



## Finding Optimal Formulae for Bilinear Maps

Razvan Barbulescu, Jérémie Detrey, Nicolas Estibals, Paul Zimmermann

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Arén<sup>^</sup>H<sup>^</sup>H<sup>^</sup>HRIC seminar, LIP — March 22, 2012

# Finding Optimal Formulae for Bilinear Maps

(slides courtesy of Nicolas Estibals)

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Nicolas Estibals

Paul Zimmermann



# A bit of history

- ▶ Multiplication is an **expensive** arithmetic operation
- ▶ Well-studied problem
  - **Karatsuba** (1962)
  - Toom–Cook (1963), **evaluation-interpolation** schemes
  - **CRT**-based algorithms
  - **Schönhage–Strassen** algorithm (1971)
  - ...
- ▶ *Five-, six-, and seven-term Karatsuba-like formulae*, P. Montgomery (2005)
  - **ad-hoc** formulae
  - **exhaustive search** for five-term multiplication
  - **non-exhaustive search** for six- and seven-term multiplications
  - (January 2011) start a task group to reproduce this work

# Outline of the talk

- ▶ Formulae for polynomial multiplication
- ▶ Enumerating formulae
- ▶ Further improvements and heuristics
- ▶ Some results

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# Example: a 2-term polynomial times a 3-term one

► We want to compute

$$\begin{aligned} C(X) &= (a_1 \cdot X + a_0) \times (b_2 \cdot X^2 + b_1 \cdot X + b_0) \\ &= a_1 b_2 \cdot X^3 + (a_1 b_1 + a_0 b_2) \cdot X^2 + (a_0 b_1 + a_1 b_0) \cdot X + a_0 b_0 \end{aligned}$$

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- ▶ Only 5 products required instead of 6
  - use Karatsuba's trick

$$C(X) = a_1 b_2 \cdot X^3 + (a_1 b_1 + a_0 b_2) \cdot X^2 + ((a_0 + a_1)(b_0 + b_1) - a_1 b_1 - a_0 b_0) \cdot X + a_0 b_0$$

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- compute the products

$$\begin{aligned}g_0 &= a_0 \cdot b_0, \\ g_1 &= a_0 \cdot b_2, \\ g_2 &= a_1 \cdot b_1, \\ g_3 &= a_1 \cdot b_2, \text{ and} \\ g_4 &= (a_0 + a_1) \cdot (b_0 + b_1).\end{aligned}$$

- reconstruct the result

$$C(X) = g_3 \cdot X^3 + (g_1 + g_2) \cdot X^2 + (g_4 - g_2 - g_0) \cdot X + g_0$$



# General form of a multiplication formula

- ▶ We want to compute, over a given field  $K$  (or any  $K$ -algebra  $\mathbf{K}$ ),

$$(a_{n-1} \cdot X^{n-1} + \cdots + a_0) \times (b_{m-1} \cdot X^{m-1} + \cdots + b_0) = c_{n+m-2} \cdot X^{n+m-2} + \cdots + c_0$$

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- ▶ This is also valid for any bilinear map

$$F : \quad K^n \quad \times \quad K^m \quad \longrightarrow \quad K^\ell \\ ((a_0, \dots, a_{n-1}) , (b_0, \dots, b_{m-1})) \longmapsto (c_0, \dots, c_{\ell-1})$$



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$$\mathcal{G} = \{(\sum \alpha_i a_i) \cdot (\sum \beta_j b_j) \mid \forall i, \alpha_i \in K \text{ and } \forall j, \beta_j \in K\} \setminus \{0\}$$

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We only consider products modulo a nonzero scalar factor

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where  $g \sim g'$  when  $\exists \lambda \in K, \lambda \neq 0$  such that  $g = \lambda g'$

