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Konstantin Avrachenkov, Giovanni Neglia, Vikas Singh

Manuscript submitted to
AIMS' Journals
Volume **X**, Number **0X**, XX **200X**

doi:10.3934/xx.xx.xx.xx

pp. **X–XX**

NETWORK FORMATION GAMES WITH TEAMS*

Konstantin Avrachenkov

INRIA Sophia-Antipolis
2004 route des Lucioles, 06902 Sophia-Antipolis, France

Giovanni Neglia

INRIA Sophia-Antipolis
2004 route des Lucioles, 06902 Sophia-Antipolis, France

Vikas Vikram Singh

INRIA Sophia-Antipolis
2004 route des Lucioles, 06902 Sophia-Antipolis, France

(Communicated by Lorens Imhof)

Abstract. Network formation games have been proposed as a tool to explain the topological characteristics of existing networks. They assume that each node is an autonomous decision-maker, ignoring that in many cases different nodes are under the control of the same authority (e.g. an Autonomous System) and then they operate as a team. In this paper we introduce the concept of network formation games for teams of nodes and show how very different network structures can arise also for some simple games studied in the literature. Beside extending the usual definition of pairwise stable networks to this new setting, we define a more general concept of stability toward deviations from a specific set $\mathcal{C}$ of teams' coalitions ($\mathcal{C}$-stability). We study then a trembling-hand dynamics, where at each time a coalition of teams can create or sever links in order to reduce its cost, but it can also take wrong decisions with some small probability. We show that this stochastic dynamics selects $\mathcal{C}$-stable networks or networks from closed cycles in the long run as the error probability vanishes.

1. **Introduction.**

Network formation games are nowadays a consolidated branch of game theory [7, 8, 15, 16]. They study which networks' structures arise when the nodes are selfish rational players, who can sever or create some links in order to increase the utility they perceive from the network. In particular, it is usually assumed that each node can unilaterally sever a link to one of its neighbors, while the creation of a new link requires the approval of both the participating nodes. This idea has lead to the concept of pairwise-stable networks [19], i.e., networks for which every existent link is beneficial to both the connected nodes and every inexistent link is not beneficial to at least one of the two nodes it would connect. Different dynamics for links' creation/destruction have been studied. Specific network formation games have been proposed to explain the topological characteristics of existing networks,

2010 *Mathematics Subject Classification.* Primary: 91A25; 91A43; 91A06; ; Secondary 93E15; 60J10.

Key words and phrases. Network Formation Games, Teams, $\mathcal{C}$-stability, Strong stability, Stochastic stability.

*The authors are listed in alphabetical order including the Autonomous Systems' (ASs') network taking for example into account both costs for routing traffic and for a lack of end-to-end connectivity [1, 5, 20]. They have also been used to investigate the distributed formation of overlay topologies on top of the Internet [9], or routing topologies among relay stations in future cellular networks [28], or among nodes in multi-hop wireless networks to prevent eavesdropping [29].

Network formation games consider that each node is an autonomous decision-maker, ignoring that in many cases different nodes are under the control of the same authority (e.g. the routers in an AS) and then they constitute a *team*. We observe that introducing teams in a simple variant of a network formation game proposed in [6] leads to different stable network structures. This motivates us to study network formation games with teams and define new stability notions for this new class of games. Although the concept of teams is not new to game theory, e.g., there is a recent work by Boncinelli and Pin [4] where they analyze team formation process in game theoretic models. To the best of our knowledge this paper is the first to introduce teams in network formation games and aims to open a new and, in our opinion, interesting research direction.

Our first contribution is then to define network formation games for teams of nodes and define team-pairwise stability analogously to pairwise stability in standard network formation games. The second contribution of our paper is to provide a more general concept of stability toward the deviations from a specific set $\mathcal{C}$ of teams' coalitions (we talk then about $\mathcal{C}$-stability). This idea can capture the fact that some groups of teams are more likely to cooperate than others. For example, ASs located in far away regions may have no incentive to connect directly. In addition to the above notions of static stability, we also consider the dynamic formation of networks also in the presence of random errors (trembling-hand dynamics). We then discuss the dynamic stability notion called stochastic stability, i.e., which networks are selected by the dynamics as the error probability vanishes. So, as a final contribution, we prove that under this trembling-hand dynamics only $\mathcal{C}$-stable networks or networks from closed cycles are selected with non-zero probability in the long term when trembling vanishes. The case where there is no restriction on the coalition formation among teams, the $\mathcal{C}$-stability is called strong stability. We show that our trembling-hand dynamics selects all strongly stable networks and networks from closed cycles as the error probability vanishes.

Now, we describe some existing works related to our stochastic stability result. Jackson and Watts [18] are the first ones to consider trembling-hand dynamics in standard network formation games for the case of pairwise interaction between nodes. At each time during pairwise interaction considered in [18], a pair of nodes are selected and only a link between them is altered. The mutations are present in the dynamics due to a wrong decision taken by the selected pair of nodes at each time. They showed that a stochastically stable network is a pairwise stable network or a network from a closed cycle. Jackson and Watts [18] also considered the case where a pair of nodes can add a new link between them and at the same time each node of selected pair can sever existing links with other nodes of network. They applied these models into matching problems such as marriage problem and college admission problem [14, 27]. For some recent stochastic stability results on matching problems see [22, 26]. We generalize the results from [18] in both static and dynamic settings. We first introduce teams in network formation games and define team-pairwise stability analogous to pairwise stability. We allow teams to form all possible coalitions and define the notion of $\mathcal{C}$-stability. In dynamic formation of networks, we assume that at each time a teams' coalition is formed. The coalition makes a move from the current network to a new network, by adding or severing some links in the current network, if at new network all the teams in the coalition are at least as well off and at least one team is strictly better off. At each time the coalition makes a mistake with small probability and as a result the obtained network need not be beneficial for all the teams in the coalition. We show that a stochastically stable network is a $\mathcal{C}$-stable network or a network belonging to a closed cycle of networks. We also show that if a network of a closed cycle is stochastically stable, all the networks of the closed cycle are stochastically stable. The stochastic dynamics based on the coalitional deviations in general finite games, using different dynamics rules, have been considered before [23, 30] . Sawa [30] studied the stochastic stability in general finite games where the mutations are present through a logit choice rule. Newton [23] considers the situation where profitable coalitional deviations are given greater importance than unprofitable single player deviations. In general, the stochastic stability results depend on the way actions being chosen during the infinite play. Some other famous works on stochastic stability in different settings include [4, 11, 12, 13, 21, 24, 25, 31].

