum oracle – than with any classical algorithm [5]. The Grover search algorithm has also been linked to quantum walks. More generally, on various graphs the mixing performance – i.e. the time needed to get close to an invariant classical distribution over the graph nodes from any starting state – can be quadratically improved with respect to classical random walks, if discrete-time quantum walks are employed [3, 7, 8].

However, in order to build fast-mixing discrete-time quantum walks, these methods enlarge the original Hilbert space, whose basis was indexed by the graph nodes: to each node, they add another ("coin" or "spin") degree of freedom, see [9] and the example below. These quantum walks thus appear to add two ingredients with respect to classical walks: (i) the quantum coherences among basis vectors; and (ii) the new "coin" degrees of freedom at each node. Both induce memory effects on the evolution of interest. This warrants the question: What element is truly responsible for the acceleration? Can we have equivalent performance with just a *classical* coin-like memory, or is the reported acceleration an inherent quantum effect? Underlying this question is the deeper objective to identify parts of quantum processes (algorithms or controlled dynamical systems) that can or cannot be replaced or simulated by classical counterparts with access to the same inputs and similar resources.

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A motivating example: In fact "classical walks with coin" have already been introduced and studied, under the name of *lifted walks* or Markov chains [10, 11]. In order to illustrate how they work, let us focus on a simple example. Take the N-cycle graph, in which node k is connected to nodes $k \pm 1$ modulo N, for $k = 1, 2, ..., N$. A balanced classical random walk on the cycle, as illustrated on Fig.a, has the transition matrix $P_0 = (P^+ + P^-)/2$ where $P^\pm$ are cyclic permutations, bringing node k to $k \pm 1$ modulo N. The lifted walk proposed in [10] and shown in Figure b, sets up a walk among the $2N$ sites $\{+, -\} \times \{1, 2, ..., N\}$, with stochastic transition matrix

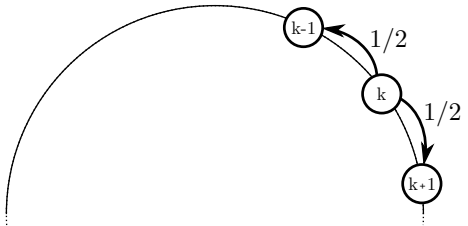
$$P = \begin{bmatrix} P^+ & 0 \\ 0 & P^- \end{bmatrix} \cdot \left(\begin{bmatrix} 1-\alpha & \alpha \\ \alpha & 1-\alpha \end{bmatrix} \otimes \text{Identity}_N \right) \tag{1}$$

for some $\alpha < 1/2$. Namely, each node of the original graph is doubled, with the new degree of freedom memorizing the previous motion direction; the walker will more likely proceed in the same direction than reversing it. For α of order $1/N$, the mixing time of P is of order N, while the one of P_0 is of order N^2 [10]. This is the same order of speedup provided by the discrete-time unitary quantum walk proposed for the same graph in [3]. The latter also takes a very similar form, with unitary transition matrix

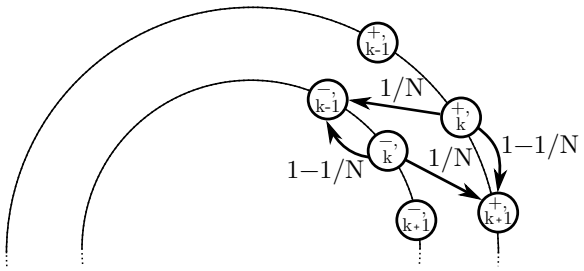
$$U = \begin{bmatrix} P^+ & 0 \\ 0 & P^- \end{bmatrix} \cdot (C_2 \otimes \text{Identity}_N) \ , \tag{2}$$

where C_2 now is a *unitary* on $\mathcal{H}_{Coin} = \text{span}\{|+\rangle, |-\rangle\}$, e.g. $C_2 = [\sqrt{1-\alpha},\ \sqrt{\alpha}\ ;\ -\sqrt{\alpha},\ \sqrt{1-\alpha}]$. Notice that if we projectively measure the quantum state at each step, we obtain exactly the P of equation (1). However, for $\alpha = 1/2$ in C_2 we get the standard Hadamard quantum walk, which is known to mix fast [3], whereas taking $\alpha = 1/2$ in (1) would in fact render the momentum memory irrelevant for the walker's motion and effectively yield back the slow P_0.

