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# A level-set based image assimilation method: applications for predicting the movement of oil spills

Long Li, François-Xavier Le Dimet, Jianwei Ma, and Arthur Vidard

## Abstract

In this paper, we present a novel method for assimilating geometric information from observed images. Image assimilation technology fully utilizes structural information from the dynamics of the images to retrieve the state of a system, and thus to better predict its evolution. Additionally, the level set method, which describes the evolution of the geometry shapes of a given system, is taken into account to include the dynamics of the images. This method differs from previous methods of image assimilation in that it takes advantage of Lagrangian information in an Eulerian numerical framework. In our numerical experiments, we apply this technique of image assimilation based on the level set method to an oil pollution problem, to calibrate the initial contours of oil pollutants and to identify diffusion coefficients of the model. Topological merging and breaking of oil slicks are well defined and easily performed by this proposed approach. The results show good agreement between simulated values and observed images.

## Index Terms

Image assimilation, Structural information, Level set method, Oil spills.

## I. INTRODUCTION

OCEANIC oil spills can have a deep impact on both the biology of the ocean as well as the environment on nearby shores. Because of this, it is extremely important to be able to predict the evolution of an oil leak not only to optimize rescue operations both on the sea and on the ground but also to efficiently guide vessels pumping oil on the ocean's surface. Currently, the movement and evolutions of these oil slicks are simulated by many disparate mathematical models. These models vary in their characteristics, for instance some of them makes use of an Eulerian method while others uses a Lagrangian approach to solve the advection-diffusion equation.

Although the equations governing geophysical fluids are well known, the use of a model to predict oil spill movement requires knowledge of the spill's initial condition, boundary condition and its own model parameters. Without an accurate knowledge of these data, the model cannot provide quality forecast, because these systems are highly sensitive to small perturbations. Therefore it presents an important scientific challenge. Indeed to generate accurate predictions, one has to acquire several distinct data sets, heterogeneous in their nature, quality and density. Variational data assimilation (VDA) is a class of techniques that combines all available heterogeneous information (i.e., mathematical models based on physical laws, observations and *a priori* knowledge) in an optimality system to retrieve the state of a system [1], [2]. VDA techniques are widely used in operational meteorology for instance.

Image assimilation for pollution can build over pas experience with more general geophysical applications. Indeed over the past several decades, a large number of satellites have orbited and observed the Earth, recording a great deal of information.

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33 This information is possibly recorded in the form of pixels representing static images or sequences of such images. For example,  
34 these images record evolutions of large weather bodies, such as hurricanes, which can travel great distances and cause large  
35 amounts of damages. It is therefore vital to couple these images with numerical models to be able to predict the evolution of  
36 events.

37 First, it was necessary to define what is recorded in these images. It is clear that a large part of the dynamical information  
38 is contained in the discontinuities or edges of the images, such as fronts, vortexes, and filaments. These discrepancies are  
39 linked to physical properties within a threshold, such as the condensation of water vapor, or the salinity level in an estuary. It  
40 is important to use images to reveal the dynamics of a system and identify the underlying physics. For example, in the field  
41 of meteorology, it is common to consider small cumulus clouds, mainly at tropical latitudes, as Lagrangian tracers, and to  
42 use estimates of their drift rates to in turn estimate wind speed. This can actually be misleading, for instance “Lenticularis  
43 Clouds” (as shown in Fig. 1) which are common on mountains, characteristically occur in strong winds, but look almost steady  
44 state because they are created by a change of phase in the water particles. Therefore if the drift of these clouds were used  
45 as Lagrangian markers to evaluate winds, the results would be in full disagreement with the true situation. Before coupling  
46 images and numerical models, it is necessary to identify what physical properties are produced within the images and to ensure  
47 that they are included in the numerical model.

48 Second, it was necessary to mathematically define the images in order to make them compatible with the physical model.  
49 This task has two constraints:

- 50 1) The discretized images must be compatible with the discretization of the model. In the process of data assimilation, one  
51 should compare recorded images with artificial images created through the solution of the numerical model. Therefore,  
52 we must construct mappings from the model toward the images.
- 53 2) To extract information from the evolutions of these images, one needs to include information about the system’s dynamics.

54 In recent years, several methods of assimilation making full use of structural information contained within images have  
55 been proposed. These methods can be divided into two basic categories. The first method estimates motion velocity fields by  
56 assimilating pseudo-observations of velocity fields produced from image sequences [3], [4]. The drawback to this approach is  
57 that it only takes into account very little physical information about the underlying processes controlling features in the image.  
58 The second method directly assimilates data from an image into a dynamic numerical model in a process called direct image  
59 assimilation (DIA). In DIA, image’s pixels can be assimilated as a tracer concentration measurement [5], alternatively, the  
60 structural information can be extracted from the image for instance by a multiscale sparse transform [6]–[8], and assimilated  
61 through a model-to-structure operator [9]–[12].

62 Going back to the specific topic of this paper, it is well established that geometry is an important aspect of prediction  
63 models because it can be used to identify polluted areas. This geometry has both a Lagrangian character because the pollution  
64 spot is advected with the flow and an Eulerian character because the proper physics of the pollutant is taken into account and  
65 interact with the environment. The geometry of an oil slick can be added to state variables of the model (including velocity,  
66 sea surface temperature, and elevation of the surface) as well as the concentration of oil. However, images of the ocean’s  
67 surface can often be obfuscated (by a cloud for instance), in that case one would need to employ an inpainting method to  
68 reconstruct the shape of oil slick. The geometry of oil slicks is complex. For instance a unique source slick can be divided  
69 into several slicks, due to the effects of wind and local turbulence on both the atmosphere and the ocean. Lagrangian methods,  
70 which define an individual interface as an ensemble of particles following their own trajectories, do not adequately describe the  
71 complex topology of many oil slicks. Currently, the level set method is more suitable for this purpose, as it uses Lagrangian



Fig. 1: Lenticular clouds image of Eulerian type (Photo by Dahlia Rudolph at Mt. Rainer, October 5, 2011).

72 information within an Eulerian numerical framework. With this method, topological merging and breaking are well defined  
 73 and easily performed. The level set method was introduced by Osher and Sethian [13] for the purposes of computing and  
 74 analyzing the motion of an interface in two or three dimensions. Since then, the level set method has been successfully used  
 75 to implicitly track moving interfaces and automatically detect changes in topology [13]–[17].

76 In this paper, we attempt to use the level set method to assimilate recorded images. To study the feasibility of this method,  
 77 we have chosen to model oil pollution in the ocean. This is an ideal site of study because, to a first order, oil remains on  
 78 the surface of the ocean, and thus a 2D model can be used to model oil spots. For other oceanographic or meteorological  
 79 applications, what is seen is the integral of the concentration of the contaminant and a 3D information will be necessary.

80 The paper is organized as follows. Section 2 describes the theory behind the method, including the application of a  
 81 conventional formulation of variational data assimilation to a pollution transport problem, as well as the level set method  
 82 and the framework of the level-set based DIA method. Section 3 details our use of the proposed method to address a global  
 83 oil spill problem, including the details of our numerical implementation and results. Finally, Section 4 summarizes the main  
 84 results and outlines possible directions of future research.

## 85 II. THEORY

86 Since the aim of this paper is to combine variational data assimilation and level set methods, both are presented separately  
 87 in the first two parts of this section. The last part proposes a combined approach tailored for oil spill monitoring.

### 88 A. Formulation of VDA to address a pollution transport problem

89 As mentioned in the introduction, VDA is a class of techniques that combines the mathematical information provided by  
 90 equations with the physical information yielded by observational data in an optimal way to retrieve the state of a system. To  
 91 use a VDA method for solving a pollution transport problem, the following components are required [18], [19]:

92 **A model** describing the evolution of the state variable  $X$ , which governs flow, and of the concentration  $C$  of a given  
 93 contaminant which is defined as:

$$\left\{ \begin{array}{l} \frac{dX}{dt} = \mathcal{F}(X, V), \quad t \in [t_0, t_f] \\ X(t_0) = U, \\ \frac{dC}{dt} = \mathcal{G}(X, C, Z), \quad t \in [t_0, t_f] \\ C(t_0) = W. \end{array} \right. \quad (1)$$

94 **A set of control variables:**  $U, V, W, Z$ .  $U, W$  represent the initial conditions, and  $V, Z$  are parameters of the models.

