# Minimax observer for sliding mode control design

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*Chapter 1*

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We consider the classical reaching problem of sliding mode control design, that is to find a control law which steers the state of a Linear Time-Invariant (LTI) system towards a given hyperplane in a finite time. Since the LTI system is subject to unknown but bounded disturbances we apply the minimax observer which provides the best possible estimate of the system's state. The reaching problem is then solved in observer's state space by constructing a feedback control law. The cases of discontinuous and continuous admissible feedbacks are studied. The theoretical results are illustrated by numerical simulations.

## 1.1 Introduction

Sliding mode is the oldest robust control technique introduced more than 50 years ago (see, for example, [14] and references therein). This method had opened new research areas from purely theoretical domains to practical applications. The key theoretical advantage of sliding mode control is that it is insensitive to the so-called matched disturbances and uncertainties, see [5], [15], [13].

We stress that in control practice it is often required to design sliding modes for systems with mismatched uncertainties [1], [10], [16]. The same holds true for output-based feedback control application [4], [13]. These practical issues prompt for new developments in sliding mode control methodology.

This chapter treats the problem of output-based sliding mode control design for an LTI system with additive exogenous disturbances and bounded deterministic measurement noises. In this case, ensuring of the ideal sliding mode in the state space of the original system is impossible due to incomplete and noised measurements. The control law, which provides the motion of the closed-loop system as close as possible to the selected sliding surface, can only be designed. The conventional (first order) sliding mode control design principles are studied in the chapter for the case of $\mathbb{L}^2$-bounded noises and disturbances. It is known (see, for example, [2]) that realization of the reaching phase of the sliding surface can be formulated as a special optimization problem. This chapter shows that for the case of incomplete and noisy output measurements, the sliding mode control algorithm is just one possible solution of the corresponding optimization problem, which, in fact, admits both continuous and discontinuous optimal control laws.

The control design relies upon minimax state estimation framework [8, 3, 7] and duality argument [17, 20, 19]. The minimax state estimator or observer constructs the best linear estimate of the system's state provided the uncertain parameters (model disturbance, observation error, error in the initial condition) belong to a given bounding set. Statistically, the latter assumption implies uniform distributions for uncertain parameters and, under these assumptions, the minimax filter is designed so that for any realisation of uncertain parameters the estimation error is minimal. Given the best linear estimate of the state we apply the linear separation principle and transform the problem of sliding mode control design in the state space of the original system to the optimal control problem for the observer's variables.

It is worth noting that the minimax observer (in the form of a linear functional of observations), used in this paper, is optimal among all observers represented by measurable functionals of observations [22]. Thus, at least theoretically, the proposed control design can not be further improved by using observers realized by non-linear functionals of the observations. We refer the reader to [13], [5], [15], [11] for more information about nonlinear sliding mode observers. Numerical methods designed for minimax observers may be found in [21, 6].

The chapter is organized as follows. The next section present the notations used in the chapter. Then the problem statement and basic assumptions are considered. The minimax observed design is given in the section 4. Next the control design algorithms are discussed. Finally, the numerical simulations and conclusions are presented.

## 1.2 Notation

Throughout the paper the following notations will be used:

- $\mathbb{R}_+ = \{x \in \mathbb{R} : x > 0\}, \mathbb{R}_- = \{x \in \mathbb{R} : x < 0\}$, where $\mathbb{R}$ is the set of real number;
- $\|\cdot\|$ is the Euclidian norm in $\mathbb{R}^n$, i.e. $\|x\| = \sqrt{x_1^2 + \ldots + x_n^2}$ for $x = (x_1, \ldots, x_n)^T \in \mathbb{R}^n$;
- if $P \in \mathbb{R}^{n\times n}$ then the inequality $P > 0$ ($P \geq 0$, $P < 0$, $P \leq 0$) means that $P$ is symmetric and positive definite (positive semidefinite, negative definite, negative semidefinite).
- $\mathbb{L}^2_{[a,b]}$ is a set of Lebesgue quadratically integrable functions defined on $[a,b]$.

