

# Trefftz-DG Approximation for the Elasto-Acoustics

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# Trefftz-DG Approximation for the Elasto-Acoustics

DIP workshop

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December 14, 2016

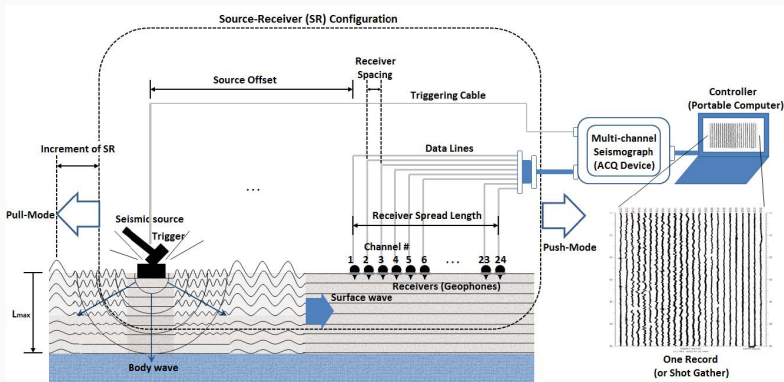
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# Abstract

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# Seismic survey

Figure 1: Schematic of overall field setup for a seismic survey<sup>1</sup>.



<sup>1</sup> Park Seismic LLC. <http://www.parkseismic.com>. Internet resource.

# Basic numerical methods

Table 1: Generic properties of the most widely used numerical methods<sup>2</sup>.

Numerical method	Complex geometries	High-order accuracy and <i>hp</i> -adaptivity	Explicit semi-discrete form	Conservation laws	Elliptic problems
FDM	●	●	●	●	●
FVM	●	●	●	●	⊙
FEM	●	●	●	⊙	●
DG-FEM	●	●	●	●	⊙

<sup>2</sup> J.S.Hesthaven T.Warburton. Nodal discontinuous galerkin methods. Algorithms, analysis, and applications. Texts in Applied Mathematics, (54):1-370, 2007.

## DG-FEM :



- Adapted to the complex geometries ;  
High-order accuracy and  $hp$ -adaptivity ;  
Explicit semi-discrete form ;  
Conservation laws ;



- Higher number of degrees of freedom  
comparing to the methods with continuous approximation.

## Trefftz method <sup>3</sup> :

Given a region of an Euclidean space of some partitions of that region, a **Trefftz Method** is any procedure for solving boundary value problems of partial differential equations or systems of such equations, on such region, **using solutions of that differential equation or its adjoint**, defined in its subregions.

<sup>3</sup> I.Herrera. Trefftz method: a general theory. Instituto de Investigaciones en Matematicas Aplicadas y en Sistemas (IIMAS), pages 562-580, 2000.

## **Time-harmonic formulations :**

O.Cassenat, B.Despres (1998);

C.Farhat, I.Harari, U.Hetmaniuk (2003); G.Gabard (2007);

T.Huttunen, P.Monk, J.P.Kaipo (2002);

R.Tezaur, C.Farhat (2006);

## **Time-domain formulations :**

F.Kretzschmar, A.Moiola, I.Perugia (2015);

F.Kretzschmar, S.M.Schnepp, I.Tsukerman, T.Weiland(2014);

H.Egger, F.Kretzschmar, S.M.Schnepp, T.Weiland (2014).



## Expected advantages of Trefftz method :



Better orders of convergence ;

Flexibility in the choice of basis functions ;

Low dispersion ;

Incorporation of wave propagation directions in the discrete space ;

Adaptivity and local space-time mesh refinement.

<sup>3</sup> I.Herrera. Trefftz method: a general theory. Instituto de Investigaciones en Matematicas Aplicadas y en Sistemas (IIMAS), pages 562-580, 2000.

# **Mathematical formulation.**

## **Fluid case**

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# Acoustic system. Problem equation

$\Omega_F \subset R^n$  - space domain ;

$I = (0, T)$  - time domain ;

$Q_F := \Omega_F \times I$  ;

$n_{Q_F} = (n_{Q_F}^x, n_{Q_F}^t)$  - o.p. unit normal vector on  $\partial Q_F$  ;

## Acoustic system

$$\frac{1}{c_f^2 \rho_f} \frac{\partial p}{\partial t} + \operatorname{div} v_f = f, \text{ in } Q_F;$$

$$\rho_f \frac{\partial v_f}{\partial t} + \nabla p = 0, \text{ in } Q_F;$$

$$v_f(\cdot, 0) = v_{f0}, \quad p(\cdot, 0) = p_0, \text{ on } \Omega_F;$$

$$v_f = g_f^D, \text{ on } \partial \Omega_F \times I.$$

$$K_F \subset Q_F (c_f, \rho_f \equiv \text{const. in } K_F) ;$$

$$n_{K_F} = (n_{K_F}^x, n_{K_F}^t) - \text{o.p. normal vector on } \partial K_F ;$$

