



Finding Optimal Formulae for Bilinear Maps

Razvan Barbulescu, Jérémie Detrey, Nicolas Estibals, Paul Zimmermann

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Arén^H^H^HRIC seminar, LIP — March 22, 2012

Finding Optimal Formulae for Bilinear Maps

(slides courtesy of Nicolas Estibals)

Jérémy Detrey

CAMEL project-team, LORIA
INRIA Nancy – Grand Est
Jeremie.Detrey@loria.fr

Joint work with:

Răzvan Bărbulescu

Nicolas Estibals

Paul Zimmermann



A bit of history

- ▶ Multiplication is an **expensive** arithmetic operation
- ▶ Well-studied problem
 - **Karatsuba** (1962)
 - Toom–Cook (1963), **evaluation-interpolation** schemes
 - **CRT**-based algorithms
 - **Schönhage–Strassen** algorithm (1971)
 - ...
- ▶ *Five-, six-, and seven-term Karatsuba-like formulae*, P. Montgomery (2005)
 - **ad-hoc** formulae
 - **exhaustive search** for five-term multiplication
 - **non-exhaustive search** for six- and seven-term multiplications
 - (January 2011) start a task group to reproduce this work

Outline of the talk

- ▶ Formulae for polynomial multiplication
- ▶ Enumerating formulae
- ▶ Further improvements and heuristics
- ▶ Some results

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Example: a 2-term polynomial times a 3-term one

► We want to compute

$$\begin{aligned} C(X) &= (a_1 \cdot X + a_0) \times (b_2 \cdot X^2 + b_1 \cdot X + b_0) \\ &= a_1 b_2 \cdot X^3 + (a_1 b_1 + a_0 b_2) \cdot X^2 + (a_0 b_1 + a_1 b_0) \cdot X + a_0 b_0 \end{aligned}$$

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- ▶ Only 5 products required instead of 6
 - use Karatsuba's trick

$$C(X) = a_1 b_2 \cdot X^3 + (a_1 b_1 + a_0 b_2) \cdot X^2 + ((a_0 + a_1)(b_0 + b_1) - a_1 b_1 - a_0 b_0) \cdot X + a_0 b_0$$

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- compute the products

$$\begin{aligned}g_0 &= a_0 \cdot b_0, \\ g_1 &= a_0 \cdot b_2, \\ g_2 &= a_1 \cdot b_1, \\ g_3 &= a_1 \cdot b_2, \text{ and} \\ g_4 &= (a_0 + a_1) \cdot (b_0 + b_1).\end{aligned}$$

- reconstruct the result

$$C(X) = g_3 \cdot X^3 + (g_1 + g_2) \cdot X^2 + (g_4 - g_2 - g_0) \cdot X + g_0$$

General form of a multiplication formula

- ▶ We want to compute, over a given field K (or any K -algebra \mathbf{K}),

$$(a_{n-1} \cdot X^{n-1} + \cdots + a_0) \times (b_{m-1} \cdot X^{m-1} + \cdots + b_0) = c_{n+m-2} \cdot X^{n+m-2} + \cdots + c_0$$

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- compute some linear combinations of the a_i 's

$$a'_j = \sum \alpha_{i,j} \cdot a_i$$

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- ▶ This is also valid for any bilinear map

$$F : \quad K^n \quad \times \quad K^m \quad \longrightarrow \quad K^\ell \\ ((a_0, \dots, a_{n-1}) , (b_0, \dots, b_{m-1})) \longmapsto (c_0, \dots, c_{\ell-1})$$

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$$F : K^n \times K^m \rightarrow K^\ell$$

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- ▶ Represent the products and the coefficients of the result as elements of the nm -dimensional K -vector space

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$$\mathcal{T} = \{c_i\}_{0 \leq i < \ell} \subset V$$