The paper is organized as follows. We start providing the basic definitions for network formation games with teams in Section 2. Then, in Section 3 we provide a simple variant of a network formation game proposed in [6] showing that different network structures arise in the presence of teams. We move then to introduce the general concept of $\mathcal{C}$-stability in Section 4. Section 5 describes the trembling-hand dynamics. We show some numerical results for specific games in Section 6. Finally, we conclude our paper in Section 7.

2. **Network formation games with teams: Basic definitions.**

In this section we define network formation games for teams and we start extending some usual concepts in network formation games' literature to our setting. In particular, we define the notion of team-pairwise stability to characterize meaningful equilibria. Let $N = \{1, 2, \cdots, n\}$ be a finite set of nodes. Let $\mathcal{P} = \{T_1, T_2, \cdots, T_m\}$, $m \leq n$ be a partition of N. The undirected edges can be formed both between the nodes belonging to a same set T_i (called internal links) as well as belonging to the different sets T_i and T_j (called external links). The edge between the nodes k and l is denoted by kl. We say that there is a link between T_i and T_j if there is at least one link kl with $k \in T_i$ and $l \in T_j$. The collection of edges defines a network g.

In standard network formation games, each node k has its own cost $c_k(g)$. In network formation games with teams we consider that the nodes belonging to each set T_i share the same cost $c(T_i, g)$, which depends in general on the structure of the whole network g. As a consequence, they will form or sever links only if this is beneficial for the whole set T_i, i.e., if the cost of T_i is reduced. For this reason we refer to each T_i, $i = 1, 2 \ldots, m$, as a *team*. In particular, the nodes in a team will agree to create an internal link kh with $k, h \in T_i$, if $c(T_i, g \cup kh) < c(T_i, g)$, and will agree to sever an internal link kh if $c(T_i, g \setminus kh) < c(T_i, g)$. This leads to following definition of internal stability:

Definition 2.1 (Internal stability)**.** A network g is internally stable in team T_i if no further links can be created or severed within the nodes from T_i, i.e., if g' is a network which is obtained from g via the addition of new links within the nodes of

T_i or destruction of existing links within the nodes of T_i, then $c(T_i, g') > c(T_i, g)$. A network is internally stable if it is internally stable in all the teams.

The destruction of an external link is unilateral: any of the two nodes can sever the link if this is beneficial for its team. On the contrary, the creation of an external link requires the agreement of both the teams involved. The link will then be created only if it does not increase the cost of any of the two teams and it decreases the cost of at least one of them. These link formation rules lead to the following notion of equilibrium for network formation games with teams:

Definition 2.2 (Team-pairwise stability)**.** A network g is said to be team-pairwise stable if

(i) g is internally stable, and
(ii) for all the pairs (T_i, T_j) if $kl \in g$ such that $k \in T_i$, $l \in T_j$ then $c(T_i, g) \leq c(T_i, g \setminus kl)$ and $c(T_j, g) \leq c(T_j, g \setminus kl)$, and
(iii) for all the pairs (T_i, T_j) if $kl \notin g$, such that $k \in T_i$, $l \in T_j$ and if $c(T_i, g \cup kl) < c(T_i, g)$ then $c(T_j, g \cup kl) > c(T_j, g)$.

3. **Network formation games with teams: A motivating example.** Fabrikant et al. [10] introduced a network formation game where each node can add as well as sever the links unilaterally. Corbo and Parkes [6] considered the case when link destruction is unilateral but link creation requires the agreement of both the nodes involved. The two games are called in [6], respectively the *Unilateral Connection Game* and the *Bilateral Connection Game* (BCG). We focus on BCG because we are interested in pairwise interaction among the teams. We introduce a simple variant of BCG, that we call the *Heterogeneous Bilateral Connection Game* (HBCG) and study it in the presence or absence of teams. We observe that, even in a very simple case, a stable network structure in the presence of teams is not stable when each node is an independent rational player. This shows the usefulness of the new stability notions defined in the Section 2 and it motivates us to study network formation games with teams. Note that while in this section we consider a specific cost function from [6], our definitions and results in the the rest of the paper hold for general cost functions.