## 1.3 Problem statement

Consider the linear output control system

$$\dot{x} = Ax + Bu + Dg(t), \tag{1.1}$$

$$y = Cx + w(t), \tag{1.2}$$

$$t \in [0,T), \quad x(0) = x_0 \in \mathbb{R}^n,$$

where

- $T \in R_+$ is a finite instant of time or $T = +\infty$,
- $x \in \mathbb{R}^n$ is the vector of system state,
- $u \in \mathbb{R}$ is the scalar control input,
- $y \in \mathbb{R}^k$ is the measured output,
- the function $g : \mathbb{R}_+ \to \mathbb{R}^p, p \leq n$ describes the matched external disturbances and
- the function $w : \mathbb{R}_+ \to \mathbb{R}^k$ is a deterministic measurement noise,
- the system parameters $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}, C \in \mathbb{R}^{k\times n}, D \in \mathbb{R}^{n\times p}$ are assumed to be known and time-invariant.

We study this system under the standard assumptions (see [15, 5]).

**Assumption 1.** *The pair $(A,C)$ is observable, the pair $(A,B)$ is controllable.*

In general, we do not assume that the disturbances are matched, but the case range$D \subseteq$ range$(B)$ is also possible.

The noise measurements $w \in \mathbb{L}^2_{[0,T]}$ and exogenous disturbances $g \in \mathbb{L}^2_{[0,T]}$ are assumed to be deterministic and satisfy the following inequality

$$x_0^T P_0^{-1} x_0 + \int_0^T (w^T(\tau) R w(\tau) + g^T(\tau) Q g(\tau)) d\tau \leq 1\,, \tag{1.3}$$

where $P_0 \in \mathbb{R}^{n\times n}$, $R \in \mathbb{R}^{k\times k}$ and $Q \in \mathbb{R}^{p\times p}$ are symmetric positive definite matrices. The above inequality defines an ellipsoid in the corresponding space. We note that each uncertain parameter above ($x_0$, $w$ and $g$) may belong to a separate ellipsoid and, in this case, the above ellipsoid represents an approximation of three independent ellipsoids, provided it has been rescaled appropriately.

The admissible control law is assumed to belong to $\mathbb{L}_{[0,T]}$-space, which contains both continuous and discontinuous functions.

The **classical control problem** (see [15, 5]) for the system (1.1) is to design the control algorithm, which realizes finite-time reaching of a given linear plane like

$$F^T x = 0, \quad F \in \mathbb{R}^{m\times n}, \quad det(FB) \neq 0,$$

and further sliding on this plane. It is worth to stress that the condition $\det(FB) \neq 0$ is necessary for realization of the *first order* sliding mode control principles [14].

Let us consider the sliding mode control design problem for the reaching phase, i.e. we need to find the control law $u$ such that $Fx(T) = 0$. The considered problem can be equivalently rewritten

$$\begin{array}{l} \|Fx(T)\| \to \min \\ \text{s.t. (1.1) - (1.3).} \end{array} \tag{1.4}$$

Indeed, obtaining a solution of this optimization problem with zero value of the cost functional guarantees the successful realization of the reaching phase. Due to measurement noises and system disturbances, the sliding mode of the given surface $Fx = 0$ may not be guaranteed. In this case it is important to know, which sort of feedback control will be optimal in order to provide the system motion as close as possible to the surface.

In what follows we only study the observer-based feedback design assuming that static output-based sliding mode control (see, for example, [13]) can not be applied, i.e. range$(F^T) \nsubseteq$ range$(C^T)$.

## 1.4 Min-Max Optimal State Observer Design

According the classical methodology of the sliding mode control design, the precise knowledge of the so-called sliding variable $s(t) := Fx(t)$ is required in order to ensure the motion of the system (1.1) on the surface $Fx = 0$. We stress that this information is not available as the given output $y(t)$ is incomplete and noisy. In this situation, the best available information about the value of $Fx(t)$ is represented by the minimax estimate of the state.

