A Context free language associated with interval maps

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A Context free language associated with interval maps

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For every interval map with finitely many periodic points of periods 1 and 2, we associate a word by taking the periods of these points from left to right. It is natural to ask which words arise in this manner. In this paper we give two different characterizations of the language obtained in this way.

Keywords: interval map, periodic point, period

1 Introduction

A topological dynamical system is a pair (X, f), where f is a continuous self map on the topological space X. An element $x \in X$ is called periodic if $f^n(x)= x$ for some positive integer n and the least such n is called the $f-$period of x. Here we consider dynamical systems on the interval $[0, 1]$, which are generally called interval maps. For an interval map f with finitely many periodic points, say $x_1 < x_2 < .... < x_n$ we define $w(f)$ = $p_1p_2...p_n$, where p_i is the f-period of x_i. A known theorem of Šarkovskii[6] implies that each p_i has to be a power of 2. A known fixed point theorem guarantees that one of the p_i's has to be 1 and infact, by a corollary of intermediate value theorem, it has to be in between the first 2 and the last 2 in this word (if at all there are 2's). If f and g are order conjugate then $w(f)$ = $w(g)$.

Now, it is natural to ask which words arise as $w(f)$. The main objective of this paper is to answer the above question. Our main theorem is one about the location of fixed points vis a vis points of period 2 for interval maps. The results of this paper form the starting point of answering a grand question: Which dynamical systems arise as the restriction of interval maps f to the set of all f-periodic points?

2 Main Results

Let $\mathcal{G}$ be the collection of continuous self maps on I such that the periodic points of period 1 and 2 are finitely many. Let $\mathcal{F}$ be the sub collection of $\mathcal{G}$ consisting of all those maps for which all periodic points are of periods 1 or 2. We denote by L_p (the language of periods) the set of all words $w(f)$ where $f \in \mathcal{F}$. This L_p is a language over $\{1, 2\}$ and we seek a description of L_p. Let L be the set of all words $w(f)$ where $f \in \mathcal{G}$. In this paper we prove that it coincides with L_p.

Some easy observations:

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- The number of 2's has to be even in every word.
- If there is only one 1 in a word, then it has to be exactly in the middle.
- It is less obvious but true that the words like 222121 and 212212 are not in L_p.

Main theorem:
The following are equivalent for a word over $\{1,2\}$:

1. $w \in L_p$. *That is, there is an interval map* $f \in \mathcal{F}$ *such that* $w(f) = w$.
2. w *belongs to the smallest language, say* L_g *satisfying:*
   (G_1) $1 \in L_g$.
   (G_2) $u \in L_g \Rightarrow \{1u, u1, 2u2\} \subset L_g$.
   (G_3) $\{u, v\} \subset L_g \Rightarrow u1v \in L_g$.

Theorem 1 *The language over* $\{1,2\}$ *is context free, but the similarly defined language over* $\{1,2,4\}$ *is not context free.*

3 A Context free language

To prove the main theorem, we introduce another language L_c, the set of words over $\{1,2\}$, satisfying the condition below:

For $w \in L_c$, there exists a permutation σ from $\{1,2,3,...,|w|\}$ to itself (where $|w|$ is the length of w) satisfying
(C_1) $\sigma(i) = i$ if and only if w_i = 1 (w_i is the i^{th} term of w).
(C_2) $\sigma(\sigma(i)) = i \; \forall \; i$.
(C_3) If $\sigma(i) < i$ and $j < \sigma(j)$ then there exists a l between i and j such that $\sigma(l) = l$. The map σ in the above theorem is called a *companion map*; $\sigma(i)$ is called the *companion* of i. Condition C_3 can be restated as follows: there should be a '1' between any two 2's whose companions are in opposite directions.