95 **A set of observations**  $X_{obs}, C_{obs} \in \mathcal{O}_{obs}$ . For the sake of simplicity, we will assume that our observations are continuous  
 96 in time. They can be obtained by physical measurements of the system state. However, these observations typically comprise  
 97 partial or indirect measurements of the state variables. For example, The observation space  $\mathcal{O}$  is not necessarily the same as  
 98 the state space  $\mathcal{X}$ , and the observation operator  $\mathcal{H}(\cdot)$  is defined as the mapping operator from space  $\mathcal{X}$  onto space  $\mathcal{O}$ . Similarly,  
 99 the operator of observation  $\mathcal{D}(\cdot)$  maps the space of the concentration variable  $\mathcal{C}$  onto the other observation space. Note that in  
 100 the specific case of image observation,  $\mathcal{H}(\cdot)$  and  $\mathcal{D}(\cdot)$  may include structure extraction operators and become what is called  
 101 model-to-structure operator in the introduction.

102 **Background estimations**  $U^b, V^b, W^b, Z^b$  of the initial states of model parameters  $U, V, W, Z$ . For meteorological  
 103 applications, this background estimation can be deduced from previous forecasts.

104 **Statistical information.** Some estimates of the statistics of errors take the form of a covariance matrix, as with the error  
 105 covariance matrices  $Q_X, Q_C$  of the observation error and the covariance matrices  $B_U, B_V, B_W, B_Z$  of the background  
 106 estimation.

107 **A cost function,  $J$ .** The cost function contains two terms: the first one,  $J^o$ , measures the discrepancy between observations  
 108 and the solution of the model associated with  $U, V, W, Z$ . The second term,  $J^b$ , is the background term, which requires the  
 109 solution to be located in the vicinity of  $U^b, V^b, W^b, Z^b$ . This term can be referred to as the Tikhonov regularization, which  
 110 is applied to remedy ill-posed inverse problems [20]. The cost function to be minimized is defined in the continuous case as:

$$J(U, V, W, Z) = J^o + J^b, \quad (2)$$

$$J^o = \frac{1}{2} \int_{t_0}^{t_f} \|\mathcal{H}(X, t) - X_{obs}(t)\|_{Q_X^{-1}}^2 dt + \frac{1}{2} \int_{t_0}^{t_f} \|\mathcal{D}(C, t) - C_{obs}(t)\|_{Q_C^{-1}}^2 dt, \quad (3)$$

$$J^b = \frac{1}{2} \|U - U^b\|_{B_U^{-1}}^2 + \frac{1}{2} \|V - V^b\|_{B_V^{-1}}^2 + \frac{1}{2} \|W - W^b\|_{B_W^{-1}}^2 + \frac{1}{2} \|Z - Z^b\|_{B_Z^{-1}}^2, \quad (4)$$

113 where the unknown parameters  $V, Z$  are space-dependent control variables. The norm takes the form of  $\|X\|_M^2 = \langle MX, X \rangle$ ,  
 114 where  $\langle \cdot, \cdot \rangle$  is an inner product operator in Hilbert space.

115 Variational data assimilation defines the optimal initial conditions  $U^a, W^a$  and model parameters  $V^a, Z^a$  which best fit  
 116 observations acquired over the time range  $[t_0, t_f]$ . To minimize the equation (2), A necessary condition for this optimality is  
 117 expressed by the Lagrange-Euler equation, which requests the gradient of  $J$  with respect to  $U, V, W, Z$  to be equal to 0.  
 118 However, evaluating this condition (*i.e.* computing  $\nabla J$ ) is made difficult by the facts that  $J$  can be an implicit function of  
 119 these variables and that the size of the problem can be large. Therefore, during the process of variational data assimilation,  
 120  $\nabla J$  is often computed using the so-called adjoint model:

$$\left\{ \begin{array}{l} \frac{dP}{dt} + \left[ \frac{\partial \mathcal{F}}{\partial X} \right]^* P + \left[ \frac{\partial \mathcal{G}}{\partial X} \right]^* R = \left[ \frac{\partial \mathcal{H}}{\partial X} \right]^* Q_X^{-1} (\mathcal{H}(X) - X_{obs}), \quad t \in [t_0, t_f] \\ P(t_f) = 0, \\ \frac{dR}{dt} + \left[ \frac{\partial \mathcal{G}}{\partial C} \right]^* R = \left[ \frac{\partial \mathcal{D}}{\partial C} \right]^* Q_C^{-1} (\mathcal{D}(C) - C_{obs}), \quad t \in [t_0, t_f] \\ R(t_f) = 0, \end{array} \right. \quad (5)$$

121 where  $*$  denotes the transpose, and  $P, R$  represent the so-called adjoint variables. The gradient of the cost function with

122 respect to these control variables is then given by the following equation:

$$\begin{pmatrix} \nabla J(U) \\ \nabla J(V) \\ \nabla J(W) \\ \nabla J(Z) \end{pmatrix} = \begin{pmatrix} -P(t_0) + B_U^{-1}(U - U^b) \\ -\int_{t_0}^{t_f} [\frac{\partial \mathcal{F}}{\partial V}]^* P dt + B_V^{-1}(V - V^b) \\ -R(t_0) + B_W^{-1}(W - W^b) \\ -\int_{t_0}^{t_f} [\frac{\partial \mathcal{G}}{\partial Z}]^* R dt + B_Z^{-1}(Z - Z^b) \end{pmatrix}. \quad (6)$$

123 Finally,  $U^a$ ,  $V^a$ ,  $W^a$ ,  $Z^a$  are determined by running a descent-type optimization algorithm of the Newton-type or L-BFGS  
124 variety [21].

### 125 B. Basic level set method

126 Mathematically, a given shape (e.g., an oil slick) can be represented by a subdomain  $\Omega(t) \subset R^d$ . Let to define the boundary  
127  $\partial\Omega(t)$  as the zero level set of a mapping  $\phi : R^d \rightarrow R$ :

$$\forall \mathbf{x} \in R^d, \quad \begin{cases} \phi(\mathbf{x}) < 0, & \mathbf{x} \in \Omega \\ \phi(\mathbf{x}) = 0, & \mathbf{x} \in \partial\Omega \\ \phi(\mathbf{x}) > 0, & \mathbf{x} \notin \bar{\Omega}. \end{cases} \quad (7)$$

128 The function  $\phi(t, \mathbf{x})$  could be the signed distance from  $(t, \mathbf{x})$  to  $\partial\Omega(t, \mathbf{x})$ . It defines the evolution of the shape whose movement  
129 is governed by some velocity field  $\vec{V}(t, \mathbf{x}) : R^d \rightarrow R^d$ . The motion of  $\Omega(t, \mathbf{x})$  follows the advection equation:

$$\frac{\partial \phi}{\partial t}(t, \mathbf{x}) + \vec{V}(t, \mathbf{x}) \cdot \nabla \phi(t, \mathbf{x}) = 0, \quad \text{on } [t_0, t_f] \times R^d. \quad (8)$$

130 In fact, the motion of  $\Omega(t, \mathbf{x})$  depends only on  $v(t, \mathbf{x})$ , the component of  $\vec{V}(t, \mathbf{x})$  that is normal to the boundary of  $\Omega(t, \mathbf{x})$ .  
131 Therefore, the motion of the boundary verifies the Hamilton-Jacobi equation:

$$\frac{\partial \phi}{\partial t}(t, \mathbf{x}) + v(t, \mathbf{x}) \|\nabla \phi(t, \mathbf{x})\| = 0, \quad \text{on } [t_0, t_f] \times R^d. \quad (9)$$

132 for which we need to provide an initial condition:

$$\phi(t_0, \mathbf{x}) = \phi_0(\mathbf{x}). \quad (10)$$

133 A more general form of this equation will be introduced in Section II-C2. However, during the evolution of the level set  
134 equation, the function  $\phi(t)$  can become very flat or very steep, both of which are undesirable effects that can lead to high  
135 numerical errors either in the reconstruction of the zero level set or elsewhere in the numerical schemes. An optimal control  
136 based approach of this important reinitialization aspect is discussed in Appendix B.

### 137 C. Level-set based direct image assimilation

138 1) *Edge capturing and image observation operators*: For considering the problem of pollution within a fluid, the variable  $\Omega$   
139 can be used to represent one or more patches of the pollutant. For adapting our theoretical methodology to analyze pollution,  
140 we express the size of the visible pollutant spot in terms of the pollutant's concentration, in which case  $\Omega$  and the threshold  
141 function  $\theta(\cdot)$  are defined as:

$$\Omega(t, \mathbf{x}) = \{\mathbf{x}(t) \in \tilde{\Omega} | C(t, \mathbf{x}) \geq \epsilon\}, \quad (11)$$

$$\theta(\phi_{obs}(t)) = \begin{cases} \phi_{obs}(t), & C(t, \mathbf{x}) \geq \epsilon \\ 0, & C(t, \mathbf{x}) < \epsilon \end{cases} \quad (12)$$

142 where  $C$  is the concentration of the pollutant,  $\tilde{\Omega} \subset R^2$  is the domain taken into account,  $\phi_{obs}$  is the observation image and  $\epsilon$   
143 represents the concentration threshold above which the spots of pollutant are visible.

