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Stability results for the parameter identification inverse problem in cardiac electrophysiology

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Abstract. In this paper we prove a stability estimate of the parameter identification problem in cardiac electrophysiology modeling. We use the monodomain model which is a reaction diffusion parabolic equation where the reaction term is obtained by solving an ordinary differential equation. We are interested in proving the stability of the identification of the parameter τ_{in} which is the parameter that multiplies the cubic term in the reaction term. The proof of the result is based on a new Carleman-type estimate for both the PDE and ODE problems. As a consequence of the stability result we prove the uniqueness of the parameter τ_{in} giving some observations of both state variables at a given time t_0 in the whole domain and the PDE variable in a non empty open subset w_0 of the domain.

Keywords: Parameter identification, Carleman inequality, stability result, Monodomain system, Mitchell-Schaeffer model, Ionic parameters. Submitted to: *Inverse Problems*

1. Introduction

The electric wave propagation in the heart can be represented by non-linear reaction-diffusion system coupled with an ordinary differential equation called bidomain model [16, 32, 33, 37]. The coupled system describes the evolution of the electrical potential v_m in the heart. In the particular monodomain model case, the intra and extracellular conductivities have similar anisotropic ratio (See [15, 19]). The membrane model used to describe the ionic current is Mitchell Shaeffer, was introduced by C. Mitchell and D. Schaeffer in 2003 [30]. The model consists of two ordinary differential equations, one on the transmembrane potential v_m and an other on a gating variable w , (representing the opening/closing of the ion channels). This model contains four time constant parameters $\tau_{in}, \tau_{out}, \tau_{close}$ and τ_{open} . These parameters affect the shape of the action

potential traces and consequently they affect the electrocardiograms. The sensitivity analysis of the electrocardiograms (ECGs) to the parameters of the model on the correctness of the ECGs have been investigated [5, 6, 38]. In [9], authors used a reduced order method for the identification of these parameters. In a recent work [13], we proved that in case of parameter estimation framework it is recommended to use a proper orthogonal decomposition in order to estimate τ_{close} , τ_{open} and τ_{out} . But to estimate τ_{in} , the data from which the proper orthogonal decomposition basis is computed should be sufficiently rich in order to maintain a good accuracy of the results. The main reason is that the transmembrane potential is too much sensitive to this parameter. Thus stability results for the identification of the parameter τ_{in} are very important.

In this paper, we study the stability of the inverse problem of identification of the parameter τ_{in} : the reconstruction of the coefficient τ_{in} from the measurement of solution v_m over a space-time domain $\omega \times (0, T)$ and some measurement at fixed time t_0 . The key ingredient to these stability results is a new global Carleman estimate for nonlinear coupled system with one observation.

Such problems were not considered widely in the past. However, for non linear parabolic problem, some uniqueness results were published for the one-dimensional case. In [26], the author proved a global uniqueness theorem for the $1D$ parabolic case, using the Bukhgeim Klibanov method based on the Carleman estimates [11, 12, 23, 24, 25, 27, 28]. Kugler [29] has proved uniqueness for a $1D$ inverse problem for a quasilinear elliptic equation. Muzylev has published a uniqueness theorem for a piecewise analytic unknown coefficient $q(u)$ in a parabolic operator [31]. Pilant and Rundell have established uniqueness under a smallness condition for an (nD) forward- $(1D)$ inverse problem [34]. That is, in [34] the unknown source function $q(u)$ is a part of an nD parabolic operator, and the data are given at a single point of the boundary. Klibanov and Bukhgeim [12] has proved a global uniqueness result by the method of Carleman estimates in n -dimensional, where the unknown coefficient q depends on the solution u and $(n - 1)$ spatial variables. For a reaction-diffusion linear system, the Carleman estimate obtained in [2] is used to solve the inverse problem of identification of one coefficient and initial condition but the weight functions are different in the left and right hand side of their estimate. Also, in [17] a simultaneous stability result for one coefficient and for the initial conditions are obtained for a two by two reaction-diffusion linear system. In 2008, Bellassoued and al. [3] show a Carleman estimate for the Lamé system without integral term. In [18], Boulakia and Schenone show Lipschitz stability inequalities for the identification of some parameters of the Fitz Hugh-Nagumo model from measurements on the cardiac potential and the ionic variable. The main difference between our work and [8, 18] is that the gating variable w satisfies an other ionic model for which we have a discontinuity of rate function [30] for which we will establish a new Carleman estimate.

Let $\Omega \in \mathbb{R}^n$ be a bounded domain of \mathbb{R}^n with $n \leq 3$. We denote by ν the outward unit normal to Ω on $\partial\Omega$. Let $T > 0$ and we consider the following monodomain model

system:

$$\begin{cases} \partial_t v_m - \operatorname{div}(\boldsymbol{\sigma} \nabla v_m) = I_{app} + I_{ion}(v_m, w, \tau_{in}, \tau_{out}) & \text{in } \Omega \times (0, T) \\ \partial_t w = g(v_m, w, \tau_{close}, \tau_{open}) & \text{in } \Omega \times (0, T) \end{cases} \quad (1)$$

We complete this system by the initial conditions

$$v_m(x, 0) = v_0(x) \quad \text{and} \quad w(x, 0) = w_0(x), \quad \forall x \in \Omega,$$

and by Neumann boundary condition on v_m

$$\nabla v_m \cdot \nu = 0 \quad \text{on} \quad \partial\Omega \times (0, T),$$

where I_{app} , I_{ion} and g are defined by the Mitchell-Schaeffer model (see Section 2).

Our problem is to establish a stability result of the parameter identification inverse problem from the knowledge of

$$v_m|_{Q_{\omega_0}}, \quad \nabla v_m|_{Q_{\omega_0}} \quad \text{and} \quad v_m(\cdot, t_0), \quad \frac{\partial^2 v_m}{\partial x_i \partial x_j}(\cdot, t_0), \quad \forall i, j = 1, \dots, 3, \quad w(\cdot, t_0)$$

where ω_0 be a subdomain of Ω , $t_0 \in (0, T)$ and $Q_{\omega_0} = \omega_0 \times (0, T)$.

Our main result will be presented in Theorem 4.1, where we establish a new Global Carleman estimate for the pair of solution (v_m, w) . The uniqueness of a solution of the identifiability problem is a consequence of this main result and would be given in Corollary 4.1

The overview of the present article is as follows: Section 2, is devoted to notation and preliminary well-posedness results. In Section 3, we derive global Carleman inequality for the monodomain system with one observation. The main result is presented in Section 4. Finally, some conclusions are given in Section 5.

2. Regularity of the solution

In this section, we introduce the notations and basic results concerning the monodomain system coupled to the Mitchell-Schaeffer ionic model. We recall some regularity results that would be useful for proving the stability in the parameter identification problem.

2.1. Monodomain system with Mitchell-Schaeffer model

We consider the monodomain system that describes the propagation of the electric wave in the heart gives by

$$\begin{cases} \partial_t v_m - \operatorname{div}(\boldsymbol{\sigma} \nabla v_m) = I_{app} + I_{ion}(v_m, w) & \text{in } \Omega \times (0, T), \\ \partial_t w = g(v_m, w) & \text{in } \Omega \times (0, T), \\ \boldsymbol{\sigma} \nabla v_m \cdot \nu = 0 & \text{on } \Sigma, \\ v_m(x, 0) = v_0(x), w(x, 0) = w_0(x) & \text{for } a.a. x \in \Omega, \end{cases} \quad (2)$$

where Ω and Σ denote respectively the heart domain and boundary. The time domain is given by $[0, T]$. The variable v_m , denotes the action potential and $\boldsymbol{\sigma} \stackrel{def}{=} \boldsymbol{\sigma}_i(\boldsymbol{\sigma}_i + \boldsymbol{\sigma}_e)^{-1} \boldsymbol{\sigma}_e$

is the bulk conductivity where σ_i and σ_e are the intra- and extracellular conductivity tensors. The term I_{app} is a given source function and the ionic current I_{ion} and the function g depends of the considering ionic model.

In this work we use the regularized version of the Mitchell-Schaeffer cardiac cell membrane model to define the ionic current I_{ion} and the function g . Then the dynamics of w and I_{ion} are described by:

$$I_{ion}(v, w) = \frac{w}{\tau_{in}} v^2 (1 - v) - \frac{v}{\tau_{out}},$$

$$g(v, w) = \left(\frac{1}{\tau_{close}} + \frac{\tau_{close} - \tau_{open}}{\tau_{close} \tau_{open}} h_{\infty}(v) \right) (w - h_{\infty}(v)),$$

where

$$h_{\infty}(v) = \frac{1}{2} \left[1 - \tanh\left(\frac{v - v_{gate}}{\alpha_{gate}}\right) \right] = \frac{1}{2} \left[1 - \frac{1 - e^{-2(\frac{v - v_{gate}}{\alpha_{gate}})}}{1 + e^{-2(\frac{v - v_{gate}}{\alpha_{gate}})}} \right].$$

Here, the parameters α_{gate} , $\tau_{in} < \tau_{out} < \tau_{open}$, τ_{close} and $0 < v_{gate} < 1$ are given positive constants.

2.2. Notations and definitions

Let $\Omega \subset \mathbb{R}^3$ be a bounded strongly Lipschitz domain. We denote by $L^p(\Omega)$ the space of functions which are in the p th power integral ($1 \leq p < \infty$), or are measurable and essentially bounded ($p = \infty$), and by $W^{1,p}(\Omega)$ the Sobolev space of functions $\psi : \Omega \rightarrow \mathbb{R}$ which, together with their first-order weak partial derivatives, belong to the space $L^p(\Omega, \mathbb{R})$ ($1 \leq p < \infty$). For X a Banach space, denote $C([0, T]; X)$ the space of continuous functions from $[0, T]$ into X equipped with the uniform convergence norm, and

$$\|f\|_{L^p(0,T;X)} = \left(\int_0^T \|f(t)\|_X^p dt \right)^{1/p} \quad \left(= \sup_{0 < t < T} \text{ess} \|f(t)\|^p \quad \text{if } p = \infty \right).$$

By definition $L^p(0, T; X)$, $p < \infty$, is the separated space of $C([0, T]; X)$ for this norm; for $p = \infty$, $L^\infty(0, T; X)$ is the subset of $L^1(0, T; X)$ on which the L^∞ norm is finite. It is a Banach space for $0 \leq p \leq \infty$. For $p = 2$ and $X = W^{1,2}(\Omega)$, we say space of Bochner integrable mappings, see [22]. Further, we set (see [10])

$$W^{1,2}(\Omega)/_{\mathbb{R}} = \{u \in W^{1,2}(\Omega), \int_{\Omega} u dx = 0\} \subset W^{1,2}(\Omega),$$

that is a Banach space with the norm $\|u\|_{W^{1,2}(\Omega)/_{\mathbb{R}}} = \|u\|_{W^{1,2}(\Omega)}$. We have the Poincaré-Wirtinger inequality

$$\exists C > 0, \quad \forall u \in W^{1,2}(\Omega)/_{\mathbb{R}}, \quad \int_{\Omega} |u|^2 dx \leq C \int_{\Omega} |\nabla u|^2 dx. \quad (3)$$

We denote $Q = \Omega \times [0, T]$, $Q_{\omega_0} = \omega_0 \times (0, T)$ and $\Sigma = \partial\Omega \times (0, T)$. Let $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ and $t \geq 0$, respectively, as the spatial and time variables,

$$\partial_t = \frac{\partial}{\partial t}, \quad \partial_j = \frac{\partial}{\partial x_j}.$$

The gradient ∇ is always taken only with respect to the spatial variables x . We assume that the conductivities of the intracellular and extracellular $\sigma_i, \sigma_e \in [L^\infty(\Omega)]^{3 \times 3}$ are symmetric and uniformly positive definite, i.e, there exist $\alpha_i > 0$ and $\alpha_e > 0$ such that,

$$\xi^T \sigma_i(x) \xi \geq \alpha_i |\xi|^2, \quad \xi^T \sigma_e(x) \xi \geq \alpha_e |\xi|^2, \quad \forall \xi \in \mathbb{R}^3. \quad (4)$$

If we do not specially state, then we always assume that $\sigma \in (C^1(\bar{\Omega}))^{3 \times 3}$, and that the coefficients σ_{ij} , $i, j = 1, 2, 3$, satisfy the uniform ellipticity: there exists a constant $\mu > 0$ such that

$$\mu |\xi|^2 \leq \xi^T \sigma \xi, \quad \forall \xi \in \mathbb{R}^3. \quad (5)$$

We set

$$|\nabla u|_\sigma^2 := \sigma \nabla u \cdot \nabla u = \sum_{i,j=1}^3 \sigma_{ij} \partial_j u \partial_i u \quad \text{in } Q.$$

For the monodomain system (2), we introduce the following notion of a strong and weak solutions (see [7, 38]):

Definition 2.1. *A weak solution of the monodomain system (2) is a couple of functions (v_m, w) with the regularity*

$$\begin{aligned} v_m &\in C^0([0, T], L^2(\Omega)) \cap L^2(0, T; W^{1,2}(\Omega)) \cap L^p(Q) \quad \text{with } 2 \leq p \leq 6, \\ w &\in W^{1,2}(0, T; L^2(\Omega)) \cap C^0([0, T]; L^2(\Omega)), \end{aligned}$$

and in the distributional sense satisfying, $\forall t \in [0, T]$

$$\begin{cases} \int_{\Omega} (\partial_t v_m - I_{ion}(v_m, w)) \psi dx + \int_{\Omega} \nabla \psi \sigma \nabla v_m dx = \int_{\Omega} I_{app} \psi dx & \forall \psi \in W^{1,2}(\Omega) \\ \int_{\Omega} (\partial_t w - g(v_m, w)) \psi dx = 0 & \forall \psi \in L^2(\Omega) \end{cases} \quad (6)$$

and obey, almost everywhere, the initial conditions

$$v_m(x, 0) = v_0(x), \quad w(x, 0) = w_0(x) \quad \forall x \in \Omega. \quad (7)$$

In the following proposition, we determine the appropriate spaces of the solution of system (6).