3.1. **The bilateral connection game** [6]**.** Let $N = \{1, 2, \cdots, n\}$ be a finite set of players (nodes). The cost of a link is 2α which is equally shared between both the corresponding nodes. Let $k_g(i)$ denote the degree of node i in network g which is the total number of links connected to i. A node also incurs the cost due to the distance from all other nodes. The distance between the nodes i and j in the network g is defined as the minimal number of hops along a path connecting them and it is denoted by $d_g(i, j)$. If there is no path connecting node i and node j, the distance $d_g(i, j) = \infty$. The cost incurred by node i in network g is

$$c_i(g) = \alpha \cdot k_g(i) + \sum_{j \in N; j \neq i} d_g(i, j). \tag{1}$$

3.2. **The heterogeneous bilateral connection game.** We introduce a variant of BCG that we call the Heterogeneous Bilateral Connection Game (HBCG). In HBCG nodes are divided into the disjoint sets $\{T_i\}_{i=1}^m$. The cost of a link between two nodes from the same set is $\frac{2\alpha}{\beta}$, $\beta > 0$, and the cost of a link between two nodes from the different sets T_i and T_j is $\frac{4\alpha}{\beta}$. If all the nodes are independent rational players, the cost of a link formation is equally shared between the corresponding nodes, i.e., a node incurs cost $\frac{\alpha}{\beta}$ by forming a link with a node from a same set and it incurs cost $\frac{2\alpha}{\beta}$ by forming a link with a node from a different set. In this case, the total cost incurred by a node i belonging to set T in a given network g is given by

$$c_i(g) = \frac{\alpha}{\beta} \cdot k_g(i|T) + \frac{2\alpha}{\beta} \cdot k_g(i|T^c) + \sum_{j \in N; j \neq i} d_g(i,j),$$

where $k_g(i|T)$ denotes the number of neighbors of i inside set T and $k_g(i|T^c)$ is the number of i's neighbors outside the set T. We observe that if each set T_i includes only one node and $\beta = 2$, HBCG reduces to BCG. In what follows we consider that each set T_i has cardinality at least 2.

If the nodes belonging to each set T_i, $i = 1, 2, \cdots, m$, form a team, the cost experienced by each member is shared by the whole team. In this case, each team incurs cost $\frac{2\alpha}{\beta}$ for forming an internal as well as an external link.

For a given network g, the cost of a team T_i is then defined as

$$c(T_i, g) = \sum_{k \in T_i} c_k(g) = \frac{2\alpha}{\beta}\left(E(T_i) + \sum_{\substack{T_j \in \mathcal{P} \\ j \neq i}} E(T_iT_j)\right) + \sum_{k \in T_i} \sum_{\substack{j \in N \\ j \neq k}} d_g(k,j), \qquad (2)$$

where $E(T_i)$ is the total number of internal links of T_i and $E(T_iT_j)$ is the total number of external links between T_i and T_j. Finally the cost of a network g is denoted by $c(g)$ and it is defined as the sum of the costs of all teams.

We now study the effect of teams on the stability of network topologies. We show that even for a simple case $\frac{\beta}{2} < \alpha \leq \beta$ a team-pairwise stable network is not a pairwise stable network.

Proposition 1. *Consider HBCG with teams and $\frac{\beta}{2} < \alpha \leq \beta$. In a network g if all the nodes from each team form a complete network internally, the network g is internally stable.*

Proof. Let g be a network where all the nodes from each team form a complete network internally. It is not possible to decrease the cost of a team by deleting internal links because deleting one internal link will increase the cost of each team by at least 2 due to distance, while the cost reduction due to deletion of link is only $\frac{2\alpha}{\beta} \leq 2$. Hence, g is internally stable.

□

Similarly to the star network in standard network formation games, we define a star network at team level which we call a *team-star*. A network is a team-star if i) each team has at least two nodes, ii) all the nodes of each team form a complete network internally, iii) there is a central team with a node called central node such that all the nodes of other teams are connected to the central node and iv) there is no other external link. An example of a team-star is in Figure 1

Proposition 2. *For $\frac{\beta}{2} < \alpha \leq \beta$, a team-star network is team-pairwise stable in HBCG with teams but it is not pairwise stable in HBCG without teams.*

Proof. Let g be a team-star network, then by definition g is internally stable. From the definition of g it is clear that the distance between any two nodes of g is at most 2. A new link between any two non-central teams cannot be created because it costs each team $\frac{2\alpha}{\beta} > 1$ while the distance reduction for a team is only 1. Similarly, a new link between the central team and a non-central team cannot be created. Also, no existing link between the central team and a non-central team can be deleted because deleting one such link will cost at least 2 to each team (because there are at least two nodes in the team whose distance to some other node increases by at least one), while the cost due to the destruction of the link is reduced only by $\frac{2\alpha}{\beta} \leq 2$. So, a team-star is team-pairwise stable.

Assume that there is no team, i.e., the nodes belonging to each set T_i, $i = 1, 2, \cdots, m$, are independent rational players. Then a team-star network is not pairwise stable because the central node k from the central set T_i will form at most one link to each set T_j, $j \neq i$. ☐

Proposition 3. *For $\alpha > \beta$, a star network is a team-pairwise stable in HBCG with teams as well as pairwise stable in HBCG without teams.*

Proof. Let g be a star network. In the presence of teams, no internal links can be formed in g because forming an internal link will cost $\frac{2\alpha}{\beta} > 2$ to a team while the distance reduction for a team is only 2. An external link in g between two teams cannot be formed because it will cost each team $\frac{2\alpha}{\beta} > 2$ while the distance reduction for a team is only 1. No existing links of g can be deleted because in that case the cost of each team due to distance would be ∞. Hence, g is a team-pairwise stable.

In the absence of teams, a new link in g between two nodes from the same set cannot be formed because it will cost each node $\frac{\alpha}{\beta} > 1$, while the distance reduction for a node is only 1. A new link in g between two nodes from different sets cannot be formed because it will cost each node $\frac{2\alpha}{\beta} > 2$, while the distance reduction for a node is only 1. No existing links of g can be deleted due to the same reason given for the case of teams. ☐

Now there is an interesting question: for $\frac{\beta}{2} < \alpha \leq \beta$ which network is pairwise stable when nodes belonging to sets $\{T_i\}_{i=1}^{m}$ do not form a team. To get an idea how a pairwise stable network looks like, we consider an instance of HBCG with 9 nodes that are divided into 3 sets, i.e., $N = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and $T_1 = \{1, 2, 3\}$, $T_2 = \{4, 5, 6\}$, $T_3 = \{7, 8, 9\}$. The team-star network given in Figure 1 is team-pairwise stable but not pairwise stable from Proposition 2. A pairwise stable network in this case when nodes from T_1, T_2 and T_3 do not form a team is given in Figure 2.