144 In this paper, we use the wavelet modulus maximum method [22]–[25] to exploit image edge detection after undergoing  
145 thresholding. The result obtained after edge detection is expressed as  $\mathcal{W}(\theta(\phi_{obs}))$ , from which we can distinguish the inside  
146 and outside edges clearly. Some edge detection methods based on wavelet can refer to [26]–[28]. Following this procedure,  
147 we apply an edge-based signum operator  $\mathcal{H}_{\mathcal{W} \rightarrow \mathcal{S}}(\cdot)$  to the edge-detected images  $\mathcal{W}(\theta(\phi_{obs}))$ . The procedure is as follows:

$$\mathcal{H}_{\mathcal{O} \rightarrow \mathcal{S}}(\phi_{obs}) = \mathcal{H}_{\mathcal{W} \rightarrow \mathcal{S}}(\mathcal{W}(\theta(\phi_{obs}))) = \begin{cases} -1, & \mathbf{x} \in \Omega \\ 0, & \mathbf{x} \in \partial\Omega \\ 1, & \mathbf{x} \in \tilde{\Omega} \setminus \bar{\Omega} \end{cases} \quad (13)$$

148 where  $\mathcal{H}_{\mathcal{O} \rightarrow \mathcal{S}}(\cdot)$  is the image-to-structure operator, and the structure image is expressed as  $\phi^{\mathcal{S}}(t) = \mathcal{H}_{\mathcal{O} \rightarrow \mathcal{S}}(\phi_{obs}(t))$ . This  
149 construction procedure is presented in Fig. 4a-4d. Additionally, we constructed another adapted observation operator, called  
150 model-to-structure operator. This operator maps the signed distance function toward the signed function, which links variables  
151 of the level set equation with the structural information of images. This model-to-structure operator is defined as:

$$\mathcal{H}_{\phi \rightarrow \mathcal{S}}(\phi) = \begin{cases} -1, & \phi < 0 \\ 0, & \phi = 0 \\ 1, & \phi > 0 \end{cases} \quad (14)$$

152 This procedure is shown in Fig. 4e-4f.

153 2) *General level set model*: In order to apply the model-to-structure operator one has to know the temporal evolution of  
154 the edge mapping  $\phi$ . This can be obtain by application of level set techniques. In general, the motion of the interface  $\partial\Omega(t)$   
155 is analyzed by convecting the level set function  $\phi(t, x, y)$  with the external velocity field  $\vec{V} = (u, v)$  and diffusing along both  
156 the normal direction by a constant value,  $a$ , as well as proportionally to the mean curvature,  $\kappa$ . This elementary equation can  
157 be expressed as a fully nonlinear second-order partial differential equations:

$$\begin{cases} \frac{\partial \phi}{\partial t} + \vec{V} \cdot \nabla \phi + a \|\nabla \phi\| - \sigma \kappa \|\nabla \phi\| = 0, & \text{on } [t_0, t_f] \times \tilde{\Omega} \\ \phi(t_0) = \phi_0, \end{cases} \quad (15)$$

158

$$\kappa = \text{div}\left(\frac{\nabla \phi}{\|\nabla \phi\|}\right) = \frac{\left(\frac{\partial \phi}{\partial x}\right)^2 \frac{\partial^2 \phi}{\partial y^2} - 2 \frac{\partial \phi}{\partial x} \frac{\partial \phi}{\partial y} \frac{\partial^2 \phi}{\partial x \partial y} + \left(\frac{\partial \phi}{\partial y}\right)^2 \frac{\partial^2 \phi}{\partial x^2}}{\|\nabla \phi\|^3}. \quad (16)$$

159 If the term contained curvature  $\sigma$  is present, closed curves of arbitrary shape will collapse to a circle and then disappear. Unlike  
160 the curvature, as it constantly evolves under  $a$ , a relatively closed smooth curve will contain sharp edges and the topological  
161 structures of the curve may change. In practical cases, the values of the diffusion coefficients  $\sigma$ ,  $a$  depend on the physical  
162 characteristics of the pollutant. In this model, we adopt a periodic boundary condition.

163 3) *Continuous variational model and the adjoint model*: We use level set equation (15) as the governing equation to predict  
164 the evolution of oil slicks over time. The method used to determine the structure of the images,  $\phi^{\mathcal{S}}(t)$ , is described in (13).  
165 The control variables are the initial condition and diffusion coefficients  $\sigma$ ,  $a$ . In this case, the total cost function associated

with the observed images  $J^o$ , the backgrounds for initial conditions  $J_{\phi(t_0)}^b$ , and their coefficients  $J_{\sigma,a}^b$ , can be expressed as:

$$J = J^o + J_{\phi(t_0)}^b + J_{\sigma,a}^b, \quad (17)$$

$$J^o + J_{\phi(t_0)}^b = \frac{1}{2} \int_{t_0}^{t_f} \|\mathcal{H}_{\phi \rightarrow \mathcal{S}}[\mathcal{K}_t(\phi(t_0), \sigma, a)] - \mathcal{H}_{\mathcal{O} \rightarrow \mathcal{S}}[\phi_{obs}(t)]\|_{L^2(\tilde{\Omega})}^2 dt + \frac{\alpha}{2} \|\phi(t_0) - \phi(t_0)^b\|_{L^2(\tilde{\Omega})}^2, \quad (18)$$

$$J_{\sigma,a}^b = \frac{\beta}{2} (\mathcal{R}_{p_1, q_1}(\sigma) + \mathcal{R}_{p_2, q_2}(a)), \quad (19)$$

where  $\mathcal{K}_t(\cdot)$  presents the forward operator of the differential equation (15) from  $t_0$  to  $t$ , and the assimilation window is set as  $[t_0, t_f]$ . The variables  $\alpha, \beta$  denote regularization parameters, and the expression  $\phi(t_0)^b$  represents the background fields. When constructing an adjoint model, an additional difficulty is that of the non-differentiability inherent in mapping  $\mathcal{H}_{\phi \rightarrow \mathcal{S}}(\cdot)$ . Here, we modify the edge-based signum function to its differentiable form:

$$\mathcal{H}_{\phi \rightarrow \mathcal{S}}(\phi) = \frac{\phi}{\sqrt{\phi^2 + \varepsilon}}, \quad (20)$$

where  $\varepsilon$  is a small constant, and Section III-A details the method for determining it. The  $\mathcal{R}_{p,q}(x)$  is a regularization term for  $\sigma$  and  $a$ . If the coefficients are defined as being constant values, the regularization term takes the following form:

$$\mathcal{R}_{p,q}(x) = \begin{cases} \frac{1}{2}(x - q)^2, & x > q \\ 0, & p \leq x \leq q \\ \frac{1}{2}(x - p)^2, & x < p \end{cases} \quad (21)$$

where  $p, q$  represent the lower and upper bounds of coefficients, respectively.

The new cost functions (17)-(19) are the main contribution of this paper. In the following, we will focus on how to compute and numerically implement above optimization problem of the cost functions, and apply the new method to oil spill VDA.

According to the definition of the adjoint operator in Hilbert space and the method of integration by parts, we can obtain a continuous adjoint model from the continuous forward model (15):

$$\begin{cases} \frac{\partial R}{\partial t} = \frac{\partial(A-u-F)R}{\partial x} + \frac{\partial(B-v-G)R}{\partial y} - \frac{\partial^2 CR}{\partial x^2} - \frac{\partial^2 DR}{\partial y^2} + \frac{\partial^2 ER}{\partial x \partial y} \\ \quad + [\frac{\partial \mathcal{H}_{\phi \rightarrow \mathcal{S}}}{\partial \phi}]^* (\mathcal{H}_{\phi \rightarrow \mathcal{S}}(\phi) - \mathcal{H}_{\mathcal{O} \rightarrow \mathcal{S}}(\phi_{obs})), \quad \text{on } [t_0, t_f] \times \tilde{\Omega} \\ R(t_f) = 0, \\ R|_{\partial \tilde{\Omega}} = 0, \end{cases} \quad (22)$$