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- ▶ **Generators:** the set $\mathcal{G} \subset V$ of the **potential products** to use in a formula

$$\mathcal{G} = \{(\sum \alpha_i a_i) \cdot (\sum \beta_j b_j) \mid \forall i, \alpha_i \in K \text{ and } \forall j, \beta_j \in K\} \setminus \{0\}$$

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We only consider products modulo a nonzero scalar factor

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where $g \sim g'$ when $\exists \lambda \in K, \lambda \neq 0$ such that $g = \lambda g'$

Example (cont'd)

Consider the previous example: 2×3 -term polynomial product in $\mathbb{F}_2[X]$

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- ▶ The set of generators contains 21 products:

$$\mathcal{G} = \left\{ \begin{array}{lll} a_0 \cdot b_0, & a_1 \cdot b_0, & (a_0 + a_1) \cdot b_0, \\ a_0 \cdot b_1, & a_1 \cdot b_1, & (a_0 + a_1) \cdot b_1, \\ a_0 \cdot (b_0 + b_1), & a_1 \cdot (b_0 + b_1), & (a_0 + a_1) \cdot (b_0 + b_1), \\ a_0 \cdot b_2, & a_1 \cdot b_2, & (a_0 + a_1) \cdot b_2, \\ a_0 \cdot (b_0 + b_2), & a_1 \cdot (b_0 + b_2), & (a_0 + a_1) \cdot (b_0 + b_2), \\ a_0 \cdot (b_1 + b_2), & a_1 \cdot (b_1 + b_2), & (a_0 + a_1) \cdot (b_1 + b_2), \\ a_0 \cdot (b_0 + b_1 + b_2), & a_1 \cdot (b_0 + b_1 + b_2), & (a_0 + a_1) \cdot (b_0 + b_1 + b_2) \end{array} \right\}$$

Naive algorithm

- ▶ Goal: find the optimal formulae (*i.e.*, with a minimum number of products)
 - enumerate the subsets $\mathcal{W} \subset \mathcal{G}$ of exactly k products which yield a valid formula
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 - \mathcal{W} is a set of k generators:

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 - enumerate the $\binom{\#\mathcal{G}}{k}$ subsets \mathcal{W} of size k
 - for each subset, test whether it generates \mathcal{T} or not

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- ▶ Drawback: Distinct subsets may span the same subspace

Improved algorithm: subspaces instead of subsets

- ▶ Look instead for subspaces W of V such that
 - W can be generated by products only: $\text{Span}(W \cap \mathcal{G}) = W$
 - only k products are required: $\dim W = k$
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- ▶ **Algorithm:**
 - 1: **procedure** expand_subspace(W)
 - 2: **if** $\dim W = k$ **and** $T \subset W$ **then**
 - 3: W is a solution
 - 4: **else if** $\dim W < k$ **then**
 - 5: **for each** $g \in \mathcal{G} \setminus W$ **do**
 - 6: expand_subspace($W \oplus \text{Span}(g)$)
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- ▶ **Several formulae** can correspond to a **single solution subspace** W
 - each **basis of** W comprising only **elements of** \mathcal{G} gives a **k -product formula**

Improved algorithm: incomplete basis

- ▶ We already know part of W !
 - target space T is a subspace of every solution space W
 - find each W by constructing $\mathcal{W}' \subset \mathcal{G}$ such that $W = T \oplus \text{Span } \mathcal{W}'$