Proposition 2.1. *If v_0 belongs to $L^2(\Omega)$, Then*

$$v_m \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega)) \cap L^4(0, T; L^4(\Omega));$$

And if v_0 belongs to $H^2(\Omega)$, Then

$$v_m \in W^{1,\infty}(0, T; L^2(\Omega)) \cap H^1(0, T; H^1(\Omega)) \cap L^\infty(0, T; H^2(\Omega)).$$

Proof. The proof of this regularity result could be found in [18] (Proposition 3.2. page 85). ■

3. Global Carleman inequality for Mitchell-Schaeffer model

In this section we establish Carleman estimate for the monodomain system coupled to the Mitchell-Schaeffer model. This Carleman estimate would be used later for the stability and uniqueness of the solution of the parameter identification problem. We are interested in identifying the parameter τ_{in} . By denoting $a = 1/\tau_{in}$, identifying a is equivalent to identifying τ_{in} .

We first have to define the weight function that we will use. This weight is fundamental in the sense that, roughly speaking, information will propagate in space along the gradient lines of this function.

3.1. Weight functions

Let ω be a subdomain of Ω . We have the following

Lemma 3.1. *Let ω_0 be an open set such that $\overline{\omega_0} \subset \omega$ (for example ω_0 can be a small open ball). Then, under the symmetric hypothesis on σ and (5), there exists a function $\beta \in C^2(\overline{\Omega})$ such that*

$$\beta(x) > 0 \quad \forall x \in \Omega, \quad \beta|_{\partial\Omega} = 0 \quad \text{and} \quad |\nabla\beta(x)| > 0 \quad \forall x \in \overline{\Omega} \setminus \omega_0.$$

If, moreover $\sigma \in (C^1(\overline{\Omega}))^{3 \times 3}$, we have also

$$\sigma(x)\nabla\beta(x) \cdot \nu(x) \leq 0, \quad x \in \partial\Omega. \quad (8)$$

The proof of Lemma 3.1 is given in [21].

Remark 1. *The construction of the function β uses Morse functions and the associated approximation theorem, such a weight function is introduced in [20].*

We will now use the function β given by Lemma 3.1 to build new weight functions. Let λ be a sufficiently large positive constant that only depends on Ω and ω . For $t \in (0, T)$ we introduce the following functions:

$$\varphi(x, t) = \frac{e^{\lambda\beta(x)}}{t(T-t)}, \quad \eta(x, t) = \frac{e^{2\lambda\|\beta\|_\infty} - e^{\lambda\beta(x)}}{t(T-t)} \quad (9)$$

and

$$\overline{\varphi}(x, t) = \frac{e^{-\lambda\beta(x)}}{t(T-t)}, \quad \overline{\eta}(x, t) = \frac{e^{2\lambda\|\beta\|_\infty} - e^{-\lambda\beta(x)}}{t(T-t)}. \quad (10)$$

Notice that

$$\eta(x, t) \leq \overline{\eta}(x, t) \quad \forall (x, t) \in Q. \quad (11)$$

We now have for every $\lambda > 0$ the following properties which will be helpful for our calculations

$$\nabla\varphi = \lambda\varphi\nabla\beta, \quad \nabla\eta = -\lambda\varphi\nabla\beta \quad (12)$$

$$\nabla\overline{\varphi} = -\lambda\overline{\varphi}\nabla\beta, \quad \nabla\overline{\eta} = \lambda\overline{\varphi}\nabla\beta. \quad (13)$$

$$1 \leq \left(\frac{T}{2}\right)^2 \varphi; \quad \varphi \leq \left(\frac{T}{2}\right)^2 \varphi^2; \quad \varphi \leq \left(\frac{T}{2}\right)^4 \varphi^3 \quad (14)$$

$$|\partial_t \varphi| \leq T \varphi^2; \quad |\partial_{tt}^2 \varphi| \leq 2T^2 \varphi^3, \quad (15)$$

$$|\partial_t \eta| \leq T \varphi^2; \quad |\partial_{tt}^2 \eta| \leq 2T^2 \varphi^3, \quad (16)$$

We can notice that η tends rapidly to $+\infty$ when $t \rightarrow T$ or $t \rightarrow 0$ but that η is uniformly bounded in $\Omega \times [\delta, T - \delta]$ if $\delta > 0$. Our last weight function will depend on a second parameter s and will be of the form $e^{-s\eta(x,t)}$. We can see that, for fixed s , this function tends very rapidly to 0 when $t \rightarrow T$ or $t \rightarrow 0$.

3.2. Global Carleman inequality for non linear parabolic equation

This sub-section is dedicated to prove some Carleman estimate for the solution of the monodomain system with Mitchell-Schaeffer ionic model.

We consider the solutions (v_m, w) and (\tilde{v}_m, \tilde{w}) to the following systems

$$\begin{cases} \partial_t v_m - \operatorname{div}(\boldsymbol{\sigma} \nabla v_m) = I_{app} + I_{ion}(a, v_m, w), & \text{in } Q, \\ \partial_t w = g(v_m, w), & \text{in } Q, \\ \boldsymbol{\sigma} \nabla v_m \cdot \nu = 0, & \text{on } \Sigma, \\ v_m(x, 0) = v_m^0(x), \quad w(x, 0) = w_m^0(x), & \text{in } \Omega, \end{cases} \quad (17)$$

and

$$\begin{cases} \partial_t \tilde{v}_m - \operatorname{div}(\boldsymbol{\sigma} \nabla \tilde{v}_m) = I_{app} + I_{ion}(\tilde{a}, \tilde{v}_m, \tilde{w}), & \text{in } Q, \\ \partial_t \tilde{w} = g(\tilde{v}_m, \tilde{w}), & \text{in } Q, \\ \boldsymbol{\sigma} \nabla \tilde{v}_m \cdot \nu = 0, & \text{on } \Sigma, \\ \tilde{v}_m(x, 0) = \tilde{v}_m^0(x), \quad \tilde{w}(x, 0) = \tilde{w}_m^0(x), & \text{in } \Omega. \end{cases} \quad (18)$$

We set $V = v_m - \tilde{v}_m$, $W = w - \tilde{w}$ and $q = a - \tilde{a}$, where $a = 1/\tau_{in}$ and $\tilde{a} = 1/\tilde{\tau}_{in}$. Then (V, W) is solution to the following problem

$$\begin{cases} \partial_t V - \operatorname{div}(\boldsymbol{\sigma} \nabla V) + kV^3 = G_1(v_m, w, \tilde{v}_m, \tilde{w}) & \text{in } \Omega \times (0, T) \\ \partial_t W = G_2(v_m, w, \tilde{v}_m, \tilde{w}) & \text{in } \Omega \times (0, T) \\ \boldsymbol{\sigma} \nabla V \cdot \nu = 0 & \text{on } \Sigma \\ V(x, 0) = V_0(x), \quad W(x, 0) = W_0(x) & \text{in } \Omega, \end{cases} \quad (19)$$

here $k(x, t) = aw(x, t)$, $G_1(v_m, w, \tilde{v}_m, \tilde{w}) = qH(\tilde{v}_m, \tilde{w}) + R(V, W, \tilde{v}_m, \tilde{w})$, where

$$H(\tilde{v}_m, \tilde{w}) = \tilde{w} \tilde{v}_m^2 (1 - \tilde{v}_m)$$

and

$$\begin{aligned} R(V, W, \tilde{v}_m, \tilde{w}) &= a\tilde{w}(1 - 3\tilde{v}_m)V^2 + a(\tilde{w}\tilde{v}_m(2 - 3\tilde{v}_m) - 1/\tau_{out})V \\ &+ a(1 - 3\tilde{v}_m)WV^2 + a\tilde{v}_m(2 - 3\tilde{v}_m)WV + a\tilde{v}_m^2(1 - \tilde{v}_m)W. \end{aligned} \quad (20)$$

The function G_2 is given by $G_2(v_m, w, \tilde{v}_m, \tilde{w}) = g(v_m, w) - g(\tilde{v}_m, \tilde{w})$,

We can now state the global Carleman inequality

Theorem 3.1. *Suppose that σ satisfy the condition (5). Then, there exists $\lambda_0 = \lambda_0(\Omega, \omega) \geq 1$, $s_0 = s_0(\lambda_0, T) > 1$ and a positive constant $C_0 = C_0(\Omega, \omega, \sigma, T)$ such that, for any $\lambda \geq \lambda_0$ and any $s \geq s_0$, the following estimate holds*

$$\begin{aligned} & \|e^{-s\eta}(s\varphi)^{-1/2}\partial_t V\|_2^2 + \|e^{-s\eta}(s\varphi)^{-1/2}\operatorname{div}(\sigma\nabla V)\|_2^2 + s^3\lambda^4 \|e^{-s\eta}\varphi^{3/2}V\|_2^2 \\ & \quad + s\lambda^2 \|e^{-s\eta}\varphi^{1/2}\nabla V\|_2^2 + s^2\lambda^2 \|e^{-s\eta}\varphi V\|_4^4 + \|e^{-s\eta}V\|_6^6 \\ & \leq C(\|e^{-s\eta}G_1(v_m, w, \tilde{v}_m, \tilde{w})\|_2^2 + s^3\lambda^4 \int_{Q_{\omega_0}} e^{-2s\eta}\varphi^3 |V|^2 dx dt \\ & \quad + s\lambda^2 \int_{Q_{\omega_0}} e^{-2s\eta}\varphi |\nabla V|^2 dx dt + s^2\lambda^2 \int_{Q_{\omega_0}} \varphi^2 e^{-2s\eta} |V|^4 dx dt). \end{aligned} \quad (21)$$

for any V solution to the first equation of (19) and where $\|\cdot\|_p$, $p = 2, 4, 6$, is the $L^p(Q)$ -norm.

The proof of Theorem 3.1 is very much technical. It will be done by steps, following Fursikov-Imanovilov [20].

Step 1. For $s > 0$, we define $\psi = e^{-s\eta}V$, we replace in equation (19) V by $e^{s\eta}\psi$, and we multiply the equation by $e^{-s\eta}$, we then obtain,

$$e^{-s\eta}(\partial_t(e^{s\eta}\psi) - \operatorname{div}(\sigma\nabla(e^{s\eta}\psi)) + a\omega e^{3s\eta}\psi^3) = e^{-s\eta}G_1(v_m, w, \tilde{v}_m, \tilde{w}) \quad (22)$$

By computing (22), we obtain an equality with the following symmetric and anti-symmetric operators

$$L_1(\psi(x, t)) + L_2(\psi(x, t)) = F_1(\psi(x, t)) \quad (x, t) \in Q \quad (23)$$

where

$$L_1(\psi) = s\partial_t\eta\psi - s^2\lambda^2\varphi^2 |\nabla\beta|_{\sigma}^2 \psi - \operatorname{div}(\sigma\nabla\psi) + \frac{3}{4}ke^{2s\eta}\psi^3 \quad (24)$$

$$L_2(\psi) = 2s\lambda^2\varphi |\nabla\beta|_{\sigma}^2 \psi + \partial_t\psi + 2s\lambda\varphi\sigma\nabla\beta\nabla\psi + \frac{1}{4}ke^{2s\eta}\psi^3 \quad (25)$$

$$F_1(\psi) = e^{-s\eta}G_1(v_m, w, \tilde{v}_m, \tilde{w}) - s\lambda\varphi\operatorname{div}(\sigma\nabla\beta)\psi + s\lambda^2\varphi |\nabla\beta|_{\sigma}^2 \psi. \quad (26)$$

Besides, by virtue of (9) and properties of η we have

$$\psi(x, 0) = \psi(x, T) = 0. \quad (27)$$

Applying the $L^2(Q)$ norm on equation (23), we obtain

$$\|L_1(\psi)\|_2^2 + \|L_2(\psi)\|_2^2 + 2(L_1(\psi), L_2(\psi)) = \|F_1(\psi)\|_2^2 \quad (28)$$

where (\cdot, \cdot) denotes the scalar product in $L^2(Q)$.

In virtue of (24) and (25), let us compute the scalar product in the left hand side of (28), we get

$$(L_1(\psi), L_2(\psi)) = \sum_{i,j=1}^4 I_{ij}. \quad (29)$$

In (29), all the terms I_{ij} represent the scalar products of the four terms in $L_1(\psi)$ by the four terms in $L_2(\psi)$. In the sequel, by C we mean various constants independent of

s , λ and T as we want to keep track of the powers of s , λ and T involved. In order to organize the calculations we will give particular importance to terms

$$J_1 = s^3 \lambda^4 \int_Q \varphi^3 |\psi|^2 dx dt, \quad J_2 = s \lambda^2 \int_Q \varphi |\nabla \psi|^2 dx dt, \quad J_3 = s^2 \lambda^2 \int_Q e^{2s\eta} \varphi^2 |\psi|^4 dx dt,$$

$$\text{and } J_4 = \int_Q e^{4s\eta} |\psi|^6 dx dt.$$

Step 2. Now, to prove Theorem 3.1, we will start by establishing some preliminary lemmas.