Let $x_u, x_g$ denote the solutions of the following ODEs:

$$\begin{aligned} \tfrac{dx_u}{dt} &= Ax_u + Bu, \quad x_u(0) = 0\,, \\ \tfrac{dx_g}{dt} &= Ax_g + Dg, \quad x_g(0) = x_0\,. \end{aligned} \tag{1.5}$$

Then, clearly, $x(t) = x_u(t) + x_g(t)$ and

$$y_g(t) := y(t) - Cx_u(t) = Cx_g(t) + w(t). \tag{1.6}$$

The function $x_g$ may be considered as a noisy part of $x$ corresponding to disturbances from the ellipsoid (1.3) and $x_u$ represents its "mean" value corresponding to the case of zero disturbances $x_0 = 0$ and $g = 0$, which forms (together with $w = 0$) the central point of the ellipsoid (1.3). Since $x_g(t)$ does not depend on the control parameter $u$ we may first construct an estimate of the noisy part. Let us introduce the following definition.

**Definition 1.** *Assume that $l \in \mathbb{R}^n$ and $\widehat{U} \in L^2(0,t^*)$. A linear functional*

$$\widehat{\mathscr{U}_l}(y) := \int_0^{t^*} \widehat{U}^T(\tau) y(\tau) d\tau$$

*is called a* ***minimax estimate*** *of $l^T x(t)$ iff*

$$\begin{aligned} \sigma(\widehat{U}, l, t^*) &:= \sup\nolimits_{(x_0,g,w)\in\Omega^*} (l^T x(t^*) - \widehat{\mathscr{U}_l}(y))^2 \\ &\le \sigma(U, l, t^*), \quad \forall U \in L^2(0,t^*)\,, \end{aligned}$$

*where $\Omega^*$ is defined by* (1.3) *with $T = t^*$.*

*The number $\hat{\sigma}(l,t^*) := \sigma(\widehat{U}, l, t^*)$ is called the* ***minimax estimation error****.*

Let $\mathscr{G}$ denote a set of all continuous mappings of $\mathbb{L}_2$ into $\mathbb{R}$ and let $g \in \mathscr{G}$. Then it can be proven that

$$\sigma(\widehat{U}, l, t^*) = \inf_{g\in\mathscr{G}} \sup_{(x_0,d_i,w)\in\Omega^*} |l^\top x(t^*) - g(y)|\,.$$

In other words, the worst-case estimation error of any continuous mapping $g$ defined by the right hand side of the above formula is greater or equal to the minimax error of the linear functional $\widehat{\mathscr{U}_l}(y)$ which has minimal worst-case estimation error $\hat{\sigma}$.

**Proposition 1.** *Let $\hat{x}_g(t) \in \mathbb{R}^n$ be the solution of the following ODE:*

$$\begin{cases} \frac{d\hat{x}_g(t)}{dt} = A\hat{x}_g + P(t)C^T R(y_g(t) - C\hat{x}_g(t), \\ \hat{x}_g(0) = 0, \end{cases}$$

*where $P(t) \in \mathbb{R}^{n\times n}$ is the solution of the following differential Riccati equation:*

$$\begin{cases} \dot{P}(t) = AP(t) + P(t)A^T + DQ^{-1}D^T - P(t)C^T RCP(t), \\ P(0) = P_0\,. \end{cases} \tag{1.7}$$

*Then $\widehat{\mathscr{U}_l}(y_g) = l^T\hat{x}_g(t^*)$ and $\hat{\sigma}(l,t^*) = (l^T P(t^*)l)^{\frac{1}{2}}$, where $y_g$ is defined by* (1.6).

The detailed proof of this proposition is available in the literature (see for instance [9, 19]).

Let us stress that the proposed observer is *stable*, that is $A - P^\infty C^T RC$ is a Hurwitz matrix, where $P^\infty$ solves the following algebraic Riccati equation:

$$0 = AP + PA^T + DQ^{-1}D^T - PC^T RCP\,.$$

The latter has the unique symmetric nonnegative definite solution provided $\{A,C\}$ is observable or, more generally, detectable and $\{A,D\}$ is controllable or, more generally, stabilizable [12].

