Signal differentiation by means of algebraic techniques and sliding mode techniques

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Abstract

The numerical differentiation has been considered an important problem to deal in control, signal processing, applied mathematics, among others. Several techniques based on Luenberger observers, homogeneity, algebraic techniques have been developed considering the problems that can be associated to a real signal. In this manuscript, a qualitative study was made to show the behavior of different differentiation techniques. The differentiation algorithms were based mainly in the algebraic techniques and polynomial approximation. A second approach was selected as the well known homogeneous differentiator. Four different differentiator techniques were studied: the algebraic differentiator, a differentiator based on Legendre polynomials, a differentiator based on Jacobi polynomials and the Homogeneous differentiator. The problems inherit to signal differentiation included in the analysis are the algorithm complexity and algorithm accuracy under noise and quantization.

Contents

1. Introduction

Numerical differentiation (ND) is the computation of values of the derivative of a function f only from some given values of this function. It has been well known that numerical differentiation should be avoided whenever possible because it tends to make matters rough and generally give values of $\frac{d}{dt}f$ much less accurate than those of f. It has been reported that unstable characteristics of numerical differentiation process may lead to large errors in the computed results [Choi, 2009]. Unfortunately, in many applications it is necessary to estimate the derivative of a function given the noisy values of it. Actually, in real-time applications most of errors come from round-off and truncation.

ND is ill-posed in the sense that a small error in measurement data can induce a large error in the approximate derivatives. Therefore, various numerical methods have been developed to obtain stable algorithms more or less sensitive to additive noise. Several results has been obtained from different theories like observer design in the control literature, digital filtering in signal processing and system identification [Dayan, 2011]. According to [Dayan, 2011], most of signal differentiators fall into 8 main categories, that is, the finite difference methods, the Savitzky Golay methods, the wavelet differentiation methods, the Fourier transform methods [Jauberteau and Jauberteau, 2009], the mollification methods, the Tikhonov regularization, the algebraic methods and the differentiation by integration methods. By this kind of problems, an analysis of different differentiation techniques under the presences of noises and with the problem of quantization is made in this manuscript [Levant, 1998], [Liu *et al.*, 2014], [Mboup *et al.*, 2007]. The rest of the manuscript is organized as follow, in the next section, the differentiation techniques are explained, including, the Euler approximation, the homogeneous differentiator, algebraic differentiators and the differentiators based on the weak derivative. Then, in section III, simulation results are presented, the different differentiators are compared in terms of computational aspects (like complexity, memory size among others) and accuracy and robustness in the presence of noisy measurements and quantization. Finally in section IV, some conclusions are given.

2. Design of differentiators

The problem of numerical differentiation can be summarized as:

Problem 1 *Given a signal* $y(t)$*, to estimate* $\dot{y}(t)$ *with* $t \in \mathbb{R}$*, using the information on the signal* $y(s)$ *for* $s \in [t-T, t]$*, where* T *is the size of the window.*

- The information on $y(t)$ is obtained on-line in a discrete time, i.e. $y(t_i)$ is known for $t_i = ih$, $i = 1, 2, 3...$ and $h > 0$ is the sampled period.
- The signal $y(t)$ is corrupted by measurements noises whose statistics are unknown
- The model of the system that generates the output signal $y(t)$ is unavailable.

2.1 Euler Differentiation

The simplest method for obtaining the first order time derivative of a signal $y(t)$, is the so called finite difference method. It consist in the following approximation to the derivative $\dot{y}(t)$ in the time instant t_i

$$\dot{y}_e(t_i) = \frac{y(t) - y(t_i)}{t - t_i}$$

This method is quite sensitive to the presence of noise perturbations in the signal to be processed.

2.2 Homogeneous differentiator

For obtaining the first derivative of a signal, the Homogeneous differentiator[Perruquetti and [Perruquetti *et al.*, 2008], [Polyakov *et al.*, 2014] is described by the following equation

$$\frac{d\hat{x}_1}{dt} = k_1 \left|\hat{x}_1 - y\right|^{1-\frac{\mu}{1+\mu}} sign\left(\hat{x}_1 - y\right) + \hat{x}_2$$
$$\frac{d\hat{x}_2}{dt} = k_2 \left|\hat{x}_1 - y\right|^{1-\frac{2\mu}{1+\mu}} sign\left(\hat{x}_1 - y\right)$$

The well known linear high-gain differentiator and the exact high order sliding mode differentiator [Levant, 1998] can be obtained as a partial cases of the homogeneous differentiator, when the homogeneity degree tends to 0 and -1 respectively.

2.3 Algebraic Signal differentiation (Jacobi approximation)

The algebraic differentiator [Mboup *et al.*, 2007], [Liu *et al.*, 2014] involving Jacobi Polynomials and algebraic methods is describred as follows. Consider a noise-corrupted measurement $x^{\varpi} : I \rightarrow \mathbb{R}$, $x^{\varpi}(t) = x(t) + \varpi(t)$, where I is a finite time open interval of $\mathbb{R}^{+}$, $x \in C^{n}(I)$ with $n \in \mathbb{N}$, and ϖ is an additive corrupting noise. The objective is to estimate the n^{th} derivative of x using x^{ϖ}. In the following, a class of algebraic differentiator involving Jacobi polynomials is introduced.

For any $t_o \in I$, $D_{t_0} := \left\{t \in \mathbb{R}_{+}^{*}; t_0 - t \in I\right\}$, x can be locally given on $[t_o - T, t_o]$ with $T \in D_{t_0}$ by the following Jacobi orthogonal series expansion

$$D_{\kappa,\mu,T,q}x\left(t_0 - T\xi\right) = \sum_{i=0}^{q} \frac{\left\langle P_i^{(\mu+1,\kappa+1)}(\cdot), \dot{x}(t_0 - T\cdot)\right\rangle_{\mu+n,\kappa+n}}{\left\| P_i^{(\mu+1,\kappa+1)} \right\|_{\mu+1,\kappa+1}^{2}} P_i^{(\mu+1,\kappa+1)}(\xi). \qquad (2.1)$$

where $P_i^{(\mu,\kappa)}(\cdot)$ is the i^{th} order shifted Jacobi orthogonal polynomial (See the appendix).

In order to approximate x on $[t_o - T, t_o]$, the classical polynomial approximation by taking the $N+1$ first terms in the Jacobi series expansion given in (2.1) was considered, the obtained N^{th} order polynomial is denoted by $D_{\kappa,\mu,T,N}^{(0)}x\left(t_o - T\cdot\right)$. Thus,

$$\forall \xi \in [0,1],\ D_{\kappa,\mu,T,N}^{(0)}x\left(t_0 - T\xi\right) = \sum_{i=0}^{N} \frac{\left\langle P_i^{(\mu,\kappa)}(\cdot), x(t_0 - T\cdot)\right\rangle_{\mu,\kappa}}{\left\| P_i^{(\mu,\kappa)} \right\|_{\mu,\kappa}^{2}} P_i^{(\mu,\kappa)}(\xi). \qquad (2.2)$$

Hence, the n^{th} order derivative of x can be approximated by the one of $D_{\kappa,\mu,T,N}^{(0)}x\left(t_0 - T\cdot\right)$. Then, the (causal) Jacobi differentiator is defined as follows

$$D_{\kappa,\mu,T,q}^{(n)}x\left(t_0 - T\xi\right) := \frac{1}{(-T)^{n}} \frac{d^{n}}{d\xi^{n}} \left\{ D_{\kappa,\mu,T,N}^{(0)}x\left(t_0 - T\xi\right) \right\},$$

where $q = N - n \in \mathbb{N}$. This differentiator can also be obtained by taking the $q+1$ first terms in the Jacobi series expansion of $x^{(n)}$ by a q^{th} order polynomial on $[t_0 - T, t_0]$:

$$\forall \xi \in [0,1]\,,\; D^{(n)}_{\kappa,\mu,T,q} x\left(t_0 - T\xi\right) = \sum_{i=0}^{q} \frac{\left\langle P_i^{(\mu+n,\kappa+n)}(\cdot), x^{(n)}(t_0 - T\cdot)\right\rangle_{\mu+n,\kappa+n}}{\left\| P_i^{(\mu+n,\kappa+n)} \right\|^2_{\mu+n,\kappa+n}} P_i^{(\mu+n,\kappa+n)}\left(\xi\right).$$

Moreover, it can be given by the following integral formula:

$$\forall \xi \in [0,1]\,,\; D^{(n)}_{\kappa,\mu,T,q} x\left(t_0 - T\xi\right) = \tfrac{1}{(-T)^n} \int_0^1 Q_{\mu,\kappa,n,q,\xi}\left(\tau\right) x\left(t_0 - T\xi\right) d\tau, \quad (2.3)$$

where

$$Q_{\kappa,\mu,n,q,\xi}\left(\tau\right) = \omega_{\mu,\kappa}\left(\tau\right) \sum_{i=0}^{q} C_{\kappa,\mu,n,i} P_i^{(\mu+n,\kappa+n)}\left(\xi\right) P_{n+i}^{(\mu,\kappa)}\left(\tau\right)$$

with $C_{\kappa,\mu,n,i} = \dfrac{(\mu+\kappa+2n+i)\,\Gamma(\mu+\kappa+2n+i)\,\Gamma(n+i+1)}{\Gamma(\kappa+n+i+1)\,\Gamma(\mu+n+i+1)}$.

Finally, we substitute x in (2.3) by x^{ϖ} so as to obtain the Jacobi differentiator $D^{(\varpi)}_{\kappa,\mu,T,q} x\left(t_0 - T\xi\right)$ in the noisy case.

2.4 Weak derivative based approach

In the following definition the concept of weak derivative is described,

Definition 2 (Clasiccal) *$D_y \in \mathcal{L}_{2_{[a,b]}}$ is the weak derivative of $y(t) \in \mathcal{L}_{2_{[a,b]}}$ if and only if*

$$\begin{gathered} \left\langle D_y, \zeta \right\rangle = -\left\langle y, \dot{\zeta} \right\rangle, \quad \forall \zeta \in C^{\infty}_{[a,b]} \\ \zeta(a) = \zeta(b) = 0 \end{gathered} \quad (2.4)$$

Remark 3 *From integration by parts*

$$\left\langle D_y, \zeta \right\rangle = y(\tau)\,\zeta(\tau)\big|_a^b - \left\langle y, \dot{\zeta} \right\rangle$$

Note. $\langle \cdot, \cdot \rangle$ is the scalar product in $\mathcal{L}^2$.