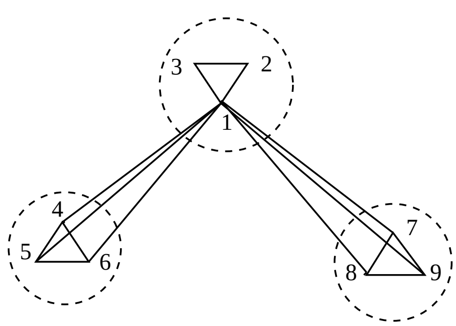

Figure 1. Team-star: A team-pairwise stable network

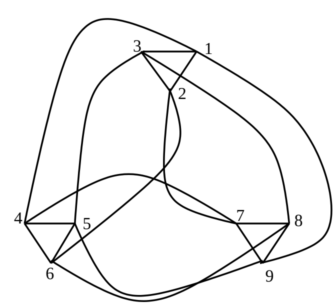

Figure 2. Pairwise stable network

The cost of the team-star network from Figure 1 is $c(\text{team-star}) = \frac{42\alpha}{\beta} + 114$ and the cost of the network g from Figure 2 is $c(g) = \frac{54\alpha}{\beta} + 108$. For $\alpha > \frac{\beta}{2}$,

$c(\text{team-star}) < c(g)$. In the simple case of HBCG, introducing the teams give us a team-pairwise stable network which is not pairwise stable but is less costly than the pairwise stable network.

4. **Stability of networks against teams' coalitions.** In this section, we introduce the concept of stability against the teams' coalitions. This notion of stability generalizes team-pairwise stability in two directions. First, the creation of a link between two teams requires the agreement of the whole coalition the two teams belong to (if any). Second, a coalition can create/severe simultaneously multiple links among its members (while team-pairwise stability considers only the possibility to either add or remove a single link at a time). In order to formally define this new stability concept we introduce the following definitions analogous to those in [17] for standard network formation games. Let $\mathcal{C} \subset 2^{\mathcal{P}}$ denote the set of teams' coalitions.

Definition 4.1. A network g' is obtainable from g via deviation by a coalition $S \in \mathcal{C}$ as denoted by $g \to_S g'$, if

(i) if $kl \in g'$, $kl \notin g$, $k \in T_i$ and $l \in T_j$, then $\{T_i, T_j\} \subset S$.
(ii) if $kl \in g$, $kl \notin g'$, $k \in T_i$ and $l \in T_j$ then $\{T_i, T_j\} \cap S \neq \phi$.

The condition (i) requires that new links can be added only between teams that are part of the coalition S. The condition (ii) requires that at least one team of any deleted link has to be part of S. The case $T_i = T_j$ corresponds to the creation/destruction of internal links from a team of the coalition. We denote $G(S, g)$ as a set of all networks which are obtainable from g via a deviation by S.

Definition 4.2. A deviation by a coalition S from a network g to a network g' is said to be improving if

(i) $g \to_S g'$,
(ii) $c(T, g') \leq c(T, g)$, $\forall\ T \in S$ (with at least one strict inequality).

Let $G_I(S, g)$ be the set of all networks g' which are obtainable from g by an improving deviation of S, i.e.,

$$G_I(S,g) = \big\{g' \in G(S,g) | c(T,g') \leq c(T,g),\ \forall\ T \in S,\\ c(T',g') < c(T',g) \text{ for some } T' \in S\big\}.$$

Let $G_{nI}(S) = G(S, g) \setminus G_I(S, g)$ be the set of networks g' which are obtainable from g but do not lead to an improvement (they would be chosen by "error"). It is clear that $g \in G_{nI}(S, g)$. It is reasonable to consider that a coalition would only accept improving deviations. This leads to the following definition of stability.

Definition 4.3. A network g is $\mathcal{C}$-stable if it is not possible for any coalition $S \in \mathcal{C}$ to make an improving deviation from g to some other network g'.

A $\mathcal{C}$-stable network may not always exist, but in that case there exists some set of networks lies on a closed cycle and all the networks in a closed cycle can be reached from each other via an improving path. We next give the definitions of an improving path and closed cycle.

Definition 4.4 (Improving Path)**.** An improving path from g to g' is a sequence of networks and coalitions $g_1, S_1, g_2, \cdots, g_{n-1}, S_{n-1}, g_n$ such that $g_1 = g$, $g_n = g'$ and $g_{k+1} \in G_I(S_k, g_k)$ for all $k = 1, 2, \cdots, n-1$.

Definition 4.5 (Cycles)**.** A set of networks Q form a cycle if for any $g, g' \in Q$ there exists an improving path connecting g and g'. A cycle Q is said to be a closed cycle when no network in Q lies on an improving path leading to a network that is not in Q.