The definition of the minimax estimate $\widehat{U}$ implies that

$$(l^T x_g(t^*) - l^T\hat{x}_g(t^*))^2 \leq l^T P(t^*)l.$$

Now we recall that $x(t^*) = x_u(t^*) + x_g(t^*)$ and so

$$x(t^*) = x_u(t^*) + \hat{x}_g(t^*) + e(t^*).$$

where $e(t^*) = x_g(t^*) - \hat{x}_g(t^*)$ and $l^T e(t^*) \leq (l^T P(t^*)l)^{\frac{1}{2}}$ **does not depend** on $u$. Define

$$\hat{x} := x_u + \hat{x}_g.$$

Then it is straightforward to check that:

$$\begin{cases} \frac{d\hat{x}(t)}{dt} = A\hat{x} + P(t)C^T R(y(t) - C\hat{x}(t) + Bu(t), \\ \hat{x}(0) = 0. \end{cases} \tag{1.8}$$

Since the calculations above hold true for any $0 < t < t^*$, we obtain:

$$x(t) = \hat{x}(t) + e(t),$$

where the estimation error satisfies the inequality

$$l^T e(t) \leq (l^T P(t) l)^{\frac{1}{2}} \tag{1.9}$$

and the latter estimate does not depend on the control $u$. The inequality (1.9) holds for all $l \in \mathbb{R}^n$ proving the following optimal (in the minimax sense) guaranteed estimate of the system state

$$x(t) \in \{z \in \mathbb{R}^n : \quad z = \hat{x}(t) + e, \quad e^T P^{-1}(t) e \leq 1\}, \tag{1.10}$$

i.e. the state vector $x(t)$ belongs to the ellipsoid centered at $\hat{x}(t)$ with the shape matrix $P(t)$. Recall that the formula (1.9) with $\forall l \in \mathbb{R}^n$ is just a way to define the ellipsoid (see, for example, [7]).

It is worth to stress that the minimax approach to observer design provides the "exact" estimate of all admissible system's states, namely, for any $e_* \in \mathbb{R}^n$ belonging to the estimating ellipsoid (i.e. $e_*^T P^{-1}(t) e_* \leq 1$) and for any $t \in [0, T]$ there exist $x_0^* \in \mathbb{R}^n$, $w^* \in \mathbb{L}^2$ and $g^* \in \mathbb{L}^2$ satisfying (1.3) such that the equality $x(t) = \hat{x}_u(t) + e^*$ holds. In addition, one may further improve the aforementioned estimate by filtering out the states which are incompatible with observations [18], provided the realisation of $y$ is available beforehand.

Note that $P$ does not depend on the control parameter explicitly. This suggests to design a controller as a function of the center of the ellipsoid, that is $\hat{x}_u$. The next section presents the controller design.

## 1.5 Control Design

Now let us consider the problem of the reaching phase realization of the sliding mode control for the system (1.1)-(1.3), which is equivalently rewritten in the form (1.4).

Denote the sliding variable by

$$\sigma = Fx.$$

Using the formula (1.10) we derive

$$\sigma(T) = Fx(T) = F\hat{x}(T) + Fe(T),$$

where the state estimate $\hat{x}$ satisfies (1.8) and $e$ is the observation error, which is not depended on the control input $u$. Recall that the ellipsoid

$\{e \in \mathbb{R}^n : e^T P(T) e \leq 1\}$ describes the best possible estimate of the observation error at time instant $t = T$. Since $Fe(T)$ also belongs to some ellipsoid in $\mathbb{R}^m$ with the centre at the origin, then any control $u$, which guarantees

$$F\hat{x}(T) = 0, \tag{1.11}$$

is the solution of the optimization problem (1.4).

### 1.5.1 The conventional sliding mode feedback

Following the classical methodology of the sliding mode control design for linear plants [15, 5] we should define the control law in the form

$$u(t) = u_{eq}(t) + u_d(t), \tag{1.12}$$

where

$$u_{eq}(t) = -(FB)^{-1} FAx(t)$$

is the so-called equivalent control part and

$$u_d(t) = -(FB)^{-1} K(t) \text{sign}[\sigma(t)],$$

is the discontinuous (relay) term with sufficiently large gain $K(t) > 0$.