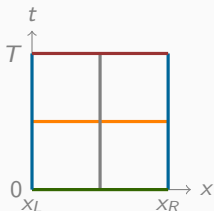
$$v_f, p \in H^1(K_F) ;$$

$$\omega_f, q \in H^1(K_F) ;$$

## Space-time integration

$$\begin{aligned} & - \int_{K_F} (v_f \cdot (\rho_f \frac{\partial \omega_f}{\partial t} + \nabla q) + p (\frac{1}{c_f^2 \rho_f} \frac{\partial q}{\partial t} + \text{div} \omega_f)) dv + \\ & + \int_{\partial K_F} ((p \omega_f + v_f q) \cdot n_{K_F}^x + (\frac{1}{c_f^2 \rho_f} p q + \rho_f v_f \cdot \omega_f) n_{K_F}^t) ds = \iint_{K_F} f q dv. \end{aligned}$$

## Mesh and DG-notation



$\mathcal{T}_h$  - mesh on  $Q_F := \Omega_F \times I$

$\mathcal{F}_h = \cup_{K_F \in \mathcal{T}_h} \partial K_F$  - mesh skeleton :

$\mathcal{F}_h^{\Omega_F} :=$  union of the internal  $\Omega$ -like faces, ( $t \equiv \text{const.}$ ) ;

$\mathcal{F}_h^{I_F} :=$  union of the internal  $I$ -like faces, ( $x \equiv \text{const.}$ ) ;

$\mathcal{F}_h^{0_F} := \Omega_F \times \{0\}$  ;

$\mathcal{F}_h^{T_F} := \Omega_F \times \{T\}$  ;

$\mathcal{F}_h^{D_F} := \partial\Omega_F \times [0, T]$  - union of the Dirichlet boundary faces.

# Acoustic system. Space-time DG formulation

Seek  $(v_{fh}, p_h) \in \mathbf{V}(\mathcal{T}_h) \subset H^1(\mathcal{T}_h)^2$ , s.t. for all  $K_F \in \mathcal{T}_h$ ,  $(\omega_f, q) \in \mathbf{V}(\mathcal{T}_h)$  it holds:

## DG formulation

$$\begin{aligned} & - \int_{K_F} \left[ p_h \left( \frac{1}{c_f^2 \rho_f} \frac{\partial q}{\partial t} + \operatorname{div} \omega_f \right) + v_{fh} \cdot \left( \rho_f \frac{\partial \omega_f}{\partial t} + \nabla q \right) \right] dv \\ & + \int_{\partial K_F} \left[ \left( \frac{1}{c_f^2 \rho_f} \hat{p}_h q + \rho_f \hat{v}_{fh} \cdot \omega_f \right) n_{K_F}^t + (\hat{p}_h \omega_f + \hat{v}_{fh} q) \cdot n_{K_F}^x \right] ds = \int_{K_F} f q dv. \end{aligned}$$

## Trefftz space

$$\mathbf{T}_F(\mathcal{T}_h) := \left\{ (\omega_f, q) \in H^1(\mathcal{T}_h)^2, \text{ s. t. in all } K_F \in \mathcal{T}_h \right. \\ \left. \rho_f \frac{\partial \omega_f}{\partial t} + \nabla q = 0, \frac{1}{c_f^2 \rho_f} \frac{\partial q}{\partial t} + \operatorname{div} \omega_f = 0 \right\}.$$

# Acoustic system. Space-time Trefftz-DG formulation

Seek  $(v_{fh}, p_h) \in \mathbf{V}(\mathcal{T}_h) \subset H^1(\mathcal{T}_h)^2$ , s.t. for all  $K_F \in \mathcal{T}_h$ ,  $(\omega_f, q) \in \mathbf{V}(\mathcal{T}_h)$  it holds:

**Trefftz-DG formulation (  $f \equiv 0$  )**

$$\int_{\partial K_F} \left[ \left( \frac{1}{c_f^2 \rho_f} \hat{p}_h q + \rho_f \hat{v}_{fh} \cdot \omega_f \right) n_{K_F}^t + (\hat{p}_h \omega_f + \hat{v}_{fh} q) \cdot n_{K_F}^x \right] ds = 0$$



Numerical flux  $\hat{v}_{fh}$  and  $\hat{\rho}_h$  :

$$\text{on } \mathcal{F}_h^{IF}: \quad \begin{pmatrix} \hat{v}_{fh} \\ \hat{\rho}_h \end{pmatrix} := \begin{pmatrix} \{v_{fh}\} + \beta \llbracket \rho_h \rrbracket_x \\ \{\rho_h\} + \alpha \llbracket v_{fh} \rrbracket_x \end{pmatrix},$$

$$\text{on } \mathcal{F}_h^{\Omega F}: \quad \begin{pmatrix} \hat{v}_{fh} \\ \hat{\rho}_h \end{pmatrix} := \begin{pmatrix} v_{fh}^- \\ \rho_h^- \end{pmatrix},$$

$$\text{on } \mathcal{F}_h^{TF}: \quad \begin{pmatrix} \hat{v}_{fh} \\ \hat{\rho}_h \end{pmatrix} := \begin{pmatrix} v_{fh} \\ \rho_h \end{pmatrix},$$

$$\text{on } \mathcal{F}_h^{0F}: \quad \begin{pmatrix} \hat{v}_{fh} \\ \hat{\rho}_h \end{pmatrix} := \begin{pmatrix} v_{f0} \\ \rho_0 \end{pmatrix},$$

$$\text{on } \mathcal{F}_h^{DF}: \quad \begin{pmatrix} \hat{v}_{fh} \\ \hat{\rho}_h \end{pmatrix} := \begin{pmatrix} g_{DF} \\ \rho_h + \alpha(v_{fh} - g_{DF}) \cdot n_{K_F}^x \end{pmatrix},$$