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- ▶ The set of generators contains 21 products:

$$\mathcal{G} = \left\{ \begin{array}{lll} a_0 \cdot b_0, & a_1 \cdot b_0, & (a_0 + a_1) \cdot b_0, \\ a_0 \cdot b_1, & a_1 \cdot b_1, & (a_0 + a_1) \cdot b_1, \\ a_0 \cdot (b_0 + b_1), & a_1 \cdot (b_0 + b_1), & (a_0 + a_1) \cdot (b_0 + b_1), \\ a_0 \cdot b_2, & a_1 \cdot b_2, & (a_0 + a_1) \cdot b_2, \\ a_0 \cdot (b_0 + b_2), & a_1 \cdot (b_0 + b_2), & (a_0 + a_1) \cdot (b_0 + b_2), \\ a_0 \cdot (b_1 + b_2), & a_1 \cdot (b_1 + b_2), & (a_0 + a_1) \cdot (b_1 + b_2), \\ a_0 \cdot (b_0 + b_1 + b_2), & a_1 \cdot (b_0 + b_1 + b_2), & (a_0 + a_1) \cdot (b_0 + b_1 + b_2) \end{array} \right\}$$

# Naive algorithm

- ▶ Goal: find the optimal formulae (*i.e.*, with a minimum number of products)
  - enumerate the subsets  $\mathcal{W} \subset \mathcal{G}$  of exactly  $k$  products which yield a valid formula
  - starting with  $k = \text{rk}(\mathcal{T})$ , increase  $k$  until a solution is found

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  - restrict to a finite subset of the generators  $\mathcal{G}$  (but search not exhaustive)

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  - restrict to a finite subset of the generators  $\mathcal{G}$  (but search not exhaustive)
- ▶ Drawback: Distinct subsets may span the same subspace

# Improved algorithm: subspaces instead of subsets

- ▶ Look instead for subspaces  $W$  of  $V$  such that
  - $W$  can be generated by products only:  $\text{Span}(W \cap \mathcal{G}) = W$
  - only  $k$  products are required:  $\dim W = k$
  - $W$  contains the target space spanned by the target vectors:  $T = \text{Span } \mathcal{T} \subset W$



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- ▶ **Algorithm:**
  - 1: **procedure** expand\_subspace( $W$ )
  - 2:   **if**  $\dim W = k$  **and**  $T \subset W$  **then**
  - 3:      $W$  is a solution
  - 4:   **else if**  $\dim W < k$  **then**
  - 5:     **for each**  $g \in \mathcal{G} \setminus W$  **do**
  - 6:       expand\_subspace( $W \oplus \text{Span}(g)$ )
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- ▶ **Several formulae** can correspond to a **single solution subspace**  $W$ 
  - each **basis of**  $W$  comprising only **elements of**  $\mathcal{G}$  gives a  $k$ -**product formula**

# Improved algorithm: incomplete basis

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  - target space  $T$  is a subspace of every solution space  $W$
  - find each  $W$  by constructing  $\mathcal{W}' \subset \mathcal{G}$  such that  $W = T \oplus \text{Span } \mathcal{W}'$

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  - target space  $T$  is a subspace of every solution space  $W$
  - find each  $W$  by constructing  $\mathcal{W}' \subset \mathcal{G}$  such that  $W = T \oplus \text{Span } \mathcal{W}'$
  
- ▶ Modified algorithm:
  - 1: **procedure** expand\_subspace( $W$ )
  - 2:     **if**  $\dim W = k$  ~~and  $T \in W$~~  **then**
  - 3:          $W$  is a solution
  - 4:     **else if**  $\dim W < k$  **then**
  - 5:         **for each**  $g \in \mathcal{G} \setminus W$  **do**
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- ▶ Complexity now depends on

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  - there are **3 such solution spaces** which yield a total of **162 formulae**



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- example: **squaring of a 2-term polynomial** in  $\mathbb{F}_3[X]$

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# Outline of the talk

- ▶ Formulae for polynomial multiplication
- ▶ Enumerating formulae
- ▶ Further improvements and heuristics
- ▶ Some results

# Symmetric bilinear maps

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  - same number of  $a_i$ 's and  $b_j$ 's:  $F : K^n \times K^n \rightarrow K^\ell$
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- ▶ Idea: group generators  $g \in \mathcal{G}$  according to their  $\sigma$ -orbit