Since the only observed state is admissible then the following laws must be applied

$$\begin{aligned} u_{eq}(t) &= -(FB)^{-1} FA\hat{x}(t), \\ u_d(t) &= -(FB)^{-1} K(t) \text{sign}[\hat{\sigma}(t)], \end{aligned} \tag{1.13}$$

where

$$\hat{\sigma}(t) := F\hat{x}(t). \tag{1.14}$$

Multiplying both sides of the system (1.8) by $F$ we obtain the following equation

$$\frac{d\hat{\sigma}(t)}{dt} = FA\hat{x}(t) + FP(t)C^T R(y(t) - C\hat{x}(t)) + FBu(t), \tag{1.15}$$

which defines the dynamic of sliding variable $\hat{\sigma}$ for the observer state space. Substituting the representation (1.12) for the control law we derive

$$\frac{d\hat{\sigma}(t)}{dt} = FP(t)C^T R(y(t) - C\hat{x}(t)) - K(t) \text{sign}[\hat{\sigma}(t)]$$

Taking into account $\hat{x}(0) = 0$ we derive that for any

$$K(t) > |FP(t)C^T R(y(t) - C\hat{x}(t))| \tag{1.16}$$

the control (1.12) guarantees achievement of the aim (1.11).

Therefore, *the convectional sliding mode control is the solution of the optimization problem* (1.4) *if the design technique is based on minimax observer application*. The formula (1.16) represents the rule for selection of the relay feedback gain.

### 1.5.2 The optimal continuous control

Evidently, the continuous control can also be designed in order to guarantee the condition (1.11). For instance, the continuous feedback law

$$u_c(t) = -(FB)^{-1} F \left[A\hat{x}(t) + P(t)C^T R(y(t) - C\hat{x}(t))\right] \tag{1.17}$$

is also optimal for the problem (1.4). Indeed, this feedback provides

$$F\frac{d\hat{x}}{dt} = 0.$$

So, taking into account $\hat{x}(0) = 0$ we obtain $\hat{x}(T) = 0$.

In fact, *the optimal continuous control law is the equivalent control designed for the observer equation* (1.8).

## 1.6 Numerical Simulations

Let us consider the model of the linear oscillator with

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad B = D = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 \end{pmatrix}$$

and select the desired sliding surface $Fx = 0$ with

$$F = \begin{pmatrix} 1 & 1 \end{pmatrix}$$

The restrictions to energy measurement noises, exogenous disturbances and uncertainty of initial conditions is represented by the inequality (1.3) with the following parameters

$$P_0 = \frac{3\pi^2}{4}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad Q = R = 40/3, \quad T = 10.$$

We compare the *linear* continuous feedback (1.17) with the *sliding mode* control of the form (1.12) with $K(t) = 10$.

The initial condition selected for simulation is

$$x_0 = \begin{pmatrix} 0 & \frac{\pi}{2} \end{pmatrix}^T.$$

The deterministic noise and disturbance functions are defined by

$$w(t) = 0.05\text{sign}[\sin(10t)] \quad \text{and} \quad g(t) = 0.05\text{sign}[\cos(10t)].$$

The numerical simulations has been made using explicit Euler method with fixed step size $h = 0.01$.

The figures 1.1-1.3 presents the simulation results. The sliding mode control is subjected to chattering during simulation. Due to this the continuous control provides better results.

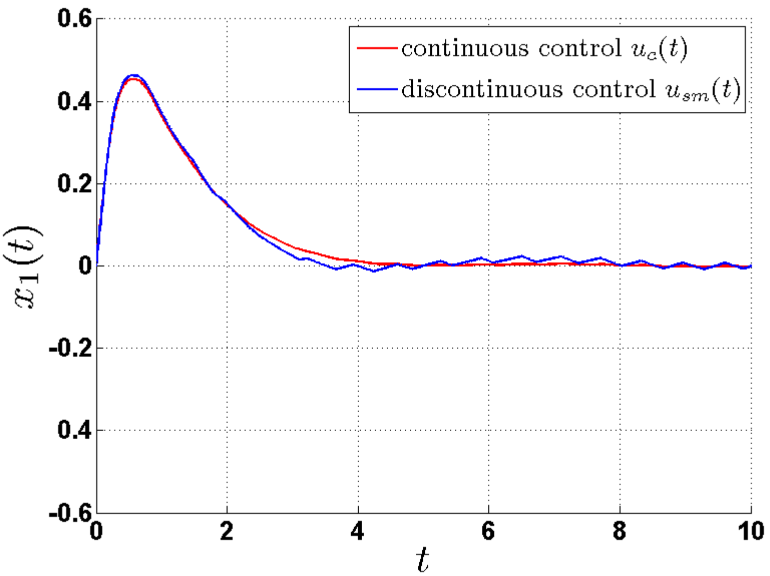


*Figure 1.1: Evolution of the state $x_1$.*

## 1.7 Conclusion

The problem of the output-based sliding mode control design in the reaching phase is studied for the linear time-invariant system with $\mathbb{L}^2_{[0,T]}$-bounded additive exogenous disturbances and the noised measurements of the output. The control providing optimal reaching (as close as possible) of the