$(\alpha \in L^\infty(\mathcal{F}_h^{IF} \cup \mathcal{F}_h^{DF}), \beta \in L^\infty(\mathcal{F}_h^{IF})$  - positive flux parameters)

## Trefftz-DG formulation :

$$\begin{aligned}
 & \int_{\mathcal{F}_h^{\Omega_F}} \left[ \frac{1}{c_f^2 \rho_f} p_h^- [[q]]_t + \rho_f v_{fh}^- [[\omega_f]]_t \right] ds \\
 & + \int_{\mathcal{F}_h^{IF}} \left[ \{p_h\} [[\omega_f]]_x + \{v_{fh}\} [[q]]_x + \alpha [[v_{fh}]_x] [[\omega_f]]_x + \beta [[p_h]_x] [[q]]_x \right] ds \\
 & + \int_{\mathcal{F}_h^{TF}} \left[ \frac{1}{c_f^2 \rho_f} p_h q + \rho_f v_{fh} \cdot \omega_f \right] ds - \frac{1}{2} \int_{\mathcal{F}_h^{0F}} \left[ \frac{1}{c_f^2 \rho_f} p_h q + \rho_f v_{fh} \cdot \omega_f \right] ds \\
 & + \int_{\mathcal{F}_h^{DF}} \left[ \omega_f \cdot (pn_{K_F}^x + \alpha v_{fh}) \right] ds = \\
 & \frac{1}{2} \int_{\mathcal{F}_h^{0F}} \left[ \frac{1}{c_f^2 \rho_f} p_h q + \rho_f v_{fh} \cdot \omega_f \right] ds + \int_{\mathcal{F}_h^{DF}} \left[ g_{DF} (\alpha \omega_f - q \cdot n_{K_F}^x) \right] ds.
 \end{aligned}$$

Seek  $(v_{fh}, p_h) \in \mathbf{V}(\mathcal{T}_h) \subset \mathbf{T}_F(\mathcal{T}_h)$  s.t. for all  $(\omega_f, q) \in \mathbf{V}(\mathcal{T}_h)$  it holds:

### Trefftz-DG formulation

$$\mathcal{A}_{TDG_F}((v_{fh}, p_h); (\omega_f, q)) = \ell_{TDG_F}(\omega_f, q).$$

## Trefftz-DG formulation in terms of $L^2(\mathcal{T}_h)$ -norms

$$\begin{aligned} \mathcal{A}_{TDG_F}((\omega_f, q); (\omega_f, q)) := & \\ & \frac{1}{2} \left\| \left( \frac{1}{c_f^2 \rho_f} \right)^{1/2} \llbracket \mathbf{q} \rrbracket_t \right\|_{L^2(\mathcal{F}_h^{\Omega F})}^2 + \frac{1}{2} \left\| \rho_f^{1/2} \llbracket \omega_f \rrbracket_t \right\|_{L^2(\mathcal{F}_h^{\Omega F})}^2 \\ & + \left\| \alpha^{1/2} \llbracket \omega_f \rrbracket_x \right\|_{L^2(\mathcal{F}_h^{IF})}^2 + \left\| \beta^{1/2} \llbracket \mathbf{q} \rrbracket_x \right\|_{L^2(\mathcal{F}_h^{IF})}^2 \\ & + \frac{1}{2} \left\| \left( \frac{1}{c_f^2 \rho_f} \right)^{1/2} \mathbf{q} \right\|_{L^2(\mathcal{F}_h^{TF})}^2 + \frac{1}{2} \left\| \rho_f^{1/2} \omega_f \right\|_{L^2(\mathcal{F}_h^{TF})}^2 \\ & + \left\| \alpha^{1/2} \omega_f \right\|_{L^2(\mathcal{F}_h^{DF})}^2. \end{aligned}$$