Theorem 4.6. *There exists at least a $\mathcal{C}$-stable network or a closed cycle of networks.*

Proof. The proof follows similar lines to the proof for pairwise stable network given in [18]. Here we sketch the proof for completeness. A network is $\mathcal{C}$-stable if and only if it does not lie on an improving path leading to some other network. So, start at any arbitrary network g. If there exists no improving path from g then g is $\mathcal{C}$-stable otherwise g lies on an improving path leading to some other network. In the first case result is established. We consider the second case. Follow the improving path. Since, there are finite number of networks, then either the improving path ends at some network which has no improving paths and in that case the resulting network must be $\mathcal{C}$-stable or it will form a cycle. That is, there exists either a $\mathcal{C}$-stable network or a cycle. Consider the case where there are no $\mathcal{C}$-stable networks. Given the finite number of networks and non-existence of $\mathcal{C}$-stable networks there must exist a closed cycle. □

5. **Dynamic formation of networks under coalitions.** In this section we consider a scenario where networks are formed dynamically over time and coalitions may or may not make errors. We first consider the case when the coalitions of teams do not make errors over time. At each time $t = 1, 2, \cdots$, a coalition $S_t \in \mathcal{C}$ is randomly selected with probability $p_{S_t} > 0$. If possible, the coalition selects an improving deviation $g_{t+1} \in G_I(S_t, g_t)$ with probability $p_I(g_{t+1}|S_t, g_t)$. If there is no improving deviation then the coalition does not modify the network, i.e. $g_{t+1} = g_t$. The above situation can be modeled as a Markov chain over the set of possible networks, whose transition law P^0 is defined as:

$$P^0(g'|g) = \sum_{S\in\mathcal{C};G_I(S,g)\neq\phi} p_S \; p_I(g'|S,g)1_{G_I(S,g)}(g') + \sum_{S\in\mathcal{C};G_I(S,g)=\phi} p_S 1_{\{g'=g\}}(g'), \tag{3}$$

for all g, g', where $p_I(\cdot|S,g)$ is a probability distribution over $G_I(S,g)$, and $1_B(\cdot)$ is an indicator function on a given set B. It is clear that the $\mathcal{C}$-stable networks and the closed cycles of networks are recurrent classes of P^0. A $\mathcal{C}$-stable network corresponds to an absorbing state of P^0 and a closed cycle corresponds to a recurrent class of P^0 containing more than one state (i.e. more than one network).

We now introduce the possibility that coalitions make errors, i.e., they may not select at a given step an improving coalition and then they can make worse off one of their members. This may take into account the incertitude about the actual costs of a network topology or a bounded rationality. In particular we adapt the usual trembling-hand model [18]. At each step the coalition chooses a non-improving deviation with probability $\varepsilon f(S_t, g_t) \in (0,1)$, where $f(S_t, g_t)$ takes into account the fact that some coalitions can be more prone to make errors than others and that some network configurations may lead to wrong choices more often than others. The parameter ε allows us to tune the frequency of errors. So, at time $t+1$ the coalition S_t selects an improving deviation with probability $\Big(1 - \varepsilon f(S_t, g_t)\Big)$. In particular it chooses $g_{t+1} \in G_I(S_t, g_t)$ according to the distribution $p_I(\cdot)$ defined above. By combining the probabilities we obtain that the coalition selects

$g_{t+1} \in G_I(S_t, g_t)$ with probability $\Big(1 - \varepsilon f(S_t, g_t)\Big) p_I(g_{t+1}|S_t, g_t)$. The coalition selects a non-improving deviation with probability $\varepsilon f(S_t, g_t)$. Let $p_{nI}(\cdot|S, g)$ be a distribution over $G_{nI}(S, g)$, the coalition chooses $g_{t+1} \in G_{nI}(S_t, g_t)$ with probability $\varepsilon f(S_t, g_t) p_{nI}(g_{t+1}|S_t, g_t)$. If there is no improving deviation, then with probability $\Big(1 - \varepsilon f(S_t, g_t)\Big)$ the coalition does not modify the network, i.e., $g_{t+1} = g_t$, and with the complementary probability selects a network in $G_{nI}(S_t, g_t)$ according to the distribution $p_{nI}(\cdot|S_t, g_t)$. These rules define a perturbed Markov chain whose transition law P^ε is:

$$\begin{aligned} &P^\varepsilon(g'|g) = \\ &\sum_{\substack{S \in \mathcal{C} \\ G_I(S,g) \neq \phi}} p_S\big[(1 - f(S,g)\varepsilon) p_I(g'|S,g) 1_{G_I(S,g)}(g') \\ &+ f(S,g)\varepsilon p_{nI}(g'|S,g) 1_{G_{nI}(S,g)}(g')\big] + \sum_{S \in \mathcal{C}; G_I(S,g) = \phi} p_S\big[(1 - f(S,g)\varepsilon) 1_{\{g'=g\}}(g') \\ &+ f(S,g)\varepsilon p_{nI}(g'|S,g) 1_{G_{nI}(S,g)}(g')\big] \end{aligned} \tag{4}$$

for all g, g'. It is easy to check that if $\varepsilon = 0$, P^ε reduces to P^0 defined above.

The perturbed Markov chain P^ε is irreducible because given nonzero errors it is possible to reach all the networks starting from any network in a finite number of steps. It is also aperiodic because with positive probability the state does not change. Hence, there exists a unique stationary distribution μ^ε of the perturbed Markov chain. However, when $\varepsilon = 0$, there can be several stationary distributions corresponding to different $\mathcal{C}$-stable networks or closed cycles. Such Markov chains are called singularly perturbed Markov chains [2], [3]. The probability of making error can be sufficiently small. So, we are interested in the networks to which the stationary distribution μ^ε assigns positive probability as $\varepsilon \to 0$. This leads to the definition of stochastic stability:

Definition 5.1. A network g is stochastically stable relatively to the process P^ε if $\lim_{\varepsilon \to 0} \mu_g^\varepsilon > 0$.