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 - target space T is a *subspace* of *every* solution space W
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- ▶ **Modified algorithm:**

```
1: procedure expand_subspace( $W$ )
2:   if  $\dim W = k$  and  $\text{rk}(W \cap \mathcal{G}) = k$  then
3:      $W$  is a solution
4:   else if  $\dim W < k$  then
5:     for each  $g \in \mathcal{G} \setminus W$  do
6:       expand_subspace( $W \oplus \text{Span}(g)$ )
7:   end procedure
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- ▶ Complexity now depends on

$$\binom{\#\mathcal{G}}{k - \text{rk } \mathcal{T}}$$

Example (cont'd)

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 - there are **3 such solution spaces** which yield a total of **162 formulae**

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Outline of the talk

- ▶ Formulae for polynomial multiplication
- ▶ Enumerating formulae
- ▶ Further improvements and heuristics
- ▶ Some results

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- ▶ Going further: instead of a single $\sigma \in \text{Aut}(V)$, take a whole subgroup $S \subset \text{Aut}(V)$

Orbits of a group action: Example

2×3 -term polynomial product in $\mathbb{F}_2[X]$

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2×3 -term polynomial product in $\mathbb{F}_2[X]$

- \mathcal{G} and T are fixed by $S = \langle \sigma_0, \sigma_1 \rangle \subset \text{Aut}(V)$ with
- $\sigma_0 : a_i b_j \mapsto a_{1-i} b_{2-j}$ (i.e., replace X by $1/X$)
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 - 3 orbits of size 3:

$$\begin{aligned}\overline{g_0} &= \{ a_0 \cdot b_0, & a_1 \cdot b_2, & (a_0 + a_1) \cdot (b_0 + b_1 + b_2) \}, \\ \overline{g_1} &= \{ a_0 \cdot b_1, & a_1 \cdot b_1, & (a_0 + a_1) \cdot b_1 \}, \\ \overline{g_2} &= \{ a_0 \cdot (b_0 + b_1), & a_1 \cdot (b_1 + b_2), & (a_0 + a_1) \cdot (b_0 + b_2) \}, \text{ and}\end{aligned}$$

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2×3 -term polynomial product in $\mathbb{F}_2[X]$

- ▶ First, consider adding to T a generator from orbit $\overline{g_1}$:

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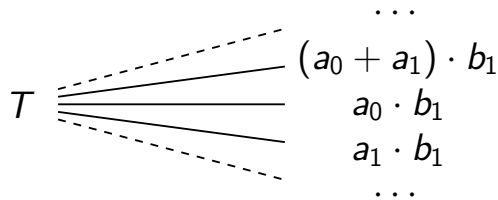
T

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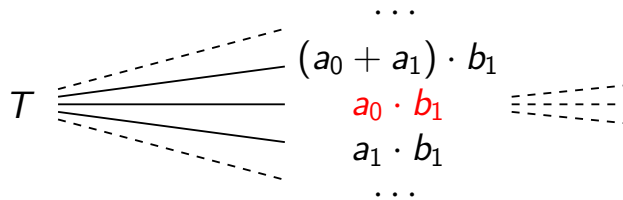


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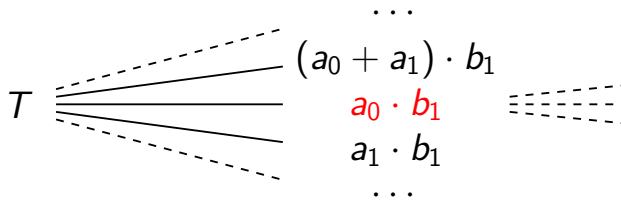
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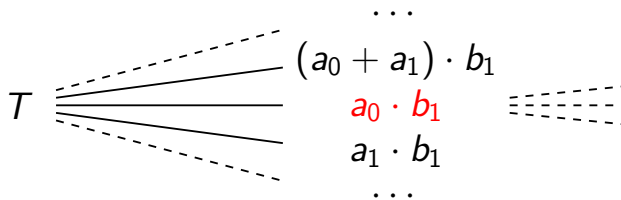
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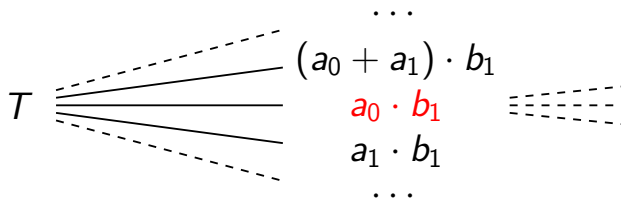
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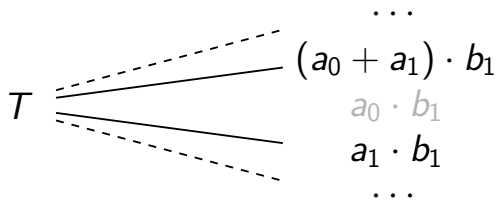
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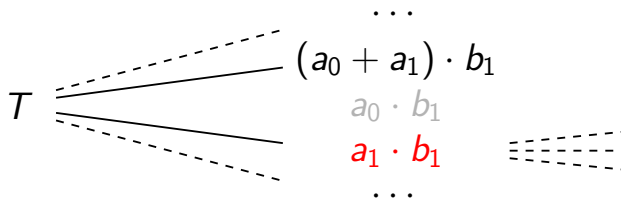
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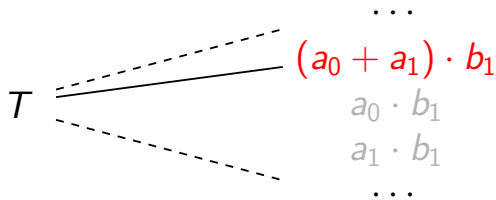
- ▶ For any solution of the form $W = T \oplus \text{Span}(a_0 \cdot b_1) \oplus \text{Span } \mathcal{W}'$, with $\mathcal{W}' \subset \mathcal{G}$:
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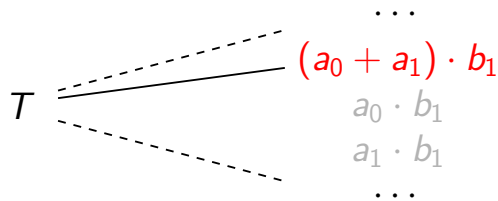


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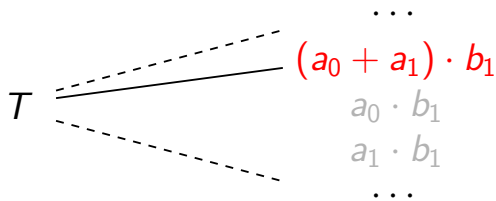


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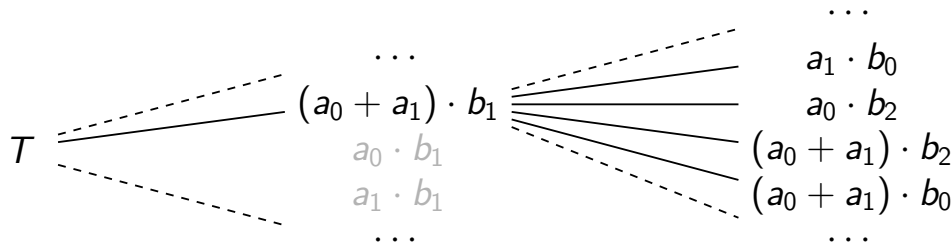


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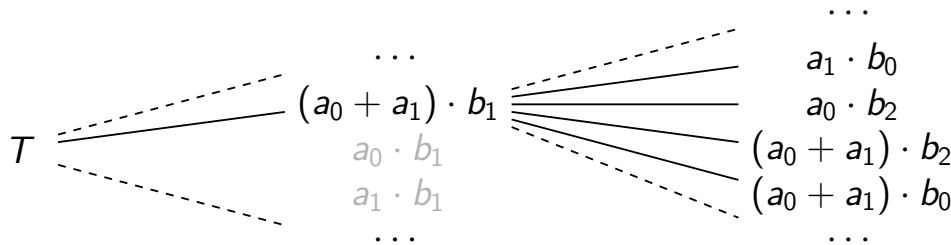
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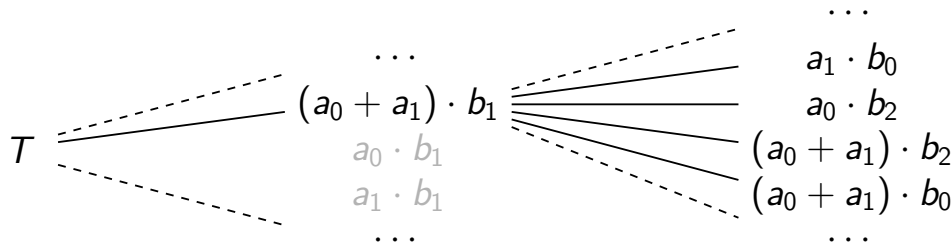
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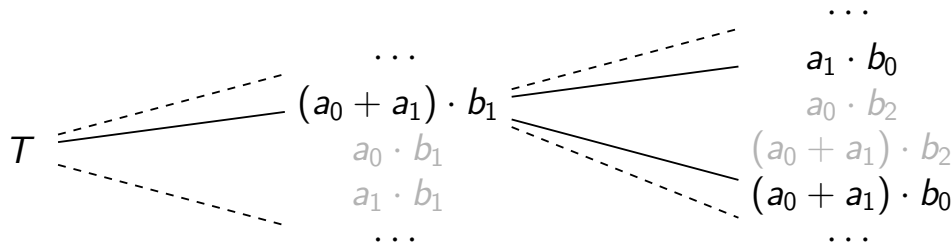
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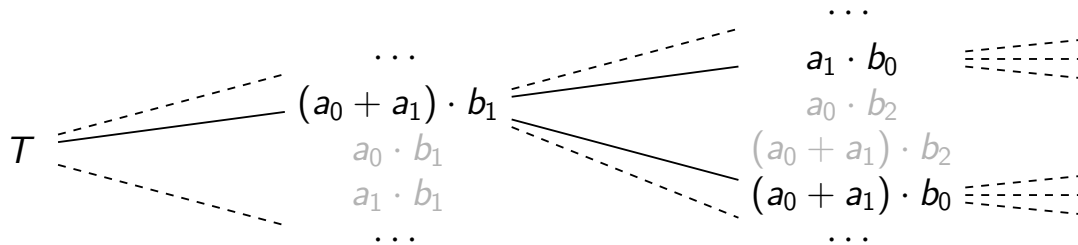
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Outline of the talk

- ▶ Formulae for polynomial multiplication
- ▶ Enumerating formulae
- ▶ Further improvements and heuristics
- ▶ **Some results**

$n \times m$ -term polynomial multiplication

Ring	$n \times m$	$\#\mathcal{G}$	k	# of tests	# of solutions	# of formulae	Calculation time [s]
$F_2[X]$	2×2	9	3	1	1	1	0.00
	3×3	49	6	9	3	9	0.00
	4×4	225	9	$6.60 \cdot 10^3$	4	4	0.03
	5×5	961	13	$9.65 \cdot 10^9$	27	27	$2.28 \cdot 10^5$
	6×6	3969	14	$4.37 \cdot 10^9$	—	—	$6.03 \cdot 10^5$
		(Sym.) 63	17	$8.08 \cdot 10^6$	6	54	17.7