Lemma 1 *If* $w \in L_c$ *then there exists a* $\tilde{\sigma} : \{1,2,...,|w|\} \to \{1,2,...,|w|\}$ *satisfying* C_1, C_2, C_3 *and the following*
C_4*: If* $i < j < \tilde{\sigma}(i)$ *then* $i < \tilde{\sigma}(j) < \tilde{\sigma}(i) \; \forall i, j \leq |w|$*(this says that no two cycles are twined).*

Proof: Let $w \in L_c$. If C_4 fails, then there exist i, k such that $i < k < \sigma(i) < \sigma(k)$; we say that C_4 fails at each of the four points. We also say that i and k form a twining pair. Let i be the smallest point where C_4 fails and j be the largest point where C_4 fails. Then we call $j - i$ as twining length. For the given word w, there may be many permutations satisfying C_1, C_2, C_3. Choose σ among them such that its twining length is minimal. Let $i_1 > i$ be the smallest point forming a twining pair with i. Define σ_1 as $\sigma_1(i_1) = \sigma(i)$, $\sigma_1(i) = \sigma(i_1)$, ${\sigma_1}^2(i) = i$, ${\sigma_1}^2(i_1) = i_1$, $\sigma_1(k) = \sigma(k) \; \forall k \neq i, i_1, \sigma(i), \sigma(i_1)$.

It is clear that σ_1 satisfies C_1, C_2 and C_3. Thus, we have removed the twining in the first pair of twining cycles. Repeat the process to define $\sigma_2, \sigma_3, ..., \sigma_k$, each time removing one twining at i, until the condition $\sigma_k(i) > \sigma(l)$ for every $i < l < \sigma_k(i)$ is satisfied for some $k \in \mathbb{N}$. Since at every stage, the image of i is increasing and $|w|$ is finite, this shall take only a finite number of steps. In this way we get $\tilde{\sigma}$ such that its twining length is less than $j - i$, which contradicts the minimality of twining length. Therefore for every $w \in L_c$ there exists a $\tilde{\sigma}$ satisfying C_4. □

Lemma 2 *$u \in L_p$ if and only if $u1, 1u \in L_p$.*

Proof: Let $u \in L_p$. Then there exists a continuous function $g : [0,1] \to [0,1]$ such that $w(g) = u$. Define $f : [0,1] \to [0,1]$ as

$$f(x) = \begin{cases} \frac{g(2x)}{2} & \text{if } x \in [0, \frac{1}{2}] \\ \frac{1}{8}[5 + (4x-3)(5-4g(1))] & \text{if } x \in [\frac{1}{2}, \frac{3}{4}] \\ \frac{3x-1}{2} & \text{if } x \in [\frac{3}{4}, 1] \end{cases}$$

In other words f is the *transfer* of g on $[0, \frac{1}{2}]$, $f(\frac{1}{2}) = \frac{g(1)}{2}$, $f(\frac{3}{4}) = \frac{5}{8}$, $f(1) = 1$, and f is linearly extended elsewhere.

$P(f) = \{\frac{x}{2} | x \in P(g)\} \cup \{1\}$. Graph of f is below the diagonal in $(\frac{1}{2}, 1)$ and $[0, \frac{1}{2}]$ is invariant. Therefore there is no periodic point strictly between $\frac{1}{2}$ and 1. $w(f) = u1$. Therefore $u1 \in L_p$. Similarly we can prove that $1u \in L_p$.

In fact we prove that if either $1u \in L_p$ or $u1 \in L_p$ then $u \in L_p$. Let $u1 \in L_p$. Then there exists a function $f : [0,1] \to [0,1]$ such that $w(f) = u1$. Let $x_1 < x_2 < ... < x_n$ be the periodic points of f and it is clear that $f(x_n) = x_n$.

Define $g : [0,1] \to [0,1]$ as

$$g(x) = \begin{cases} f(x) & \text{if } x \in [0, x_{n-1}] \\ f(x_{n-1}) & \text{if } x \in [x_{n-1}, 1] \end{cases}$$

Then $P(g) = \{x_1, x_2, ...x_{n-1}\}$ and $w(g) = u$. Therefore $u \in L_p$. Similarly we can prove that if $1u \in L_p$ then $u \in L_p$.