Proposition 4 *The weak derivative of a signal $y(t) \in \mathcal{L}_{2_{[a,b]}}$ is given by*

$$Dy = \sum_{i=0}^{\infty} c_i \tilde{\varphi}_i^{(1)}(t)$$

Proof. Define a function $S_N = \sum_{i=0}^{N} c_i \tilde{\varphi}_i^{(1)}(t)$, where, if $N \to \infty$, $S_N \to Dy$ and

$$\langle S_N, \zeta \rangle \to \langle D_y, \zeta \rangle, \forall \zeta \in C^{\infty}_{[-1,1]}$$

Then, by the definition of the internal product [Hildebrand, 1987], one have

$$\left\langle \sum_{i=0}^{N} c_i \tilde{\varphi}_i^{(1)}, \zeta \right\rangle = \sum_{i=0}^{N} c_i \left\langle \tilde{\varphi}_i^{(1)}, \zeta \right\rangle$$

Integrating by parts the next result is obtained

$$\sum_{i=0}^{N} c_i \left\langle \tilde{\varphi}_i^{(1)}, \zeta \right\rangle = \sum_{i=0}^{N} c_i \left(\tilde{\varphi}_i^{(1)} \zeta \right) \Big|_a^b - \sum_{i=0}^{N} c_i \left\langle \tilde{\varphi}_i, \zeta^{(1)} \right\rangle$$

By the definition of $\zeta(\cdot)$ in (2.4) the result is obtained. ■

2.4.1 Differentiation of a signal from its discrete measurements by Legendre polynomials

Consider the function $y(t) \in \mathcal{L}^2_{[-1,1]}$, the main problem to deal is to approximate the first order derivative of $y(t)$ taking at discrete instans of time by means of a series of orthonormal functions defined as

$$\tilde{\varphi}_i(t) \in \mathcal{L}^2_{[-1,1]}, \quad i = 1, 2, ... \tag{2.5}$$

The function $y(t)$ could be approximated by

$$y(t) \approx \sum_{i=0}^{\infty} c_i \tilde{\varphi}_i(t) \tag{2.6}$$

The derivative of the function $y(t)$ by definition is given by

$$\frac{d}{dt} y(t) := \sum_{i=0}^{\infty} c_i \frac{d}{dt} \tilde{\varphi}_i(t) = \sum_{i=0}^{\infty} \langle y, \tilde{\varphi}_i \rangle \frac{d}{dt} \tilde{\varphi}_i(t) \tag{2.7}$$

where, the dot product is given as

$$\langle x, y\rangle = \int\limits_{-1}^{1} x(\tau)\, y(\tau)\, d\tau \tag{2.8}$$

with this definition, equation (2.7) can be rewritten as

$$\frac{d}{dt}y(t) = \sum_{i=0}^{\infty} \int\limits_{-1}^{1} y(\tau)\, \tilde{\varphi}_i(\tau)\, \frac{d}{dt}\tilde{\varphi}_i(t)\, d\tau$$

$$\frac{d}{dt}y(t) = \int\limits_{-1}^{1} y(\tau) \sum_{i=0}^{\infty} \tilde{\varphi}_i(\tau)\, \frac{d}{dt}\tilde{\varphi}_i(t)\, d\tau$$

if the function is approximated with N orthonormal functions, one has

$$\frac{d}{dt}y(t) \approx \int\limits_{-1}^{1} y(\tau) \sum_{i=0}^{N} \tilde{\varphi}_i(\tau)\, \frac{d}{dt}\tilde{\varphi}_i(t)\, d\tau$$

where

$$\tilde{\varphi}_i = \frac{1}{\langle \varphi_i, \varphi_i\rangle^{1/2}} \varphi_i \tag{2.9}$$

where φ_i can be any orthogonal basis. The derivative approximation in the point $y(1)$ (by the windowing to be described below) is given by

$$\frac{d}{dt}y(1) = \int\limits_{-1}^{1} y(\tau) \sum_{i=0}^{N} \tilde{\varphi}_i(\tau)\, \frac{d}{dt}\tilde{\varphi}_i(1)\, d\tau \tag{2.10}$$

Defining $\alpha_i = \dfrac{d}{dt}\tilde{\varphi}_i(1)$, equation (2.10) takes the form

$$\frac{d}{dt}y(1) = \int\limits_{-1}^{1} y(\tau) \sum_{i=1}^{N} \alpha_i \tilde{\varphi}_i(\tau)\, d\tau \tag{2.11}$$

The polynomials used in this approximation were selected as Legendre polynomials that are orthogonal functions over the interval $[-1, 1]$. In this way, for signals defined in a different interval $[a, b]$, the following transformation in time must be implemented

$$t = \frac{s-a}{b-a} - \frac{s-b}{b-a} \tag{2.12}$$

Then,

$$s = \frac{(b-a)\,t + a + b}{2} \tag{2.13}$$

Applying the last equation, the following approximation is obtained

$$\frac{d}{dt} y\,(1) \approx \int_{-1}^{1} y\,(s) \sum_{i=1}^{N} \alpha_i \tilde{\varphi}_i\,(\tau)\, d\tau \tag{2.14}$$

For the discrete-time domain, taking a windowing of $\tilde{N}$ samples the last equation is approximated by

$$\begin{aligned} \frac{d}{dt} y\,(1) &\approx \sum_{j=1}^{\tilde{N}-1} \int_{\tau_j}^{\tau_{j+1}} \tilde{y}\,(s) \sum_{i=1}^{N} \alpha_i \tilde{\varphi}_i\,(\tau)\, d\tau = \sum_{j=1}^{\tilde{N}-1} y\,(s) \int_{\tau_j}^{\tau_{j+1}} \sum_{i=1}^{N} \alpha_i \tilde{\varphi}_i\,(\tau)\, d\tau \\ &= \sum_{j=1}^{\tilde{N}-1} y_j \sum_{i=1}^{N} \alpha_i \int_{\tau_j}^{\tau_{j+1}} \tilde{\varphi}_i\,(\tau)\, d\tau \end{aligned} \tag{2.15}$$

Defining

$$\beta_{ij} = \int_{\tau_j}^{\tau_{j+1}} \tilde{\varphi}_i\,(\tau)\, d\tau \tag{2.16}$$

valid for $\tau_i \leq \tau \leq \tau_{i+1}$ with

$$\tau_i = -1 + \frac{2h}{b-a} i = -1 + \frac{2}{\tilde{N}} i \tag{2.17}$$

h is the sampled period and q_j is defined as

$$q_j = \sum_{i=1}^{N} \alpha_i \beta_{ij} \tag{2.18}$$

Finally, the derivative approximation takes de form

$$\frac{d}{dt}y(1) \approx \sum_{j=0}^{\tilde{N}-1} y_j q_j \tag{2.19}$$

2.4.2 Example for $N = 3$

From equation (2.9) the parameters $\tilde{\varphi}_i$ were obtaining for $N = 3$, for φ_0 one gets

$$\langle \varphi_0, \varphi_0 \rangle = \int_{-1}^{1} d\tau = 2, \qquad \tilde{\varphi}_0 = \frac{1}{\sqrt{2}} \tag{2.20}$$

For φ_1 one gets

$$\begin{gathered} \langle \varphi_1, \varphi_1 \rangle = \int_{-1}^{1} \tau^2 d\tau = \left.\frac{\tau^3}{3}\right|_{-1}^{1} = \frac{2}{3} \\ \tilde{\varphi}_1 = \sqrt{\frac{3}{2}}\tau \end{gathered} \tag{2.21}$$

For φ_2 one gets

$$\begin{gathered} \langle \varphi_2, \varphi_2 \rangle = \int_{-1}^{1} \frac{1}{4}\left(3\tau^2 - 1\right)^2 d\tau = \frac{1}{4}\int_{-1}^{1} \left(9\tau^4 - 6\tau^2 + 1\right) d\tau = \\ = \frac{1}{4}\left.\left(\frac{9}{5}\tau^5 - 2\tau^3 + \tau\right)\right|_{-1}^{1} \\ = \frac{1}{4}\left(\frac{9}{5}(1)^5 - 2(1)^3 + 1\right) - \frac{1}{4}\left(\frac{9}{5}(-1)^5 - 2(-1)^3 - 1\right) = \frac{2}{5} \\ \tilde{\varphi}_2 = \sqrt{\frac{5}{2}}\frac{1}{2}\left(3\tau^2 - 1\right) \end{gathered} \tag{2.22}$$

Finally, for φ_3 one gets

$$
\begin{aligned}
\langle\varphi_3,\varphi_3\rangle = \int_{-1}^{1}\frac{1}{4}\left(5\tau^3-3\tau\right)^2 d\tau = \\
\frac{1}{4}\int_{-1}^{1}\left(25\tau^6-30\tau^4+9\tau^2\right)d\tau = \\
\frac{1}{4}\left(\frac{25}{7}\tau^7-6\tau^5+3\tau^3\Big|_{-1}^{1}\right) \\
=\frac{1}{4}\left(\frac{25}{7}-6+3\right)-\frac{1}{4}\left(-\frac{25}{7}+6-3\right) \\
=\frac{1}{4}\left(\frac{50}{7}-12+6\right)=\frac{2}{7} \\
\tilde{\varphi}_3=\sqrt{\frac{7}{2}}\varphi_2\left(\tau\right)=\sqrt{\frac{7}{2}}\frac{1}{2}\left(5\tau^3-3\tau\right)
\end{aligned}
\tag{2.23}
$$

The above calculations let us to the values of α_i

$$
\alpha_0=0 \quad \alpha_1=\sqrt{\frac{3}{2}} \quad \alpha_2=3\sqrt{\frac{5}{2}} \quad \alpha_3=6\sqrt{\frac{7}{2}} \tag{2.24}
$$

2.5 Weak-derivatives of periodic functions

Let $\tilde{W}^{1,2}\subset\mathcal{L}^2_{[a,b]}$ be the Sobolev space of absolute continuous periodic functions: $y\left(a\right)=y\left(b\right)$ if there exist $\xi\in C^{\infty}_{[a,b]}:\xi\left(a\right)=\xi\left(b\right)$, by the definition of (2.8) and integrating by parts we have

$$
\begin{aligned}
\langle\dot{y},\xi\rangle &= y\left(\tau\right)\xi\left(\tau\right)\big|_a^b-\int_a^b y\left(\tau\right)\dot{\xi}\left(\tau\right)d\tau \\
&= y\left(\tau\right)\xi\left(\tau\right)\big|_a^b-\left\langle y,\dot{\xi}\right\rangle \\
&= -\left\langle y,\dot{\xi}\right\rangle
\end{aligned}
$$

Definition 5 *If $Dy\in\mathcal{L}^2_{[a,b]}$, satisfies*

$$
\langle Dy,\xi\rangle=-\left\langle y,\dot{\xi}\right\rangle,
$$

$$
\forall\xi\in C^{\infty}_{[a,b]}:\xi\left(a\right)=\xi\left(b\right)
$$

then, Dy is called the weak derivative of periodic function $y \in \tilde{W}^{1,2}_{[a,b]} : y(a) = y(b)$.