**Norm**  $||| \cdot |||_{TDGF}$  in  $\mathbf{T}_F(\mathcal{T}_h)$

$$\begin{aligned} |||(\omega_f, \mathbf{q})|||_{TDGF}^2 &:= \frac{1}{2} \left\| \left( \frac{1}{c_f^2 \rho_f} \right)^{1/2} \llbracket \mathbf{q} \rrbracket_t \right\|_{L^2(\mathcal{F}_h^{\Omega F})}^2 + \frac{1}{2} \left\| \rho_f^{1/2} \llbracket \omega_f \rrbracket_t \right\|_{L^2(\mathcal{F}_h^{\Omega F})}^2 \\ &+ \left\| \alpha^{1/2} \llbracket \omega_f \rrbracket_x \right\|_{L^2(\mathcal{F}_h^{IF})}^2 + \left\| \beta^{1/2} \llbracket \mathbf{q} \rrbracket_x \right\|_{L^2(\mathcal{F}_h^{IF})}^2 \\ &+ \frac{1}{2} \left\| \left( \frac{1}{c_f^2 \rho_f} \right)^{1/2} \mathbf{q} \right\|_{L^2(\mathcal{F}_h^{TF})}^2 + \frac{1}{2} \left\| \rho_f^{1/2} \omega_f \right\|_{L^2(\mathcal{F}_h^{TF})}^2 \\ &+ \left\| \alpha^{1/2} \omega_f \right\|_{L^2(\mathcal{F}_h^{DF})}^2. \end{aligned}$$

## Coercivity of Trefftz-DG formulation

$$\mathcal{A}_{TDG_F}((\omega_f, q); (\omega_f, q)) = |||(\omega_f, q)|||_{TDG_F}^2, \quad \forall (\omega_f, q) \in \mathbf{T}_F(\mathcal{T}_h).$$

**Add-on norm  $||| \cdot |||_{TDG_F^*}$  in  $\mathbf{T}_F(\mathcal{T}_h)$**

$$\begin{aligned} |||(\omega_f, \mathbf{q})|||_{TDG_F^*}^2 &:= |||(\omega_f, \mathbf{q})|||_{TDG_F}^2 \\ &+ \|\rho_f^{1/2} \omega_f^-\|_{L^2(\mathcal{F}_h^{\Omega_F})}^2 + \left\| \left( \frac{1}{c_f^2 \rho_f} \right)^{1/2} \mathbf{q}^- \right\|_{L^2(\mathcal{F}_h^{\Omega_F})}^2 \\ &+ \|\beta^{-1/2} \{\omega_f\}\|_{L^2(\mathcal{F}_h^{I_F})}^2 + \|\alpha^{-1/2} \{\mathbf{q}\}\|_{L^2(\mathcal{F}_h^{I_F})}^2 \\ &+ \|\alpha^{-1/2} \omega_f\|_{L^2(\mathcal{F}_h^{D_F})}^2. \end{aligned}$$

## Continuity of Trefftz-DG formulation

$$|\mathcal{A}_{TDG_F}((v_f, p); (\omega_f, q))| \leq 2 \| (v_f, p) \|_{TDG_F^*} \| (\omega_f, q) \|_{TDG_F},$$

$$|\ell_{TDG_F}(\omega_f, q)| \leq \sqrt{2} \left[ \|\rho_f^{1/2} v_{f0}\|_{L^2(\mathcal{F}_h^{0F})}^2 + \left\| \left( \frac{1}{c_f^2 \rho_f} \right)^{1/2} p_0 \right\|_{L^2(\mathcal{F}_h^{0F})}^2 \right]^{1/2}.$$



## Well-posedness

$$\| (v_f - v_{fh}, p - p_h) \|_{TDG_F} \leq 3 \inf_{(\omega_f, q) \in \mathbf{V}(T_h)} \| (v_f - \omega_f, p - q) \|_{TDG_F^*}.$$

# **Mathematical formulation.**

## **Solid case**

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# Elastodynamic system. Problem equation

$\Omega_S \subset R^n$  - space domain ;

$I = (0, T)$  - time domain ;

$Q_S := \Omega_S \times I$  ;

$n_{Q_S} = (n_{Q_S}^x, n_{Q_S}^t)$  - o.p. unit normal vector on  $\partial Q_S$  ;

## Elastodynamic system

$$\underline{\underline{\frac{\partial \sigma}{\partial t}}} - \underline{\underline{C}} \underline{\underline{\varepsilon}}(v_s) = 0 \text{ in } Q_S;$$

$$\rho_s \frac{\partial v_s}{\partial t} - \operatorname{div} \underline{\underline{\sigma}} = 0 \text{ in } Q_S;$$

$$v_s(\cdot, 0) = v_{s0}, \quad \underline{\underline{\sigma}}(\cdot, 0) = \underline{\underline{\sigma}}_0 \text{ on } \Omega_S;$$

$$v_s = g_{D_S} \text{ on } \partial \Omega_S \times I.$$

# Elastodynamic system. Problem equation

$\Omega_S \subset R^n$  - space domain ;

$I = (0, T)$  - time domain ;

$Q_S := \Omega_S \times I$  ;

$n_{Q_S} = (n_{Q_S}^x, n_{Q_S}^t)$  - o.p. unit normal vector on  $\partial Q_S$  ;