where

$$A = \sigma \frac{2 \frac{\partial \phi}{\partial y} [(\frac{\partial \phi}{\partial x})^2 \frac{\partial^2 \phi}{\partial x \partial y} - (\frac{\partial \phi}{\partial y})^2 \frac{\partial^2 \phi}{\partial x \partial y} + \frac{\partial \phi}{\partial x} \frac{\partial \phi}{\partial y} \frac{\partial^2 \phi}{\partial y^2} - \frac{\partial \phi}{\partial x} \frac{\partial \phi}{\partial y} \frac{\partial^2 \phi}{\partial x^2}]}{\|\nabla \phi\|^4}, \quad (23)$$

$$B = \sigma \frac{2 \frac{\partial \phi}{\partial x} [-(\frac{\partial \phi}{\partial x})^2 \frac{\partial^2 \phi}{\partial x \partial y} + (\frac{\partial \phi}{\partial y})^2 \frac{\partial^2 \phi}{\partial x \partial y} - \frac{\partial \phi}{\partial x} \frac{\partial \phi}{\partial y} \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial \phi}{\partial x} \frac{\partial \phi}{\partial y} \frac{\partial^2 \phi}{\partial x^2}]}{\|\nabla \phi\|^4}, \quad (24)$$

$$C = \sigma \frac{(\frac{\partial \phi}{\partial y})^2}{\|\nabla \phi\|^2}, \quad D = \sigma \frac{(\frac{\partial \phi}{\partial x})^2}{\|\nabla \phi\|^2}, \quad E = \sigma \frac{2 \frac{\partial \phi}{\partial x} \frac{\partial \phi}{\partial y}}{\|\nabla \phi\|^2}, \quad F = a \frac{\frac{\partial \phi}{\partial x}}{\|\nabla \phi\|}, \quad G = a \frac{\frac{\partial \phi}{\partial y}}{\|\nabla \phi\|}, \quad (25)$$

$$[\frac{\partial \mathcal{H}_{\phi \rightarrow \mathcal{S}}}{\partial \phi}]^*(\cdot) = (\frac{1}{\sqrt{\phi^2 + \varepsilon}} - \frac{\phi^2}{(\phi^2 + \varepsilon)^{\frac{3}{2}}})(\cdot). \quad (26)$$



181 Then, the gradient can be expressed as:

$$\begin{pmatrix} \nabla J(\phi_0) \\ \nabla J(\sigma) \\ \nabla J(a) \end{pmatrix} = \begin{pmatrix} -R(t_0) + (\phi(t_0) - \phi(t_0)^b) \\ -\int_{t_0}^{t_f} \int_{\tilde{\Omega}} \kappa R d\Omega dt + \nabla R_{p_1, q_1}(\sigma) \\ \int_{t_0}^{t_f} \int_{\tilde{\Omega}} ((\frac{\partial \phi}{\partial x})^2 + (\frac{\partial \phi}{\partial y})^2)^{\frac{1}{2}} R d\Omega dt + \nabla R_{p_2, q_2}(a) \end{pmatrix}. \quad (27)$$

### 182 III. APPLICATIONS TO EVOLUTION OF OIL SPILLS

#### 183 A. Numerical implementation

184 The evolution of the fluid can be modeled using the state variable  $X = (u, v, h)$ , whose components verify the shallow  
185 water equations in the conservative form expressed as:

$$\begin{cases} \frac{\partial h}{\partial t} + \frac{\partial uh}{\partial x} + \frac{\partial vh}{\partial y} = 0, & \text{on } [t_0, t_f] \times \tilde{\Omega} \\ \frac{\partial uh}{\partial t} + \frac{\partial}{\partial x}(u^2h + \frac{gh^2}{2}) + \frac{\partial uwh}{\partial y} = h(fv - g\frac{\partial H}{\partial x}), & \text{on } [t_0, t_f] \times \tilde{\Omega} \\ \frac{\partial vh}{\partial t} + \frac{\partial uwh}{\partial x} + \frac{\partial}{\partial y}(v^2h + \frac{gh^2}{2}) = h(-fu - g\frac{\partial H}{\partial y}), & \text{on } [t_0, t_f] \times \tilde{\Omega} \end{cases} \quad (28)$$

186 where a numerical simulation is performed on a rectangular domain  $\tilde{\Omega} = [0, 2] \times [0, 2]$ . Additionally,  $H$  is the height of the  
187 orography,  $h = z - H$  is the water elevation, and  $z$  is the free surface elevation. The variables  $u(t, \mathbf{x})$  and  $v(t, \mathbf{x})$  represent the  
188 zonal and meridional components of the current velocity, respectively, and  $g$  represents the acceleration due to gravity. In this  
189 model, the Coriolis parameter on the  $\beta$ -plane is defined as  $f = f_0 + \beta y$ . The following numerical values are used for these  
190 experiments:  $f_0 = 0.25s^{-1}$ ,  $\beta = 0.0406m^{-1}s^{-1}$ ,  $g = 9.81ms^{-2}$  and  $H = 0$ . We integrate this scheme forwards-in-time using  
191 the Lax-Wendroff scheme [29], which is accurate to a second-order.

192 To discretize the level set equation, we use an upwind scheme for convection term discretization, and employ a central  
193 difference method for diffusion terms. For the sake of simplicity, time integration is performed using the Euler scheme.  
194 In applied cases, a third-order accurate TVD Runge-Kutta scheme could also be used. Detailed descriptions of high-order  
195 discretization schemes for level set equations can be found in the literature [13], [14], [30], [31]. In our model, the domain is  
196 discretized on a  $N \times M = 81 \times 81$  uniform square grid. There are a total of 201 time steps. Each time step  $\Delta t$  is set to 0.0005s,  
197 and the assimilation window is  $[0, 0.1s]$ . The variable  $\varepsilon$ , as expressed in (20), is usually defined as  $(\Delta x)^2$ . Additionally, in the  
198 assimilation experiments, we assume that the current velocity fields  $u, v$  are known.

199 Because the forward model and the observations are both discrete, we can use the discrete adjoint model in this test to obtain  
200 the gradient of the control variables. The validation of both the tangent linear and the adjoint is of the utmost importance for  
201 nonlinear models. The methods of producing the discrete adjoint model and testing the gradient are shown in detail in the  
202 Appendix A.

203 In this section, we have contributed directly assimilate the images of the structures of oil pollutants  $\phi^S(t)$  into numerical  
204 models to correct the contours of initial oil pollutants and diffusion coefficients of models. True structure images  $\phi^S(t)$  are  
205 taken at each time step and at every spatial grid point using the image-to-structure operator  $\mathcal{H}_{\mathcal{O} \rightarrow \mathcal{S}}(\cdot)$ . In this test, the sequences  
206 of structure images proceeds according to  $\mathcal{H}_{\phi \rightarrow \mathcal{S}}(\phi)$ , where the level set function  $\phi$  is produced from the forward model. The  
207 sequences of synthetic structure images can thus be expressed as:

$$\phi^S = \mathcal{H}_{\phi \rightarrow \mathcal{S}}[\mathcal{K}_t(\phi(0)^t, \sigma^t, a^t)], \quad (29)$$

208 where  $\phi(0)^t$  represents the true initial state of the system. To test the efficiency of this proposed method, we have contributed  
 209 a missing data point at  $t = 0$  to generate the initial image of the structure of the oil slicks,  $\phi^S(0)$ , which is shown in Fig. 5a.  
 210 To assess the background fields of initial conditions, we assumed the missing oil slick began as a rectangular domain whose  
 211 zero level set is shown as the solid red line in Fig. 6a. The true model parameters are  $\sigma^t = 0.1$  and  $a^t = 0.5$ , and weights are  
 212 set at  $\alpha = 1$  and  $\beta = 2$ .

213 To assess the performance of the numerical test, we computed the root mean square error (*RMSE*) of the solution estimated  
 214 relative to the true value. The *RMSE* is defined as

$$RMSE_{\phi(0)^a} = \sqrt{\frac{1}{|\tilde{\Omega}|} \sum_{\mathbf{x} \in \tilde{\Omega}} (\phi(0, \mathbf{x})^a - \phi(0, \mathbf{x})^t)^2}, \quad (30)$$

$$RMSE_{\sigma^a} = |\sigma^a - \sigma^t|, \quad (31)$$

$$RMSE_{a^a} = |a^a - a^t|, \quad (32)$$

217 where  $\phi(0)^a$ ,  $\sigma^a$ ,  $a^a$  represent optimal solutions.