Remark 6 *If $y \in C^{k}_{[a,b]}$, then $Dy = \dot{y}$.*

2.5.1 Optimal estimation in $\mathcal{L}^2$

The objective is to minimize

$$\left\| Dy - \tilde{D}y \right\|_{\mathcal{L}^2} \rightarrow \min_{\tilde{D}y \in M} \tag{2.25}$$

where M is a linear subspace of $\mathcal{L}^2$.

Remark 7 *The optimal estimate of $\tilde{D}y$ is the unique solution of (2.25) if and only if*

$$\left\langle Dy - \tilde{D}y, z \right\rangle = 0, \qquad \forall z \in M$$

Hence, if $M = span\left\{\varphi_0, \ldots, \varphi_N\right\}$ where φ_i are orthonormal, that is

$$\left\langle \varphi_i, \varphi_j \right\rangle = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

then

$$\tilde{D}y = \sum_{i=0}^{N} \left\langle \tilde{D}y, \varphi_i \right\rangle \varphi_i$$

2.5.2 Estimation of weak derivatives of periodic functions

1. $y \in \tilde{W}^{k,2}_{[a,b]}$, then $\exists D^{(k)}y$ in the sense of definition 2.
2. If $\{\varphi_i\}_{i=0}^{\infty}$ is orthonormal basis in $\mathcal{L}^2_{[a,b]}$, such that $\varphi_i \in C^{\infty}_{[a,b]}$ and $\varphi_i^{(j)}(a) = \varphi_i^{(j)}(b)$ for $i = 1, \ldots, k-1$, then

$$\tilde{D}^{(k)}y = \sum_{i=0}^{N} \left\langle \tilde{D}^{(k)}y, \varphi_i \right\rangle \varphi_i \tag{2.26}$$

   is optimal estimation of $D^{(k)}y$ in the subspace $span\left\{\varphi_0, \ldots, \varphi_N\right\}$.

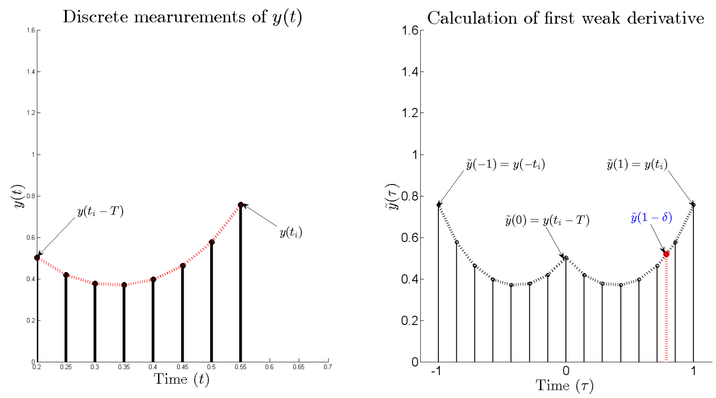


Figure 2.1: Transformation of signal y into the $\tilde{y}(\tau)$ in the interval $[-1, 1]$

3. Since $\varphi_i^{(k-1)}(a) = \varphi_i^{(k-1)}(b)$, then (2.26) can be rewritten as

$$\tilde{D}^{(k)} y = (-1)^k \sum_{i=0}^{N} \left\langle y, \varphi_i^{(k)} \right\rangle \varphi_i \tag{2.27}$$

2.5.3 Estimation of the first derivative

Consider the function

$$y \in W^{1,2} \subset \mathcal{L}^2_{[0,1]}$$

Assume that y can be extended into a periodic function $\tilde{y} \in \tilde{W}^{1,2}_{[-1,1]}$ defined as (see figure)

$$\tilde{y}(\tau) = \begin{cases} y(-\tau) & \tau \in [-1, 0] \\ y(\tau) & \tau \in [0, 1] \end{cases} \tag{2.28}$$

Let φ_i be orthonormal basis of periodic functions in $\mathcal{L}^2_{[-1,1]}$ such that

$$\varphi_i \in C^{\infty}_{[-1,1]} : \varphi(-1) = \varphi(1)$$

Then by (2.27) we have

$$\tilde{D} y(\tau) = -\sum_{i=0}^{N} \langle y, \dot{\varphi}_i \rangle \varphi_i(\tau), \tau \in [-1, 1]$$

is the optimal estimate of $D\tilde{y}$ in the subspace $span\{\varphi_0, \ldots, \varphi_N\}$, $N \geq 0$.

Remark 8 *If $\varphi_i(-1) = \varphi_i(1) = 0$, $\tilde{D}y(1) = 0$, then $\tilde{D}y(1-\delta)$ is the estimate of the derivative of $\tau = 1$.*

Remark If $\varphi_i(-1) = \varphi_i(1)$, $\tilde{D}y(1) = 0$, then $\tilde{D}y(1-\delta)$ is the estimate of the derivative of $\tau = 1$.

Hence

$$\tilde{D}y(1-\delta) = -\int_{-1}^{1} \sum_{i=0}^{N} \tilde{y}(\tau)\, \dot{\varphi}_i(\tau)\, \varphi_i(1-\delta)\, d\tau$$

$$= -\int_{-1}^{1} \tilde{y}(\tau) \sum_{i=0}^{N} \varphi_i(1-\delta)\, \dot{\varphi}_i(\tau)\, d\tau$$

and by the property given in (2.28), the next result is obtained

$$\tilde{D}y(1-\delta) = -\int_{-1}^{0} y(-\tau) \sum_{i=0}^{N} \varphi_i(1-\delta)\, \dot{\varphi}_i(\tau)\, d\tau$$
$$-\int_{0}^{1} y(\tau) \sum_{i=0}^{N} \varphi_i(1-\delta)\, \dot{\varphi}_i(\tau)\, d\tau$$

$$= -\int_{0}^{1} y(\tau) \sum_{i=0}^{N} \varphi_i(1-\delta)\left(\dot{\varphi}_i(\tau) + \dot{\varphi}_i(-\tau)\right) d\tau$$

Discrete estimation of the first derivative

Consider the equation

$$\begin{aligned} \tilde{D}y(1-\delta) = & -\int_{0}^{1} y(\tau) \sum_{i=0}^{N} \varphi_i(1-\delta)\left(\dot{\varphi}_i(\tau) + \dot{\varphi}_i(-\tau)\right) d\tau \\ = & -\int_{0}^{1} y(\tau) \sum_{i=0}^{N} \alpha_i\left(\dot{\varphi}_i(\tau) + \dot{\varphi}_i(-\tau)\right) d\tau \end{aligned} \tag{2.29}$$

where $\alpha_i = \varphi_i(1-\delta)$. In this way, for signals defined in a different interval $[a,b]$, the following transformation in time must be implemented

$$t = \frac{s-a}{b-a} - \frac{s-b}{b-a} \tag{2.30}$$

Then,

$$s = \frac{(b-a)\,t + a + b}{2} \tag{2.31}$$

Equation (2.29) becomes

$$\tilde{D}y\,(1-\delta) = -\int_0^1 y\,(s)\sum_{i=0}^{N}\alpha_i\left(\dot{\varphi}_i\,(\tau) + \dot{\varphi}_i\,(-\tau)\right)d\tau$$

for the discrete time domain, taking a windowing of $\tilde{N}$ samples, the last equation is approximated by

$$\tilde{D}y\,(1-\delta) \approx -\sum_{j=0}^{\tilde{N}}\int_{\tau_j}^{\tau_{j+1}} y\,(s)\sum_{i=0}^{N}\alpha_i\left(\dot{\varphi}_i\,(\tau) + \dot{\varphi}_i\,(-\tau)\right)d\tau$$

$$= -\sum_{j=0}^{\tilde{N}} y\,(s)\sum_{i=0}^{N}\alpha_i\int_{\tau_j}^{\tau_{j+1}}\left(\dot{\varphi}_i\,(\tau) + \dot{\varphi}_i\,(-\tau)\right)d\tau$$

Defining

$$\beta_{i,j} = \int_{\tau_j}^{\tau_{j+1}}\left(\dot{\varphi}_i\,(\tau) + \dot{\varphi}_i\,(-\tau)\right)d\tau$$

valid for $\tau_i \leq \tau \leq \tau_{i+1}$ with

$$\tau_i = \frac{h}{b-a}i \tag{2.32}$$

h is the sampled period. With q_j defined as

$$q_j = \sum_{i=1}^{N}\alpha_i\beta_{ij} \tag{2.33}$$

The optimal derivative approximation takes the form

$$\tilde{D}y\,(1-\delta) \approx -\sum_{j=0}^{\tilde{N}-1} y_j q_j$$

First Order Approximation

Consider the approximation given by

$$
\begin{aligned}
Dy(\delta) &= -\int_0^1 y(\tau)\,\varphi_i(1)\left(\dot{\varphi}_i(\tau)+\dot{\varphi}_i(-\tau)\right)d\tau \\
&= -2\int_0^1 y(\tau)\sum_{i=0}^{N}\varphi_i(\delta)\,\dot{\varphi}_i(\tau)\,d\tau
\end{aligned}
\tag{2.34}
$$

where the function $y(\tau)$ is defined as

$$
y(\tau) = y(\tau_j)\,\frac{\tau_{j+1}-\tau}{\tau_{j+1}-\tau_j} + y(\tau_{j+1})\,\frac{\tau-\tau_j}{\tau_{j+1}-\tau_j}
$$
$$
\tau \in [\tau_j, \tau_{j+1}]
$$