We recall a few definitions from Young [31]. If $P^\varepsilon(g'|g) > 0$, $g \neq g'$, the *one step resistance* $r(g, g')$ from network g to g' is defined as the minimum number of errors that are required for the Markov chain to make a transition from g to g' in one step. In our case it is clear that $r(g, g') \in \{0, 1\}$ for any pair (g, g'). One can view the networks as the nodes of a directed graph that has no self loops. The weight of a directed edge is represented by the one step resistance between the corresponding networks and the resistance of a directed path (/tree) is the sum of the weights of its edges. We can then define the resistance from g to g' as the resistance of the minimum-resistance path from g to g' (there exists at least one because P^ε is irreducible). We now introduce a *stochastic potential* of recurrent classes of P^0 (remember these are important because they are the $\mathcal{C}$-stable networks or the closed cycles of networks). It can be computed by restricting to a reduced graph $\mathcal{G}$ where the total number of nodes are the number of recurrent classes of P^0 (one representative $g_i \in G_i$ for each recurrent class G_i). Given two nodes g_i, g_j of $\mathcal{G}$, the weight of the directed edge from g_i to g_j is the resistance from g_i to g_j. Given a node $g_i \in \mathcal{G}$, consider all the directed spanning trees such that from every node $g_j \in \mathcal{G}$, $g_j \neq g_i$, there is a unique path directed from g_j to g_i. Such spanning trees are called g_i-trees. The stochastic potential of g_i (and then of G_i) is the resistance of the g_i-tree having minimum resistance among all the g_i-trees. The resistance between any two networks in the same recurrent class is null, because no error is required to move from one to the other. It follows that the stochastic potential of any network $g_i \in G_i$ is the same and then the definition of the stochastic potential of the class G_i does not depend on the specific choice of its representative. We have now provided the background required to state and prove the following result.

Theorem 5.2. *All $\mathcal{C}$-stable networks and the networks belonging to closed cycles, that have minimum stochastic potential, are stochastically stable networks . Furthermore, if one network in a closed cycle is stochastically stable then all the networks in that closed cycle are stochastically stable.*

Proof. We know that the Markov chain P^ε is aperiodic and irreducible. From (3) and (4) it is easy to see that

$$\lim_{\varepsilon \to 0} P^\varepsilon(g'|g) = P^0(g'|g), \ \forall\, g, g'.$$

From (4) it is clear that, if $P^\varepsilon(g'|g) > 0$ for some $\varepsilon \in \ (0, \varepsilon_0]$, we have

$$0 < \varepsilon^{-r(g,g')} P^\varepsilon(g'|g) < \infty.$$

Therefore, the Markov chain P^ε is a regular perturbed process because it satisfies all the three conditions given in [31]. Hence, it follows from Theorem 4 of Young [31] that as $\varepsilon \to 0$, μ^ε converges to a stationary distribution μ^0 of P^0 and a network g is stochastically stable if and only if g is contained in a recurrent class of P^0 having minimum stochastic potential. Therefore, all $\mathcal{C}$-stable networks and the networks belonging to closed cycles having minimum stochastic potential are stochastically stable. The proof of last part follows from the fact that the stochastic potential of each network in a closed cycle is the same. ☐

Remark 1. The stochastic stability results do not depend on the function $f(\cdot)$ or the distributions of $p_I(\cdot)$, $p_{nI}(\cdot)$ and $p = (p_S)_{S \in \mathcal{C}}$.

5.1. **Strong stability:** We consider the case where there is no restriction on the formation of teams' coalitions, i.e., $\mathcal{C} = 2^{\mathcal{P}}$. In this case $\mathcal{C}$-stability is called *strong stability*.

Corollary 1. *If there is no restriction on the formation of teams' coalitions, all the strongly stable networks and the networks belonging to every closed cycle are stochastically stable.*

Proof. Now, it is always possible to reach one network from another network by at most one error due to the formation of grand coalition. Then, the resistance between any two distinct recurrent classes G_i and G_j is always 1. Hence, the stochastic potential of each recurrent class of P^0 is J-1, where J is the number of recurrent classes of P^0. In fact, a spanning tree in graph $\mathcal{G}$ includes only $J-1$ links and each of them has resistance 1. The proof then follows from Theorem 5.2. ☐

6. **Simulation results.** In the previous section we have characterized which networks are selected in the long run by the trembling-hand dynamics, when the error probability ε converges to 0. Studying the dynamics itself for a finite ε is a harder problem, so we resort to simulations to investigate i) which networks appear more frequently during the dynamics and ii) which *quasi-stable* networks arise and how fast. By a quasi-stable network we mean a network that appears for a long period of time before a random error makes it disappear. We cannot be sure that such

networks are indeed stable with respect to the set of coalitions we consider in the specific experiment (in general only a direct inspection of all the possible deviations could reach such conclusion). Nevertheless, specially for the larger values of ε the quasi-stable networks can appear over time durations comparable with those of the stable networks, so that practically speaking, they are as important as stable networks. The dynamics clearly depends on the value of ε, but also on the specific set $\mathcal{C}$ of the coalitions we consider: we are going to explore the effect of both. As case study, we consider a simple instance of HBCG with teams defined in Section 3.2. Let the set of nodes be $N = \{1,2,3,4,5,6,7,8,9\}$. The nodes are divided into three teams, $T_1 = \{1,2,3\}$, $T_2 = \{4,5,6\}$ and $T_3 = \{7,8,9\}$. We take $\alpha = 1.5$ and $\beta = 2$. For any experiment the initial network topology is selected uniformly at random among all the possible networks. At each iteration, $p = (p_S)_{S\in\mathcal{C}}$, $p_I(\cdot)$, $p_{nI}(\cdot)$ are uniformly distributed. The internal stability is known for this game, so, in the dynamics we update the networks in such a way that the internal stability of the networks is preserved.

The case of all coalitions. We start considering a very small error probability $\varepsilon = 0.00001$. Figure 3 shows the time-evolution of the cost of the network selected by the dynamics. The curve suggests that the network dynamics has selected (very fast) a quasi-stable network of cost 145.5, that is the cost of the team-star for these specific values of α and β. An inspection of the topologies of the networks show that the dynamics reaches a specific team-star network and then nothing changes until the end of the simulation. In this simulation, the probability of appearing a specific team-star network is 0.9979. Note that we do not know if the team-star is stable with respect to the set of all the possible coalitions, but this experiment (and many others we carried on) suggest that this is the case, because we never observed an improving deviation.