## Elastodynamic system

$$\underline{A} \frac{\partial \underline{\underline{\sigma}}}{\partial t} - \underline{\underline{\varepsilon}}(v_s) = 0 \text{ in } Q_S;$$

$$\rho_s \frac{\partial v_s}{\partial t} - \operatorname{div} \underline{\underline{\sigma}} = 0 \text{ in } Q_S;$$

$$v_s(\cdot, 0) = v_{s0}, \underline{\underline{\sigma}}(\cdot, 0) = \underline{\underline{\sigma}}_0 \text{ on } \Omega_S;$$

$$v_s = g_{D_S} \text{ on } \partial \Omega_S \times I.$$

$$K_S \subset Q_S \quad (\underline{A}, \rho_s \equiv \text{const. in } K_S) ;$$

$$n_{K_S} = (n_{K_S}^x, n_{K_S}^t) - \text{o.p. unit normal vector on } \partial K_S ;$$

$$v_s, \underline{\sigma} \in H^1(K_S) ;$$

$$\omega_s, \underline{\xi} \in H^1(K_S) ;$$

## Space-time integration

$$\begin{aligned}
 & - \int_{K_S} \left[ \underline{\sigma} : \left( \underline{A} \frac{\partial \underline{\xi}}{\partial t} - \underline{\varepsilon}(\omega_s) \right) + v_s \cdot \left( \rho_s \frac{\partial \omega_s}{\partial t} - \text{div} \underline{\xi} \right) \right] dv \\
 & + \int_{\partial K_S} \left[ \left( \underline{A} \underline{\sigma} : \underline{\xi} + \rho_s v_s \cdot \omega_s \right) \cdot n_{K_S}^t - \left( v_s \cdot \underline{\xi} + \underline{\sigma} \cdot \omega_s \right) \cdot n_{K_S}^x \right] ds = 0.
 \end{aligned}$$

## Trefftz space

$$\mathbf{T}_S(\mathcal{T}_h) := \left\{ (\omega_s, \underline{\underline{\xi}}) \in H^1(\mathcal{T}_h)^2, \text{ s. t. in all } K_S \in \mathcal{T}_h \right. \\ \left. \rho_s \frac{\partial \omega_s}{\partial t} - \operatorname{div} \underline{\underline{\xi}} = 0, \underline{\underline{A}} \frac{\partial \underline{\underline{\xi}}}{\partial t} - \underline{\underline{\varepsilon}}(\omega_s) = 0 \right\}.$$

Seek  $(v_{sh}, \underline{\underline{\sigma}}_h) \in \mathbf{V}(\mathcal{T}_h) \subset H^1(\mathcal{T}_h)^2$ , s.t. for all  $K_S \in \mathcal{T}_h$ ,  $(\omega_s, \underline{\underline{\xi}}) \in \mathbf{V}(\mathcal{T}_h)$  it holds:

## Space-time Trefftz-DG formulation

$$\int_{\partial K_S} \left[ (\underline{\underline{A}} \hat{\underline{\underline{\sigma}}}_h : \underline{\underline{\xi}} + \rho_s \hat{v}_{sh} \cdot w_s) \cdot n_K^t - (\hat{v}_{sh} \underline{\underline{\xi}} + \hat{\underline{\underline{\sigma}}}_h \omega_s) \cdot n_K^x \right] ds = 0$$

## Trefftz-DG formulation

$$\begin{aligned}
 & \int_{\mathcal{F}_h^{\Omega_S}} \left[ \underline{\underline{A}} \underline{\underline{\sigma}}_h^- : \underline{\underline{[\xi]}}_t + \rho_s v_{sh}^- \underline{\underline{[\omega_s]}}_t \right] ds \\
 & - \int_{\mathcal{F}_h^{I_S}} \left[ \{ \underline{\underline{\sigma}}_h \} \underline{\underline{[\omega_s]}}_x + \{ v_{sh} \} \underline{\underline{[\xi]}}_x - \gamma \underline{\underline{[v_{sh}]}}_x \underline{\underline{[\omega_s]}}_x - \delta \underline{\underline{[\sigma_h]}}_x \underline{\underline{[\xi]}}_x \right] ds \\
 & + \int_{\mathcal{F}_h^{T_S}} \left[ \underline{\underline{A}} \underline{\underline{\sigma}}_h : \underline{\underline{\xi}} + \rho_s v_{sh} \cdot \omega_s \right] ds - \frac{1}{2} \int_{\mathcal{F}_h^{0_S}} \left[ \underline{\underline{A}} \underline{\underline{\sigma}}_h : \underline{\underline{\xi}} + \rho_s v_{sh} \cdot \omega_s \right] ds \\
 & - \int_{\mathcal{F}_h^{D_S}} \left[ \underline{\underline{\xi}} (v_{sh} \cdot n_{K_S}^x - \delta \underline{\underline{\sigma}}_h) \right] ds = \\
 & \frac{1}{2} \int_{\mathcal{F}_h^{0_S}} \left[ \underline{\underline{A}} \underline{\underline{\sigma}}_h : \underline{\underline{\xi}} + \rho_s v_{sh} \cdot \omega_s \right] ds + \int_{\mathcal{F}_h^{D_S}} \left[ g_{D_S} (\omega_s \cdot n_{K_S}^x + \delta \underline{\underline{\xi}}) \right] ds.
 \end{aligned}$$