Applying this output as the information of the differentiator in the discrete time domain, the equation (2.34) becomes

$$
Dy(1-\delta) = -2\sum_{j=0}^{\tilde{N}-1}\frac{1}{\tau_{j+1}-\tau_j}\int_{\tau_j}^{\tau_{j+1}}\left[y(\tau_j)(\tau_{j+1}-\tau)+y(\tau_{j+1})(\tau-\tau_j)\right]\sum_{i=0}^{N}\varphi_i(1-\delta)\,\dot{\varphi}_i(\tau)\,d
$$

because $\varphi_i(\delta)$ do not depend of τ one has

$$
Dy(1-\delta) = -2\sum_{j=0}^{\tilde{N}-1}\tfrac{1}{\tau_{j+1}-\tau_j}\sum_{i=0}^{N}\varphi_i(1-\delta)\int_{\tau_j}^{\tau_{j+1}} y(\tau_j)(\tau_{j+1}-\tau)\,\dot{\varphi}_i(\tau)\,d\tau -
$$

$$
2\sum_{j=0}^{\tilde{N}-1}\tfrac{1}{\tau_{j+1}-\tau_j}\sum_{i=0}^{N}\varphi_i(1-\delta)\int_{\tau_j}^{\tau_{j+1}} y(\tau_{j+1})(\tau-\tau_j)\,\dot{\varphi}_i(\tau)\,d\tau
$$

then,

$$
\begin{aligned}
Dy(1-\delta) = -2\sum_{j=0}^{\tilde{N}-1}\tfrac{1}{\tau_{j+1}-\tau_j}\sum_{i=0}^{N}\varphi_i(1-\delta)\int_{\tau_j}^{\tau_{j+1}} y(\tau_j)(\tau_{j+1}-\tau)\,\dot{\varphi}_i(\tau)\,d\tau \\
-2\sum_{j=0}^{\tilde{N}-1}\tfrac{1}{\tau_{j+1}-\tau_j}\sum_{i=0}^{N}\varphi_i(1-\delta) + \int_{\tau_j}^{\tau_{j+1}} y(\tau_{j+1})(\tau-\tau_j)\,\dot{\varphi}_i(\tau)\,d\tau
\end{aligned}
$$

$y(\tau_j)$ and $y(\tau_{j+1})$ are constants during the interval $[\tau_j, \tau_{j+1}]$ and they do not depend of the variable τ, then

$$Dy(1-\delta) = -2\sum_{j=0}^{\tilde{N}-1} \frac{1}{\tau_{j+1}-\tau_j}\left[y(\tau_j)\,\alpha_{ij} + y(\tau_{j+1})\,\beta_{ij}\right] \tag{2.35}$$

where

$$\alpha_{ij} := \sum_{i=0}^{N} \varphi_i(1-\delta) \int_{\tau_j}^{\tau_{j+1}} (\tau_{j+1}-\tau)\,\dot{\varphi}_i(\tau)\,d\tau$$

$$\beta_{ij} := \sum_{i=0}^{N} \varphi_i(1-\delta) \int_{\tau_j}^{\tau_{j+1}} (\tau-\tau_j)\,\dot{\varphi}_i(\tau)\,d\tau$$

and they can be obtained integrating by parts, that is,

$$\begin{aligned}
\alpha_{ij} &:= \sum_{i=0}^{N} \varphi_i(1-\delta) \int_{\tau_j}^{\tau_{j+1}} (\tau_{j+1}-\tau)\,\dot{\varphi}_i(\tau)\,d\tau \\
&= \sum_{i=0}^{N} \varphi_i(1-\delta) \left[(\tau_{j+1}-\tau)\,\varphi_i(\tau)\big|_{\tau_j}^{\tau_{j+1}} + \int_{\tau_j}^{\tau_{j+1}} \varphi_i(\tau)\,d\tau \right] \\
&= \sum_{i=0}^{N} \varphi_i(1-\delta) \left[-(\tau_{j+1}-\tau_j)\,\varphi_i(\tau_j) + \int_{\tau_j}^{\tau_{j+1}} \varphi_i(\tau)\,d\tau \right]
\end{aligned}$$

For β_{ij} we have

$$\begin{aligned}
\beta_{ij} &:= \sum_{i=0}^{N} \varphi_i(1-\delta) \int_{\tau_j}^{\tau_{j+1}} (\tau-\tau_j)\,\dot{\varphi}_i(\tau)\,d\tau \\
&= \sum_{i=0}^{N} \varphi_i(1-\delta) \left[(\tau-\tau_j)\,\varphi_i(\tau)\big|_{\tau_j}^{\tau_{j+1}} - \int_{\tau_j}^{\tau_{j+1}} \varphi_i(\tau)\,d\tau \right] \\
&= \sum_{i=0}^{N} \varphi_i(1-\delta) \left[(\tau_{j+1}-\tau_j)\,\varphi_i(\tau_{j+1}) - \int_{\tau_j}^{\tau_{j+1}} \varphi_i(\tau)\,d\tau \right]
\end{aligned}$$

3. Comparison between signal differentiators

The function used in simulations (figure 3.1) was chosen as

$$y(t) = e^{\sin(15t)}$$

the available output was considered. i. e.

- Different sampled times:
- Different levels of quantization
- Two kinds of noise in the output
  - Uniform distributed noise from $[-1, 1]$
  - Deterministic noise ($\eta(t) = 0.005 \sin 900t + 0.01 \cos 600t$)

The parameters used in simulation for each differentiator were chosen as:

Homogeneous Differentiator
$\mu = \frac{1}{2} \quad k_1 = -180 \quad k_2 = -350$

Algebraic Differentiator (Jacobi)
$\mu = 11 \quad \kappa = -0.1 \quad q = 4$

where q is the number of polinomios used. The parameter selection was done using the notation presented in table 1 in the appendix.

Algebraic Differentiator (Legendre)
$q = 4$

and

Weak differentiator
$\varphi_i(\tau) = \sin(i\pi\tau), \; i = 1 : q \quad q = 8 \quad \delta = 3$

in the same way, q is the number of elements in the cosines basis, δ is the delay for computing the derivative. For the differentiators based on algebraic

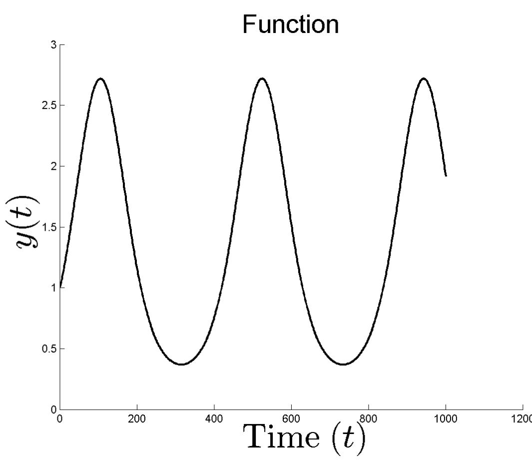


Figure 3.1: Function chosen in simulation

techniques and weak differentiation the window (T) was selected according the sampling period. For sampled period 0, $T = 100$, for $h = 0.002$, $T = 50$. The parameters for the Jacobi and weak differentiators were chosen in order to obtain a good approximation of the signal derivative without having a big delay. By this reason the number of elements in the basis is not equal in all the cases.

For comparing the behavior of each differentiation technique, several scripts were run in Matlab in a computer with the following characteristics: *Intel Core i7-4700MQ @ 2.40 MHz. 16 Gb RAM, Operating system of 64 bits.* The first analysis was done in terms of computational aspects like complexity, memory size and complexity tuning of each differentiators The second part of the analysis was done in terms of accuracy, where several sampled times were tested in order to know what differentiator has a better performance in presence of bounded perturbations and quantization. The objective of all the simulations was to approximate the derivative of an arbitrary unknown signal.

3.1 Computational aspects

The computational aspects involve complexity of the algorithm, the memory required to approximated one point of the signal in time and the complexity of tuning the algorithm.

3.1.1 Complexity

In this study it was defined as the number of basic operations that have to be implemented in order to obtain the derivative in an instant of time. In the following table the number of operations required for each differentiator to calculate one sampled time of simulation is summarized.

Differ	+	*	Adc [1]
Euler	2	0	Div
Homog.	4	7	4 [2]
Algebraic (Legendre)	$m-1$	m	0
Algebraic (Jacobi)	$m-1$	m	0
Weak	$m-1$	m	0

A second characteristic to know the complexity of the algorithm was selected as the time that the computer needed to obtain one second of simulation. In the following table, this analysis is described with τ being the sampled period. The results are given for whole interval

Differ	h=0.001	h=0.01
Euler	5.32x10^{-4}	3.35x10^{-5}
Homog.	7.35x10^{-4}	1.53x10^{-4}
Algebraic (Legendre)	0. 015	0.005
Algebraic (Jacobi)	0.019	0.005
Weak	0.014	0.005

3.1.2 Memory size.

Number of values stored in memory in order to calculate the signal derivative in one sampled time.

Differentiator	Values
Euler	2
Homog.	3
Algebraic (Jacobi)	m
Algebraic (Legendre)	m
Weak	m

[1] Aditional operations
[2] 2 square roots and 2 signum functions

3.1.3 Complexity tuning.

Difficult for changing the free parameters of each algorithm.

Differentiator	Values
Euler	Not tuning
Homog.	k_1, k_2, and μ
Algebraic (Jacobi)	q, μ, κ
Algebraic (Legendre)	q
Weak	q and δ

3.2 Accuracy

3.2.1 Noisy free.

1. Sampled time $h = 0.001$, $m = 100$.

   In figure (3.2), the simulations results obtained for each differentiator are shown. When the signal is not affected by any noise in its output, the four differentiators can estimate the derivative of the nonlinear signal when it is sampled with small sampled time. The homogeneous approximation and the Euler approximation converged faster than the other algorithms. The best accuracy was obtained when the Euler methods is applied for small sampled period. In figure (3.3) the estimation error defined as the absolute value of the error is shown. Clearly, a less error is appreciated with the Euler approximation.

2. Sampled time $h = 0.002$, $T = 50$.

   When the sampled time is selected as 0.002 and the window is reduced according to this new sampled time figure (3.4), the different differentiators still reproduce the derivative error. However, error in the Euler approximation increased, The estimation errors are shown in figure (3.5). The Jacobi differentiator without any modification in its parameters becomes to be inaccurate. The three differentiators based on weak derivative still having good approximation properties.