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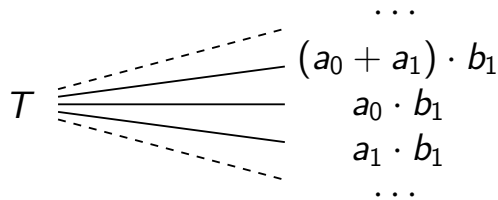
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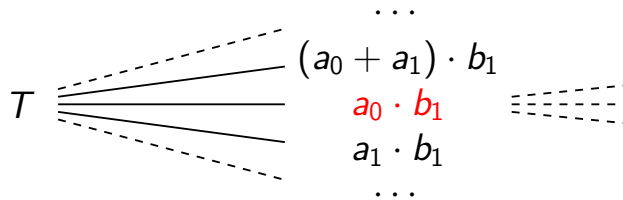


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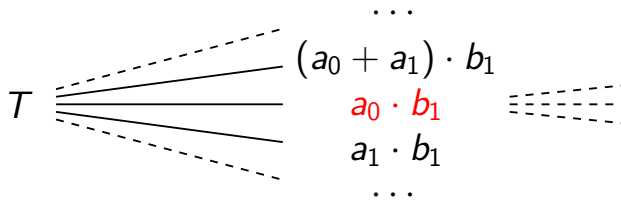
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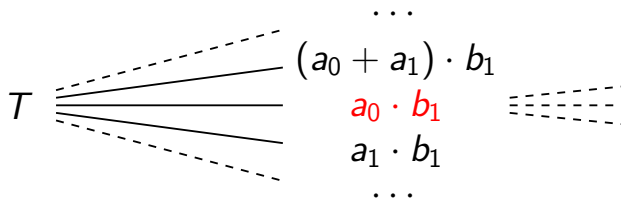
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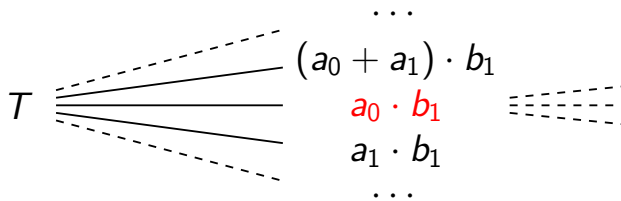
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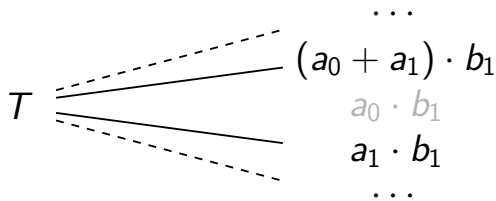
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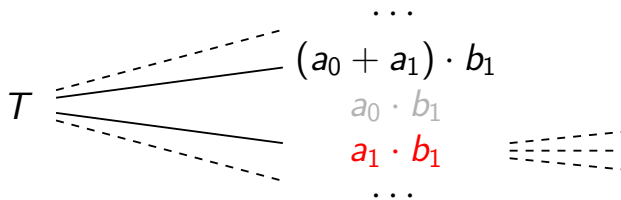
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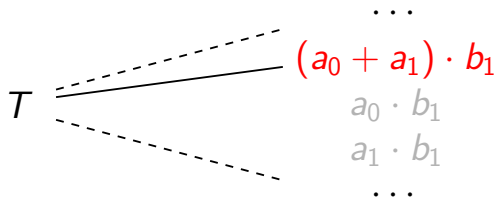
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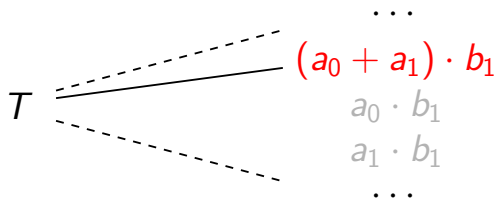


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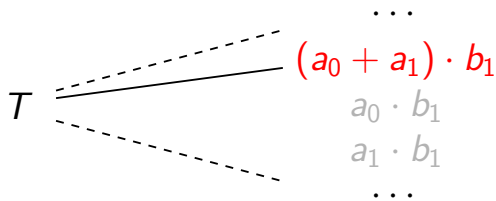