	7×7	(Sym.) 127	22	$3.38 \cdot 10^{12}$	2 618	19 550	$1.59 \cdot 10^7$
$F_3[X]$	2×2	16	3	1	1	4	0.00
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	5×5	14 641	11	$4.89 \cdot 10^7$	—	—	$4.02 \cdot 10^4$
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$F_2[X]$	2×2	9	3	1	1	1	0.00
	3×3	49	6	9	3	9	0.00
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	5×5	961	13	$9.65 \cdot 10^9$	27	27	$2.28 \cdot 10^5$
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Multiplication in small finite-field extensions

Finite field	$\#\mathcal{G}$	k	# of tests	# of solutions	# of formulae	Calculation time [s]
\mathbf{F}_{2^2}	9	3	3	3	3	0.00
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\mathbf{F}_{2^4}	225	9	$2.57 \cdot 10^9$	2025	2025	$1.13 \cdot 10^4$
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	(Sym.) 31	13	$3.49 \cdot 10^6$	2015	2015	6.24
\mathbf{F}_{2^6}	(Sym.) 63	15	$2.21 \cdot 10^{10}$	21	21	$6.63 \cdot 10^4$
\mathbf{F}_{2^7}	(Sym.) 127	15	$1.34 \cdot 10^{12}$	—	—	$6.17 \cdot 10^6$
\mathbf{F}_{3^2}	16	3	3	4	16	0.00
\mathbf{F}_{3^3}	169	6	$2.42 \cdot 10^5$	11 843	105 963	1.08
\mathbf{F}_{3^4}	1 600	8	$2.27 \cdot 10^{11}$	—	—	$1.08 \cdot 10^7$
	(Sym.) 40	9	$1.10 \cdot 10^5$	234	615 240	0.45
\mathbf{F}_{3^5}	(Sym.) 121	11	$2.66 \cdot 10^9$	121	121	$1.45 \cdot 10^4$
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- ▶ Work in progress and perspectives
 - **Lifting formulae** for higher-characteristic or characteristic-0 fields
 - Find formulae for **your** bilinear application!

Thank you for your attention

Questions?

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- machine-specific bound on N : ≤ 64 , ≤ 128 (SSE2), or ≤ 256 (AVX)

Representing subspaces

- ▶ Subspaces $W \subset V$ represented using matrices in row echelon form

$$W = \text{RowSpace} \left(\begin{array}{cccccccccc} & & & & \overbrace{\hspace{10em}}^{\dim V = N \text{ columns}} & & & & & & \\ 0 & w_{1,j_1} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \\ 0 & 0 & 0 & w_{2,j_2} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \\ \vdots & \vdots & \vdots & 0 & 0 & \ddots & & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & w_{r,j_r} & \cdots & & \end{array} \right) \left. \vphantom{\begin{array}{c} \\ \\ \\ \\ \end{array}} \right\} \dim W = r \text{ rows}$$