□

Lemma 3 *If $u \in L_p$ then $2u2 \in L_p$.*

Proof: Let $g : [0,1] \to [0,1]$ be a continuous function such that $w(g) = u$. Define $f : [0,1] \to [0,1]$ as

$$f(x) = \begin{cases} (g(0)-3)x + 1 & \text{if } x \in [0, \frac{1}{4}] \\ \frac{g(4x-1)+1}{4} & \text{if } x \in [\frac{1}{4}, \frac{1}{2}] \\ \frac{(5-6x)(1+g(1))}{8} & \text{if } x \in [\frac{1}{2}, \frac{3}{4}] \\ \frac{(1-x)(1+g(1))}{4} & \text{if } x \in [\frac{3}{4}, 1] \end{cases}$$

(In other words, f is the transfer of g on $[\frac{1}{4}, \frac{1}{2}]$, $f(0) = 1$, $f(\frac{1}{4}) = (\frac{1}{4})(g(0)+1)$, $f(\frac{1}{2}) = (\frac{g(1)+1}{4})$, $f(\frac{3}{4}) = \frac{g(1)+1}{16}$, $f(1) = 0$, and f is linearly extended elsewhere.)

$P(f) = \{\frac{x}{4} + \frac{1}{4} | x \in P(g)\} \cup \{0, 1\}$.

Choose q in the last lap such that $f(q) = \frac{1}{4}$. Then choose p in the first lap such that $f(p) = q$. Then f takes $[q, 1]$ to $[0, \frac{1}{2}]$. Therefore $f \circ f$ is monotonic on $[0, p]$ and $[q, 1]$. If $p < x < \frac{1}{4}$, then $f \circ f(x) \in [\frac{1}{4}, \frac{1}{2}]$. The interval $[\frac{1}{4}, \frac{1}{2}]$ is invariant, and so x is not periodic. If $\frac{1}{2} < x < 1$, then $f(x) \in [\frac{1}{4}, \frac{1}{2}]$ and so x is not periodic. Therefore $w(f) = 2u2$. Thus $2u2 \in L_p$.

□

Remark 1 *The converse of the above lemma is not true.* $221221212 \in L_p$ *but* $2122121 \notin L_p$.

Lemma 4 *If $u, v \in L_p$ then $u1v \in L_p$.*

Proof: Let $u, v \in L_p$ then by Lemma 2, $u1, 1v \in L_p$. Then there exists continuous functions $g : [0,1] \to [0,1]$, $h : [0,1] \to [0,1]$ and $g(1) = 1$, $h(0) = 0$ such that $w(g) = u1$, $w(h) = 1v$,. Define $f : I \to I$ by
$$f(x) = \begin{cases} \frac{g(2x)}{2} & \text{if } x \in [0, \frac{1}{2}] \\ \frac{h(2x-1)+1}{2} & \text{if } x \in [\frac{1}{2}, 1] \end{cases}$$
$P(f) = \{\frac{x}{2} | x \in P(g)\} \cup \{\frac{x+1}{2} | x \in P(h)\}$. Therefore $w(f) = u1v$. □

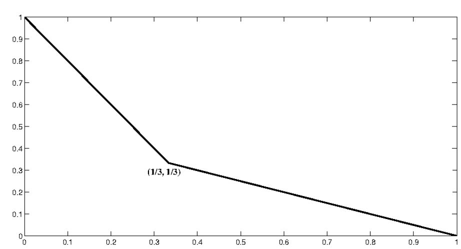
Remark 2 *The converse of the above lemma is not true.* $22122 \in L_p$ *but* $22 \notin L_p$

While constructing the proofs of the above three lemmas 2,3,4, we have introduced piecewise linear segments in such a way that the slopes 1,-1,0 are avoided.