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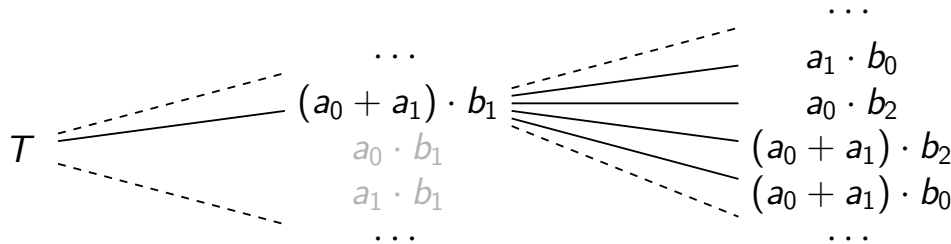


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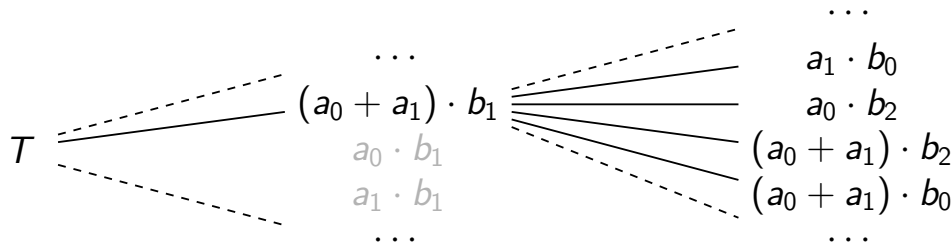
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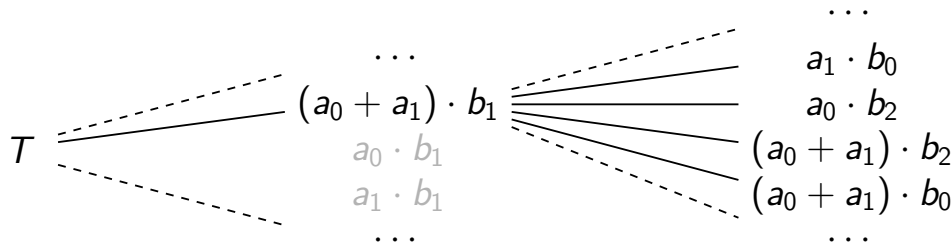
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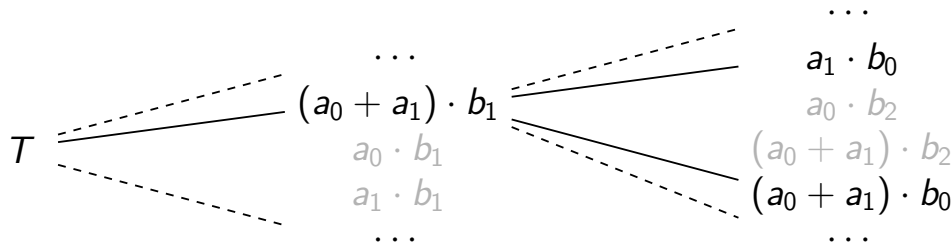
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$2 \times 3$ -term polynomial product in  $\mathbb{F}_2[X]$

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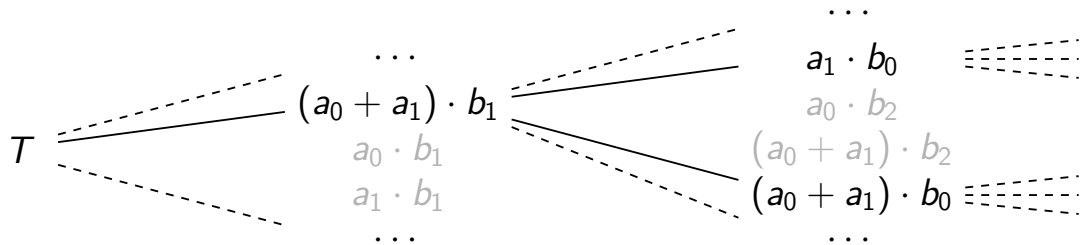
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# Outline of the talk

- ▶ Formulae for polynomial multiplication
- ▶ Enumerating formulae
- ▶ Further improvements and heuristics
- ▶ **Some results**