Lemma 3.2. *Under the same assumptions of Theorem 3.1, we have the following inequality*

$$\begin{aligned} & \|L_1(\psi)\|_2^2 + \|L_2(\psi)\|_2^2 + s^3 \lambda^4 \int_Q \varphi^3 |\nabla \beta|_{\sigma}^4 |\psi|^2 dx dt + s \lambda^2 \int_Q \varphi |\nabla \beta|_{\sigma}^2 |\nabla \psi|_{\sigma}^2 dx dt \\ & + s^2 \lambda^2 \int_Q \varphi^2 k e^{2s\eta} |\nabla \beta|_{\sigma}^2 |\psi|^4 dx dt + \int_Q k^2 e^{4s\eta} |\psi|^6 dx dt \\ & \leq C(T) \left(\|e^{-s\eta} G_1\|_2^2 + s^3 \lambda^4 \int_Q \varphi^3 |\psi|^2 dx dt + s \lambda \int_Q \varphi |\nabla \psi|^2 dx dt \right. \\ & \left. + s \lambda^2 \int_Q e^{2s\eta} \varphi^2 |\psi|^4 dx dt + \int_Q e^{4s\eta} |\psi|^6 dx dt \right) + X_9. \end{aligned} \quad (30)$$

where

$$\begin{aligned} X_9 = & -s^2 \lambda \int_{\Sigma} \partial_t \eta \varphi |\psi|^2 \sigma \nabla \beta \cdot \nu d\Sigma + 2s \lambda^2 \int_{\Sigma} \varphi |\nabla \beta|_{\sigma}^2 \psi \sigma \nabla \psi \cdot \nu d\Sigma \\ & + \int_{\Sigma} \partial_t \psi \sigma \nabla \psi \cdot \nu d\Sigma - s \lambda \int_{\Sigma} \varphi |\nabla \psi|_{\sigma}^2 \sigma \nabla \beta \cdot \nu d\Sigma \\ & + \frac{1}{4} \int_{\Sigma} e^{2s\eta} k \psi^3 \sigma \nabla \psi \cdot \nu d\Sigma - \frac{1}{4} s \lambda \int_{\Sigma} e^{2s\eta} k \varphi |\psi|^4 \sigma \nabla \beta \cdot \nu d\Sigma \\ & + s^3 \lambda^3 \int_{\Sigma} \varphi^3 |\nabla \beta|_{\sigma}^2 |\psi|^2 \sigma \nabla \beta \cdot \nu d\Sigma + 2s \lambda \int_{\Sigma} \varphi |\sigma \nabla \psi \cdot \nu|^2 \sigma \nabla \beta \cdot \nu d\Sigma. \end{aligned} \quad (31)$$

Proof. The first term in the right hand side of equation (29)

$$I_{11} = 2s^2 \lambda^2 \int_Q \partial_t \eta \varphi |\nabla \beta|_{\sigma}^2 |\psi|^2 dx dt. \quad (32)$$

By integration by parts with respect to the time variable, the second integral I_{12} can be written as follows

$$I_{12} = s \int_Q \partial_t \eta \psi \partial_t \psi dx dt = -\frac{s}{2} \int_Q \partial_{tt} \eta |\psi|^2 dx dt. \quad (33)$$

Here we have used (27) which says that $\psi(x, 0) = \psi(x, T) = 0$. From (16), we obtain

$$I_{12} \geq -C(\Omega, \omega) s T^2 \int_Q \varphi^3 |\psi|^2 dx dt. \quad (34)$$

Integrating by parts the third integral, I_{13} , we get

$$\begin{aligned}
I_{13} &= 2s^2\lambda \int_Q \partial_t \eta \varphi \psi \boldsymbol{\sigma} \nabla \beta \cdot \nabla \psi dx dt = s^2\lambda \int_Q \partial_t \eta \varphi \boldsymbol{\sigma} \nabla \beta \cdot \nabla (|\psi|^2) dx dt \\
&= -s^2\lambda \int_Q \operatorname{div}(\partial_t \eta \varphi \boldsymbol{\sigma} \nabla \beta) |\psi|^2 dx dt + s^2\lambda \int_\Sigma \partial_t \eta \varphi |\psi|^2 \boldsymbol{\sigma} \nabla \beta \cdot \boldsymbol{\nu} d\Sigma \\
&= -s^2\lambda \int_Q \varphi \boldsymbol{\sigma} \nabla \beta \nabla \partial_t \eta |\psi|^2 dx dt - s^2\lambda \int_Q \partial_t \eta \boldsymbol{\sigma} \nabla \beta \nabla \varphi |\psi|^2 dx dt \\
&\quad - s^2\lambda \int_Q \partial_t \eta \varphi \operatorname{div}(\boldsymbol{\sigma} \nabla \beta) |\psi|^2 dx dt + s^2\lambda \int_\Sigma \partial_t \eta \varphi |\psi|^2 \boldsymbol{\sigma} \nabla \beta \cdot \boldsymbol{\nu} d\Sigma \\
&= s^2\lambda^2 \int_Q \varphi (\partial_t \varphi - \partial_t \eta) |\nabla \beta|_\sigma^2 |\psi|^2 dx dt - s^2\lambda \int_Q \partial_t \eta \varphi \operatorname{div}(\boldsymbol{\sigma} \nabla \beta) |\psi|^2 dx dt \\
&\quad + s^2\lambda \int_\Sigma \partial_t \eta \varphi |\psi|^2 \boldsymbol{\sigma} \nabla \beta \cdot \boldsymbol{\nu} d\Sigma. \tag{35}
\end{aligned}$$

Besides, due to hypothesis on β of Lemma 3.1, we use that

$$|\partial_i \boldsymbol{\sigma}_{ij} \boldsymbol{\sigma}_{kl} \partial_k \beta| \leq C(\Omega, \omega, \boldsymbol{\sigma}), \tag{36}$$

$$|\boldsymbol{\sigma}_{ij} \partial_j (\boldsymbol{\sigma}_{kl} \partial_k \beta)| \leq C(\Omega, \omega, \boldsymbol{\sigma}), \tag{37}$$

$$|\operatorname{div}(\boldsymbol{\sigma} \nabla \beta)| \leq C(\Omega, \omega, \boldsymbol{\sigma}), \tag{38}$$

$$|\boldsymbol{\sigma}_{kl} \partial_l \boldsymbol{\sigma}_{ij} \partial_k \beta| \leq C(\Omega, \omega, \boldsymbol{\sigma}) \tag{39}$$

$$|\boldsymbol{\sigma}_{ij} \partial_j \beta \partial_i \beta| \leq C(\Omega, \omega, \boldsymbol{\sigma}), \tag{40}$$

then, from (15), (16), (38) and (35), we obtain

$$I_{11} + I_{13} \geq -C(\Omega, \omega, \boldsymbol{\sigma}) s^2 \lambda^2 T \int_Q \varphi^3 |\psi|^2 dx dt + s^2 \lambda \int_\Sigma \partial_t \eta \varphi |\psi|^2 \boldsymbol{\sigma} \nabla \beta \cdot \boldsymbol{\nu} d\Sigma. \tag{41}$$

We now consider the scalar product between the second term of $L_1(\psi)$ with the second one of $L_2(\psi)$. By integration by parts with respect to the time variable, the following holds,

$$\begin{aligned}
I_{22} &= -s^2\lambda^2 \int_Q \varphi^2 |\nabla \beta|_\sigma^2 \psi \partial_t \psi dx dt = -\frac{1}{2} s^2\lambda^2 \int_Q \varphi^2 |\nabla \beta|_\sigma^2 \partial_t (|\psi|^2) dx dt \\
&= s^2\lambda^2 \int_Q \varphi \partial_t \varphi |\nabla \beta|_\sigma^2 |\psi|^2 dx dt. \tag{42}
\end{aligned}$$

Here we have used (27) which say that $\psi(x, 0) = \psi(x, T) = 0$. From (15) and (40), we obtain

$$I_{22} \geq -C(\Omega, \omega, \boldsymbol{\sigma}) s^2 \lambda^2 T^2 \int_Q \varphi^3 |\psi|^2 dx dt. \tag{43}$$

Integrating now by parts in the seven term of (29) we obtain,

$$\begin{aligned}
I_{23} &= -2s^3\lambda^3 \int_Q \varphi^3 \psi |\nabla \beta|_\sigma^2 \boldsymbol{\sigma} \nabla \beta \cdot \nabla \psi dx dt = -s^3\lambda^3 \int_Q \varphi^3 |\nabla \beta|_\sigma^2 \boldsymbol{\sigma} \nabla \beta \cdot \nabla (|\psi|^2) dx dt \\
&= s^3\lambda^3 \int_Q \operatorname{div}(\varphi^3 |\nabla \beta|_\sigma^2 \boldsymbol{\sigma} \nabla \beta) |\psi|^2 dx dt - s^3\lambda^3 \int_\Sigma \varphi^3 |\nabla \beta|_\sigma^2 |\psi|^2 \boldsymbol{\sigma} \nabla \beta \cdot \boldsymbol{\nu} d\Sigma
\end{aligned}$$

$$\begin{aligned}
&= 3s^3\lambda^4 \int_Q \varphi^3 |\nabla\beta|_\sigma^4 |\psi|^2 dx dt + s^3\lambda^3 \int_Q \varphi^3 \operatorname{div}(|\nabla\beta|_\sigma^2 \boldsymbol{\sigma} \nabla\beta) |\psi|^2 dx dt \\
&\quad - s^3\lambda^3 \int_\Sigma \varphi^3 |\nabla\beta|_\sigma^2 |\psi|^2 \boldsymbol{\sigma} \nabla\beta \cdot \nu d\Sigma.
\end{aligned} \tag{44}$$

Note that $|\operatorname{div}(|\nabla\beta|_\sigma^2 \boldsymbol{\sigma} \nabla\beta)| \leq C(\Omega, \omega, \boldsymbol{\sigma})$, therefore from (44) and (40),

$$\begin{aligned}
I_{23} &\geq 3s^3\lambda^4 \int_Q \varphi^3 |\nabla\beta|_\sigma^4 |\psi|^2 dx dt - C(\Omega, \omega, \boldsymbol{\sigma}) s^3\lambda^3 \int_Q \varphi^3 |\psi|^2 dx dt \\
&\quad - s^3\lambda^3 \int_\Sigma \varphi^3 |\nabla\beta|_\sigma^2 |\psi|^2 \boldsymbol{\sigma} \nabla\beta \cdot \nu d\Sigma.
\end{aligned} \tag{45}$$

Integrating by parts the integral I_{31} with respect to the space variable, we obtain

$$\begin{aligned}
I_{31} &= -2s\lambda^2 \int_Q \varphi |\nabla\beta|_\sigma^2 \psi \operatorname{div}(\boldsymbol{\sigma} \nabla\psi) dx dt \\
&= 2s\lambda^3 \int_Q \varphi |\nabla\beta|_\sigma^2 \psi \boldsymbol{\sigma} \nabla\beta \cdot \nabla\psi dx dt + 2s\lambda^2 \int_Q \varphi |\nabla\beta|_\sigma^2 |\nabla\psi|_\sigma^2 dx dt \\
&\quad + 2s\lambda^2 \int_Q \varphi \psi \nabla(|\nabla\beta|_\sigma^2) \boldsymbol{\sigma} \nabla\psi dx dt - 2s\lambda^2 \int_\Sigma \varphi |\nabla\beta|_\sigma^2 \psi \boldsymbol{\sigma} \nabla\psi \cdot \nu d\Sigma.
\end{aligned} \tag{46}$$

Since $\nabla\varphi = \lambda\varphi\nabla\beta$. From this and Cauchy-Schwartz inequality, we obtain

$$\begin{aligned}
&\left| 2s\lambda^3 \int_Q \varphi |\nabla\beta|_\sigma^2 \psi \boldsymbol{\sigma} \nabla\beta \cdot \nabla\psi dx dt \right| \leq \\
&4s\lambda^4 \int_Q \varphi |\nabla\beta|_\sigma^4 |\psi|^2 dx dt + \frac{1}{4}s\lambda^2 \int_Q \varphi |\boldsymbol{\sigma} \nabla\beta|^2 |\nabla\psi|^2 dx dt,
\end{aligned} \tag{47}$$

and

$$\begin{aligned}
&\left| 2s\lambda^2 \int_Q \varphi \psi \nabla(|\nabla\beta|_\sigma^2) \boldsymbol{\sigma} \nabla\psi dx dt \right| = \left| 4s\lambda^2 \int_Q \varphi \psi |\nabla\beta|_\sigma \boldsymbol{\sigma} \nabla|\nabla\beta|_\sigma \cdot \nabla\psi dx dt \right| \\
&\leq 16s\lambda^2 \int_Q \varphi |\boldsymbol{\sigma} \nabla|\nabla\beta|_\sigma|^2 |\psi|^2 dx dt + \frac{1}{4}s\lambda^2 \int_Q \varphi |\nabla\beta|_\sigma^2 |\nabla\psi|^2 dx dt.
\end{aligned} \tag{48}$$

Since $|\boldsymbol{\sigma} \nabla|\nabla\beta|_\sigma|^2 < C(\Omega, \omega, \boldsymbol{\sigma})$, thus from (40), (47) and (48) we transform (46) as follows.

$$\begin{aligned}
I_{31} &\geq \frac{3}{2}s\lambda^2 \int_Q \varphi |\nabla\beta|_\sigma^2 |\nabla\psi|_\sigma^2 dx dt - C \int_Q s\varphi\lambda^4 |\psi|^2 dx dt \\
&\quad - 2s\lambda^2 \int_\Sigma \varphi |\nabla\beta|_\sigma^2 \psi \boldsymbol{\sigma} \nabla\psi \cdot \nu d\Sigma.
\end{aligned} \tag{49}$$

Integrating by parts the integral I_{32} with respect to the space variable, we obtain

$$\begin{aligned}
I_{32} &= - \int_Q \partial_t \psi \operatorname{div}(\boldsymbol{\sigma} \nabla\psi) dx dt = \int_Q \psi \operatorname{div}(\boldsymbol{\sigma} \nabla \partial_t \psi) dx dt \\
&= - \int_Q \nabla\psi \boldsymbol{\sigma} \nabla \partial_t \psi dx dt + \int_\Sigma \psi \boldsymbol{\sigma} \nabla \partial_t \psi \cdot \nu d\Sigma \\
&= - \int_\Sigma \partial_t \psi \boldsymbol{\sigma} \nabla\psi \cdot \nu d\Sigma.
\end{aligned} \tag{50}$$

Here we have used $\psi = e^{-s\eta}V$, and then we can write

$$\partial_i\psi = (-s\partial_i\eta V + \partial_i V)e^{-s\eta}$$

therefore

$$\partial_i\psi(x, 0) = \partial_i\psi(x, T) = 0 \quad \forall x \in \Omega, \quad i = 1, 2, 3.$$

and from (27) which say that $\psi(x, 0) = \psi(x, T) = 0$.