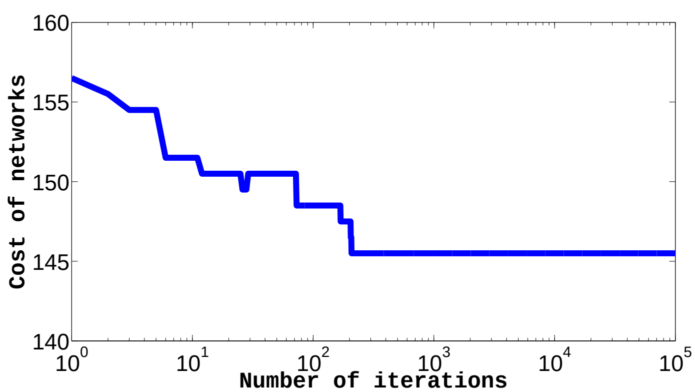

FIGURE 3. Sample path of network dynamics for $\varepsilon = 0.00001$

To check the stability of a team-star network using simulation we run a long experiment at sufficiently low $\varepsilon = 0.000001$ where we take the initial network a team-star network. We do not see any improving deviation till the end of simulation as shown by Figure 4.

Next, we consider the case of much higher error probability $\varepsilon = 0.01$. We can see from Figure 5 that the evolution of network cost is now very noisy because the dynamics jumps to a random network on average every 100 iterations due to high error

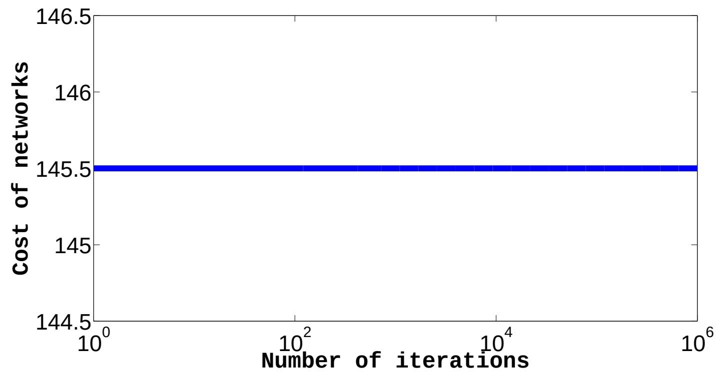

FIGURE 4. Sample path at $\varepsilon = 0.000001$ under team-star as initial network

probability. Even a stable network would not survive for a long time in this case. We run 100 simulations where each simulation starts with an initial condition that is chosen uniformly at random. Figure 6 shows the aggregate empirical distribution of network costs. The probability of appearing team-star network in this case is 0.0303. The network dynamics visits now more often the networks which are more costly than the team-star. We checked some of these networks which appear more frequently and none of them was stable. In particular, for all of them, the team-star was an improving deviation for the grand coalition. One could then wonder why these networks appear more often than the team-star during the dynamics. The reason behind is that such networks appear in much more "forms" than team-star. In fact there are only 9 team-star networks (in each of them a different node is the center of the star). For example some of the networks appearing more frequently are team-stars with a few more links. There are many different ways to place such extra links in order to have a network with the same cost. Said in other words, the classes of isomorphic networks have different sizes, and frequency with which a representant of the class appears during the network dynamics depends on its stability versus deviations but also simply on the size of the class.

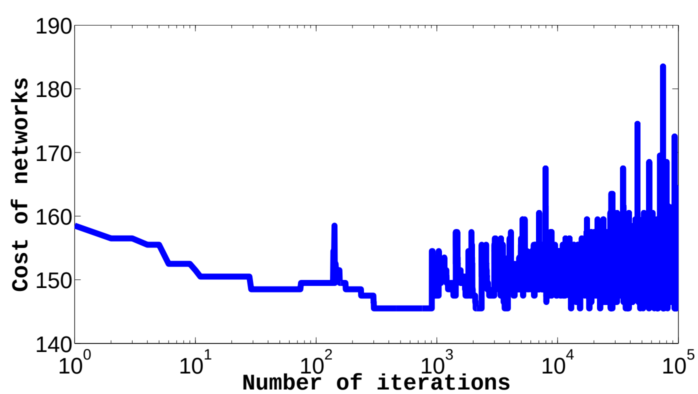

FIGURE 5. Sample path of network dynamics for $\varepsilon = 0.01$

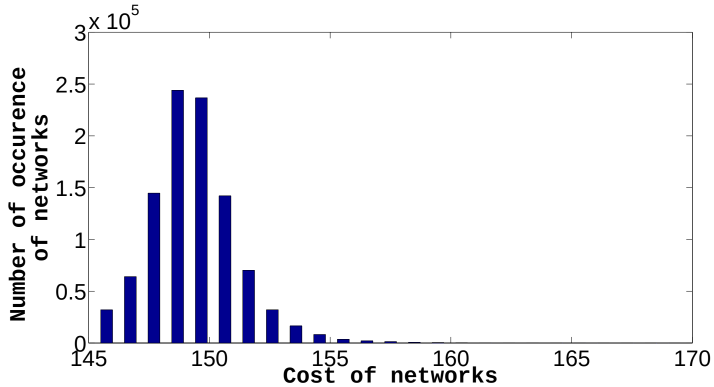

FIGURE 6. Empirical cost distribution for $\varepsilon = 0.01$

To check the fraction of time spent in team-star corresponding to different error probabilities over multiple simulations, we run 100 simulations, where initial condition is chosen uniformly at random in each simulation, for a given ε. The summary of these results are given in Figure 7. We observe that probability of appearing team-star increases as error probability decreases. Finally, we calculate the fraction of time team-star appeared, when edge additions or edge deletions occur at random, from our simulations data. The summary of these results are given in Figure 8. In this case, we observe that probability of appearing team star is almost zero corresponding to all the error probabilities considered in the experiment.