Examples:

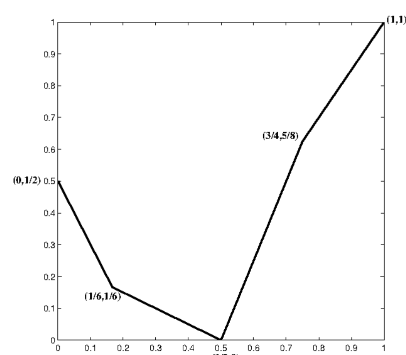
1. $g(x) = \begin{cases} -2x+1 & \text{if } x \in [0, \frac{1}{3}] \\ \frac{-x+1}{2} & \text{if } x \in [\frac{1}{3}, 1] \end{cases}$

   $w(g) = 212$



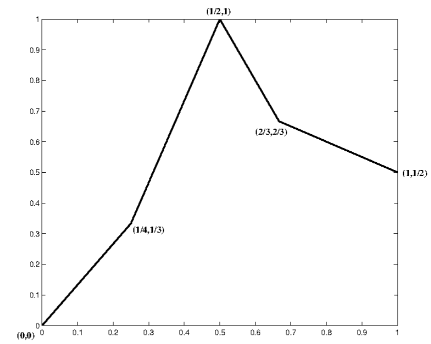
2. $f_1(x) = \begin{cases} \frac{-4x+1}{2} & \text{if } x \in [0, \frac{1}{6}] \\ \frac{-2x+1}{4} & \text{if } x \in [\frac{1}{6}, \frac{1}{2}] \\ \frac{5}{4}[2x-1] & \text{if } x \in [\frac{1}{2}, \frac{3}{4}] \\ \frac{3x-1}{2} & \text{if } x \in [\frac{3}{4}, 1] \end{cases}$

   $w(f_1) = 2121$



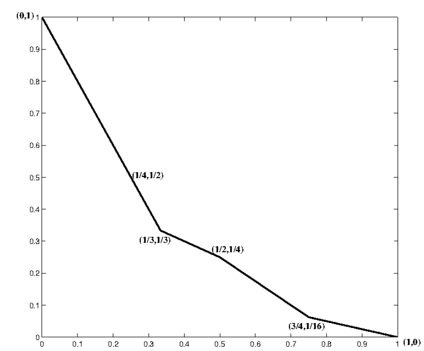
3. $f_2(x) = \begin{cases} \frac{4x}{3} & \text{if } x \in [0, \frac{1}{4}] \\ \frac{8x-1}{3} & \text{if } x \in [\frac{1}{4}, \frac{1}{2}] \\ -2x+2 & \text{if } x \in [\frac{1}{2}, \frac{2}{3}] \\ \frac{-x+2}{2} & \text{if } x \in [\frac{2}{3}, 1] \end{cases}$

   $w(f_2) = 1212$



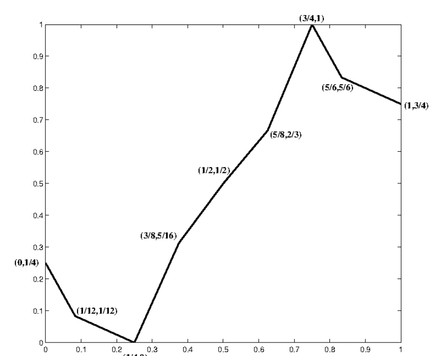
4. $f_3(x) = \begin{cases} -2x+1 & \text{if } x \in [0, \frac{1}{3}] \\ \frac{1-x}{2} & \text{if } x \in [\frac{1}{3}, \frac{1}{2}] \\ \frac{5-6x}{8} & \text{if } x \in [\frac{1}{2}, \frac{3}{4}] \\ \frac{1-x}{4} & \text{if } x \in [\frac{3}{4}, 1] \end{cases}$