3. Sampled time $h = 0.01$, $T = 10$

   A third example was chosen taking a smaller sampled time. See figure (3). Also the window was reduced to a value of 10. In this way, the Jacobi differentiator without any modification of its parameters lose its

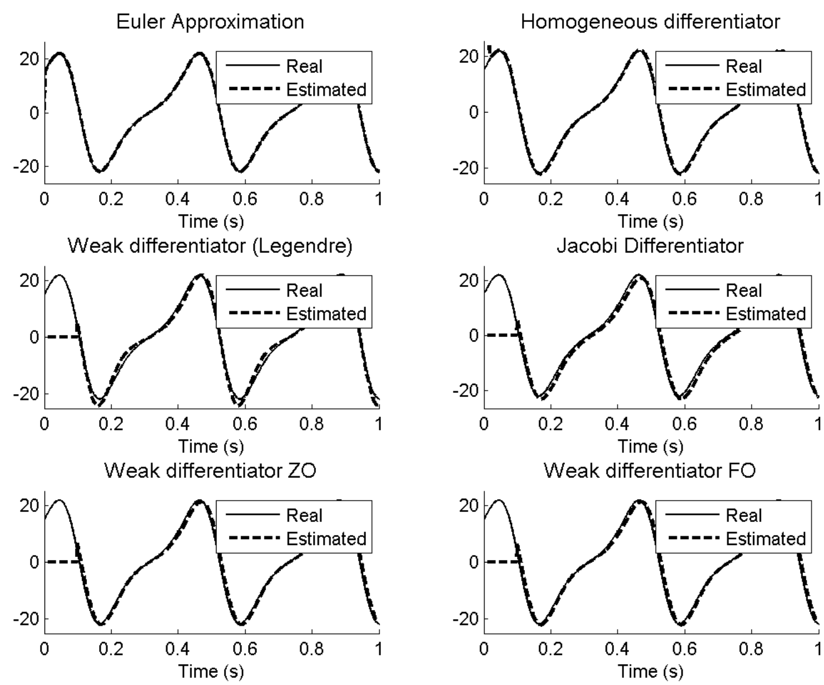


Figure 3.2: Derivative estimation sampled at 0.001 seconds.

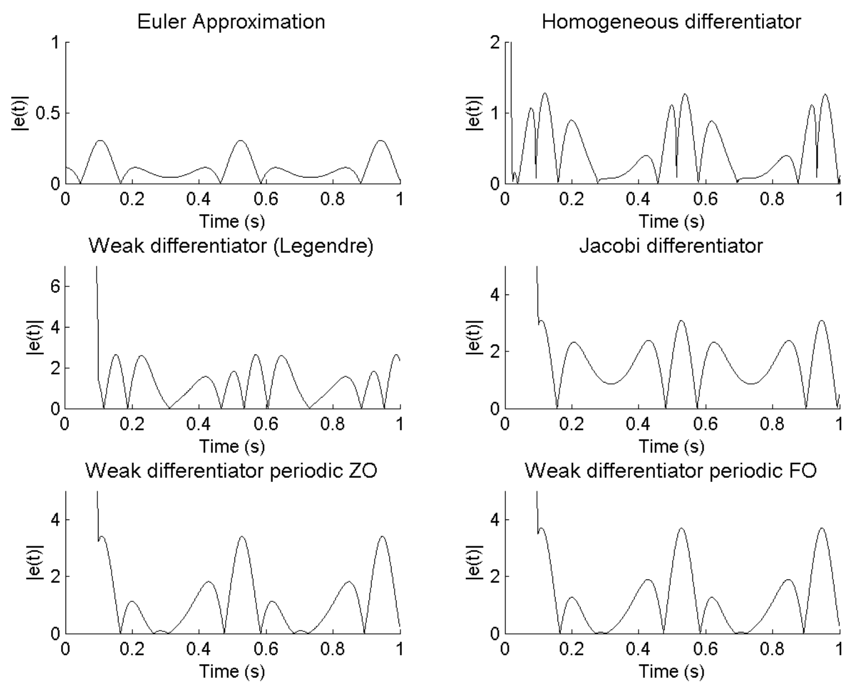


Figure 3.3: Derivative estimation error with sampled time of 0.001

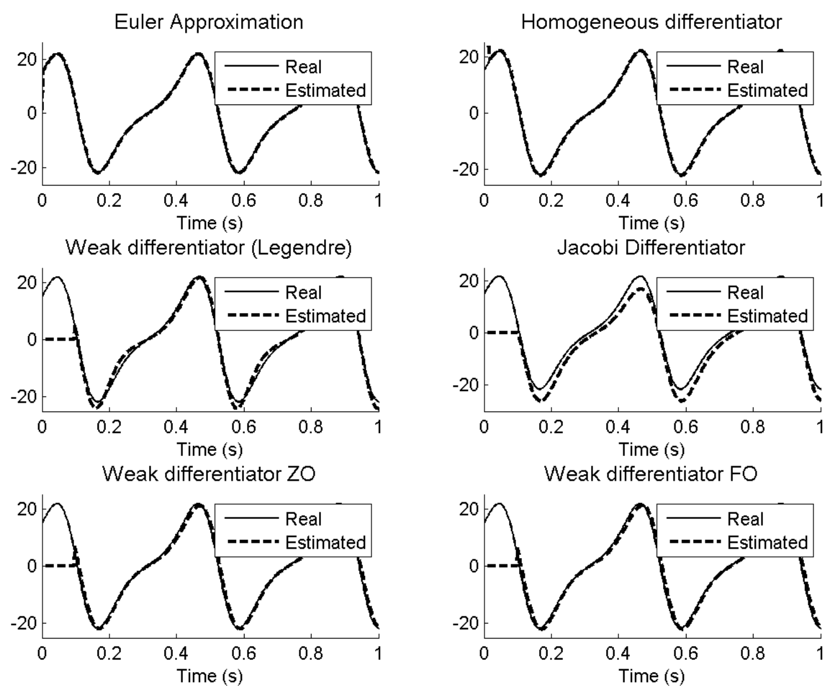


Figure 3.4: Derivative esimation for a sampled time of 0.002

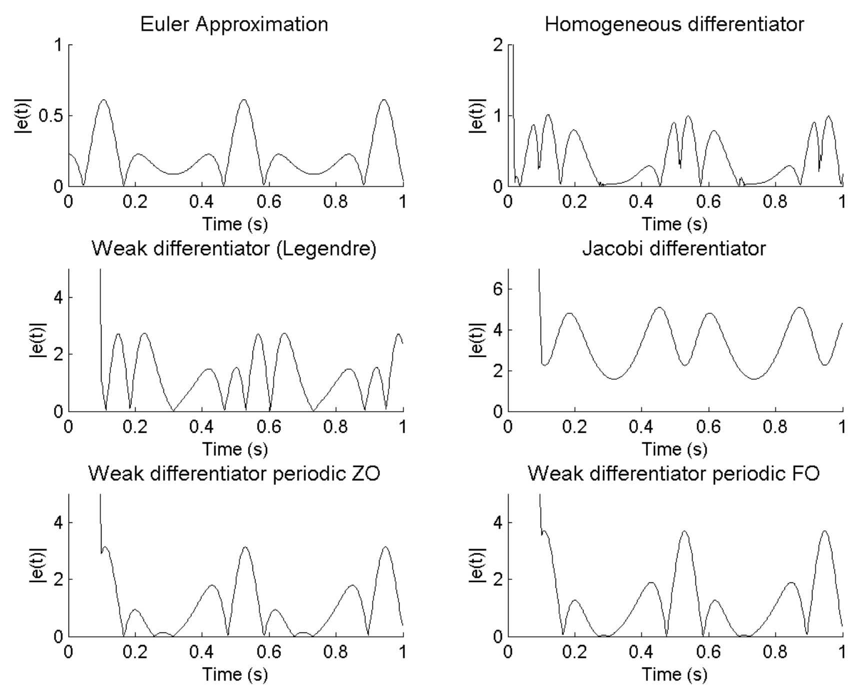
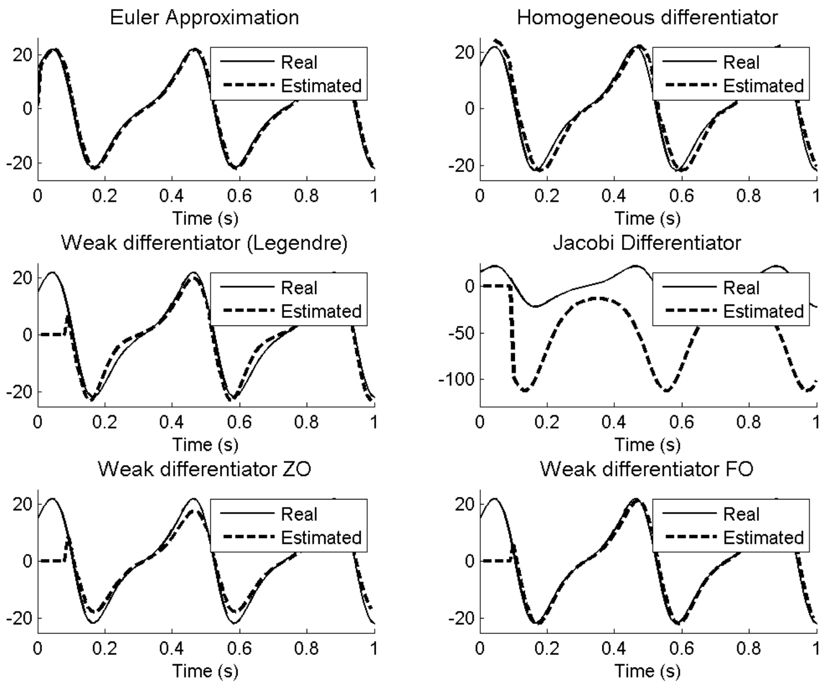


Figure 3.5: Derivative esimation error for a sampled time of 0.002



accuracy. The best estimation under these conditiones was obtained with the weak defirrentiator with periodic functions with a first order hold in the available output. The different estimation errors are described in figure (3.6)..

