A Nash-game approach to solve the Coupled problem of conductivity identification and data completion

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We consider the identification problem of the conductivity coefficient for an elliptic operator using an incomplete over-specified measurements on the surface (Cauchy data). Data completion problems are widely discussed in literature by several methods (see, e.g., for control and game oriented approaches [1, 2], and references therein). The identification of conductivity and permittivity parameters has also been investigated in many studies (see, e.g., [3, 4]). In this work, our purpose is to extend the method introduced in [1], based on a game theory approach, to develop a new algorithm for the simultaneous identification of conductivity coefficient and missing boundary data. We shall say that there are three players and we define three objective functions. Each player controls one variable and minimizes his own cost function in order to seek a Nash equilibrium which is expected to approximate the inverse problem solution. The first player solves the elliptic equation ($div(k.\nabla(u)) = 0$) with the Dirichlet part of the Cauchy data prescribed over the accessible boundary and a variable Neumann condition (which we call first player's strategy) prescribed over the inaccessible part of the boundary. The second player makes use correspondingly of the Neumann part of the Cauchy data, with a variable Dirichlet condition prescribed over the inaccessible part of the boundary. The first player then minimizes the gap related to the non used Neumann part of the Cauchy data, and so does the second player with a corresponding Dirichlet gap. The two players consider a response of the unknown conductivity of the third player. The third player controls the conductivity coefficient, and uses the over specified Dirichlet condition as well as the second's player Dirichlet condition strategy prescribed over the inaccessible part of the boundary. He minimizes then a Kohn-Vogelius type functional with respect to the conductivity parameter. This method is quite general and has wide applications ranging from bioelectrical field to mechanical engineering.

In this work, we are interested in solving the electrocardiography inverse problem which could be reduced to the data completion problem for the diffusion equation. The difficulty comes from the fact the conductivity values of the torso organs like lungs, bones, liver,... *etc*, are not known and could be patient dependent. Our goal is to construct a methodology allowing to solve both data completion and conductivity optimization problems at the same time.

We consider the following elliptic problem :

$$\left\{ \begin{array}{rcll} \nabla.(k\nabla u) & = & 0 & \text{in} \quad \Omega \\ u & = & f & \text{on} \quad \Gamma_c \\ k\nabla u.\nu & = & \Phi & \text{on} \quad \Gamma_c \end{array} \right. \tag{1}$$

where Ω is a bounded open domain in $\mathbb{R}^d$ ($d = 2, 3$) with a sufficiently smooth boundary $\partial\Omega$ composed of two connected disjoint components Γ_c and Γ_i. The functions f and Φ are the Cauchy data and ν is the unit outward normal vector on the boundary and k is a piecewise constant function representing the unknown conductivity.

For given $\eta \in H^{-\frac{1}{2}}(\Gamma_i)$, $\tau \in H^{\frac{1}{2}}(\Gamma_i)$ and $k \in L^\infty(\Omega)$, let us define $u_1(\eta, k)$, $u_2(\tau, k)$ and $u_3(\tau, k)$

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as the unique solutions in $H^1(\Omega)$ of the following elliptic boundary value problems :

$$(1)\left\{\begin{array}{rcll} \nabla.(k\nabla u_1) &=& 0 & \text{in} \quad \Omega \\ u_1 &=& f & \text{on} \quad \Gamma_c \\ k\nabla u_1.\nu &=& \eta & \text{on} \quad \Gamma_i \end{array}\right. \qquad (2)\left\{\begin{array}{rcll} \nabla.(k\nabla u_2) &=& 0 & \text{in} \quad \Omega \\ u_2 &=& \tau & \text{on} \quad \Gamma_i \\ k\nabla u_2.\nu &=& \Phi & \text{on} \quad \Gamma_c \end{array}\right. \qquad (3)\left\{\begin{array}{rcll} \nabla.(k\nabla u_3) &=& 0 & \text{in} \quad \Omega \\ u_3 &=& \tau & \text{on} \quad \Gamma_i \\ u_3 &=& f & \text{on} \quad \Gamma_c \end{array}\right.$$

The three costs are defined as follows:

$$J_1(\eta,\tau,k) = \frac{1}{2}\|k\nabla u_1.\nu - \Phi\|^2_{H^{-\frac{1}{2}}(\Gamma_c)} + \frac{1}{2}\|u_1 - u_2.\|^2_{H^{\frac{1}{2}}(\Gamma_i)}. \tag{2}$$

$$J_2(\eta,\tau,k) = \frac{1}{2}\|u_2 - f\|^2_{L^2(\Gamma_c)} + \frac{1}{2}\|u_1 - u_2\|^2_{H^{\frac{1}{2}}(\Gamma_i)}. \tag{3}$$

$$J_3(\eta,\tau,k) = \|\sqrt{k}\nabla(u_2 - u_3)\|^2_{L^2(\Omega)} \ . \tag{4}$$

From the computational viewpoint, we used the Stackelberg game algorithm to compute the Nash equilibrium. We consider an annular domain with circular boundary components Γ_i and Γ_c, both centered at $(0,0)$ and with radii $R_i = 0.6$ and $R_c = 1$, respectively. However, a third cercle of radius $R_m = 0.8$ is added between Γ_i and Γ_c. We consider a non-homogeneous conductivity k: equal to 1 (is known) in the domain between R_c and R_m, and equal to $\frac{1}{2}$ (is unknown) in the domain delimited by R_m and R_i. The figure-1 presents the potentiel and the flux reconstructed over Γ_i. The value of the reconstructed conductivity between R_m and R_i is equal to 0.48.

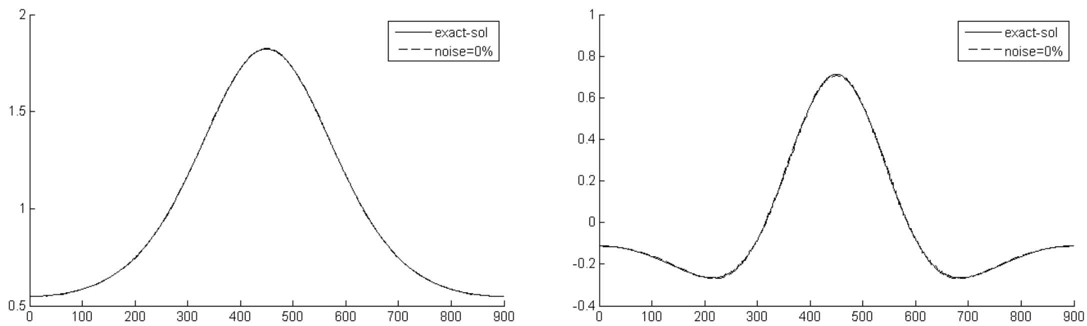

Figure 1: Reconstructed Dirchlet (left) and Neumann (right) data over Γ_i, where the reconstructed conductivity is 0.48

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