Seek  $(v_{sh}, \underline{\underline{\sigma}}_h) \in \mathbf{V}(\mathcal{T}_h) \subset \mathbf{T}_S(\mathcal{T}_h)$  s.t. for all  $(\omega_s, \underline{\underline{\xi}}) \in \mathbf{V}(\mathcal{T}_h)$  it holds:

## Trefftz-DG formulation

$$\mathcal{A}_{TDG_S}((v_{fh}, \underline{\underline{\sigma}}_h); (\omega_s, \underline{\underline{\xi}})) = \ell_{TDG_S}(\omega_s, \underline{\underline{\xi}}).$$

## Norms in $\mathbf{T}_S(\mathcal{T}_h)$

$$\begin{aligned}
 |||(\omega_s, \underline{\xi})|||_{TDG_S}^2 &:= \frac{1}{2} \left\| (\underline{\underline{A}})^{1/2} \llbracket \underline{\xi} \rrbracket_t \right\|_{L^2(\mathcal{F}_h^{\Omega_S})}^2 + \frac{1}{2} \left\| \rho_s^{1/2} \llbracket \omega_s \rrbracket_t \right\|_{L^2(\mathcal{F}_h^{\Omega_S})}^2 \\
 &+ \left\| \gamma^{1/2} \llbracket \omega_s \rrbracket_x \right\|_{L^2(\mathcal{F}_h^{I_S})}^2 + \left\| \delta^{1/2} \llbracket \underline{\xi} \rrbracket_x \right\|_{L^2(\mathcal{F}_h^{I_S})}^2 \\
 &+ \frac{1}{2} \left\| (\underline{\underline{A}})^{1/2} \underline{\xi} \right\|_{L^2(\mathcal{F}_h^{T_S})}^2 + \frac{1}{2} \left\| \rho_s^{1/2} \omega_s \right\|_{L^2(\mathcal{F}_h^{T_S})}^2 \\
 &+ \left\| \delta^{1/2} \underline{\xi} \right\|_{L^2(\mathcal{F}_h^{D_S})}^2.
 \end{aligned}$$

## Norms in $\mathbf{T}_S(\mathcal{T}_h)$

$$\begin{aligned} \|\underline{\omega}_s, \underline{\xi}\|_{TDG_S^*}^2 &:= \|\underline{\omega}_s, \underline{\xi}\|_{TDG_S}^2 \\ &+ \|\rho_s^{1/2} \omega_s^-\|_{L^2(\mathcal{F}_h^{\Omega_S})}^2 + \|(\underline{A})^{1/2} \underline{\xi}^-\|_{L^2(\mathcal{F}_h^{\Omega_S})}^2 \\ &+ \|\delta^{-1/2} \{\omega_s\}\|_{L^2(\mathcal{F}_h^{I_S})}^2 + \|\gamma^{-1/2} \{\underline{\xi}\}\|_{L^2(\mathcal{F}_h^{I_S})}^2 \\ &+ \|\delta^{-1/2} \underline{\xi}\|_{L^2(\mathcal{F}_h^{D_S})}^2. \end{aligned}$$

## Coercivity of Trefftz-DG formulation

$$\mathcal{A}_{TDG_S}((\omega_s, \underline{\underline{\xi}}); (\omega_s, \underline{\underline{\xi}})) = |||(\omega_s, \underline{\underline{\xi}})|||_{TDG_S}^2, \quad \forall (\omega_s, \underline{\underline{\xi}}) \in \mathbf{T}_S(\mathcal{T}_h).$$

## Continuity of Trefftz-DG formulation

$$|\mathcal{A}_{TDG_S}((v_s, \underline{\underline{\sigma}}); (\omega_s, \underline{\underline{\xi}}))| \leq 2 \| (v_s, \underline{\underline{\sigma}}) \|_{TDG_S^*} \| (\omega_s, \underline{\underline{\xi}}) \|_{TDG_S},$$

$$|\ell_{TDG_S}(\omega_s, \underline{\underline{\xi}})| \leq \sqrt{2} \left[ \|\rho_s^{1/2} v_{s0}\|_{L^2(\mathcal{F}_h^{0s})}^2 + \|\underline{\underline{A}}^{1/2} \underline{\underline{\sigma}}_0\|_{L^2(\mathcal{F}_h^{0s})}^2 \right]^{1/2}.$$

## Well-posedness

$$\| (v_s - v_{sh}, \underline{\underline{\sigma}} - \underline{\underline{\sigma}}_h) \|_{TDG_S} \leq 3 \inf_{(\omega_s, \underline{\underline{\xi}}) \in \mathbf{V}(\mathcal{T}_h)} \| (v_s - \omega_s, \underline{\underline{\sigma}} - \underline{\underline{\xi}}) \|_{TDG_S^*}.$$

**Mathematical formulation.**  
**Fluid-solid case**

---

Representation of the **acoustic medium** as a limit case of an **elastic isotropic medium** with shear modulus  $\mu$  tending or equal to 0 ?