   $w(f_3) = 22122$



5. $f_4(x) = \begin{cases} \frac{-8x+1}{4} & \text{if } x \in [0, \frac{1}{12}] \\ \frac{-4x+1}{8} & \text{if } x \in [\frac{1}{12}, \frac{1}{4}] \\ \frac{5}{8}[4x-1] & \text{if } x \in [\frac{1}{4}, \frac{3}{8}] \\ \frac{6x-1}{4} & \text{if } x \in [\frac{3}{8}, \frac{1}{2}] \\ \frac{8x-1}{6} & \text{if } x \in [\frac{1}{2}, \frac{5}{8}] \\ \frac{8x-3}{3} & \text{if } x \in [\frac{5}{8}, \frac{3}{4}] \\ \frac{-4x+5}{2} & \text{if } x \in [\frac{3}{4}, \frac{5}{6}] \\ \frac{-2x+5}{4} & \text{if } x \in [\frac{5}{6}, 1] \end{cases}$

   $w(f_4) = 2121212$



Theorem 2 $L_p = L_g$.

Proof: Step 1: First we prove $L_p \subset L_g$.
To prove this, we consider the language L_c introduced in the beginning of Section 3 and prove that $L_p \subset L_c \subset L_g$. Let $w \in L_p$, then there is an interval map f such that $w(f) = w$. Then x_i and $f(x_i)$ are companions to each other, for every periodic point x_i. Define $\sigma : \{1, 2, .., |w|\} \to \{1, 2, .., |w|\}$ by

$\sigma(i) = j$ if $f(x_i) = x_j$. It is easy to observe that C_1 and C_2 are satisfied. Now we prove that C_3 is satisfied.

If $\sigma(i) < i < j < \sigma(j)$, then $f(x_i) < x_i < x_j < f(x_j)$. By Intermediate value theorem there is a fixed point between x_i and x_j. So there is a 1 between w_i and w_j. Similarly, if $i > \sigma(i) > \sigma(j) > j$ or $i > \sigma(j) > \sigma(i) > j$ or $i > \sigma(j) > \sigma(i) > j$ or $i > \sigma(j) > j > \sigma(i)$, there is a fixed point between i, j. Therefore C_3 is satisfied and so $w \in L_c$. Therefore $L_p \subset L_c$.

Now we prove that $L_c \subset L_g$. Suppose it is not true.

Let $w \in L_c$ be the word of smallest length, which is not in L_g. Then there exists a $\sigma : \{1, 2, ..., |w|\} \to \{1, 2, ..., |w|\}$ such that the conditions C_1, C_2, C_3 and C_4 are satisfied (by Lemma 1). It is clear from the definition of L_c that $w \in 1L_c$ if $\sigma(1) = 1$, $w \in L_c1$ if $\sigma(|w|) = |w|$, $w \in 2L_c2$ if $\sigma(1) = \sigma(|w|)$ and $w \in L_c1L_c$ if $1 < \sigma(1) < \sigma(|w|) < |w|$. That means there exists $u \in L_c$ such that $w = 1u$ or $u1$ or $2u2$ or $u1v$ (where $v \in L_c$). By the minimality of length of w, we have that $u, v \in L_g$. Therefore $w \in L_g$, a contradiction. Hence $L_c \subset L_g$. This implies that $L_p \subset L_g$.

Step 2: $L_g \subset L_p$.

This inclusion follows from Lemma 2, 3 and 4. Therefore $L_g \subset L_p$. □

Remark 3 *Theorem 1 completely describes the location of fixed points between the points of period 2.*

For $f \in \mathcal{G}$, we define $w_{\{1,2\}}(f)$ as $p_1p_2...p_n$, where $p_1, p_2, ..p_n$ are the f-periods of the fixed points of $f \circ f$, say $x_1 < x_2 < ... < x_n$. Then we get a new language $L = \{w_{\{1,2\}} | f \in \mathcal{G}\}$ and obviously $L_p \subset L$. In fact, the following theorem says that $L = L_p$.