Integrating by parts the integral I_{33} with respect to the space variable, we obtain

$$\begin{aligned} I_{33} &= -2s\lambda \int_Q \varphi \operatorname{div}(\boldsymbol{\sigma} \nabla \psi) \boldsymbol{\sigma} \nabla \beta \cdot \nabla \psi \, dx \, dt = 2s\lambda \sum_{i,j,k,l=1}^3 \int_Q \varphi \partial_i \boldsymbol{\sigma}_{ij} \boldsymbol{\sigma}_{kl} \partial_k \beta \partial_i \psi \partial_l \psi \, dx \, dt \\ &\quad + 2s\lambda^2 \int_Q \varphi |\boldsymbol{\sigma} \nabla \beta \cdot \nabla \psi|^2 \, dx \, dt - s\lambda^2 \int_Q \varphi |\nabla \beta|_{\boldsymbol{\sigma}}^2 |\nabla \psi|_{\boldsymbol{\sigma}}^2 \, dx \, dt \\ &\quad + 2s\lambda \sum_{i,j,k,l=1}^3 \int_Q \varphi \boldsymbol{\sigma}_{ij} \partial_j (\boldsymbol{\sigma}_{kl} \partial_k \beta) \partial_i \psi \partial_l \psi \, dx \, dt - s\lambda \int_Q \varphi \operatorname{div}(\boldsymbol{\sigma} \nabla \beta) |\nabla \psi|_{\boldsymbol{\sigma}}^2 \, dx \, dt \\ &\quad - s\lambda \sum_{i,j,k,l=1}^3 \int_Q \varphi \boldsymbol{\sigma}_{kl} \partial_l \boldsymbol{\sigma}_{ij} \partial_k \beta \partial_i \psi \partial_j \psi \, dx \, dt - s\lambda \int_{\Sigma} \varphi |\nabla \psi|_{\boldsymbol{\sigma}}^2 \boldsymbol{\sigma} \nabla \beta \cdot \nu \, d\Sigma \\ &= 2s\lambda^2 \int_Q \varphi |\boldsymbol{\sigma} \nabla \beta \cdot \nabla \psi|^2 \, dx \, dt - s\lambda^2 \int_Q \varphi |\nabla \beta|_{\boldsymbol{\sigma}}^2 |\nabla \psi|_{\boldsymbol{\sigma}}^2 \, dx \, dt + \sum_{i=1}^4 X_i \\ &\quad + s\lambda \int_{\Sigma} \varphi |\nabla \psi|_{\boldsymbol{\sigma}}^2 \boldsymbol{\sigma} \nabla \beta \cdot \nu \, d\Sigma - 2s\lambda \int_{\Sigma} \varphi \boldsymbol{\sigma} \nabla \beta \cdot \nabla \psi \boldsymbol{\sigma} \nabla \psi \cdot \nu \, d\Sigma, \end{aligned} \quad (51)$$

with

$$X_1 = 2s\lambda \sum_{i,j,k,l=1}^3 \int_Q \varphi \partial_i \boldsymbol{\sigma}_{ij} \boldsymbol{\sigma}_{kl} \partial_k \beta \partial_i \psi \partial_l \psi \, dx \, dt, \quad (52)$$

$$X_2 = 2s\lambda \sum_{i,j,k,l=1}^3 \int_Q \varphi \boldsymbol{\sigma}_{ij} \partial_j (\boldsymbol{\sigma}_{kl} \partial_k \beta) \partial_i \psi \partial_l \psi \, dx \, dt, \quad (53)$$

$$X_3 = -s\lambda \int_Q \varphi \operatorname{div}(\boldsymbol{\sigma} \nabla \beta) |\nabla \psi|_{\boldsymbol{\sigma}}^2 \, dx \, dt, \quad (54)$$

and

$$X_4 = -s\lambda \sum_{i,j,k,l=1}^3 \int_Q \varphi \boldsymbol{\sigma}_{kl} \partial_l \boldsymbol{\sigma}_{ij} \partial_k \beta \partial_i \psi \partial_j \psi \, dx \, dt. \quad (55)$$

Since $\boldsymbol{\sigma}$ is symmetric and $\boldsymbol{\sigma} \nabla \beta = (\boldsymbol{\sigma} \nabla \beta \cdot \nu) \nu$, then

$$-2s\lambda \int_{\Sigma} \varphi \boldsymbol{\sigma} \nabla \beta \cdot \nabla \psi \boldsymbol{\sigma} \nabla \psi \cdot \nu \, d\Sigma = -2s\lambda \int_{\Sigma} \varphi |\boldsymbol{\sigma} \nabla \psi \cdot \nu|^2 \boldsymbol{\sigma} \nabla \beta \cdot \nu \, d\Sigma. \quad (56)$$

Thus, using (56) in (51), we obtain, in first step, for the integral I_{33} , the following estimation

$$\begin{aligned} I_{33} &\geq \sum_{i=1}^4 X_i - s\lambda^2 \int_Q \varphi |\nabla\beta|_{\sigma}^2 |\nabla\psi|_{\sigma}^2 dx dt \\ &\quad + s\lambda \int_{\Sigma} \varphi |\nabla\psi|_{\sigma}^2 \sigma \nabla\beta \cdot \nu d\Sigma - 2s\lambda \int_{\Sigma} \varphi |\sigma \nabla\psi \cdot \nu|^2 \sigma \nabla\beta \cdot \nu d\Sigma. \end{aligned} \quad (57)$$

In the second step, from (36), (37), (38), (39) and (40), we have

$$\sum_{i=1}^4 X_i \geq -C(\Omega, \omega, \sigma) s\lambda \int_Q \varphi |\nabla\psi|^2 dx dt. \quad (58)$$

Thus,

$$\begin{aligned} I_{33} &\geq -s\lambda^2 \int_Q \varphi |\nabla\beta|_{\sigma}^2 |\nabla\psi|_{\sigma}^2 dx dt - C(\Omega, \omega, \sigma) s\lambda \int_Q \varphi |\nabla\psi|^2 dx dt \\ &\quad + s\lambda \int_{\Sigma} \varphi |\nabla\psi|_{\sigma}^2 \sigma \nabla\beta \cdot \nu d\Sigma - 2s\lambda \int_{\Sigma} \varphi |\sigma \nabla\psi \cdot \nu|^2 \sigma \nabla\beta \cdot \nu d\Sigma. \end{aligned} \quad (59)$$

Applying Green's formula in I_{34} , we get

$$\begin{aligned} I_{34} &= -\frac{1}{4} \int_Q e^{2s\eta} k \psi^3 \operatorname{div}(\sigma \nabla\psi) dx dt \\ &= \frac{1}{4} \int_Q \nabla(e^{2s\eta} k \psi^3) \sigma \nabla\psi dx dt - \frac{1}{4} \int_{\Sigma} e^{2s\eta} k \psi^3 \sigma \nabla\psi \cdot \nu d\Sigma \\ &= \frac{1}{4} \int_Q e^{2s\eta} (2s\nabla\eta k \psi^3 + \nabla k \psi^3 + 3k\psi^2 \nabla\psi) \sigma \nabla\psi dx dt - \frac{1}{4} \int_{\Sigma} e^{2s\eta} k \psi^3 \sigma \nabla\psi \cdot \nu d\Sigma \\ &= -\frac{1}{2} s\lambda \int_Q e^{2s\eta} k \varphi \psi^3 \sigma \nabla\beta \cdot \nabla\psi dx dt + \frac{1}{4} \int_Q e^{2s\eta} \psi^3 \nabla k \sigma \nabla\psi dx dt \\ &\quad + \frac{3}{4} \int_Q e^{2s\eta} k \psi^2 |\nabla\psi|_{\sigma}^2 dx dt - \frac{1}{4} \int_{\Sigma} e^{2s\eta} k \psi^3 \sigma \nabla\psi \cdot \nu d\Sigma \\ &= \sum_{i=5}^7 X_i - \frac{1}{4} \int_{\Sigma} e^{2s\eta} k \psi^3 \sigma \nabla\psi \cdot \nu d\Sigma. \end{aligned} \quad (60)$$

With

$$X_5 = -\frac{1}{2} s\lambda \int_Q e^{2s\eta} k \varphi \psi^3 \sigma \nabla\beta \cdot \nabla\psi dx dt, \quad X_6 = \frac{1}{4} \int_Q e^{2s\eta} \nabla k \psi^3 \sigma \nabla\psi dx dt$$

and

$$X_7 = \frac{3}{4} \int_Q e^{2s\eta} k \psi^2 |\nabla\psi|_{\sigma}^2 dx dt.$$

Let us calculate X_5 .

$$\begin{aligned} X_5 &= -\frac{1}{2} s\lambda \int_Q e^{2s\eta} k \varphi \psi^3 \sigma \nabla\beta \cdot \nabla\psi dx dt = -\frac{1}{8} s\lambda \int_Q e^{2s\eta} k \varphi \sigma \nabla\beta \cdot \nabla(|\psi|^4) dx dt \\ &= \frac{1}{8} s\lambda \int_Q \operatorname{div}(e^{2s\eta} k \varphi \sigma \nabla\beta) |\psi|^4 dx dt - \frac{1}{8} s\lambda \int_{\Sigma} e^{2s\eta} k \varphi |\psi|^4 \sigma \nabla\beta \cdot \nu d\Sigma \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{8}s\lambda \int_Q \nabla(e^{2s\eta}k\varphi)\boldsymbol{\sigma}\nabla\beta|\psi|^4 dx dt + \frac{1}{8}s\lambda \int_Q e^{2s\eta}k\varphi \operatorname{div}(\boldsymbol{\sigma}\nabla\beta)|\psi|^4 dx dt \\
&\quad - \frac{1}{8}s\lambda \int_{\Sigma} e^{2s\eta}k\varphi|\psi|^4 \boldsymbol{\sigma}\nabla\beta.\nu d\Sigma \\
&= \frac{1}{8}s\lambda \int_Q e^{2s\eta}(2s\nabla\eta k\varphi + \nabla k\varphi + k\nabla\varphi)\boldsymbol{\sigma}\nabla\beta|\psi|^4 dx dt \\
&+ \frac{1}{8}s\lambda \int_Q e^{2s\eta}k\varphi \operatorname{div}(\boldsymbol{\sigma}\nabla\beta)|\psi|^4 dx dt - \frac{1}{8}s\lambda \int_{\Sigma} e^{2s\eta}k\varphi|\psi|^4 \boldsymbol{\sigma}\nabla\beta.\nu d\Sigma \\
&= \frac{1}{8}s\lambda \int_Q e^{2s\eta}(-2s\lambda\nabla\beta k\varphi^2 + \nabla k\varphi + k\lambda\nabla\beta\varphi)\boldsymbol{\sigma}\nabla\beta|\psi|^4 dx dt \\
&+ \frac{1}{8}s\lambda \int_Q e^{2s\eta}k\varphi \operatorname{div}(\boldsymbol{\sigma}\nabla\beta)|\psi|^4 dx dt - \frac{1}{8}s\lambda \int_{\Sigma} e^{2s\eta}k\varphi|\psi|^4 \boldsymbol{\sigma}\nabla\beta.\nu d\Sigma \\
&= -\frac{1}{4}s^2\lambda^2 \int_Q e^{2s\eta}|\nabla\beta|_{\boldsymbol{\sigma}}^2 k\varphi^2|\psi|^4 dx dt + \frac{1}{8}s\lambda \int_Q e^{2s\eta}\varphi\boldsymbol{\sigma}\nabla\beta.\nabla k|\psi|^4 dx dt \\
&+ \frac{1}{8}s\lambda^2 \int_Q e^{2s\eta}k|\nabla\beta|_{\boldsymbol{\sigma}}^2 \varphi|\psi|^4 dx dt - \frac{1}{8}s\lambda \int_{\Sigma} e^{2s\eta}k\varphi|\psi|^4 \boldsymbol{\sigma}\nabla\beta.\nu d\Sigma \\
&\quad + \frac{1}{8}s\lambda \int_Q e^{2s\eta}k\varphi \operatorname{div}(\boldsymbol{\sigma}\nabla\beta)|\psi|^4 dx dt \\
&= -\frac{1}{4}s^2\lambda^2 \int_Q e^{2s\eta}|\nabla\beta|_{\boldsymbol{\sigma}}^2 k\varphi^2|\psi|^4 dx dt - \frac{1}{8}s\lambda \int_{\Sigma} e^{2s\eta}k\varphi|\psi|^4 \boldsymbol{\sigma}\nabla\beta.\nu d\Sigma + X_8, \tag{61}
\end{aligned}$$

with

$$\begin{aligned}
X_8 &= \frac{1}{8}s\lambda \int_Q e^{2s\eta}\varphi\boldsymbol{\sigma}\nabla\beta.\nabla k|\psi|^4 dx dt + \frac{1}{8}s\lambda^2 \int_Q e^{2s\eta}k|\nabla\beta|_{\boldsymbol{\sigma}}^2 \varphi|\psi|^4 dx dt \\
&\quad + \frac{1}{8}s\lambda \int_Q e^{2s\eta}k\varphi \operatorname{div}(\boldsymbol{\sigma}\nabla\beta)|\psi|^4 dx dt. \tag{62}
\end{aligned}$$

Thus,

$$\begin{aligned}
I_{34} &= -\frac{1}{4}s^2\lambda^2 \int_Q e^{2s\eta}|\nabla\beta|_{\boldsymbol{\sigma}}^2 k\varphi^2|\psi|^4 dx dt + X_6 + X_7 + X_8 \\
&\quad - \frac{1}{4} \int_{\Sigma} e^{2s\eta}k\psi^3 \boldsymbol{\sigma}\nabla\psi.\nu d\Sigma - \frac{1}{8}s\lambda \int_{\Sigma} e^{2s\eta}k\varphi|\psi|^4 \boldsymbol{\sigma}\nabla\beta.\nu d\Sigma. \tag{63}
\end{aligned}$$

On the other hand, observing that

$$I_{43} = -3X_5, \tag{64}$$

then, inserting the above equality (64) in the integral I_{34} , it yields

$$I_{34} + I_{43} = -2X_5 + X_6 + X_7 - \frac{1}{4} \int_{\Sigma} e^{2s\eta}k\psi^3 \boldsymbol{\sigma}\nabla\psi.\nu d\Sigma. \tag{65}$$

Now, we are working with $I_{34} + I_{43}$. In fact,

$$|X_8| \leq Cs\lambda^2 T^2 \int_Q e^{2s\eta}\varphi^2|\psi|^4 dx dt. \tag{66}$$

Therefore by (61), we obtain

$$\begin{aligned} -2X_5 \geq & \frac{1}{2}s^2\lambda^2 \int_Q e^{2s\eta}\varphi^2 k |\nabla\beta|_{\sigma}^2 |\psi|^4 dx dt - Cs\lambda^2 T^2 \int_Q e^{2s\eta}\varphi^2 |\psi|^4 dx dt \\ & + \frac{1}{4}s\lambda \int_{\Sigma} e^{2s\eta} k \varphi \psi^4 \sigma \nabla\beta \cdot \nu d\Sigma. \end{aligned} \quad (67)$$

Using Young inequality $ab \leq \varepsilon a^2 + C_{\varepsilon} b^2$, $\varepsilon \ll 1$, one could check that

$$\begin{aligned} |X_6| \leq & \frac{1}{4} \int_Q e^{2s\eta} |\nabla k| |\psi|^3 |\sigma \nabla\psi| dx dt \\ \leq & \frac{1}{4} \left(\varepsilon \int_Q e^{4s\eta} |\nabla k|^2 |\psi|^6 dx dt + C_{\varepsilon} \int_Q |\sigma \nabla\psi|^2 dx dt \right) \\ \leq & C \left(\varepsilon \int_Q e^{4s\eta} |\psi|^6 dx dt + C_{\varepsilon} T^2 \int_Q \varphi |\nabla\psi|^2 dx dt \right). \end{aligned} \quad (68)$$

Here we have used that $k \in L^{\infty}(Q)$ and $\sigma \in [L^{\infty}(\Omega)]^{3 \times 3}$. Using (60), (64), (65), (61) and (68), we obtain

$$\begin{aligned} I_{34} + I_{43} \geq & \frac{1}{2}s^2\lambda^2 \int_Q e^{2s\eta}\varphi^2 k |\nabla\beta|_{\sigma}^2 |\psi|^4 dx dt - Cs\lambda^2 T^2 \int_Q e^{2s\eta}\varphi^2 |\psi|^4 dx dt \\ & - C \left(\varepsilon \int_Q e^{4s\eta} |\psi|^6 dx dt + C_{\varepsilon} T^2 \int_Q \varphi |\nabla\psi|^2 dx dt \right) \\ & - \frac{1}{4} \int_{\Sigma} e^{2s\eta} k \psi^3 \sigma \nabla\psi \cdot \nu d\Sigma + \frac{1}{4}s\lambda \int_{\Sigma} e^{2s\eta} k \varphi |\psi|^4 \sigma \nabla\beta \cdot \nu d\Sigma. \end{aligned} \quad (69)$$

Here we have used $X_7 \geq 0$ and $k = aw \geq 0$ ($0 < w < 1$).