## 218 B. Numerical results

219 To illustrate the level set method, we first show the numerical results, the level set function  $\phi(t)$ , produced from the forward  
 220 model (15) at three different moments in Fig. 2. The level set function at the first moment derives from a synthetic oil spot  
 221 image, which is obtained by constructing a signed distance function based on the oil spot edges. In this paper, we only take  
 222 into account the affect of the shear flow fields provided by the shallow water equation (28). Fig. 3 presents the zero level set  
 223 of  $\phi(t)$  at corresponding moments of Fig. 2, representing the point set  $(x(t), y(t))$  at which  $\phi(x(t), y(t), t) = 0$ . The zero  
 224 level sets express the edges of oil slicks. The changes of the contours reveal the evolution of the shapes of the polluted areas,  
 225 which drifted on the surface of the water at a velocity determined by the current and the wind in practical cases.

226 In the framework of level-set based DIA, we need two observation operators for the purpose of mapping the numerical  
 227 solutions and the observations into the same structure space. One is the so-called image-to-structure operator  $\mathcal{H}_{\mathcal{O} \rightarrow \mathcal{S}}(\cdot)$ ,  
 228 the other is the model-to-structure operator  $\mathcal{H}_{\phi \rightarrow \mathcal{S}}(\cdot)$ . This construction procedure is presented in Fig. 4. First, a synthetic  
 229 observation image  $\phi_{obs}$  with noise is given displayed in Fig. 4a. The light gray part in the image indicates the oil pollution.  
 230 Edge detection result  $\mathcal{W}(\phi_{obs})$  by the method of wavelet modulus maximum is shown in Fig. 4b. Indeed, this method can  
 231 balance noises and edges. Fig. 4c-4d depict the structure and modified structure image (i.e. the edge-based signum function  
 232 and smoothing signum function image  $\varphi^S = \mathcal{H}_{\mathcal{O} \rightarrow \mathcal{S}}(\phi_{obs})$ ). This can show the structure information of edge clearly. From  
 233 the aspects of variables from the forward model, the level set function  $\phi$  and the structure image  $\varphi^S = \mathcal{H}_{\phi \rightarrow \mathcal{S}}(\phi)$  are shown  
 234 in Fig. 4e-4f. As a result, Fig. 4d approximately equals to Fig. 4f.

235 Fig. 5 displays portions of the structure image sequences  $\phi^S(t)$  constructed by the method defined in (29), which are used  
 236 as true observations in numerical implementation. In order to test the efficiency of the proposed method, the structure image  
 237 sequences include a missing image at  $t = 0$ . In practical cases, it can be considered as an image obfuscated by clouds.

238 To clearly compare numerical results, we can examine the zero level set of the analysis value  $\phi(0)^a$  (shown as a solid  
 239 red line in Fig. 6b) and the first guess or background fields  $\phi(0)^b$  (shown as a solid red line in Fig. 6a), with respect to the  
 240 corresponding true initial states  $\phi(0)^t$  (shown as dashed black line in Fig. 6a and 6b). Fig. 6b suggests that the optimal solution  
 241 will eventually converge to true values with increasing numbers of iterations. A part of the first observation is missing, the

242 proposed method is capable of recovering it. Additionally, the evolution of the  $l_2$  norm of the gradient and its cost function  
 243 are shown in Fig. 6c, in which the  $l_2$  norm of the gradient gradually moves closer to zero.

244 Fig. 7 displays the evolution of the estimated coefficients  $\sigma$ ,  $a$ , as well as their gradients and cost function. This shows that  
 245 the optimal coefficients will be close to the true values with increasing numbers of iterations.

246 Finally, Table I and II list the results of our numerical tests, including the  $RMSE$ , gradients and cost function values, which  
 247 change with increasing numbers of iterations. These results suggest that our proposed method performs well.

## 248 IV. CONCLUSIONS AND FUTURE WORK

### 249 A. Conclusions

250 In this paper, we propose a level-set based image assimilation method for assimilating geometric information from the  
 251 observed images into numerical models. We present numerical experiments which can be applied to better predict oceanic  
 252 oil spill dispersion. In general, our method takes into account both the complex topological structures of oil slicks and the  
 253 geometric shape of the polluted areas to avoid delaying cleanup efforts by making inaccurate predictions from observations.  
 254 This method represents a key step to better predict the evolutions of oil spill. Furthermore, the adapted edge-based signum  
 255 observation operator could eventually link the data from the level set function to the structural information contained in satellite  
 256 images. Our assimilation test yielded promising results.

### 257 B. Future work

258 In this paper it is assumed that both the external velocity and the concentration field are known. In practice they have to be  
 259 estimated as well. This can be done sequentially with the proposed approach, but for the sake of consistency one may prefer  
 260 to use a more integrated approach called splitting method for global oil spill problem. We first need to define the framework  
 261 of optimal control.

#### 262 1) Model and control variables:

- 263 • A model of flow:  $\mathcal{F}(X, U) = 0$ , where  $X$  is the state variable describing the evolution of the flow and initial conditions  
 264  $X(t_0) = U$  is a control variable.
- 265 • An equation of concentration:  $\mathcal{G}(X, C, W) = 0$ , where  $C$  is the concentration of pollutant and  $C(t_0) = W$  is a control  
 266 variable.
- 267 • An equation for the evolution of an image:  $\mathcal{K}(X, C, I, \phi_0, \lambda) = 0$ , where  $I$  is the image and  $\phi_0$  is a control variable  
 268 defining the initial level set associated with the spot of pollution.  $\lambda$  is an additional parameter (e.g., the viscosity of oil  
 269 on the ocean's surface). In this paper,  $\lambda$  represents  $(\sigma, a)$ .

270 2) *Optimal control*: The goal of this problem is to determine  $(U^a, W^a, \phi_0^a, \lambda^a)$  while minimizing the cost function; this is  
 271 expressed as:

$$J(U, W, \phi_0, \lambda) = J_F + J_G + J_I = \frac{1}{2} \int_{t_0}^{t_f} \|\Gamma_X X - X_{obs}\|^2 + \|\Gamma_C C - C_{obs}\|^2 + \|\Gamma_I I - I_{obs}\|^2 dt, \quad (33)$$

272 where  $\Gamma_X$ ,  $\Gamma_C$  and  $\Gamma_I$  are observation operators. Throughout the process of quantifying minimum values,  $U_n, W_n, (\phi_0)_n$  can  
 273 be obtained from the iterative formula:

- 274 •  $U_{n+1}$  minimizes  $J_F$  with the constraint  $\mathcal{F}(X, U) = 0$ ,
- 275 •  $W_{n+1}$  minimizes  $J_G$  with the constraint  $\mathcal{G}(X_{n+1}, C, W) = 0$ ,

276 •  $((\phi_0)_{n+1}, \lambda_{n+1})$  minimizes  $J_I$  with the constraint  $\mathcal{K}(X_{n+1}, C_{n+1}, I, \phi_0, \lambda) = 0$ .

277 The two first formulae are well known in the VDA community [18], [19]. This splitting method solves large numerical problems  
278 using a sequence and iteration of smaller, simpler problems. In the numerical experiments of this paper, we focus on the third  
279 part of this algorithm.

280 Additionally, in the future we would like not only to further consider the physical properties controlling the movements of  
281 oil spills but also to utilize real satellite images of oil slicks.

## 282 APPENDIX A

### 283 THE DISCRETE ADJOINT MODEL AND THE GRADIENT OF THE CONTROL VARIABLES

284 We consider the cost function  $J$  as follows:

$$J(u_0, \nu) = \frac{1}{2} \Delta t \sum_{n=1}^N \langle (\mathcal{H}(u^n) - u_{obs}^n), (\mathcal{H}(u^n) - u_{obs}^n) \rangle + \frac{1}{2} \langle (u_0 - u_b), (u_0 - u_b) \rangle, \quad (34)$$

285 where the discrete model is defined as follows:

$$u^{n+1} = \mathcal{M}(u^n, \nu), \quad (35)$$

286 where  $u_0$  represents the initial condition, having been arranged in a column vector, and  $\nu$  is a parameter of the model. The  
287 variables  $u_{obs}^n$ ,  $n = 1, \dots, N$  represent the observations sampled at  $t = n\Delta t$ , and  $u_b$  is the background field or the first guess.  
288  $\mathcal{H}(\cdot)$  represents a nonlinear observation operator which combines the model variables with observations.  $\mathcal{M}(\cdot)$  is the discrete  
289 operator of the forward model from the moment  $t = n\Delta t$  to  $t = (n+1)\Delta t$ .