# $n \times m$ -term polynomial multiplication

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$F_2[X]$	$2 \times 2$	9	3	1	1	1	0.00
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	$4 \times 4$	225	9	$6.60 \cdot 10^3$	4	4	0.03
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	$4 \times 4$	1 600	9	$4.11 \cdot 10^5$	726	50 640	14.9
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	$6 \times 6$	(Sym.) 364	15	$2.37 \cdot 10^8$	4	1 024	$3.79 \cdot 10^3$
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# Multiplication in small finite-field extensions

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$\mathbf{F}_{2^4}$	225	9	$2.57 \cdot 10^9$	2025	2025	$1.13 \cdot 10^4$
$\mathbf{F}_{2^5}$	961	9	$3.10 \cdot 10^{10}$	—	—	$8.11 \cdot 10^5$
	(Sym.) 31	13	$3.49 \cdot 10^6$	2015	2015	6.24
$\mathbf{F}_{2^6}$	(Sym.) 63	15	$2.21 \cdot 10^{10}$	21	21	$6.63 \cdot 10^4$
$\mathbf{F}_{2^7}$	(Sym.) 127	15	$1.34 \cdot 10^{12}$	—	—	$6.17 \cdot 10^6$
$\mathbf{F}_{3^2}$	16	3	3	4	16	0.00
$\mathbf{F}_{3^3}$	169	6	$2.42 \cdot 10^5$	11 843	105 963	1.08
$\mathbf{F}_{3^4}$	1 600	8	$2.27 \cdot 10^{11}$	—	—	$1.08 \cdot 10^7$
	(Sym.) 40	9	$1.10 \cdot 10^5$	234	615 240	0.45
$\mathbf{F}_{3^5}$	(Sym.) 121	11	$2.66 \cdot 10^9$	121	121	$1.45 \cdot 10^4$
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- ▶ Work in progress and perspectives
  - **Lifting formulae** for higher-characteristic or characteristic-0 fields
  - Find formulae for **your** bilinear application!



**Thank you for your attention**

**Questions?**

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- machine-specific bound on  $N$ :  $\leq 64$ ,  $\leq 128$  (SSE2), or  $\leq 256$  (AVX)

# Representing subspaces

- ▶ Subspaces  $W \subset V$  represented using matrices in row echelon form

$$W = \text{RowSpace} \left( \begin{array}{cccccccccc} & & & & \overbrace{\hspace{10em}}^{\dim V = N \text{ columns}} & & & & & & \\ 0 & w_{1,j_1} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \\ 0 & 0 & 0 & w_{2,j_2} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \\ \vdots & \vdots & \vdots & 0 & 0 & \ddots & & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & w_{r,j_r} & \cdots & & \end{array} \right) \left. \vphantom{\begin{array}{c} \\ \\ \\ \\ \end{array}} \right\} \dim W = r \text{ rows}$$

where each pivot  $w_{i,j_i} \neq 0$ , and with  $j_1 < j_2 < \cdots < j_r$

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 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
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  - ☹ waste of memory
  - 😊 easier to insert extra rows when expanding  $W$



# Reduction by a subspace

- ▶ Given a vector  $v \in V$  and a subspace  $W$ :
  - sequentially cancel the contribution of pivot rows of  $W$  to  $v$
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  - 4:         **if**  $w_{i,i} \neq 0$  **then**  $v \leftarrow v - (v_i/w_{i,i})w_i$
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  - compute  $\tilde{g} \leftarrow \text{reduce}(g, W)$
  - let  $i$  be the iteration at which the algorithm returns  $\tilde{g}$  ( $\neq 0$ )
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  - then  $w_i \leftarrow \tilde{g}$
- ▶ Improvement: since  $W$  is constructed incrementally
  - keep the vectors of  $\mathcal{G}$  reduced by  $W$  at all times
  - when expanding  $W$  by  $\text{Span}(g)$ , reduce  $\mathcal{G}$  by  $g$  only

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- $g \in W$  if and only if  $g^T \cdot W' = g^T$
- batch 64, 128, or 256 products  $g^T \cdot (W' - \text{Id})$