3.2.2 Quantization.

A second test was made including the problem of quantization. In this analysis three different levels of quantization were tested. In this section the signal was considered without any noise. Two levels of quantization were tested: 0.005 and 0.02 were etsd

1. Level of quantization of 0.005, $h = 0.001$ and $T = 100$. In figure (3.7) the derivative estimation of the same nonlineal signal is depicted. Even when the level of quantization was chosen small enough in order to obtain a good approximation the Euler approximation and the gomogeneous differentiator becomes to have some inaccurate in the estimation

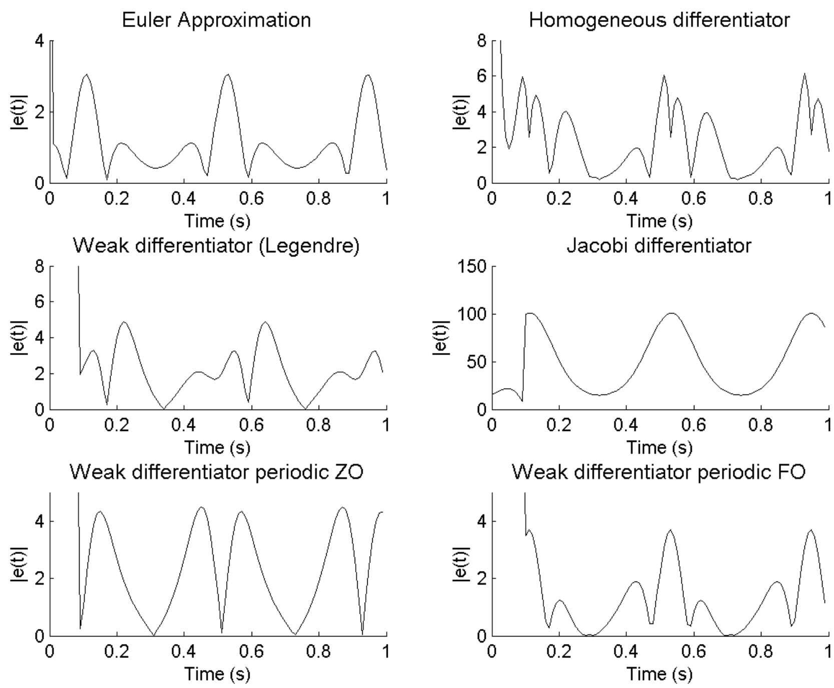


Figure 3.6: Derivative estimation error

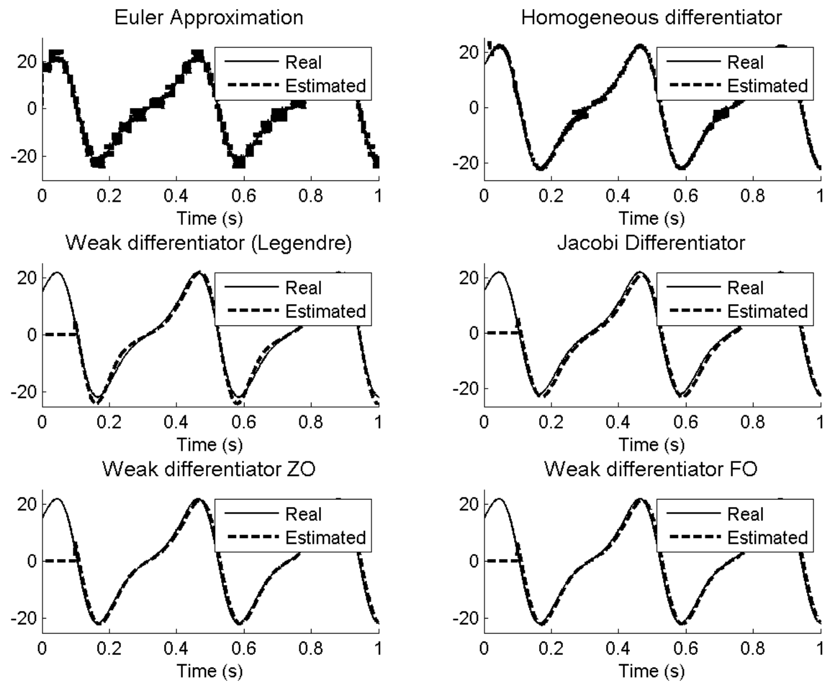


Figure 3.7: Derivative estimation with quantization level of 0.002 and sampled period of 0.001

process. In the next examples one can see how the quantization affects the quality of estimation. In figure (3.8) the estimation error can be seen. The estimation error is increased in comparison with figure (3.3), where the same parameters were simulated without the problem of quantization.

2. Quantization of 0.02, $h = 0.001$ and $T = 100$. When the signal is quantized only with 50 values between two consecutive integers, the Euler approximation and the homogeneous approximation did not estimate the trajectories of the nonlinear function (see figure 3.9). The accuracy of the algebraic approaches decreased with this level of quantization, however the approximation remains acceptable. They are more robust to this problem. This fact can be appreciated in figure (3.10).

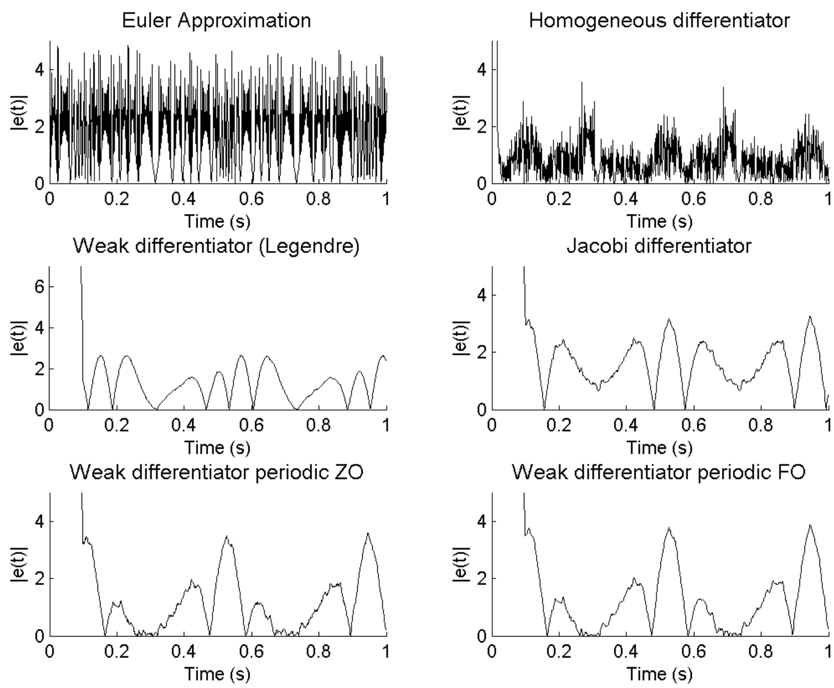


Figure 3.8: Derivative estimation error for quantization level of 0.002 and sampled period of 0.001

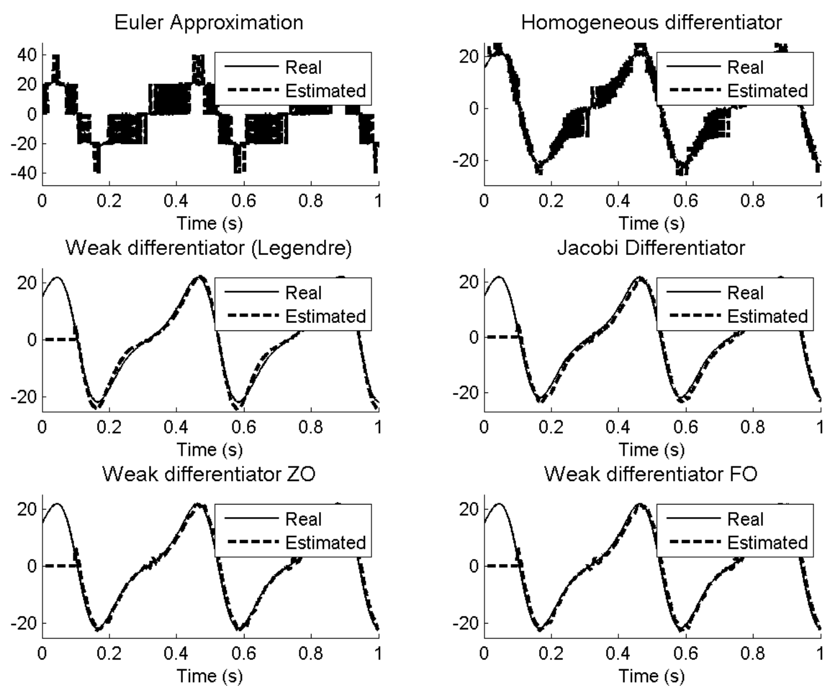


Figure 3.9: Derivative estimation with quantization level of 0.02 and sampled period of 0.001

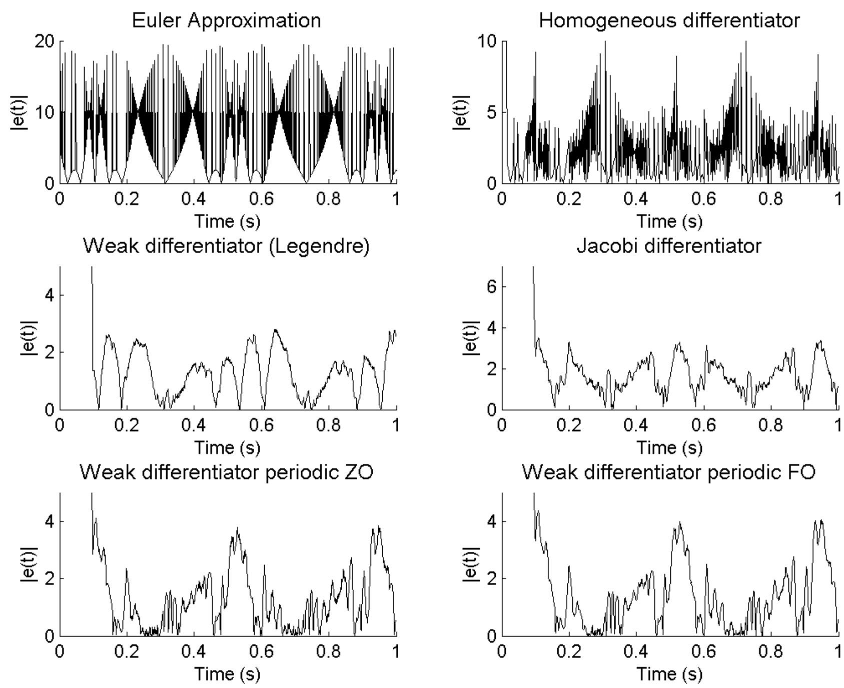


Figure 3.10: Derivative estimation with quantization level of 0.002 and sampled period of 0.001

3.2.3 Noisy signal.

Two classes of noise where tested to see the robustness of each differentiator. Distributed uniform noise and deterministic noises were applied. The simulations were obtained for different cases, first, each differentiator was tested only with noise in the measurements without considering the problem of quantization. Then, a second group of simulations was made adding the problem of quantization to the noisy signal.

Signal with an uniform distributed noise

1. Signal without quantization and $h = 0.001$. The results for each differentiator are shown in figure (3.11). In figure (3.12) the estimation errors are described. The Euler approximation lost accuracy in the presence of noises. The polynomial approximations, as they were calculated using the first 100 values to calculate one instant of time produce a less oscillated signal.