By integrating by parts the term I_{42} with respect to the time variable and using the fact that $0 < w < 1$ (see [7] Lemma 3.3) and a is positive, we have

$$\begin{aligned} I_{42} &= \frac{3}{4} \int_Q e^{2s\eta} k \psi^3 \partial_t \psi dx dt = -\frac{3}{16} \int_Q e^{2s\eta} (\partial_t k + 2s\partial_t \eta k) |\psi|^4 dx dt \\ &\geq -C(1 + sT) \int_Q e^{2s\eta} \varphi^2 |\psi|^4 dx dt. \end{aligned} \quad (70)$$

Using (16), (40) and the fact $k \in L^{\infty}(Q)$, we have

$$I_{14} = \frac{1}{4}s \int_Q \partial_t \eta k e^{2s\eta} |\psi|^4 dx dt \geq -CsT \int_Q e^{2s\eta} \varphi^2 |\psi|^4 dx dt. \quad (71)$$

and

$$I_{41} = \frac{3}{2}s\lambda^2 \int_Q k e^{2s\eta} \varphi |\nabla\beta|_{\sigma}^2 |\psi|^4 dx dt \geq -Cs\lambda^2 T^2 \int_Q e^{2s\eta} \varphi^2 |\psi|^4 dx dt. \quad (72)$$

Finally, we have

$$I_{21} = -2s^3\lambda^4 \int_Q \varphi^3 |\nabla\beta|_{\sigma}^4 |\psi|^2 dx dt, \quad I_{24} = -\frac{1}{4}s^2\lambda^2 \int_Q \varphi^2 k e^{2s\eta} |\nabla\beta|_{\sigma}^2 |\psi|^4 dx dt \quad (73)$$

and

$$I_{44} = \frac{3}{16} \int_Q k^2 e^{4s\eta} |\psi|^6 dx dt. \quad (74)$$

Making use of estimations (34), (41), (43), (45), (49), (50), (59), (65), (70), (71), (73), (72) and (74) we obtain the lower bound for $2(L_1(\psi), L_2(\psi))$, we get the following desired inequality:

$$\begin{aligned}
& \|L_1(\psi)\|^2 + \|L_2(\psi)\|^2 + s^3\lambda^4 \int_Q \varphi^3 |\nabla\beta|_{\sigma}^4 |\psi|^2 dx dt + s\lambda^2 \int_Q \varphi |\nabla\beta|_{\sigma}^2 |\nabla\psi|_{\sigma}^2 dx dt \\
& \quad + s^2\lambda^2 \int_Q \varphi^2 k e^{2s\eta} |\nabla\beta|_{\sigma}^2 |\psi|^4 dx dt + \int_Q k^2 e^{4s\eta} |\psi|^6 dx dt \\
& \leq \|F_1\|_2^2 + C(T)(s^3\lambda^4 \int_Q \varphi^3 |\psi|^2 dx dt + s\lambda \int_Q \varphi |\nabla\psi|_{\sigma}^2 dx dt \\
& \quad + s\lambda^2 \int_Q e^{2s\eta} \varphi^2 |\psi|^4 dx dt + \int_Q e^{4s\eta} |\psi|^6 dx dt) + X_9
\end{aligned} \tag{75}$$

where

$$\begin{aligned}
X_9 &= -s^2\lambda \int_{\Sigma} \partial_t \eta \varphi |\psi|^2 \sigma \nabla\beta \cdot \nu d\Sigma + 2s\lambda^2 \int_{\Sigma} \varphi |\nabla\beta|_{\sigma}^2 \psi \sigma \nabla\psi \cdot \nu d\Sigma \\
& \quad + \int_{\Sigma} \partial_t \psi \sigma \nabla\psi \cdot \nu d\Sigma - s\lambda \int_{\Sigma} \varphi |\nabla\psi|_{\sigma}^2 \sigma \nabla\beta \cdot \nu d\Sigma \\
& \quad + \frac{1}{4} \int_{\Sigma} e^{2s\eta} k \psi^3 \sigma \nabla\psi \cdot \nu d\Sigma - \frac{1}{4} s\lambda \int_{\Sigma} e^{2s\eta} k \varphi |\psi|^4 \sigma \nabla\beta \cdot \nu d\Sigma \\
& \quad + s^3\lambda^3 \int_{\Sigma} \varphi^3 |\nabla\beta|_{\sigma}^2 |\psi|^2 \sigma \nabla\beta \cdot \nu d\Sigma + 2s\lambda \int_{\Sigma} \varphi |\sigma \nabla\psi \cdot \nu|^2 \sigma \nabla\beta \cdot \nu d\Sigma,
\end{aligned} \tag{76}$$

and F_1 is given by (26). Here we used the obvious inequalities $\varphi^{-1} \leq (T/2)^2$, $\varphi^{-2} \leq (T/2)^4$ and the fact that the parameters $s > 1$, $\lambda > 1$.

In order to complete the estimations involving the terms of (28), we need to obtain the upper bounds for the L^2 integrals of $F_1(\psi)$. From the equation (26), we see that

$$\|F_1(\psi)\|_2^2 \leq \|e^{-s\eta} G_1\|_2^2 + C(s\lambda T^2 + s\lambda^2 T^2) \int_Q \varphi^3 |\psi|^2 dx dt. \tag{77}$$

Using the inequalities, (75) and (77) we prove Lemma 3.2. \blacksquare

Step 3. In this step, we aim at eliminating the boundary integrals in X_9 . By defining $\bar{\psi} = e^{-s\eta} V$, replacing V by $e^{s\eta} \bar{\psi}$ in equation (19) and by multiplying the equation by $e^{-s\eta}$, we obtain

$$\bar{L}_1(\bar{\psi}(x, t)) + \bar{L}_2(\bar{\psi}(x, t)) = \bar{F}_1(\bar{\psi}(x, t)) \quad \text{in } (x, t) \in Q \tag{78}$$

where

$$\bar{L}_1(\bar{\psi}) = s\partial_t \bar{\eta} \bar{\psi} - s^2\lambda^2 \bar{\varphi}^2 |\nabla\beta|_{\sigma}^2 \bar{\psi} - \text{div}(\sigma \nabla \bar{\psi}) + \frac{3}{4} k e^{2s\eta} \bar{\psi}^3 \tag{79}$$

$$\bar{L}_2(\bar{\psi}) = 2s\lambda^2 \bar{\varphi} |\nabla\beta|_{\sigma}^2 \bar{\psi} + \partial_t \bar{\psi} - 2s\lambda \bar{\varphi} \sigma \nabla\beta \nabla \bar{\psi} + \frac{1}{4} k e^{2s\eta} \bar{\psi}^3 \tag{80}$$

$$\bar{F}_1(\bar{\psi}) = e^{-s\eta} G_1 + s\lambda \bar{\varphi} \text{div}(\sigma \nabla\beta) \bar{\psi} + s\lambda^2 \bar{\varphi} |\nabla\beta|_{\sigma}^2 \bar{\psi} \tag{81}$$

In this step we prove the following lemma

Lemma 3.3. *Under the same assumptions of Theorem 3.1, we have the following inequality*

$$\begin{aligned}
& \|L_1(\psi)\|^2 + \|L_2(\psi)\|^2 + s^3\lambda^4 \int_Q \varphi^3 |\nabla\beta|_\sigma^4 |\psi|^2 dx dt + s\lambda^2 \int_Q \varphi |\nabla\beta|_\sigma^2 |\nabla\psi|_\sigma^2 dx dt \\
& \quad + s^2\lambda^2 \int_Q \varphi^2 k e^{2s\eta} |\nabla\beta|_\sigma^2 |\psi|^4 dx dt + \int_Q k^2 e^{4s\eta} |\psi|^6 dx dt \\
& \leq 2 \|e^{-s\eta} G_1\|_2^2 + C(T)(s^3\lambda^4 \int_Q \varphi^3 |\psi|^2 dx dt + s\lambda \int_Q \varphi |\nabla\psi|^2 dx dt \\
& \quad + s\lambda^2 \int_Q e^{2s\eta} \varphi^2 |\psi|^4 dx dt + \int_Q e^{4s\eta} |\psi|^6 dx dt). \tag{82}
\end{aligned}$$

Proof. Using the same procedure as for the proof of Lemma 3.2, but here for equations (78)-(81), we obtain the following inequality

$$\begin{aligned}
& \|\bar{L}_1(\bar{\psi})\|^2 + \|\bar{L}_2(\bar{\psi})\|^2 + s^3\lambda^4 \int_Q \bar{\varphi}^3 |\nabla\beta|_\sigma^4 |\bar{\psi}|^2 dx dt + s\lambda^2 \int_Q \bar{\varphi} |\nabla\beta|_\sigma^2 |\nabla\bar{\psi}|_\sigma^2 dx dt \\
& \quad + s^2\lambda^2 \int_Q \bar{\varphi}^2 k e^{2s\bar{\eta}} |\nabla\beta|_\sigma^2 |\bar{\psi}|^4 dx dt + \int_Q k^2 e^{4s\bar{\eta}} |\bar{\psi}|^6 dx dt \\
& \leq \|e^{-s\bar{\eta}} G_1\|_2^2 + s^3\lambda^4 \int_Q \bar{\varphi}^3 |\bar{\psi}|^2 dx dt + s\lambda \int_Q \bar{\varphi} |\nabla\bar{\psi}|^2 dx dt \\
& \quad + s\lambda^2 \int_Q e^{2s\bar{\eta}} \bar{\varphi}^2 |\bar{\psi}|^4 dx dt + \int_Q e^{4s\bar{\eta}} |\bar{\psi}|^6 dx dt + X_{10}. \tag{83}
\end{aligned}$$

with

$$\begin{aligned}
X_{10} &= s^2\lambda \int_\Sigma \partial_t \bar{\eta} \bar{\varphi} |\bar{\psi}|^2 \sigma \nabla\beta \cdot \nu d\Sigma - 2s\lambda^2 \int_\Sigma \bar{\varphi} |\nabla\beta|_\sigma^2 \bar{\psi} \sigma \nabla\bar{\psi} \cdot \nu d\Sigma \\
& \quad - \int_\Sigma \partial_t \bar{\psi} \sigma \nabla\bar{\psi} \cdot \nu d\Sigma + s\lambda \int_\Sigma \bar{\varphi} |\nabla\bar{\psi}|_\sigma^2 \sigma \nabla\beta \cdot \nu d\Sigma \\
& \quad - \frac{1}{4} \int_\Sigma e^{2s\bar{\eta}} k \bar{\psi}^3 \sigma \nabla\bar{\psi} \cdot \nu d\Sigma + \frac{1}{4} s\lambda \int_\Sigma e^{2s\bar{\eta}} k \bar{\varphi} |\bar{\psi}|^4 \sigma \nabla\beta \cdot \nu d\Sigma \\
& \quad - s^3\lambda^3 \int_\Sigma \bar{\varphi}^3 |\nabla\beta|_\sigma^2 |\bar{\psi}|^2 \sigma \nabla\beta \cdot \nu d\Sigma - 2s\lambda \int_\Sigma \bar{\varphi} |\sigma \nabla\bar{\psi} \cdot \nu|^2 \sigma \nabla\beta \cdot \nu d\Sigma. \tag{84}
\end{aligned}$$

Remark 2. *First, we notice that $X_{10} = -X_9$. From the definitions of φ , $\bar{\varphi}$, η and $\bar{\eta}$ we have*

$$\bar{\varphi} \leq \varphi, \quad \eta \leq \bar{\eta}, \quad \text{and} \quad |\bar{\psi}| \leq |\psi|, \quad \text{in } Q. \tag{85}$$

Additionally, since we have $\bar{\psi} = e^{-s(\bar{\eta}-\eta)}\psi$ then, we obtain

$$|\nabla\bar{\psi}| \leq C(|\nabla\psi| + s\lambda\varphi|\nabla\beta| |\bar{\psi}|) \leq C(|\nabla\psi| + s\lambda\varphi|\psi|), \tag{86}$$

where the constant c depends only on β .

By summing equations (75) and (82) and using the inequalities in Remark 2, we prove Lemma 3.3. ■

Step 4. Using the fundamental properties of the function β given by Lemma 3.1, we have the following result.