Theorem 3 $L = L_p$.

Proof: It is obvious that $L_p \subset L$ and now we prove that $L \subset L_p$. To prove this we recall L_c and prove that $L \subset L_c$, since $L_p = L_c$. Let $w \in L$ and let $f \in \mathcal{G}$ be such that $w = w_{\{1,2\}}(f)$. Define $\sigma : \{1, 2, .., |w|\} \to \{1, 2, .., |w|\}$ by $\sigma(i) = j$ if $f(x_i) = x_j$. It is easy to observe that C_1 and C_2 are satisfied. Now we will prove that C_3 is satisfied.

If $\sigma(i) < i < j < \sigma(j)$, then $f(x_i) < x_i < x_j < f(x_j)$. By the Intermediate value theorem, there is a fixed point between x_i and x_j. So there is a 1 between w_i and w_j. Therefore C_3 is satisfied and so $w \in L_c$. Therefore $L \subset L_c$. □

Remark 4 *From the above theorem we can observe:*

- *If $w \in L$, the new word obtained by inserting any number of 1's, anywhere in w, will also be in L. Here is a description of all words with six 2's: the three words* 2221222, 212122122, 221221212 *and all words by insertion of additional 1's any where. There are no other words in L with exactly six 2's.*

- *Here we give the list of all words in L whose length is less than 7.*

  1

  11

  111, 212

  1111, 1212, 2121, 2112

  11111, 11212, 12121, 12112, 21211, 21121, 21112, 22122

111111, 212111, 211211, 211121, 211112, 121211, 121121, 121112, 112121, 112112, 111212, 122122, 212122, 221122, 221221.

Remark 5 *For $w = w_1w_2...w_n \in L$, there exists an interval map f such that $w(f) = w$. Let $x_1 < x_2 < ... < x_n$ be the periodic points of f and w_i be the period of x_i. Now we define that w_i is equal to * when $f(x_i) = x_i$, to [when $x_i < f(x_i)$ and to] when $f(x_i) < x_i$. Note that $w(f)$ is a well-formed parentheses word as every] to the right is in correspondence with a [to the left and it is well known that well-parentheses words are exactly those with as many [as] and such that every prefix has at least as much [as]. Consider any $i < j$ such that w_i and w_j are different kinds of parentheses, either $x_i < f(x_i)$ and $f(x_j) < x_j$ or $f(x_i) < x_i$ and $x_j < f(x_j)$ thus by the intermediate value theorem f admits a fixed point in the interval $[x_i, x_j]$, i.e., there is a * somewhere in between every pair of different parentheses in $w(f)$, or equivalently the word $w(f)$ cannot contain the factor [] nor the factor][. To summarize, we have proven that $w(f)$ is included in the language defined by the following grammar: $S \to *|[S]|S*S|*S|S*$. To prove that every word in the language can be produced, one can easily proceed by induction using piecewise linear functions ($f(x) = x/2$ realizes the * and lemmas 2, 3 and 4 in the paper give short explicit constructions of the inductive steps). The language L is the projection of this language, that is the language with the following grammar: $S \to 1|2S2|S1S|1S|S1$.*(i)

L is a context free language with the grammar defined above.

Remark 6 *Using the pumping lemma for regular languages[5], we can prove that this language is not regular.*

Remark 7 *It can be proved that, for every word $w \in L_p$, there is a piecewise linear map f such that every critical point of f is a periodic point (the proofs of our lemmas can be modified to yield this). This f is post critically finite[2]. From Thurston's theorem([2]) for real polynomials there exists only one polynomial p up to affine conjugation with the same kneading data as f. Here we have the open question: Is $w(p)$ the same as $w(f)$?*