2. Signal with quantization of 0.02 and $h = 0.001$. In figure (3.13), the results for the noisy signal with quantization of 0.001 can be appreciated, the results are quite similar to the case without quantization. The less affected algorithms are the ones based in algebraic techniques. In figure (3.14) the estimation errors are shown.

Deterministic noise

1. Signal without quantization and $h = 0.001$. In figure (3.15) the simulations results for the differentiator algorithm affected by a deterministic noise described by $(\eta(t) = 0.005 \sin 900t + 0.01 \cos 600t$ are presented. The four algorithms reproduced the derivative. However, the Euler and homogeneous algorithms were less robust than the other approximators. Figure (3.16) shows the euclidean norm of the estimation errors. The Algebraic differentiators almost have the same performance.

2. Signal with quantization of 0.02 and sampling period 0.001. In figure (3.17) the simulation results for the differentiators are shown when the signal is quantized. This small quantization value affect so much the behavior of the four differentiators in comparison with the previous simulation when the signal was not quantized. The estimation error is described in figure (3.18).

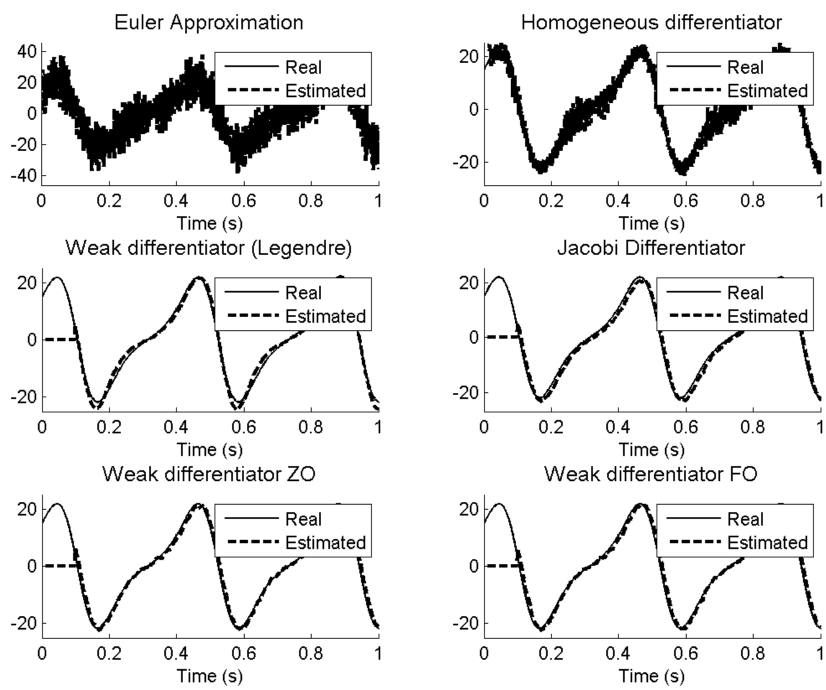


Figure 3.11: Derivative estimation with the signal corrupted by distributed uniform noise from -0.1 to 0.01 and sampled period of 0.001

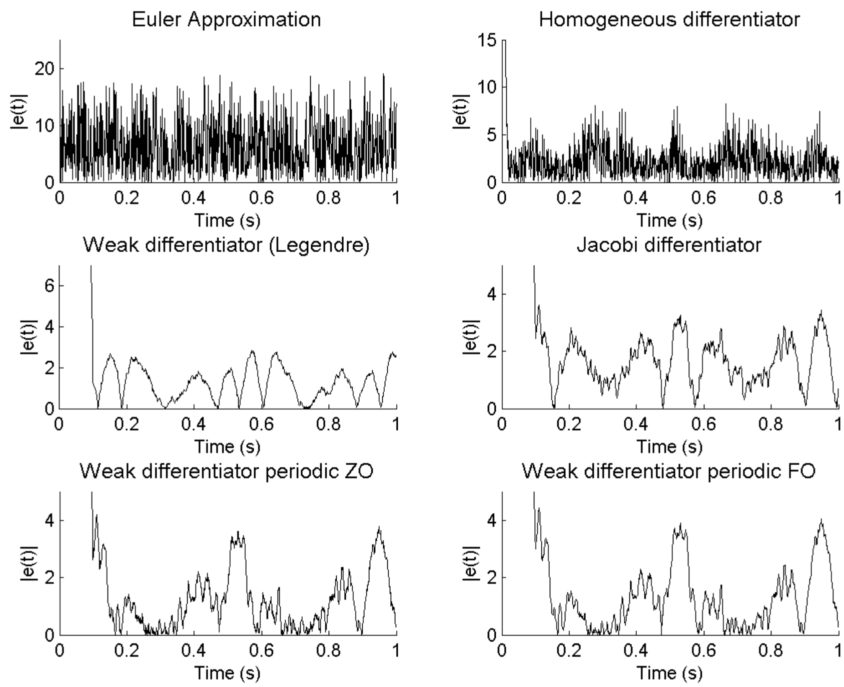


Figure 3.12: Derivative estimation error with the signal corrupted by distributed uniform noise and sampled period of 0.001

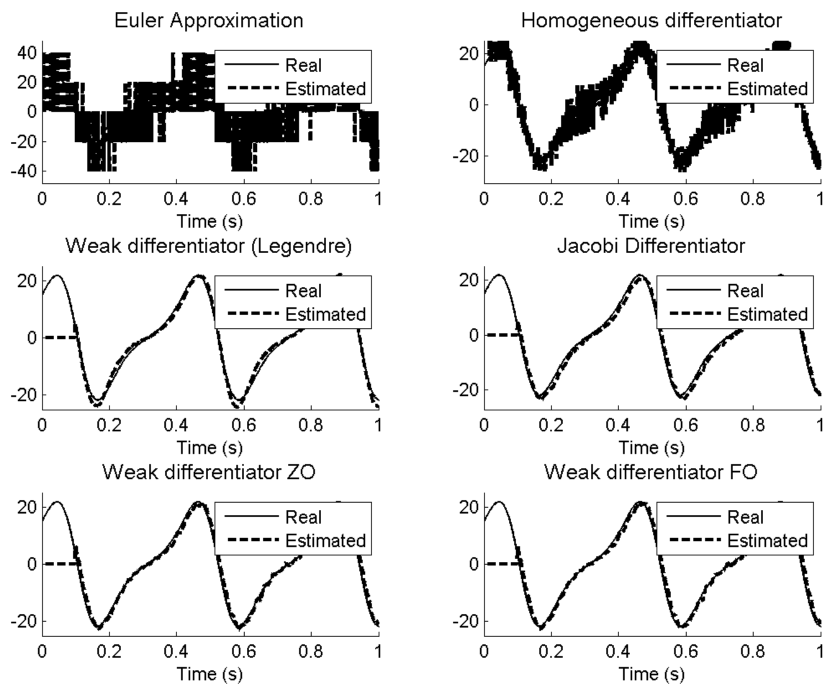


Figure 3.13: Derivative estimation with the signal corrupted by distributed uniform noise, sampled period of 0.001 and quantization of 0.02

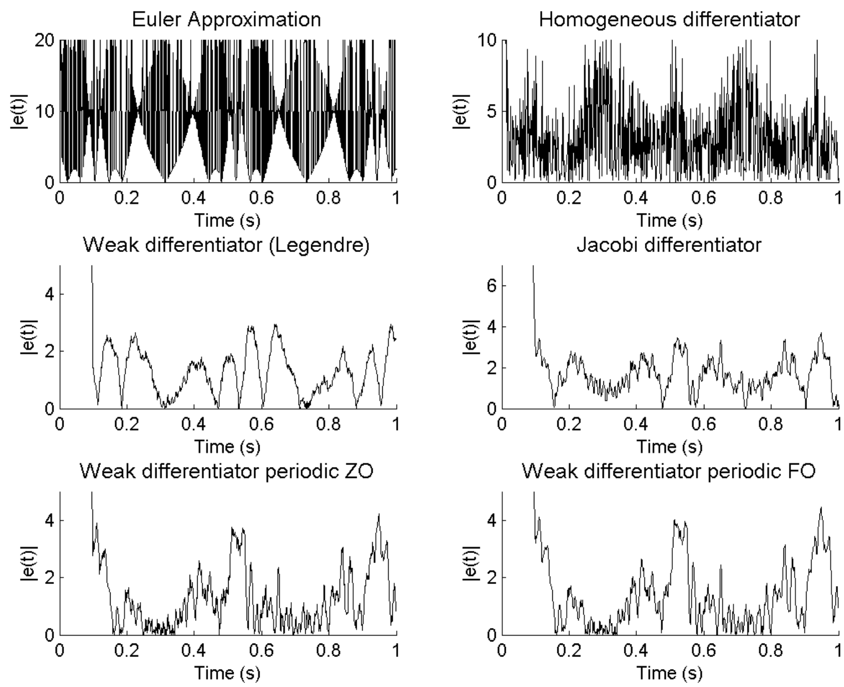


Figure 3.14: Derivative estimation error with the signal corrupted by distributed uniform noise, sampled period of 0.001 and quantization of 0.02

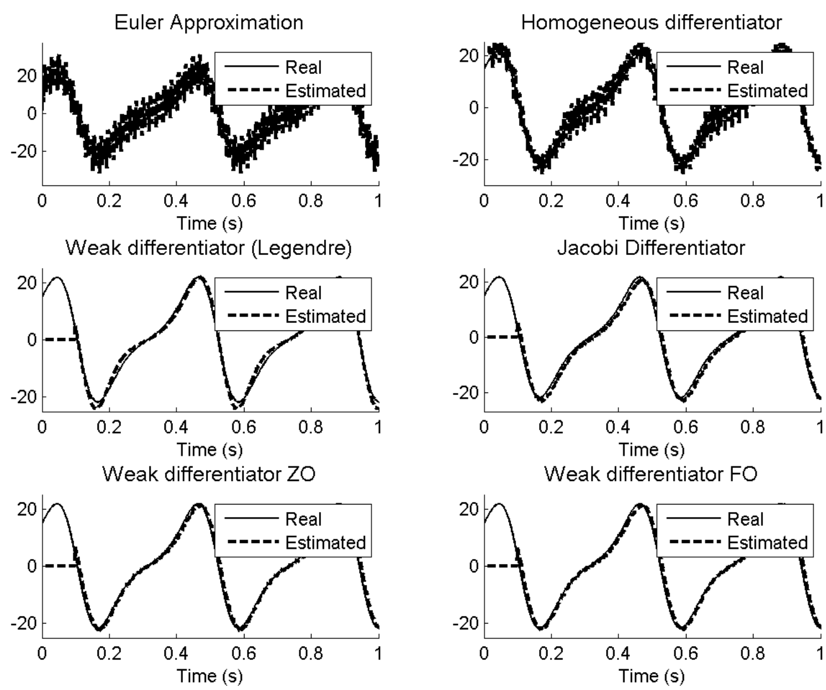


Figure 3.15: Derivative estimation with the signal corrupted by deterministic noise, sampled period of 0.001.