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 - ☹ waste of memory
 - 😊 easier to insert extra rows when expanding W

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- ▶ Test if $v \in W$:
 - $v \in W$ if and only if $\text{reduce}(v, W) = 0$

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 - 4: **if** $w_{i,i} \neq 0$ **then** $v \leftarrow v - (v_i/w_{i,i})w_i$
 - 5: **else return** v
 - 6: **end for**
- ▶ Test if $v \in W$:
 - $v \in W$ if and only if $\text{reduce}(v, W) = 0$
 - ▶ Construct $W \oplus \text{Span}(g)$, for $g \in \mathcal{G} \setminus W$:
 - compute $\tilde{g} \leftarrow \text{reduce}(g, W)$
 - let i be the iteration at which the algorithm returns \tilde{g} ($\neq 0$)
 - then $w_i \leftarrow \tilde{g}$

Reduction by a subspace

- ▶ Given a vector $v \in V$ and a subspace W :
 - sequentially cancel the contribution of pivot rows of W to v
 - stop when no pivot corresponds to the leading coefficient of v
- 1: **for** $i \leftarrow 1$ **to** N **do**
- 2: **if** $v = 0$ **then return** 0
- 3: **if** $v_i \neq 0$ **then**
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 - let i be the iteration at which the algorithm returns \tilde{g} ($\neq 0$)
 - then $w_i \leftarrow \tilde{g}$
- ▶ Improvement: since W is constructed incrementally
 - keep the vectors of \mathcal{G} reduced by W at all times
 - when expanding W by $\text{Span}(g)$, reduce \mathcal{G} by g only

Computing $W \cap \mathcal{G}$

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- ▶ Idea (courtesy of E. Thomé): put matrix W in reduced row echelon form

$$W = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & w_{j_1, j_1} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & w_{j_2, j_2} & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & w_{j_r, j_r} & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

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- batch 64, 128, or 256 products $g^T \cdot (W' - \text{Id})$