Lemma 3.4.

$$\begin{aligned}
& \|L_1(\psi)\|_2^2 + \|L_2(\psi)\|_2^2 + s^3\lambda^4 \int_Q \varphi^3 |\psi|^2 dx dt \\
& + s\lambda^2 \int_Q \varphi |\nabla\psi|^2 dx dt + s^2\lambda^2 \int_Q \varphi^2 e^{2s\eta} |\psi|^4 + \int_Q e^{4s\eta} |\psi|^6 dx dt \\
& \leq 2 \|e^{-s\eta} G_1\|_2^2 + C \left(s^3\lambda^4 \int_{Q_{\omega_0}} \varphi^3 |\psi|^2 dx dt \right. \\
& \left. + s\lambda^2 \int_{Q_{\omega_0}} \varphi |\nabla\psi|^2 dx dt + s^2\lambda^2 \int_{Q_{\omega_0}} \varphi^2 e^{2s\eta} |\psi|^4 dx dt \right). \tag{87}
\end{aligned}$$

Proof. From Lemma 3.1, we see that $|\nabla\beta|$ has a lower bound on $\Omega \setminus \omega_0$ and therefore on $Q \setminus Q_{\omega_0}$ (note that β does not depend on t), then there exists a constant $\delta > 0$ such that $|\nabla\beta| \geq \delta$ in $Q \setminus Q_{\omega_0}$, so the left-hand side terms of (82) have the following lower bounds:

$$\begin{aligned}
s^3\lambda^4 \int_Q \varphi^3 |\nabla\beta|_{\sigma}^4 |\psi|^2 dx dt & \geq s^3\lambda^4 \int_{Q \setminus Q_{\omega_0}} \varphi^3 |\nabla\beta|_{\sigma}^4 |\psi|^2 dx dt \\
& \geq \alpha_{\sigma} \delta^4 s^3\lambda^4 \int_{Q \setminus Q_{\omega_0}} \varphi^3 |\psi|^2 dx dt \\
& = \alpha_{\sigma} \delta^4 \left(s^3\lambda^4 \int_Q \varphi^3 |\psi|^2 dx dt - s^3\lambda^4 \int_{Q_{\omega_0}} \varphi^3 |\psi|^2 dx dt \right). \tag{88}
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
s\lambda^2 \int_Q \varphi |\nabla\beta|_{\sigma}^2 |\nabla\psi|_{\sigma}^2 dx dt & \geq s\lambda^2 \int_{Q \setminus Q_{\omega_0}} \varphi |\nabla\beta|_{\sigma}^2 |\nabla\psi|_{\sigma}^2 dx dt \\
& \geq \alpha_{\sigma^2} \delta^2 s\lambda^2 \int_{Q \setminus Q_{\omega_0}} \varphi |\nabla\psi|^2 dx dt \\
& = \alpha_{\sigma^2} \delta^2 \left(s\lambda^2 \int_Q \varphi |\nabla\psi|^2 dx dt - s\lambda^2 \int_{Q_{\omega_0}} \varphi |\nabla\psi|^2 dx dt \right) \tag{89}
\end{aligned}$$

and

$$\begin{aligned}
s^2\lambda^2 \int_Q \varphi^2 k e^{2s\eta} |\nabla\beta|_{\sigma}^2 |\psi|^4 dx dt & \geq s^2\lambda^2 \int_{Q \setminus Q_{\omega_0}} \varphi^2 k e^{2s\eta} |\nabla\beta|_{\sigma}^2 |\psi|^4 dx dt \\
& \geq c_k \alpha_{\sigma} \delta^2 s^2\lambda^2 \int_{Q \setminus Q_{\omega_0}} \varphi^2 e^{2s\eta} |\psi|^4 dx dt \\
& = c_k \alpha_{\sigma} \delta^2 \left(s^2\lambda^2 \int_Q \varphi^2 e^{2s\eta} |\psi|^4 dx dt - s^2\lambda^2 \int_{Q_{\omega_0}} \varphi^2 e^{2s\eta} |\psi|^4 dx dt \right), \tag{90}
\end{aligned}$$

where c_{σ} (respectively c_{σ^2}) is the coercivity coefficient of σ (respectively σ^2).

Making use of these estimates (88)-(90) in (82), we have

$$\begin{aligned}
& \|L_1(\psi)\|^2 + \|L_2(\psi)\|^2 + \alpha_\sigma \delta^4 s^3 \lambda^4 \int_Q \varphi^3 |\psi|^2 dx dt + \alpha_{\sigma^2} \delta^2 s \lambda^2 \int_Q \varphi |\nabla \psi|^2 dx dt \\
& + c_k \alpha_\sigma \delta^2 s^2 \lambda^2 \int_Q \varphi^2 e^{2s\eta} |\psi|^4 + c_{k^2} \int_Q e^{4s\eta} |\psi|^6 dx dt \\
& \leq 2 \|e^{-s\eta} G_1\|_2^2 + C \left(s^2 \lambda^4 \int_Q \varphi^3 |\psi|^2 dx dt + s \lambda \int_Q \varphi |\nabla \psi|^2 dx dt \right. \\
& \quad \left. + s \lambda^2 \int_Q e^{2s\eta} \varphi^2 |\psi|^4 dx dt \right) \\
& + \alpha_\sigma \delta^4 s^3 \lambda^4 \int_{Q_{\omega_0}} \varphi^3 |\psi|^2 dx dt + \alpha_{\sigma^2} \delta^2 s \lambda^2 \int_{Q_{\omega_0}} \varphi |\nabla \psi|^2 dx dt \\
& + c_k \alpha_\sigma \delta^2 s^2 \lambda^2 \int_{Q_{\omega_0}} \varphi^2 e^{2s\eta} |\psi|^4 dx dt. \tag{91}
\end{aligned}$$

Now we note that for any s sufficiently large such that $s > s_0 = 2 \max(\frac{C}{\alpha_\sigma \delta^4}, \frac{C}{\alpha_\sigma \delta^2 c_k})$ and for any λ such that $\lambda > \lambda_0 = 2 \frac{C}{\alpha_{\sigma^2} \delta^2}$, the all integral terms on Q and can be absorbed. Then we have

$$\begin{aligned}
& \|L_1(\psi)\|^2 + \|L_2(\psi)\|^2 + s^3 \lambda^4 \int_Q \varphi^3 |\psi|^2 dx dt + s \lambda^2 \int_Q \varphi |\nabla \psi|^2 dx dt \\
& + s^2 \lambda^2 \int_Q \varphi^2 e^{2s\eta} |\psi|^4 dx dt + \int_Q e^{4s\eta} |\psi|^6 dx dt \\
& \leq C \left(\|e^{-s\eta} G_1\|_2^2 + s^3 \lambda^4 \int_{Q_{\omega_0}} \varphi^3 |\psi|^2 dx dt + s \lambda^2 \int_{Q_{\omega_0}} \varphi |\nabla \psi|^2 dx dt \right. \\
& \quad \left. + s^2 \lambda^2 \int_{Q_{\omega_0}} \varphi^2 e^{2s\eta} |\psi|^4 dx dt \right). \tag{92}
\end{aligned}$$

■

Step 5. Back to the original variable V .

Lemma 3.5.

$$\begin{aligned}
& \|L_1(\psi)\|_2^2 + \|L_2(\psi)\|_2^2 + s^3 \lambda^4 \int_Q e^{-2s\eta} \varphi^3 |V|^2 dx dt \\
& + s \lambda^2 \int_Q e^{2s\eta} \varphi |\nabla V|^2 dx dt + s^2 \lambda^2 \int_Q e^{-2s\eta} \varphi^2 |V|^4 dx dt + \int_Q e^{-2s\eta} |V|^6 dx dt \\
& \leq C \left(\|e^{-s\eta} G_1\|_2^2 + s^3 \lambda^4 \int_{Q_{\omega_0}} e^{-2s\eta} \varphi^3 |V|^2 dx dt \right. \\
& \quad \left. + s \lambda^2 \int_{Q_{\omega_0}} e^{-2s\eta} \varphi |\nabla V|^2 dx dt + s^2 \lambda^2 \int_{Q_{\omega_0}} \varphi^2 e^{-2s\eta} |V|^4 dx dt \right). \tag{93}
\end{aligned}$$

Proof. Recall that $\psi = e^{-s\eta}V$. Then,

$$\nabla\psi = e^{-s\eta}(\nabla V - s\nabla\eta V) = e^{-s\eta}(\nabla V + s\lambda\varphi\nabla\beta V), \quad (94)$$

and it implies

$$\begin{aligned} s\lambda^2 \int_{Q_{\omega_0}} \varphi |\nabla\psi|^2 dx dt \\ \leq C \left(s^3\lambda^4 \int_{Q_{\omega_0}} \varphi^3 e^{-2s\eta} |V|^2 dx dt + s\lambda^2 \int_{Q_{\omega_0}} \varphi e^{-2s\eta} |\nabla V|^2 dx dt \right). \end{aligned} \quad (95)$$

We also have

$$|\nabla\psi|^2 = e^{-2s\eta} (|\nabla V|^2 + s^2 |\nabla\eta|^2 |V|^2 - 2sV\nabla\eta \cdot \nabla V). \quad (96)$$

Besides, we have

$$\begin{aligned} \left| 2s^2\lambda^2 \int_Q e^{-2s\eta} \varphi V \nabla\eta \cdot \nabla V \right| dx dt &\leq \int_Q (2e^{-s\eta} s^{3/2} \lambda \varphi^{1/2} |\nabla\eta| |V|) (e^{-s\eta} s^{1/2} \lambda \varphi^{1/2} |\nabla V|) dx dt \\ &\leq 2s^3\lambda^2 \int_Q e^{-2s\eta} \varphi |\nabla\eta|^2 |V|^2 dx dt + \frac{1}{2}s\lambda^2 \int_Q e^{-2s\eta} \varphi |\nabla V|^2 dx dt. \end{aligned} \quad (97)$$

Then,

$$\begin{aligned} -2s^2\lambda^2 \int_Q e^{-2s\eta} \varphi V \nabla\eta \cdot \nabla V &\geq -2s^3\lambda^2 \int_Q e^{-2s\eta} \varphi |\nabla\eta|^2 |V|^2 dx dt \\ &\quad - \frac{1}{2}s\lambda^2 \int_Q e^{-2s\eta} \varphi |\nabla V|^2 dx dt. \end{aligned} \quad (98)$$

Note that $\nabla\eta = \lambda\varphi\nabla\beta$ and $|\nabla\beta| \leq C(\Omega, \omega, \sigma)$. Therefore, using the inequality (87), we can write

$$\begin{aligned} s^3\lambda^2 \int_Q e^{-2s\eta} \varphi |\nabla\eta|^2 |V|^2 dx dt &\leq Cs^3\lambda^4 \int_Q \varphi^3 |\psi|^2 dx dt \\ &\leq C \left(\|e^{-s\eta}G_1\|_2^2 + s^3\lambda^4 \int_{Q_{\omega_0}} \varphi^3 |\psi|^2 dx dt \right. \\ &\quad \left. + s\lambda^2 \int_{Q_{\omega_0}} \varphi |\nabla\psi|^2 dx dt + s^2\lambda^2 \int_{Q_{\omega_0}} \varphi^2 e^{2s\eta} |\psi|^4 dx dt \right) \end{aligned} \quad (99)$$

Using the inequality in (95), we modify (99) to find

$$\begin{aligned} s^3\lambda^2 \int_Q e^{-2s\eta} \varphi |\nabla\eta|^2 |V|^2 dx dt &\leq C \left(\|e^{-s\eta}G_1\|_2^2 + s^2\lambda^2 \int_{Q_{\omega_0}} \varphi^2 e^{-2s\eta} |V|^4 dx dt \right. \\ &\quad \left. + s^3\lambda^4 \int_{Q_{\omega_0}} \varphi^3 e^{-2s\eta} |V|^2 dx dt + s\lambda^2 \int_{Q_{\omega_0}} \varphi e^{-2s\eta} |\nabla V|^2 dx dt \right). \end{aligned} \quad (100)$$

We conclude then Lemma 3.5. ■

Step 6.(End of the proof of Theorem 3.1)

Proof. From the first equation of (19), we have

$$\begin{aligned} \int_Q e^{-2s\eta}(s\varphi)^{-1}(\partial_t V - \operatorname{div}(\boldsymbol{\sigma}\nabla V) + kV^3)^2 dx dt &= \int_Q e^{-2s\eta}(s\varphi)^{-1} |G_1|^2 dx dt \\ &\leq C \int_Q e^{-2s\eta} |G_1|^2 dx dt. \end{aligned} \quad (101)$$

To complete the proof, we have to estimate the integral

$$\begin{aligned} \int_Q e^{-2s\eta}(s\varphi)^{-1} (|\partial_t V|^2 + |\operatorname{div}(\boldsymbol{\sigma}\nabla V)|^2 + k^2 |V|^6 - 2\partial_t V \operatorname{div}(\boldsymbol{\sigma}\nabla V) \\ + 2k\partial_t V V^3 - 2kV^3 \operatorname{div}(\boldsymbol{\sigma}\nabla V)) dx dt. \end{aligned} \quad (102)$$

Now we will examine each term of (102).

$$\begin{aligned} -2 \int_Q e^{-2s\eta}(s\varphi)^{-1} \partial_t V \operatorname{div}(\boldsymbol{\sigma}\nabla V) dx dt &= 2 \int_Q \nabla(e^{-2s\eta}(s\varphi)^{-1} \partial_t V) \boldsymbol{\sigma}\nabla V dx dt \\ &= 2 \int_Q \nabla(e^{-2s\eta}(s\varphi)^{-1}) \partial_t V \boldsymbol{\sigma}\nabla V dx dt + 2 \int_Q e^{-2s\eta}(s\varphi)^{-1} \nabla(\partial_t V) \boldsymbol{\sigma}\nabla V dx dt \end{aligned} \quad (103)$$

Here we have used $\boldsymbol{\sigma}\nabla V \cdot \boldsymbol{\nu} = 0$ on Σ .