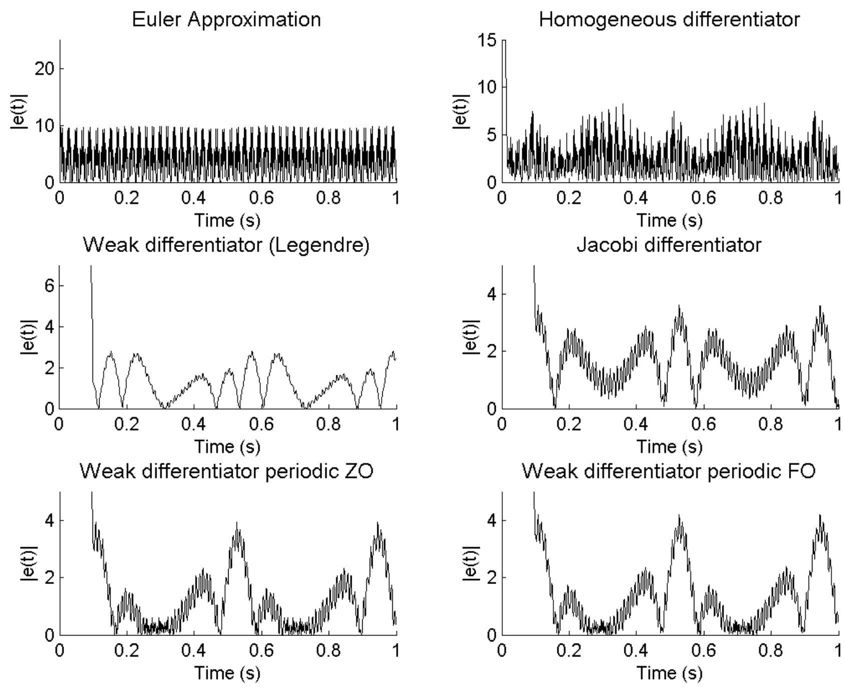


Figure 3.16: Derivative estimation error with the signal corrupted by deterministic noise, sampled period of 0.001

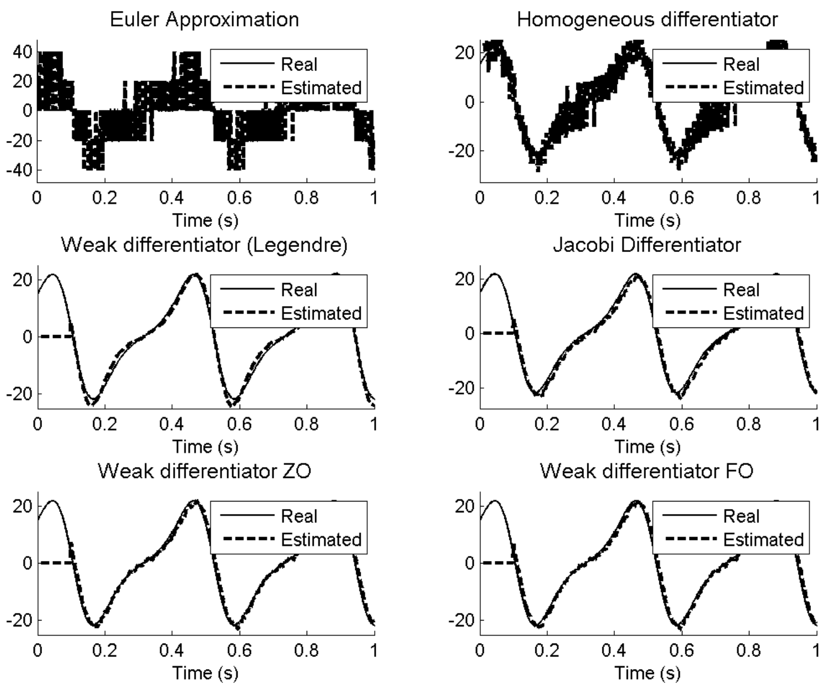


Figure 3.17: Derivative estimation with the signal corrupted by deterministic noise, sampled period of 0.001 and quantization of 0.02

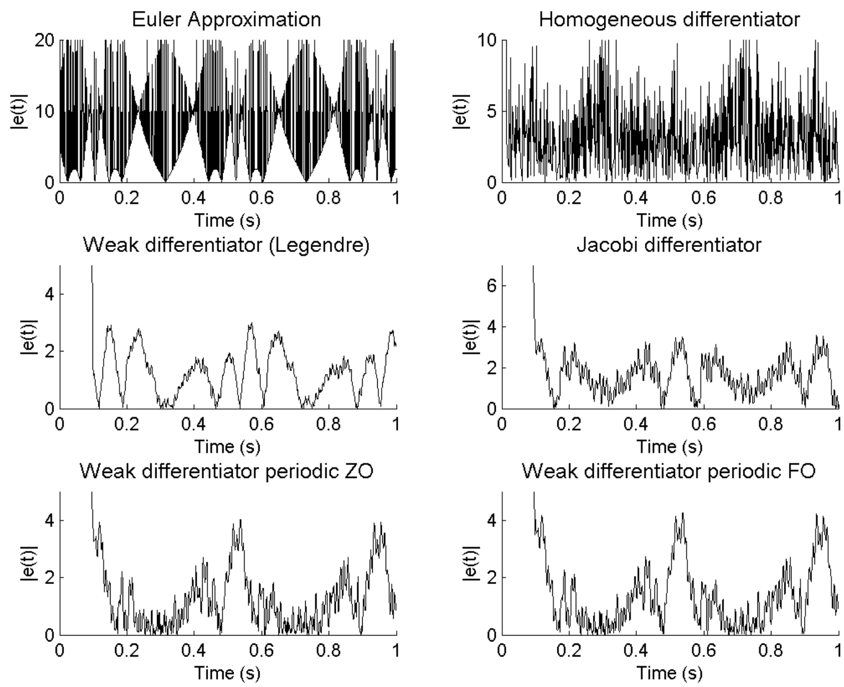


Figure 3.18: Derivative estimation error with the signal corrupted by determinisctic noise, sampled period of 0.001 and quantization of 0.02

4. Conclusions

In this manuscript, several differentiation techniques were tested in order to see the advantages and disadvantages of each technique. Four different techniques were implemented considering several problems that always are presented in real applications. The techniques based on algebraic techniques have better performance in the presence of noise in the measurements. Moreover, if the problem of quantization is included in the signal differentiation, the differentiators based on algebraic technique also bring some better results. The fourth differentiation technique was selected as the homogeneous differentiation, selecting its parameters in order to have a sliding mode differentiator.

Between the algebraic techniques, the one based on Jacobi polynomials showed better behavior, however, it is important to note, that this technique is a generalization of the Legendre differentiator and Tchevyshev differentiator. However, the μ parameter induced a delay in the differentiation process. If a Fourier basis is used, the estimation of the first order derivative presents a good approximation when the signal is not corrupted by noise. In the followin tables, the final conclusions are given.

- **Complexity**

Charac/Diff	**Euler**	**Homogeneous**	**Jacobi**	**Legendre**	**Weak**
Complexity	Low	Medium	High	High	High
Memory Size	Low	Low	High	High	High
Tuning	Null	Medium	High	Low	Medium

- **Sensitivity**

Charac/Diff	**Euler**	**Homogeneous**	**Jacobi**	**Legendre**	**Weak**
Noise	High	Medium	Low	Low	Low
Quantization	High	Medium	Low	Low	Low
Sampling	Medium	Medium	High	Low	Low

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5. Appendix

5.1 Legendre polynomials

In mathematics, an orthogonal polynomial sequence is a family of polynomials such that any two different polynomials in the sequence are orthogonal to each other under some inner product. The most widely used orthogonal polynomials are the classical orthogonal polynomials, consisting of the Hermite polynomials, the Laguerre polynomials, the Jacobi polynomials together with their special cases the Gegenbauer polynomials, the Chebyshev polynomials, and the Legendre polynomials [Hildebrand, 1987].

The Legendre polynomials are given by the relation

$$P_r(t) = \frac{1}{d^r r!} \frac{d^r}{dt^r} \left(t^2 - 1\right)^r \tag{5.1}$$

that is often called the *Rodrigues formula* for $P_r(t)$. From the preceding equation, it follows that

$$\int_{-1}^{1} P_r(t)\, P_s(t)\, dt = 0, \qquad r \neq s$$

where r and s are nonnegative integers.

5.2 Jacobi polynomials

The Jacobi orthogonal polynomial defined on $[0,1]$ as follows

$$P_i^{(\mu,\kappa)}(\tau) = \sum_{j=0}^{i} \left(\begin{array}{c} i+\mu \\ j \end{array} \right) \left(\begin{array}{c} i+\kappa \\ 1-j \end{array} \right) (\tau - 1)^{1-j} \tau^j,$$

with $\mu, \kappa \in \left]-1, +\infty\right[$, $\langle \cdot, \cdot \rangle_{\mu,\kappa}$ is a $\mathcal{L}^2([0,1])$ scalar product with the associated weight function $\omega_{\mu,\kappa}(\tau) = (1-\tau)^\mu \tau^\kappa$, and the associated norm

$\left\| P_i^{(\mu,\kappa)} \right\|_{\mu,\kappa}^2 = \frac{1}{2i+\mu+\kappa+1} \frac{\Gamma(\mu+i+1)\,\Gamma(\kappa+i+1)}{\Gamma(\mu+\kappa+i+1)\,\Gamma(i+1)}$, $\Gamma(\cdot)$ is the classical Gamma function.

In the following table the behavior of the Jacobi differentiator according the selection of its parameters

Table1. Parameter selection in Jacobi differentiator

Parameter	Amplitude Error	Time delay	Noise contribution
$\kappa \uparrow$	$\nearrow$	$\nearrow$	$\nearrow$
$\mu \uparrow$	$\searrow$	$\searrow$	$\nearrow$
$q \uparrow$	$\searrow$	$\searrow$	$\nearrow$
$T \uparrow$	$\nearrow$	$\nearrow$	$\searrow$