These observations motivate the question: how different are the mixing possibilities of quantum walks and *lifted* chains really?



a.



b.

Related work: To the best of our knowledge, a detailed comparison of quantum walks with lifted Markov chains has not been considered before. Bounds on the quantum mixing time have been established, see for instance [3] and [12]. Our results improve them in the case of free edge weights, and extend them to non-unital quantum channels. In particular, we show that, as far as mixing time is concerned, the quantum speedup as compared to lifted Markov chains can be at most a factor 2. For our main proof we use a result from hidden variables theory by Aaronson [13], stating that stochastic classical "bridges" can simulate a class of locally constrained quantum processes. By extending and using this idea to construct lifted Markov chains, we get a quite precise picture about quantum walk capabilities.

Main results and discussion of their significance

Our main result states that lifted Markov chains mix essentially as fast as quantum walks:

For any quantum walk (and in fact any quantum channel) that mixes to a stationary classical distribution over the graph nodes, we can explicitly construct a lifted Markov chain that mixes to this distribution with essentially the same mixing time. I.e.:
a. *If for any initial state, the quantum walk reaches a total variation distance of* $1/4$ *from the stationary in a time at most* τ*, then the lifted Markov chain reaches the same distance, starting from any initial lifted node, in a time at most* 2τ*.*
b. *The same lifted Markov chain will reach any distance* ϵ *from the stationary distribution in a time at most* $2\tau(1 + \log_2(1/\epsilon))$*.*

The proof is based on the possibility to construct a "stochastic bridge" which follows the evolution of the node-populations induced by the quantum walk starting from a given state, a slight extension of the results in [13]. The bridges for various initial states can then be combined into satisfying the formal definition of a lifted Markov chain. Our explicit construction requires knowledge of the entire graph, and involves coins of size $N \cdot \tau(1/4)$, with N the number of graph nodes. This may arguably not look like a practical solution. However its existence has several important implications.

Indeed, it shows that quantum walks cannot mix essentially faster than the best lifted Markov chain on a given graph. Since the mixing time of lifted Markov chains is tightly bounded by the graph conductance, this provides a new bound for the quantum walk mixing time. Viewed from a more physical angle, our result means that fast mixing on a graph is never diagnostic of a quantum effect, in the sense that in fact it does not require quantum hardware, if local memories are allowed.

To complement our existence result, we characterize a specific set of fast-mixing lifted Markov chains on *periodic lattices in arbitrary dimension* d, i.e. with nodes indexed by $\{1, 2, ..., M\}^d$.

a. *There exist lifted Markov chains on arbitrary dimensional periodic lattices which have the same "coin size" as the proposed quantum walks [14, 15] and which, like those quantum walks, mix quadratically faster than the best classical walk.*
b. *These lifted walks can only mix fast if the coin transition matrix differs by exactly* $O(1/M)$ *from a deterministic permutation matrix.*

Thus, at least for lattices, where most fast-mixing quantum walks have been discussed, a "practical", low-memory fast-mixing lifted Markov chain also exists. But at the same time, tuning it such that it beats the diffusive behavior of simple walks, requires to know the lattice size. In contrast, on the cycle at least *almost any* quantum transition matrix U of the form (2) appears to mix fast [6], and on the infinite line, a given unitary quantum walk can keep mixing fast for arbitrarily long times [7]. This seems to indicate that regarding their mixing performance, the *only* advantage of discrete-time quantum walks is that they still work with less precise or at least deferred *graph knowledge*. This connects, from a completely different approach, with the result of [5] on continuous-time quantum walks with a *local* graph oracle. In this sense, our results should help narrow down the scope in the important quest for generic, algorithmically exploitable mechanisms behind quantum acceleration.

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