We have

$$\partial_t \nabla V \cdot \boldsymbol{\sigma}\nabla V = \partial_t |\nabla V|_{\boldsymbol{\sigma}}^2 - \frac{1}{2} \partial_t \sum_i \sigma_{ii} |\partial_i V|^2, \quad (104)$$

therefore (103) becomes

$$\begin{aligned} -2 \int_Q e^{-2s\eta}(s\varphi)^{-1} \partial_t V \operatorname{div}(\boldsymbol{\sigma}\nabla V) dx dt &= 2 \int_Q \nabla(e^{-2s\eta}(s\varphi)^{-1}) \partial_t V \boldsymbol{\sigma}\nabla V dx dt \\ &\quad - \int_Q e^{-2s\eta}(s\varphi)^{-1} \partial_t \sum_i \sigma_{ii} |\partial_i V|^2 dx dt + 2 \int_Q e^{-2s\eta}(s\varphi)^{-1} \partial_t |\nabla V|_{\boldsymbol{\sigma}}^2 dx dt \\ &= 2 \int_Q e^{-2s\eta} (2\lambda - \lambda(s\varphi)^{-1}) \partial_t V \boldsymbol{\sigma}\nabla V \nabla \beta dx dt \\ &\quad + \int_Q e^{-2s\eta} (-2\partial_t \eta \varphi^{-1} - \partial_t \varphi s^{-1} \varphi^{-2}) \left(\sum_i \sigma_{ii} |\partial_i V|^2 - 2 |\nabla V|_{\boldsymbol{\sigma}}^2 \right) dx dt. \end{aligned} \quad (105)$$

From (14)-(16), we have

$$|-2\partial_t \eta \varphi^{-1} - \partial_t \varphi s^{-1} \varphi^{-2}| \leq C s \varphi, \quad (106)$$

and

$$\left| \sum_i \sigma_{ii} |\partial_i V|^2 - 2 |\nabla V|_{\boldsymbol{\sigma}}^2 \right| \leq 3C |\nabla V|^2. \quad (107)$$

Therefore

$$\begin{aligned} \int_Q e^{-2s\eta} (-2\partial_t \eta \varphi^{-1} - \partial_t \varphi s^{-1} \varphi^{-2}) \left(\sum_i \sigma_{ii} |\partial_i V|^2 - 2 |\nabla V|_{\boldsymbol{\sigma}}^2 \right) dx dt \\ \geq -C \int_Q e^{-2s\eta} s \varphi |\nabla V|^2 dx dt \end{aligned} \quad (108)$$

and using Young's inequality we have

$$\begin{aligned} & 2 \int_Q e^{-2s\eta} (2\lambda - \lambda(s\varphi)^{-1}) \partial_t V \boldsymbol{\sigma} \nabla V \nabla \beta dx dt \\ & \geq -\frac{1}{2} \int_Q e^{-2s\eta} (s\varphi)^{-1} |\partial_t V|^2 dx dt - C \int_Q e^{-2s\eta} s \lambda^2 \varphi |\nabla V|^2 dx dt \end{aligned} \quad (109)$$

Substituting (108) and (109) in (105), we obtain

$$\begin{aligned} & -2 \int_Q e^{-2s\eta} (s\varphi)^{-1} \partial_t V \operatorname{div}(\boldsymbol{\sigma} \nabla V) dx dt \\ & \geq -\frac{1}{2} \int_Q e^{-2s\eta} (s\varphi)^{-1} |\partial_t V|^2 dx dt - C \int_Q e^{-2s\eta} s \lambda^2 \varphi |\nabla V|^2 dx dt. \end{aligned} \quad (110)$$

For the fifth term in equation (102), using integrations by part on the time variable, we establish estimates for the product of $\partial_t V$ and V^3 as follows

$$\begin{aligned} 2 \int_Q e^{-2s\eta} (s\varphi)^{-1} k \partial_t V V^3 dx dt &= -\frac{1}{2} \int_Q s^{-1} \partial_t (e^{-2s\eta} (s\varphi)^{-1} k) |V|^4 dx dt \\ &= -\frac{1}{2} \int_Q e^{-2s\eta} (-2s \partial_t \eta \varphi^{-1} k + \partial_t k \varphi^{-1} - \varphi^{-2} \partial_t \varphi) |V|^4 dx dt. \end{aligned} \quad (111)$$

As consequence from (15) and (16), we have

$$|-2\partial_t \eta \varphi^{-1} k + s^{-1} \partial_t k \varphi^{-1} - s^{-1} \varphi^{-2} \partial_t \varphi| \leq C s^2 \varphi^2, \quad (112)$$

and then

$$2 \int_Q e^{-2s\eta} (s\varphi)^{-1} k \partial_t V V^3 dx dt \geq -c s^2 \int_Q \varphi^2 e^{-2s\eta} |V|^4 dx dt. \quad (113)$$

Let us now estimate the product of the term V^3 and $\operatorname{div}(\boldsymbol{\sigma} \nabla V)$ with the integral term. Applying the Green's formula and observing that $\boldsymbol{\sigma} \nabla V \cdot \boldsymbol{\nu} = 0$ on Σ , we get

$$\begin{aligned} & -2 \int_Q e^{-2s\eta} k (s\varphi)^{-1} V^3 \operatorname{div}(\boldsymbol{\sigma} \nabla V) dx dt \\ &= 2 \int_Q \nabla (e^{-2s\eta} k (s\varphi)^{-1} V^3) \boldsymbol{\sigma} \nabla V dx dt - 2 \int_{\Sigma} e^{-2s\eta} k (s\varphi)^{-1} V^3 \boldsymbol{\sigma} \nabla V \cdot \boldsymbol{\nu} d\Sigma \\ &= 2 \int_Q e^{-2s\eta} s^{-1} \nabla k \varphi^{-1} V^3 \boldsymbol{\sigma} \nabla V dx dt + 4\lambda \int_Q e^{-2s\eta} k V^3 \boldsymbol{\sigma} \nabla V \cdot \nabla \beta dx dt \\ &\quad - 2s^{-1} \lambda \int_Q e^{-2s\eta} k \varphi^{-1} V^3 \boldsymbol{\sigma} \nabla V \cdot \nabla \beta dx dt + 6s^{-1} \int_Q e^{-2s\eta} k \varphi^{-1} V^2 |\nabla V|_{\boldsymbol{\sigma}}^2 dx dt \\ &= p_1 + p_2 + p_3 + p_4. \end{aligned} \quad (114)$$

where p_i , $i = 1, \dots, 4$ correspond to the i^{th} term in the right hand side of (114) Taking account

$$c \leq \varphi \leq \frac{T^4}{16} \varphi^3, \quad \varphi^{-2} \leq \frac{T^4}{16c} \varphi,$$

we can estimate p_1 as follows

$$\begin{aligned}
|p_1| &\leq C \int_Q s^{-1} e^{-2s\eta} \varphi^{-1} |V|^3 |\nabla V| dx dt \\
&= C s^{-1} \int_Q (e^{-s\eta} |V|^3 s^{1/2}) (e^{-s\eta} s^{1/2} \varphi^{-1} |\nabla V|) dx dt \\
&\leq C \left(\int_Q e^{-2s\eta} |V|^6 dx dt + \int_Q e^{-2s\eta} \varphi^{-2} |\nabla V|^2 dx dt \right) \\
&\leq C \left(\int_Q e^{-2s\eta} |V|^6 dx dt + s \int_Q e^{-2s\eta} \varphi |\nabla V|^2 dx dt \right). \tag{115}
\end{aligned}$$

Thus we obtain

$$|p_1| \leq e^{-2s\eta} |V|^6 dx dt + s \int_Q e^{-2s\eta} \varphi |\nabla V|^2 dx dt. \tag{116}$$

We repeat the calculations as explained in the above for p_2 and p_3 , we conclude that

$$|p_1| + |p_2| + |p_3| \leq C \left(\int_Q e^{-2s\eta} |V|^6 dx dt + s \int_Q e^{-2s\eta} \varphi |\nabla V|^2 dx dt \right). \tag{117}$$

Hence

$$\begin{aligned}
&-2 \int_Q e^{-2s\eta} k(s\varphi)^{-1} V^3 \operatorname{div}(\boldsymbol{\sigma} \nabla V) dx dt \\
&\geq -C \left(\int_Q e^{-2s\eta} |V|^6 dx dt + s \int_Q e^{-2s\eta} \varphi |\nabla V|^2 dx dt \right). \tag{118}
\end{aligned}$$

Here we have used $p_4 \geq 0$.

The last term in equation (102) is treated as follows

$$\left| \int_Q e^{-2s\eta} (s\varphi)^{-1} k^2 |V|^6 dx dt \right| \leq C \int_Q e^{-2s\eta} |V|^6 dx dt. \tag{119}$$

Finally making use of the estimates (110), (113), (118) and (119) we obtain

$$\begin{aligned}
&\int_Q e^{-2s\eta} (s\varphi)^{-1} |\partial_t V|^2 dx dt + \int_Q e^{-2s\eta} (s\varphi)^{-1} |\operatorname{div}(\boldsymbol{\sigma} \nabla V)|^2 dx dt \\
&\leq C \left(\int_Q e^{-2s\eta} |G_1|^2 dx dt + s\lambda^2 \int_Q e^{-2s\eta} \varphi |\nabla V|^2 dx dt \right. \\
&\quad \left. + s^2 \lambda^2 \int_Q \varphi^2 e^{-2s\eta} |V|^4 dx dt + \int_Q e^{-2s\eta} |V|^6 dx dt \right). \tag{120}
\end{aligned}$$

The proof of Theorem 3.1 is done by combining equation (120) and Lemma 3.5. \blacksquare

4. Stability estimate of parameter τ_{in}

In this section, we establish a the main stability result and we deduce a uniqueness result for the identification problem of parameter τ_{in} . We estimate the difference between the coefficients a and \tilde{a} with an upper bound given by some Sobolev norms of the difference

between the solutions v_m , and \tilde{v}_m of (17) and (18). Recall that $V = v_m - \tilde{v}_m$, $W = w - \tilde{w}$ and $q = a - \tilde{a}$, where $a = 1/\tau_{in}$, $\tilde{a} = 1/\tilde{\tau}_{in}$ and

$$\begin{cases} \partial_t V - \operatorname{div}(\boldsymbol{\sigma} \nabla V) + kV^3 = G_1 & \text{in } \Omega \times (0, T) \\ \partial_t W = G_2 & \text{in } \Omega \times (0, T) \\ \boldsymbol{\sigma} \nabla V \cdot \boldsymbol{\nu} = 0 & \text{on } \Sigma \\ V(x, 0) = V_0(x), \quad W(x, 0) = W_0(x) & \text{in } \Omega. \end{cases} \quad (121)$$

The Carleman estimate (21) proved in the previous section will be the key ingredient in the proof of such a stability estimate.

For a fixed time $t_0 \in (0, T/2]$, let define $T_0 = 2t_0 \leq T$ and $\eta = (t(T_0 - t))^{-1}(e^{2\lambda\|\beta\|_\infty} - e^{\lambda\beta})$. The wight function η reaches its minimum at $t = T_0/2$. We have thus obtained the following stability result.

Theorem 4.1. *Let ω be a subdomain of an open set Ω of \mathbb{R}^3 . Furthermore, we assume that $\tilde{v}_0 \in H^2(\Omega)$, $\tilde{w}_0 \in L^\infty(\Omega)$ and there exists $t_0 \in (0, T)$ such that*

$$H(\tilde{v}_m(x, t_0), \tilde{w}(x, t_0)) \geq r_0 > 0 \quad \forall x \in \Omega. \quad (122)$$

Then there exists a constant C

$$C = C(\Omega, \omega, T, r_0) > 0$$

such that

$$\begin{aligned} |q|^2 \leq C & (\|V\|_{H^1(0, T; H^1(\omega_0))}^2 + \|V\|_{L^4(0, T; L^4(\omega_0))}^4 + \|V(\cdot, t_0)\|_{H^2(\Omega)}^2 \\ & + \|V(\cdot, t_0)\|_{L^4(\Omega)}^4 + \|V(\cdot, t_0)\|_{L^6(\Omega)}^6 + \|W(\cdot, t_0)\|_{L^2(\Omega)}^2). \end{aligned} \quad (123)$$

To prove Theorem 4.1, we need the following fundamental lemma.

Lemma 4.1. *Recall that $W = w - \tilde{w}$ where w and \tilde{w} verify (17) and (18) respectively. For $t_0 = T_0/2$ and $s > 0$, we have*

$$\int_{Q'} e^{-2s\eta(x, t)} |W(x, t)|^2 dx dt \leq C \left(\int_{Q'} e^{-2s\eta(x, t)} |W(x, t_0)|^2 dx dt + \frac{1}{s} \int_{Q'} e^{-2s\eta(x, t)} |\partial_t W(x, t)|^2 dx dt \right).$$

where $Q' := \Omega \times (0, T_0)$.

Proof. The proof is similar to [4] and [28]. From (19) we have $G_2 = \partial_t W$. By the Cauchy-Schwarz inequality, we obtain

$$\int_{Q'} \left| \int_{t_0}^t G_2(x, \tau) d\tau \right|^2 e^{-2s\eta(x, t)} dx dt \leq \int_{Q'} \left(\int_{t_0}^t |G_2(x, \tau)|^2 d\tau \right) (t - t_0) e^{-2s\eta(x, t)} dx dt \quad (124)$$

Since

$$\partial_t \eta(x, t) = \frac{2(t - t_0)}{t^2(T_0 - t)^2} (e^{2\lambda\|\beta\|_\infty} - e^{\lambda\beta}) \quad (125)$$

then

$$\begin{aligned}
& \int_{Q'} \left| \int_{t_0}^t G_2(x, \tau) d\tau \right|^2 e^{-2s\eta(x,t)} dx dt \\
& \leq C \int_{Q'} \left(\int_{t_0}^t |G_2(x, \tau)|^2 d\tau \right) \partial_t \eta(x, t) e^{-2s\eta(x,t)} dx dt \\
& \leq -\frac{C}{2s} \int_{Q'} \left(\int_{t_0}^t |G_2(x, \tau)|^2 d\tau \right) \partial_t (e^{-2s\eta(x,t)}) dx dt.
\end{aligned} \tag{126}$$

By noting that $e^{-2s\eta(x, T_0)} = e^{-2s\eta(x, 0)} = 0$, the integration by parts with respect to the time variable implies that the right hand side is equal to

$$\frac{C}{2s} \int_{Q'} |G_2(x, t)|^2 e^{-2s\eta(x,t)} dx dt. \tag{127}$$

We write

$$|W(x, t)|^2 \leq C(|W(x, t) - W(x, 0)|^2 + |W(x, 0)|^2), \tag{128}$$

thus the proof of Lemma 4.1 is completed. \blacksquare

Let us now prove the Theorem 4.1.

Proof. Let $p = \partial_t V$ and we consider the time derivative of the first equation of the system (19)

$$\partial_t p - \operatorname{div}(\boldsymbol{\sigma} \nabla p) = G \tag{129}$$

where $G = -\partial_t kV^3 - 3kV^2 p + q\partial_t H + \partial_t R$, the functions H, R are defined in (20).

First, we evaluate the first equation (19) at a fixed time t_0 such that $2t_0 = T_0 < T$

$$p(x, t_0) - \operatorname{div}(\boldsymbol{\sigma} \nabla V(x, t_0)) + kV^3(x, t_0) - qH(x, t_0) - R(x, t_0) = 0. \tag{130}$$

Then, we integrate on Ω the square of (130) with the weight function $e^{-2s\eta(x, t_0)}$, we obtain

$$\begin{aligned}
& \int_{\Omega} e^{-2s\eta(x, t_0)} |q|^2 |H(x, t_0)|^2 dx \leq \\
& C \left(\int_{\Omega} e^{-2s\eta(x, t_0)} |p(x, t_0)|^2 dx + \int_{\Omega} e^{-2s\eta(x, t_0)} |\operatorname{div}(\boldsymbol{\sigma} \nabla V(x, t_0))|^2 dx \right. \\
& \left. + \int_{\Omega} e^{-2s\eta(x, t_0)} |V(x, t_0)|^6 dx + \int_{\Omega} e^{-2s\eta(x, t_0)} |R(x, t_0)|^2 dx \right)
\end{aligned} \tag{131}$$

since $k \in L^\infty$.