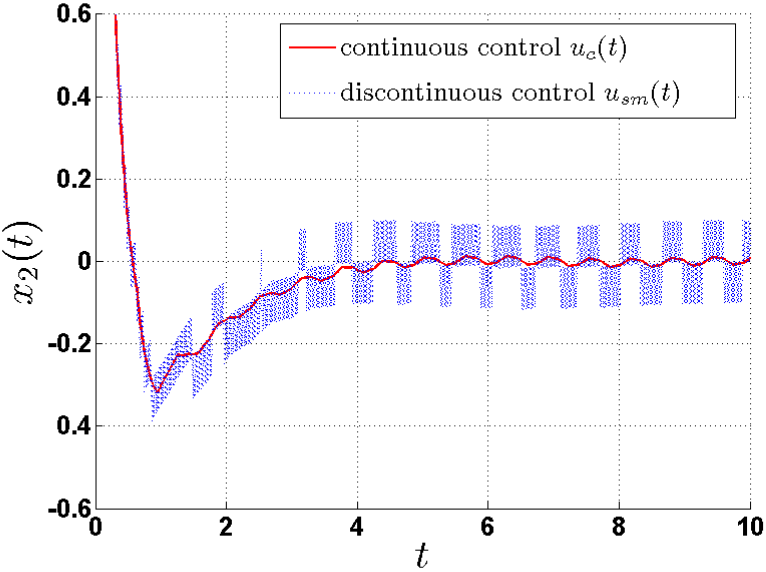


*Figure 1.2: Evolution of the state $x_2$.*

selected linear sliding surface is obtained using minimax observation approach.

The following important facts were discovered:

- The conventional sliding mode control law is an optimal solution to the considered problem.
- The optimal control law is not unique. The continuous optimal feedback is designed in the form of equivalent control for the observer equation. Therefore, discontinuity of sliding mode control may be unnecessary in the case of noised measurements.

*The last fact poses the question on consistency of existing discontinuous sliding mode control design methodology for LTI systems with noised measurements.*

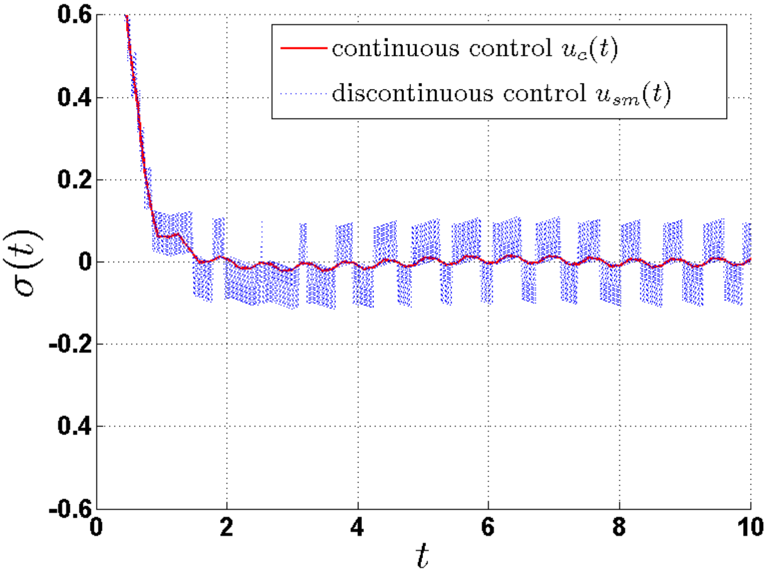


*Figure 1.3: Evolution of the sliding variable* $\sigma$.

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