290 The Gâteaux derivative of  $J$  in the direction of perturbation  $\delta u_0, \delta \nu$  is defined as:

$$\begin{aligned} \hat{J}(u_0, \nu, \delta u_0, \delta \nu) &= \lim_{\alpha \rightarrow 0} \frac{J(u_0 + \alpha \delta u_0, \nu + \alpha \delta \nu) - J(u_0, \nu)}{\alpha} \\ &= \langle \delta u_0, (u_0 - u_b) \rangle + \Delta t \sum_{n=1}^N \langle \hat{u}, \left( \frac{\partial \mathcal{H}}{\partial u}(u^n) \right)^* (\mathcal{H}(u^n) - u_{obs}^n) \rangle, \end{aligned} \quad (36)$$

291 where  $*$  denotes the transpose. We can obtain the discrete tangent linear equation (DTLE) from the discrete model given in  
292 (35):

$$\hat{u}^{n+1} = \frac{\partial \mathcal{M}}{\partial u}(u^n, \nu) \hat{u}^n + \frac{\partial \mathcal{M}}{\partial \nu}(u^n, \nu) \hat{\nu}. \quad (37)$$

293 Then, we can take the scalar products of (37) with the adjoint variable  $p^{n+1}$  and sum them all in time:

$$\sum_{n=1}^N \langle \hat{u}^{n+1}, p^{n+1} \rangle = \sum_{n=1}^N \langle \left( \frac{\partial \mathcal{M}}{\partial u}(u^n, \nu) \hat{u}^n \right), p^{n+1} \rangle + \sum_{n=1}^N \langle \left( \frac{\partial \mathcal{M}}{\partial \nu}(u^n, \nu) \hat{\nu} \right), p^{n+1} \rangle, \quad (38)$$

294 which is the same as:

$$\begin{aligned} \sum_{n=1}^N \langle \hat{u}^n, p^n \rangle - \langle \hat{u}^1, p^1 \rangle + \langle \hat{u}^{N+1}, p^{N+1} \rangle &= \sum_{n=1}^N \langle \hat{u}^n, \left( \frac{\partial \mathcal{M}}{\partial u}(u^n, \nu) \right)^* p^{n+1} \rangle \\ &+ \sum_{n=1}^N \langle \hat{\nu}, \left( \frac{\partial \mathcal{M}}{\partial \nu}(u^n, \nu) \right)^* p^{n+1} \rangle. \end{aligned} \quad (39)$$

295 Next, we can assume that  $p^{N+1}$  is 0 and use the fact that  $\hat{u}^1 = \delta u_0$ . This yields the following equation:

$$\sum_{n=1}^N \langle \hat{u}^n, (p^n - \left( \frac{\partial \mathcal{M}}{\partial u}(u^n, \nu) \right)^* p^{n+1}) \rangle = \langle \delta u_0, p^1 \rangle + \sum_{n=1}^N \langle \hat{\nu}, \left( \frac{\partial \mathcal{M}}{\partial \nu}(u^n, \nu) \right)^* p^{n+1} \rangle. \quad (40)$$

296 The Gâteaux derivative of  $J$  can be expressed as:

$$\begin{aligned}\hat{J}(u_0, \nu, \delta u_0, \delta \nu) &= \langle \delta u_0, \nabla_{u_0} J \rangle + \langle \hat{\nu}, \nabla_{\nu} J \rangle \\ &= \langle \delta u_0, (u_0 - u_b) \rangle + \Delta t \sum_{n=1}^N \langle \hat{u}, (\frac{\partial \mathcal{H}}{\partial u}(u^n))^* (\mathcal{H}(u^n) - u_{obs}^n) \rangle.\end{aligned}\quad (41)$$

297 If the so-called variable  $p$  is the solution of the following equation,

$$p^n - (\frac{\partial \mathcal{M}}{\partial u}(u^n, \nu))^* p^{n+1} = \Delta t (\frac{\partial \mathcal{H}}{\partial u}(u^n))^* (\mathcal{H}(u^n) - u_{obs}^n), \quad (42)$$

298 then

$$\begin{aligned}\langle \delta u_0, \nabla_{u_0} J \rangle + \langle \hat{\nu}, \nabla_{\nu} J \rangle &= \langle \delta u_0, (u_0 - u_b) \rangle + \langle \delta u_0, p^1 \rangle + \sum_{n=1}^N \langle \hat{\nu}, (\frac{\partial \mathcal{M}}{\partial \nu}(u^n, \nu))^* p^{n+1} \rangle \\ &= \langle \delta u_0, ((u_0 - u_b) + p^1) \rangle + \sum_{n=1}^N \langle \hat{\nu}, (\frac{\partial \mathcal{M}}{\partial \nu}(u^n, \nu))^* p^{n+1} \rangle.\end{aligned}\quad (43)$$

299 Finally, we can obtain the gradient:

$$\nabla_{u_0} J = (u_0 - u_b) + p^1, \quad (44)$$

300

$$\nabla_{\nu} J = \sum_{n=1}^N (\frac{\partial \mathcal{M}}{\partial \nu}(u^n, \nu))^* p^{n+1}. \quad (45)$$

301 Next, we can use the test of the gradient to validate both the tangent linear and the adjoint models. The test of the gradient  
302 is based on the Taylor expansion of the cost function, which is written as follows:

$$J(\mathbf{x} + \alpha \delta \mathbf{x}) = J(\mathbf{x}) + \alpha \langle \nabla_{\mathbf{x}} J(\mathbf{x}), \delta \mathbf{x} \rangle + O(\alpha \|\delta \mathbf{x}\|), \quad (46)$$

303 where  $J$  is the cost function and  $\mathbf{x}$  is a symbolic representation of the initial condition or the parameters of the models.

304 Rearranging (46) leads to the following expression:

$$\lim_{\alpha \rightarrow 0} \frac{J(\mathbf{x} + \alpha \delta \mathbf{x}) - J(\mathbf{x})}{\alpha \langle \nabla_{\mathbf{x}} J(\mathbf{x}), \delta \mathbf{x} \rangle} = 1. \quad (47)$$

305

## APPENDIX B

306

### THE REINITIALIZATION BASED OPTIMAL CONTROL

307 In this section, we take into account the process of reinitialization mentioned in Section II-B in the framework of level-set  
308 based DIA. In order to maintain the numerical stability of (9), one needs to reinitialize the level set function so that, after a  
309 period of time, it moves closer to the signed distance function of the actual shape. This reinitialization procedure is expressed  
310 as [14]:

$$\begin{cases} \frac{\partial \psi}{\partial \tau} = \text{sgn}(\phi)(1 - \|\nabla \psi\|), & \text{on } [\tau_0, \tau_f] \times R^d \\ \psi(\tau_0) = \phi(t), \end{cases} \quad (48)$$

311 where  $\text{sgn}(\cdot)$  is the signum function and the variable  $\psi$  is called renormalized level set function. This makes possible the  
312 computation of  $\psi$  by setting  $\|\nabla \psi(\tau_f)\| = 1$  as the long-time limit  $\tau_f \rightarrow \infty$ , related to the corresponding evolution equation.

313 Then  $\phi(t)$  will be substituted by  $\psi(\tau_f)$ .

314 For numerical implementation, we denote the discrete equation (15) and (48) respectively as:

$$\phi^{n+1} = \mathcal{A}(\phi^n), \quad (49)$$

$$\psi^{m+1} = \mathcal{B}(\psi^m), \quad \psi^1 = \phi^{n+1}, \quad (50)$$

where  $\mathcal{A}(\cdot)$  and  $\mathcal{B}(\cdot)$  are discrete forward operators from the moment  $t = n\Delta t$  to  $t = (n+1)\Delta t$  and  $t = m\Delta t$  to  $t = (m+1)\Delta t$ .  $R^d$  in (48) is set to  $\tilde{\Omega}$ . And the process of initialization is used for each time step. We can obtain the DTLE from the discrete model given in (49) and (50):

$$\hat{\phi}^{n+1} = \frac{\partial \mathcal{A}}{\partial \phi}(\phi^n) \hat{\phi}^n, \quad (51)$$

$$\hat{\psi}^{m+1} = \frac{\partial \mathcal{B}}{\partial \psi}(\psi^m) \hat{\psi}^m, \quad \hat{\psi}^1 = \hat{\phi}^{n+1}. \quad (52)$$

Then we have:

$$\hat{\psi}^{m+1} = \frac{\partial \mathcal{B}}{\partial \psi}(\psi^m) \cdots \frac{\partial \mathcal{B}}{\partial \psi}(\psi^1) \hat{\psi}^1 = \frac{\partial \mathcal{B}}{\partial \psi}(\psi^m) \cdots \frac{\partial \mathcal{B}}{\partial \psi}(\psi^1) \hat{\phi}^{n+1} = \frac{\partial \mathcal{B}}{\partial \psi}(\psi^m) \cdots \frac{\partial \mathcal{B}}{\partial \psi}(\psi^1) \frac{\partial \mathcal{A}}{\partial \phi}(\phi^n) \hat{\phi}^n. \quad (53)$$