Remark 8 *The main result of this paper leads us to the question: What are all the finite subsystems $f|_{P(f)}$, if $f \in \mathcal{F}$? This question can be restated as follows: Given a word from the above language L, what are all the "pairings of 2's" that are allowed for interval maps?*

4 A Language that is not context free

Let $\mathcal{C}$ be the class of interval maps f such that $P(f)$ is finite. For $f \in \mathcal{C}$, let $x_1 < x_2 < ... < x_n$ be the periodic points of f, whose period is in A. We denote $w_A(f)$ by $w_1w_2..w_n$, where $w_i \in A$ is the f-period of x_i for $1 \leq i \leq n$ and L_A denote the collection of all words $w_A(f)$ over A, where $f \in \mathcal{C}$. For $w_j \in \mathbb{N}$, w_j^k means $w_jw_j...w_j$ (k times). For $f \in C^0(I)$, P is a periodic orbit of period 2^n. We say P is simple if for any subset $\{x_1, x_2, ..., x_k\}$ of P where $k \geq 2$ divides 2^n, for any positive integer r which divides 2^n, such that $\{x_1, x_2, ..., x_k\}$ is periodic orbit of f^r with $x_1 < x_2 < ... < x_k$, we have
$f^r(\{x_1, x_2, ..., x_{\frac{k}{2}}\}) = \{x_{\frac{k}{2}+1}, ..., x_k\}$.

Theorem 4 *[4] Let $f \in C(I)$ and suppose that f has a simple periodic orbit of period 2^n. Then there exists $n+1$ simple periodic orbits of periods $1, 2, 4, ..., 2^n$ such that the 2^k periodic points of period 2^k are separated by the $1 + 2 + ... + 2^{k-1} = 2^k - 1$ fixed points of f^{2^k-1} for any $k = 1, 2, ..., n$.*

(i) We are thankful to the referee for suggesting this remark and for avoiding our inclusion of a proof of another characterization of L.

Theorem 5 $4^n24^n14^n24^n \in L_{\{1,2,4\}}$

Proof: $2^n12^n \in L_p$, then there exists a function $f \in \mathcal{F}$ on $[0,1]$ such that $w(f) = 2^n12^n$. Define $\tilde{f} : [0,3] \to [0,3]$ as

$$\tilde{f}(x) = \begin{cases} f(x) + 2 & \text{if } x \in [0,1] \\ x - 2 & \text{if } x \in [2,3] \end{cases}$$

In $[1,2]$, its graph is a line joining $(1, f(1)), (2,0)$ and $(\tilde{f} \circ \tilde{f})|_{[0,1]} = f|_{[0,1]}$, $P(\tilde{f}) = P(f) \cup \{x+2 | x \in P(f)\} \cup \{$ a fixed point in $[1,2]\}$. Therefore $w_{\{1,2,4\}}(f) = 4^n24^n14^n24^n$. □

Theorem 6 *If* $4^{n_1}24^{n_2}14^{n_3}24^{n_4} \in L_{\{1,2,4\}}$ *if and only if* $n_1 = n_2 = n_3 = n_4$.

Proof: Let $4^{n_1}24^{n_2}14^{n_3}24^{n_4} \in L_{\{1,2,4\}}$, then there exists a function $f \in \mathcal{C}$ such that $w_A(f) = 4^{n_1}24^{n_2}14^{n_3}24^{n_4}$ (since f has finitely many periodic points, according to a theorem of Block[1]). Every 4-cycle has to be separated by a 2-cycle and a 1-cycle. In every 4-cycle between any two points, a point of period 1 or 2 should be present. Here there are four blocks of 4's of length n_1, n_2, n_3, n_4. From Theorem 4 it follows that every 4-cycle meets each block once and only once. So the size of each block is the same. Therefore $n_1 = n_2 = n_3 = n_4$. □

Theorem 7 $L_{\{1,2,4\}}$ *is not a context free language.*

Proof: This follows from Theorem 5, Theorem 6 and pumping lemma[5] for context free languages. □

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