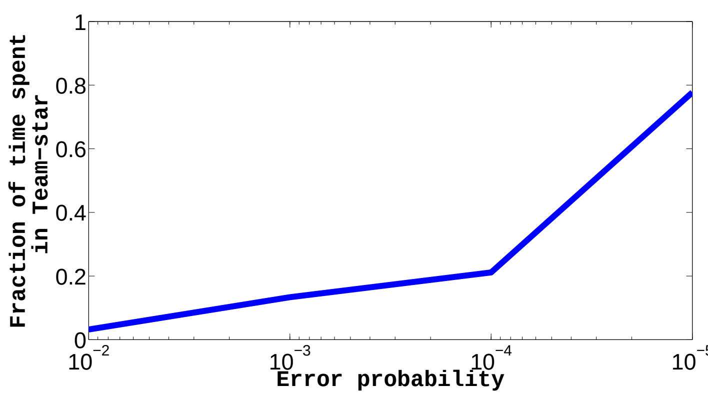

FIGURE 7. Fraction of time spent in team-star under network dynamics

Without the grand coalition. Here we consider the case where the grand coalition cannot form, but only the coalitions of size up to 2 can be formed. We first consider $\varepsilon = 0.00001$.

In Figure 9, we observe first an evolution similar to that in Figure 3 with the network dynamics selecting very fast a team-star, but in this case an error after the 70000-th iteration produces a restart. Interestingly, after the restart, the cost reaches a constant value equal to 146.5. In this case the constant cost hides in reality a continuously changing topology. In fact, there are multiple networks with

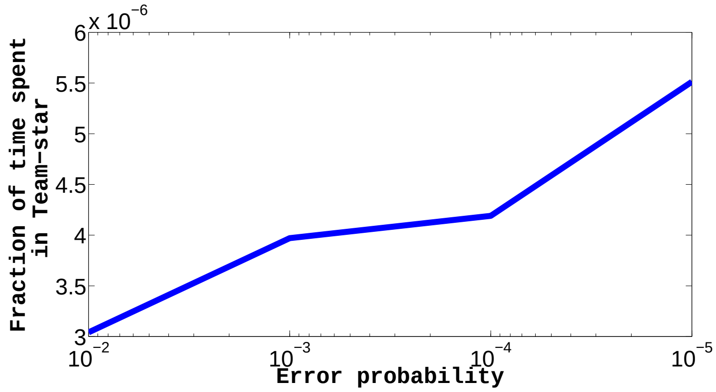

FIGURE 8. Fraction of time spent in team-star under Markov model

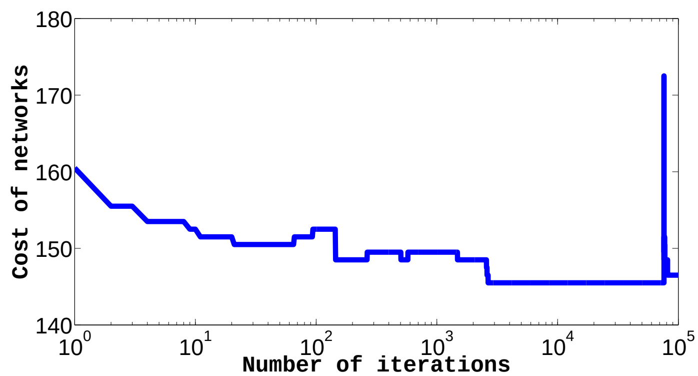

FIGURE 9. Sample path of network dynamics for $\varepsilon = 0.00001$

cost 146.5 which can be obtained one from the other by improving deviations from a coalition of size 2 and this is actually what is happening in this simulation. These networks are not even quasi-stable according to our definition, but from the point of view of the global cost, the situation does not change. As a final remark, we checked that also for these networks the team-star is an improving deviation, but it can only be enforced by the grand coalition, which is not considered in this case. For the larger error probability $\varepsilon = 0.01$ the dynamics appear to be similar to the previous case.

Only single team coalitions. Since, the creation of an external link requires a coalition of the two teams, then, in this case the only possible change is due to team severing links to other teams if it is beneficial for them or because of a random error. As soon as one of the team is disconnected from the other, there will be no more improving deviation, because the cost of each team becomes infinite. Then in this scenario the network evolves because of the random errors until all the teams are disconnected. This is the absorbing state for the network dynamics.

7. **Conclusions.** To the best of our knowledge this is the first paper where the concept of teams is introduced in network formation games. We extend the concept

of pairwise stable networks to this new class of games, but also define the concept of network stability toward deviations from a specific set $\mathcal{C}$ of teams' coalitions. We show that $\mathcal{C}$-stable networks (and closed network cycles), having minimum stochastic potential, are selected by a trembling-hand dynamics when the error probability vanishes. Finally, we resort to simulations to study the evolution of networks in a specific game by using our stochastic dynamics. We think that the idea of teams in network formation games may capture many practical phenomena such as connectivity pattern in the Internet Autonomous systems and it opens new research directions.

Acknowledgements. This research was sponsored by the European Commission within the framework of the CONGAS project FP7-ICT-2011-8-317672.

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Received October 2015; revised March 2016.

E-mail address: konstantin.avrachenkov@inria.fr
E-mail address: giovanni.neglia@inria.fr
E-mail address: vikasstar@gmail.com