The variable  $\hat{\psi}^{m+1}$  represents the updated  $\hat{\phi}^{n+1}$ . In addition, the adjoint of the operator  $\mathbf{B} = \frac{\partial \mathcal{B}}{\partial \psi}(\psi^m) \cdots \frac{\partial \mathcal{B}}{\partial \psi}(\psi^1) \frac{\partial \mathcal{A}}{\partial \phi}(\phi^n)$  used in the adjoint model (42) will be expressed as follows:

$$(\mathbf{B})^* = \left[ \frac{\partial \mathcal{B}}{\partial \psi}(\psi^m) \cdots \frac{\partial \mathcal{B}}{\partial \psi}(\psi^1) \frac{\partial \mathcal{A}}{\partial \phi}(\phi^n) \right]^* = \left[ \frac{\partial \mathcal{A}}{\partial \phi}(\phi^n) \right]^* \left[ \frac{\partial \mathcal{B}}{\partial \psi}(\psi^1) \right]^* \cdots \left[ \frac{\partial \mathcal{B}}{\partial \psi}(\psi^m) \right]^*. \quad (54)$$

Because of the linearity of the Jacobian matrix, the adjoint of the operator is equivalent to its transpose. In practical cases, the number  $m$  is fixed in each iteration, which represents the duration of the reinitialization. For the sake of simplicity, we will not take this procedure into account in this paper.

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TABLE I: ASSIMILATION RESULTS OF INITIAL CONDITIONS

Iterations	$RMSE_{\phi(0)}$	$\ \nabla J(\phi(0))\ $	$J(\phi(0))$
<b>70</b>	<b>3.5748e-02</b>	<b>1.3687e-01</b>	<b>1.0920</b>

TABLE II: ASSIMILATION RESULTS OF COEFFICIENTS

Iterations	$RMSE_{\sigma}$	$RMSE_a$	$\ \nabla J(\sigma)\ $	$\ \nabla J(a)\ $	$J(a, \sigma)$
30	1.7118e-02	9.5625e-02	6.9408e-04	3.9667e-03	3.5445e-01
60	4.9719e-03	2.7232e-02	1.9990e-04	1.1045e-03	2.2405e-01
<b>120</b>	<b>4.2778e-04</b>	<b>2.3226e-03</b>	<b>1.7104e-05</b>	<b>9.2939e-05</b>	<b>2.1261e-01</b>

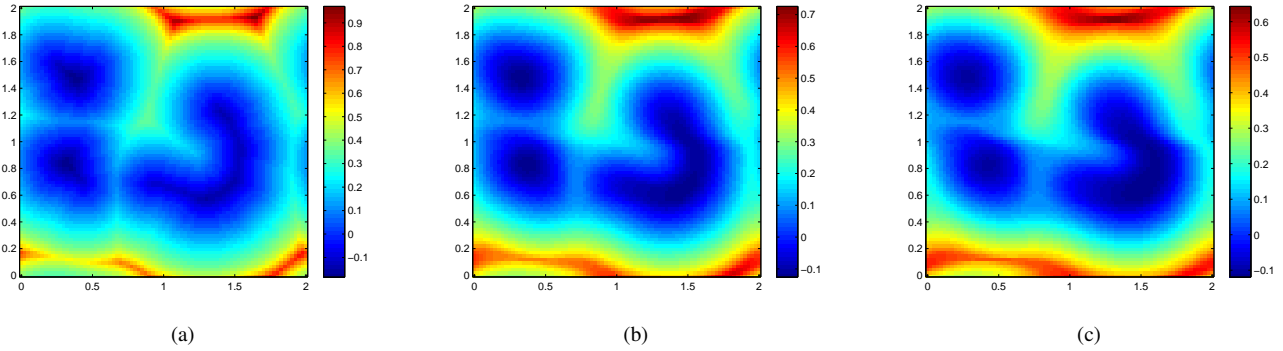


Fig. 2: Level set function  $\phi(t)$  at three moments produced by forward model. (a) Initial moment  $t = 0$ . (b) After 100 time steps. (c) Final moment  $t = 0.1s$ .

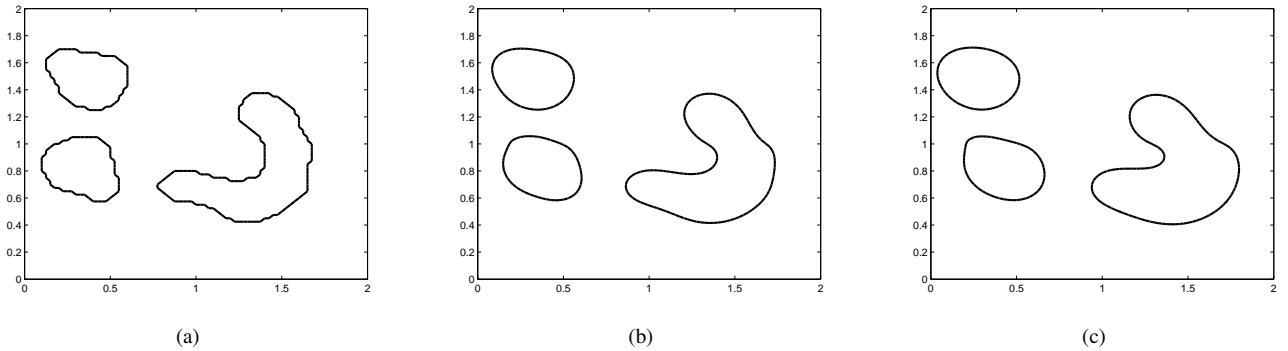


Fig. 3: Zero level set of  $\phi(t)$  at three moments produced by forward model. (a) Initial moment  $t = 0$ . (b) After 100 time steps. (c) Final moment  $t = 0.1s$ .



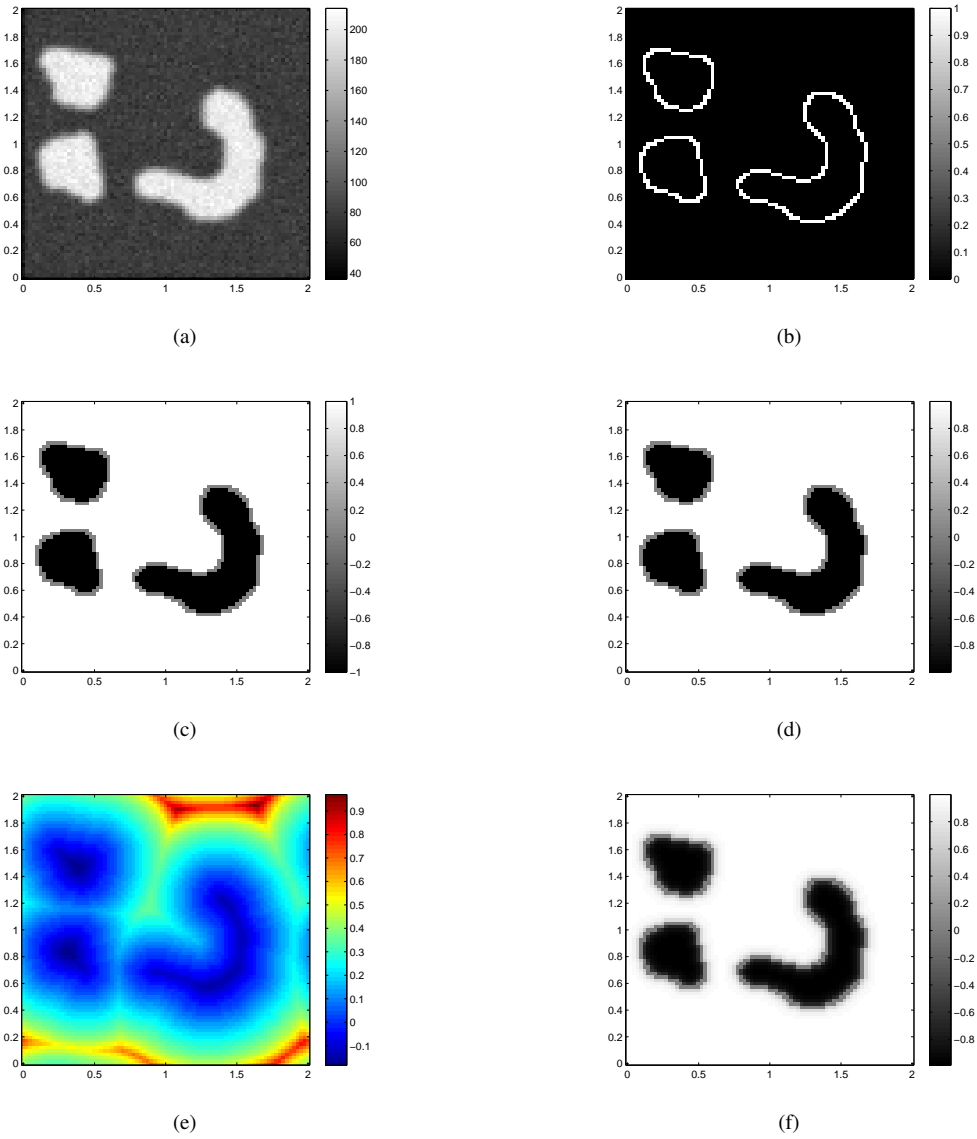


Fig. 4: (a) Synthetic observation image  $\phi_{obs}$ . (b) Edge detection result  $\mathcal{W}(\phi_{obs})$ . (c) Structure image. (d) Modified structure image  $\varphi^S = \mathcal{H}_{\mathcal{O} \rightarrow \mathcal{S}}(\phi_{obs})$ . (e) Signed distance function image  $\phi$ . (f) Modified structure image  $\varphi^S = \mathcal{H}_{\phi \rightarrow \mathcal{S}}(\phi)$ .