First, we notice that the fourth term of the right-hand side verify

$$\begin{aligned}
& \int_{\Omega} e^{-2s\eta(x, t_0)} |R(x, t_0)|^2 dx \leq C \left(\|e^{-s\eta(x, t_0)} V(x, t_0)\|_{L^2(\Omega)}^2 \right. \\
& \left. + \|e^{-s/2\eta(x, t_0)} V(x, t_0)\|_{L^4(\Omega)}^4 + \|e^{-s\eta(x, t_0)} W(x, t_0)\|_{L^2(\Omega)}^2 \right).
\end{aligned} \tag{132}$$

Let us bound the first term in the right-hand side of (131)

$$\begin{aligned} K_1 &= \int_{\Omega} e^{-2s\eta(x,t_0)} |p(x, t_0)|^2 dx = \int_0^{t_0} \int_{\Omega} \frac{d}{dt} (e^{-2s\eta(x,t)} |p|^2) dx dt \\ &= \int_0^{t_0} \int_{\Omega} (-2s\partial_t \eta(x, t) |p|^2 + 2p\partial_t p) e^{-2s\eta(x,t)} dx dt. \end{aligned} \quad (133)$$

Besides, from (16) and using Young inequality, we have

$$\begin{aligned} |-2s\partial_t \eta(x, t) |p|^2 + 2p\partial_t p| &\leq 2s |\partial_t \eta(x, t)| |p|^2 + 2 |p| |\partial_t p| \\ &\leq C (s\varphi^2 |p|^2 + 2(s\varphi^{1/2} |p|)(s^{-1}\varphi^{-1/2} |\partial_t p|)) \\ &\leq C (s\varphi^2 |p|^2 + s^2\varphi |p|^2 + s^{-2}\varphi^{-1} |\partial_t p|^2) \\ &\leq C (s^2\varphi^2 |p|^2 + s^{-2}\varphi^{-1} |\partial_t p|^2). \end{aligned} \quad (134)$$

Then, we apply the Carleman inequality given in Theorem 3.1 satisfied by p without non linear cubic term, we obtain that for s and λ

$$K_1 \leq \frac{C}{s} \|e^{-s\eta} G\|_{L^2(Q')}^2 + C(s^2\lambda^4 \int_{Q'_{\omega_0}} e^{-2s\eta} \varphi^3 |p|^2 dx dt + \lambda^2 \int_{Q'_{\omega_0}} e^{-2s\eta} \varphi |\nabla p|^2 dx dt), \quad (135)$$

where $Q' = \Omega_0 \times (0, T_0)$ and $Q'_{\omega_0} = \omega_0 \times (0, T_0)$. According to Proposition 2.1, $\tilde{v}_m \in W^{1,\infty}(0, T_0; L^\infty(\Omega))$ and $\tilde{w} \in W^{1,\infty}(0, T_0; L^\infty(\Omega))$. Then, using the definition of $G = -\partial_t k V^3 - 3kV^2 p + q\partial_t H + \partial_t R$ we obtain

$$\begin{aligned} \|e^{-s\eta} G\|_{L^2(Q')}^2 &\leq C \left(\int_{Q'} e^{-2s\eta} (|V|^6 + |V|^4 + |V|^2 + |p|^2 + |W|^2 + |\partial_t W|^2) dx dt \right. \\ &\quad \left. + \int_{Q'} e^{-2s\eta} |q|^2 dx dt \right) \end{aligned} \quad (136)$$

since $\tilde{v}_m \in L^\infty(Q')$. Then, we apply again the Carleman inequality given in Theorem 3.1 satisfied by p without non linear cubic term,

$$\begin{aligned} \int_{Q'} e^{-2s\eta} |p|^2 dx dt &\leq C \left(s^{-3} \|e^{-s\eta} G\|_{L^2(Q')}^2 + \lambda^4 \int_{Q'_{\omega_0}} e^{-2s\eta} \varphi^3 |p|^2 dx dt \right. \\ &\quad \left. + s^{-2}\lambda^2 \int_{Q'_{\omega_0}} e^{-2s\eta} \varphi |\nabla p|^2 dx dt \right). \end{aligned} \quad (137)$$

Let us now estimate the following term $B = \int_{Q'} e^{-2s\eta} |\partial_t W|^2 dx dt$. From system (19), we have

$$\partial_t W = G_2 = g(v_m, w) - g(\tilde{v}_m, \tilde{w}). \quad (138)$$

We write the Taylor-Lagrange development of the function g at point (\tilde{v}_m, \tilde{w}) , there exists $\theta \in [0, 1]$ such that

$$\begin{aligned} G_2 &= \sum_{i=1}^2 \left(\frac{V^i}{i!} \frac{\partial^i g(\tilde{v}_m, \tilde{w})}{\partial v^i} + \frac{V^{i-1} W}{(i-1)!} \frac{\partial^i g(\tilde{v}_m, \tilde{w})}{\partial v^{i-1} \partial w} \right) \\ &\quad + \frac{V^3}{6} \frac{\partial^3 g(\tilde{v}_m + \theta h_1, \tilde{w} + \theta h_2)}{\partial v^3} + \frac{V^2 W}{2} \frac{\partial^3 g(\tilde{v}_m + \theta h_1, \tilde{w} + \theta h_2)}{\partial v^2 \partial w}. \end{aligned} \quad (139)$$

Now, since the function g and its all partial derivatives are expressed as a function of the hyperbolic tangent function, which is uniformly bounded, then we deduce that

$$|\partial_t W|^2 = |G_2|^2 \leq |V|^6 + |V|^4 + |V|^2 + |W|^2. \quad (140)$$

Substituting (137), (140) in (136), we obtain

$$\begin{aligned} (1 - Cs^{-3}) \|e^{-s\eta} G\|_{L^2(Q')}^2 &\leq C \left(\int_{Q'} e^{-2s\eta} (|V|^6 + |V|^4 + |V|^2 + |W|^2) dx dt \right. \\ &\quad + s^{-2} \lambda^2 \int_{Q'_{\omega_0}} e^{-2s\eta} \varphi |\nabla p|^2 dx dt + \lambda^4 \int_{Q'_{\omega_0}} e^{-2s\eta} \varphi^3 |p|^2 dx dt \\ &\quad \left. + \int_{Q'} e^{-2s\eta} |q|^2 dx dt \right). \end{aligned} \quad (141)$$

Using the Carleman inequality given by Theorem 3.1 for the estimation of $|V|^6 + |V|^4 + |V|^2$ and using Lemma 4.1 for $|W|^2$, we get, for s and λ large enough

$$\begin{aligned} (1 - s^{-3}) \|e^{-s\eta} G\|_{L^2(Q')}^2 &\leq C \left(\int_{Q'} e^{-2s\eta} |G_1|^2 dx dt + \int_{Q'} e^{-2s\eta(x,t)} |G_2|^2 dx dt + \|W(\cdot, t_0)\|_{L^2(\Omega)}^2 \right. \\ &\quad + s^3 \lambda^4 \int_{Q'_{\omega_0}} e^{-2s\eta} \varphi^3 |V|^2 dx dt + s \lambda^2 \int_{Q'_{\omega_0}} e^{-2s\eta} \varphi |\nabla V|^2 dx dt \\ &\quad \left. + s^2 \lambda^2 \int_{Q'_{\omega_0}} \varphi^2 e^{-2s\eta} |V|^4 dx dt + \int_{Q'} e^{-2s\eta} |q|^2 dx dt \right). \end{aligned} \quad (142)$$

Setting $D := \int_{Q'} e^{-2s\eta} |G_1|^2 + \int_{Q'} e^{-2s\eta} |G_2|^2 dx dt$, and observing that $G_1 = qH(\tilde{v}_m, \tilde{w}) + R(V, W)$, we obtain

$$D \leq C \int_{Q'} e^{-2s\eta} |q|^2 dx dt + \int_{Q'} e^{-2s\eta} (|V|^4 + |V|^2 + |W|^2) dx dt \quad (143)$$

since $v_m, \tilde{v}_m, \tilde{w} \in L^\infty(Q)$ and from (140). Using again the Carleman inequality given by Theorem 3.1 to $|V|^4 + |V|^2$ and using Lemma 4.1 to $|W|^2$, we get, for s and λ large enough

$$\begin{aligned} (1 - s^{-1})D &\leq C \int_{Q'} e^{-2s\eta} |q|^2 dx dt + C \left(s \lambda^4 \int_{Q'_{\omega_0}} e^{-2s\eta} \varphi^3 |V|^2 dx dt + \|W(\cdot, t_0)\|_{L^2(\Omega)}^2 \right. \\ &\quad \left. + s^{-1} \lambda^2 \int_{Q'_{\omega_0}} e^{-2s\eta} \varphi |\nabla V|^2 dx dt + \lambda^2 \int_{Q'_{\omega_0}} \varphi^2 e^{-2s\eta} |V|^4 dx dt \right). \end{aligned} \quad (144)$$

Substituting (144) in (142), we obtain

$$\begin{aligned} (1 - s^{-1})(1 - s^{-3}) \|e^{-s\eta} G\|_{L^2(Q')}^2 &\leq C \left(\int_{Q'} e^{-2s\eta} |q|^2 dx dt + s^3 \lambda^4 \int_{Q'_{\omega_0}} e^{-2s\eta} \varphi^3 |V|^2 dx dt \right. \\ &\quad \left. + s \lambda^2 \int_{Q'_{\omega_0}} e^{-2s\eta} \varphi |\nabla V|^2 dx dt + s^2 \lambda^2 \int_{Q'_{\omega_0}} \varphi^2 e^{-2s\eta} |V|^4 dx dt + \|W(\cdot, t_0)\|_{L^2(\Omega)}^2 \right) \end{aligned} \quad (145)$$

Now back to the estimate (135), we can write

$$\begin{aligned}
s(1-s^{-1})(1-s^{-3})K_1 \leq & C \left(\int_{Q'} e^{-2s\eta} |q|^2 dx dt + s^3 \lambda^4 \int_{Q'_{\omega_0}} e^{-2s\eta} \varphi^3 |V|^2 dx dt \right. \\
& + s \lambda^2 \int_{Q'_{\omega_0}} e^{-2s\eta} \varphi |\nabla V|^2 dx dt + s^2 \lambda^2 \int_{Q'_{\omega_0}} \varphi^2 e^{-2s\eta} |V|^4 dx dt \\
& \left. + s^3 \lambda^4 \int_{Q'_{\omega_0}} e^{-2s\eta} \varphi^3 |p|^2 dx dt + s \lambda^2 \int_{Q'_{\omega_0}} e^{-2s\eta} \varphi |\nabla p|^2 dx dt + \|W(\cdot, t_0)\|_{L^2(\Omega)}^2 \right). \quad (146)
\end{aligned}$$

At last, combining hypothesis (122), inequalities (132) and (146), estimate (131) becomes:

$$\begin{aligned}
\int_{\Omega} e^{-2s\eta(x,t_0)} r_0^2 |q|^2 \leq & C \left(s^3 (s-1)^{-1} (s^3-1)^{-1} \int_{Q'} e^{-2s\eta} |q|^2 dx dt + \|V\|_{H^1(0,T;H^1(\omega_0))}^2 \right. \\
& \left. + \|V\|_{L^4(0,T;L^4(\omega_0))}^4 + \|V(\cdot, t_0)\|_{H^2(\Omega)}^2 + \|V(\cdot, t_0)\|_{L^4(\Omega)}^4 + \|W(\cdot, t_0)\|_{L^2(\Omega)}^2 \right). \quad (147)
\end{aligned}$$

Hence, for s and λ large enough (such that $r_0 s (s-1) (s^3-1) - s^3 > 0$)

$$\begin{aligned}
\int_{\Omega} e^{-2s\eta(x,t_0)} |q|^2 \leq & C \left(\|V\|_{H^1(0,T;H^1(\omega_0))}^2 + \|V\|_{L^4(0,T;L^4(\omega_0))}^4 + \|V(\cdot, t_0)\|_{H^2(\Omega)}^2 \right. \\
& \left. + \|V(\cdot, t_0)\|_{L^4(\Omega)}^4 + \|W(\cdot, t_0)\|_{L^2(\Omega)}^2 \right). \quad (148)
\end{aligned}$$

This ends the proof of Theorem 4.1. ■

With Theorem 4.1 we have the following uniqueness result.

Corollary 4.1. *Under the same assumptions as in Theorem 4.1 and if*

$$\begin{aligned}
v_m &= \tilde{v}_m \quad \text{in } \omega_0 \times (0, T), \\
v_m(x, t_0) &= \tilde{v}_m(x, t_0) \quad \text{in } \Omega, \\
w(x, t_0) &= \tilde{w}(x, t_0) \quad \text{in } \Omega, \quad (149)
\end{aligned}$$

then $\tau_{in} = \tilde{\tau}_{in}$.

5. Discussion and conclusions

The Mitchell-Schaeffer ionic model coupled to the monodomain equation has been used for model personalization based on optical imaging and magnetic resonance data in [36]. It has been used also for personalizing the model to ventricular tachycardia using X-ray/magnetic resonance imaging and non-contact mapping procedure on a patient with heart failure in [35]. The parameter τ_{in} in both studies plays an important role in estimating the velocity of the electrical wave. The monodomain equation combined with the Mitchell-Schaeffer ionic model has also been used for detecting ischemia region using an adjoint approach combined with a level set method in two dimensional and three dimensional frameworks [1, 14]. In these works a low value of the parameter τ_{in} has been identified to be one of the major characteristics of the ischemic region. The estimation of the parameter τ_{in} is then crucial in these applications.

In this paper, we established a stability estimates for the parameter identification problem in cardiac electrophysiology modeling. Our concern is about the estimation of the parameter τ_{in} to which the solution is the most sensitive. The novelty of our work comes from the fact that, we established a new Carleman inequality for a reaction diffusion equation coupled to an ordinary differential equation. The Carleman inequality that we established for the ODE was fundamental in order to prove the global Carleman estimate for non linear parabolic equation coupled with an ordinary differential equation and solving the parameter stability problem.

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