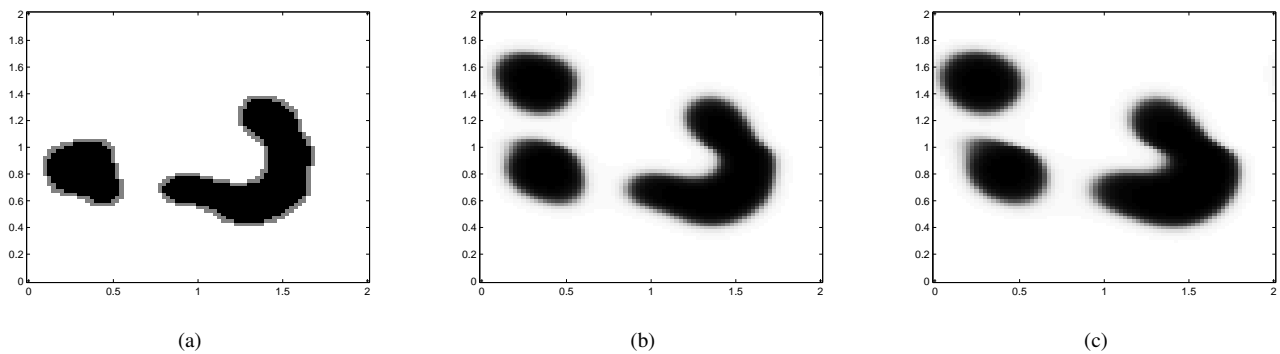


Fig. 5: Portions of the sequences of structure images  $\phi^S(t)$  including a missing image at  $t = 0$ . (a) Initial moment  $t = 0$ . (b)  $t = 0.05s$ . (c) Final moment  $t = 0.1s$ .

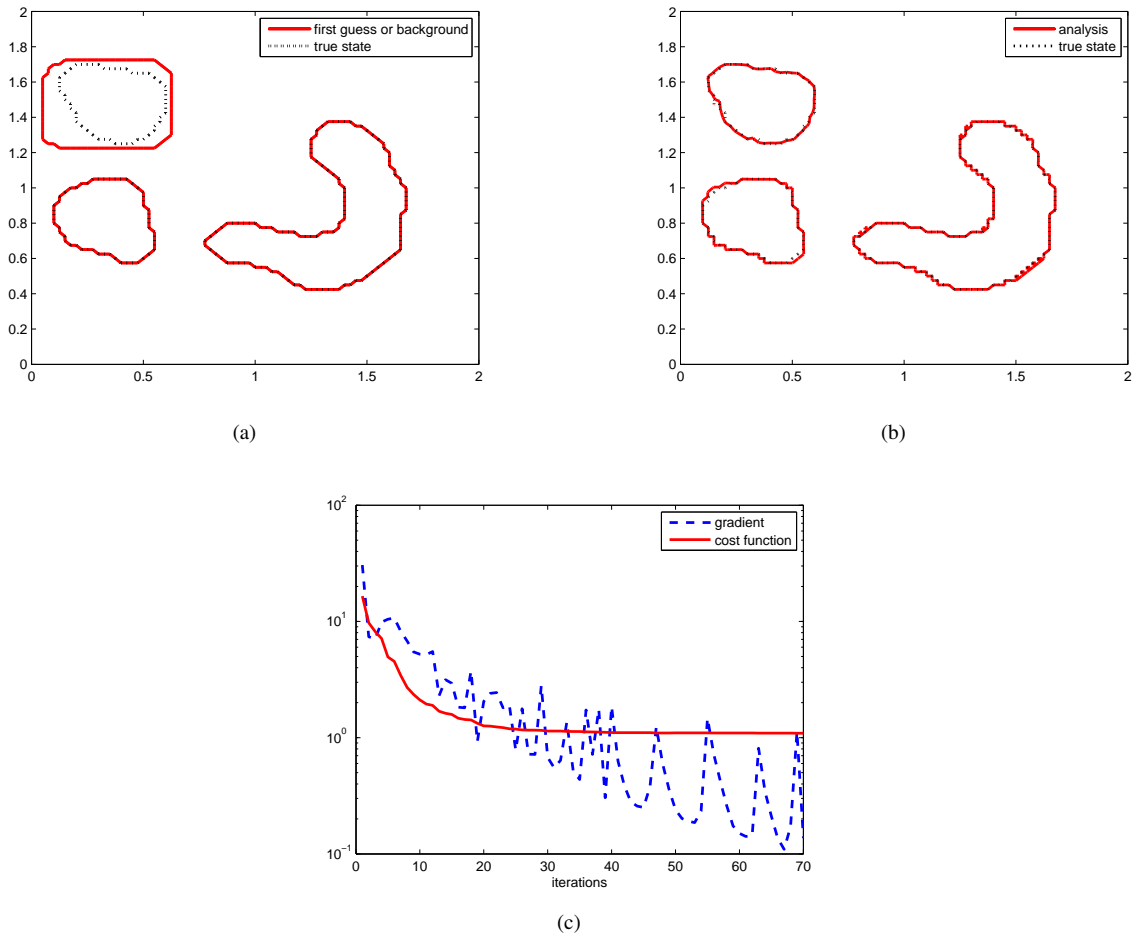


Fig. 6: Assimilation results of initial conditions  $\phi(0)$ . (a) Comparison before assimilation. (b) Comparison after assimilation. (c) Evolution of the  $l_2$  norm of the gradient and its cost function.

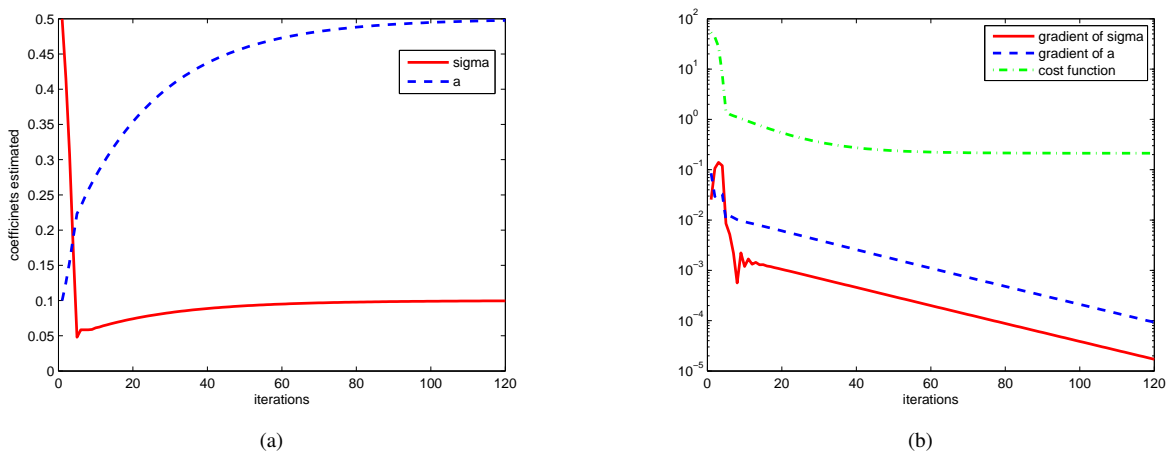


Fig. 7: Assimilation results of coefficients. (a) Evolution of coefficients  $\sigma$ ,  $a$  estimated with the iterations ( $\sigma^t = 0.1$ ,  $a^t = 0.5$ ). (b) Evolution of the  $l_2$  norm of the gradients and cost function.

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