## Advantages of the numerical coupling:



Computing 1 unknown scalar pressure instead of 6 components of stress tensor ;

Avoiding presence of the numerical artifacts caused of the slow  $S$ -waves appearance.

## Acoustic system

$$\frac{1}{c_f^2 \rho_f} \frac{\partial p}{\partial t} + \operatorname{div} v_f = f, \text{ in } Q_F;$$

$$\rho_f \frac{\partial v_f}{\partial t} + \nabla p = 0, \text{ in } Q_F;$$

$$v_f = g_{D_F}, \text{ on } \partial\Omega_F \times I;$$

## Elastodynamic system

$$\underline{\underline{A}} \frac{\partial \underline{\underline{\sigma}}}{\partial t} - \underline{\underline{\varepsilon}}(v_s) = 0 \text{ in } Q_S;$$

$$\rho_s \frac{\partial v_s}{\partial t} - \operatorname{div} \underline{\underline{\sigma}} = 0 \text{ in } Q_S;$$

$$v_s = g_{D_S} \text{ on } \partial\Omega_S \times I;$$

### Fluid-solid transmission conditions through the $\Gamma_{FS}$

$$v_f \cdot n_{\Omega}^x = v_s \cdot n_{\Omega}^x \text{ on } \Gamma_{FS};$$

$$\underline{\underline{\sigma}} n_{\Omega}^x = -pn_{\Omega}^x \text{ on } \Gamma_{FS}.$$

## Trefftz space

$$\begin{aligned} \mathbf{T}(\mathcal{T}_h) := & \left\{ (\omega_f, q, \omega_s, \underline{\underline{\xi}}) \in H^1(\mathcal{T}_h)^4, \text{ s. t.} \right. \\ & \rho_f \frac{\partial \omega_f}{\partial t} + \nabla q = 0, \quad \frac{1}{c_f^2 \rho_f} \frac{\partial q}{\partial t} + \operatorname{div} \omega_f = 0, \quad \forall K_F \in \mathcal{T}_h, \\ & \left. \rho_s \frac{\partial \omega_s}{\partial t} - \operatorname{div} \underline{\underline{\xi}} = 0, \quad \frac{\partial A \underline{\underline{\xi}}}{\partial t} - \underline{\underline{\xi}}(\omega_s) = 0, \quad \forall K_S \in \mathcal{T}_h \right\}. \end{aligned}$$

## Coupled system. Space-time Trefftz-DG formulation

Seek  $(v_{fh}, p_h, v_{sh}, \underline{\underline{\sigma}}_h) \in \mathbf{V}(\mathcal{T}_h) \subset \mathbf{T}(\mathcal{T}_h)$ , s. t. for all  $K_F, K_S \in \mathcal{T}_h$ ,  
 $(\omega_f, q, \omega_s, \underline{\underline{\xi}}) \in \mathbf{V}(\mathcal{T}_h)$  it holds :

### Trefftz-DG formulation ( $f \equiv 0$ )

$$\int_{\partial K_F} \left[ \left( \frac{1}{c_f^2 \rho_f} \hat{p}_h q + \rho_f \hat{v}_{fh} \cdot \omega_f \right) n_{K_F}^t + (\hat{p}_h \omega_f + \hat{v}_{fh} q) \cdot n_{K_F}^x \right] ds = 0,$$

$$\int_{\partial K_S} \left[ (\underline{\underline{A}} \hat{\underline{\underline{\sigma}}}_h : \underline{\underline{\xi}} + \rho_s \hat{v}_{sh} \cdot w_s) \cdot n_K^t - (\hat{v}_{sh} \underline{\underline{\xi}} + \hat{\underline{\underline{\sigma}}}_h \omega_s) \cdot n_K^x \right] ds = 0$$

Numerical flux  $\hat{v}_{fh}$ ,  $\hat{v}_{sh}$ ,  $\hat{p}_h$  and  $\hat{\underline{\sigma}}_h$  through  $\mathcal{F}_h^{FS}$ :

$$\text{on } \mathcal{F}_h^{FS} : \quad \begin{pmatrix} \hat{v}_{fh} \cdot n_{K_F}^x \\ \hat{p}_h \\ \hat{v}_{sh} \\ \hat{\underline{\sigma}}_h \cdot n_{K_S}^x \end{pmatrix} := \begin{pmatrix} v_{sh} \cdot n_{K_F}^x \\ p + \alpha(v_{fh} - v_{sh}) \cdot n_{K_F}^x \\ v_{sh} - \delta(\underline{\underline{\sigma}}_h - p_h) \cdot n_{K_S}^x \\ -p_h n_{K_S}^x \end{pmatrix}$$

Seek  $(v_{fh}, p_h, v_{sh}, \underline{\underline{\sigma}}_h) \in \mathbf{V}(\mathcal{T}_h) \subset \mathbf{T}(\mathcal{T}_h)$  s.t.  $\forall (\omega_f, q, \omega_s, \underline{\underline{\xi}}) \in \mathbf{V}(\mathcal{T}_h)$  it holds:

### Trefftz-DG formulation

$$\mathcal{A}_{TDG}((v_{fh}, p_h, v_{sh}, \underline{\underline{\sigma}}_h); (\omega_f, q, \omega_s, \underline{\underline{\xi}})) = \ell_{TDG}(\omega_f, q, \omega_s, \underline{\underline{\xi}})$$

## Norms in $\mathbf{T}(\mathcal{T}_h)$

$$|||(\omega_f, \mathbf{q}, \omega_s, \underline{\underline{\xi}})|||_{TDG}^2 := |||(\omega_f, \mathbf{q})|||_{TDG_F}^2 + |||(\omega_s, \underline{\underline{\xi}})|||_{TDG_S}^2 + 2\|\delta^{1/2}\underline{\underline{\xi}}\|_{L^2(\mathcal{F}_h^{FS})}^2$$

$$|||(\omega_f, \mathbf{q}, \omega_s, \underline{\underline{\xi}})|||_{TDG^*}^2 := |||(\omega_f, \mathbf{q})|||_{TDG_F^*}^2 + |||(\omega_s, \underline{\underline{\xi}})|||_{TDG_S^*}^2 + \frac{1}{2}\|\delta^{-1/2}\underline{\underline{\xi}}\|_{L^2(\mathcal{F}_h^{FS})}^2$$



## Coercivity of Trefftz-DG formulation

$$\mathcal{A}_{TDG}((\omega_f, \mathbf{q}, \omega_s, \underline{\underline{\xi}}); (\omega_f, \mathbf{q}, \omega_s, \underline{\underline{\xi}})) = |||(\omega_f, \mathbf{q}, \omega_s, \underline{\underline{\xi}})|||_{TDG}^2$$

## Continuity of Trefftz-DG formulation

$$|\mathcal{A}_{TDG}((\mathbf{v}_f, \mathbf{p}, \mathbf{v}_s, \underline{\underline{\sigma}}); (\omega_f, \mathbf{q}, \omega_s, \underline{\underline{\xi}}))| \leq 2 |||(\mathbf{v}_f, \mathbf{p}, \mathbf{v}_s, \underline{\underline{\sigma}})|||_{TDG} * |||(\omega_f, \mathbf{q}, \omega_s, \underline{\underline{\xi}})|||_{TDG},$$

$$|\ell_{TDG}(\omega_f, \mathbf{q}, \omega_s, \underline{\underline{\xi}})| \leq \sqrt{2} \left[ \|\rho_f^{1/2} \mathbf{v}_{f0}\|_{L^2(\mathcal{F}_h^{0F})}^2 + \left\| \left( \frac{1}{c_f^2 \rho_f} \right)^{1/2} \mathbf{p}_0 \right\|_{L^2(\mathcal{F}_h^{0F})}^2 \right. \\ \left. + \|\rho_s^{1/2} \mathbf{v}_{s0}\|_{L^2(\mathcal{F}_h^{0S})}^2 + \|\underline{\underline{A}}^{1/2} \underline{\underline{\sigma}}_0\|_{L^2(\mathcal{F}_h^{0S})}^2 \right]^{1/2}.$$

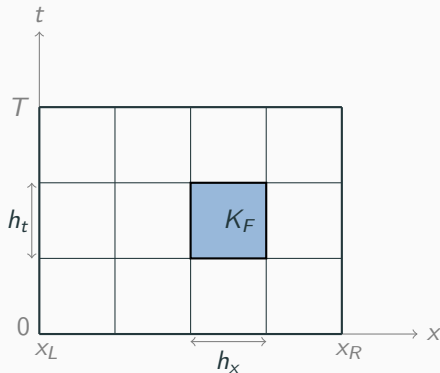
## Well-posedness

$$\begin{aligned} & |||(v_f - v_{fh}, p - p_h, v_s - v_{sh}, \underline{\underline{\sigma}} - \underline{\underline{\sigma}}_h)|||_{TDG} \\ & \leq 3 \inf_{(\omega_f, q, \omega_s, \underline{\underline{\xi}}) \in \mathbf{v}(\mathcal{T}_h)} |||(v_f - \omega_f, p - q, v_s - \omega_s, \underline{\underline{\sigma}} - \underline{\underline{\xi}})|||_{TDG^*}. \end{aligned}$$

# **Implementation of the algorithm. Acoustic system**

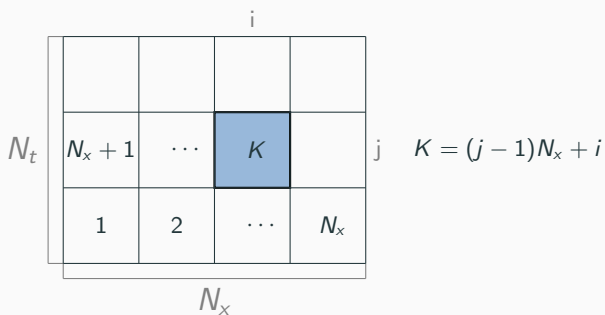
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# Implementation of the algorithm



**Figure 2:** Rectangular mesh on  $Q_F := [x_L, x_R] \times [0, T]$ .

# Implementation of the algorithm



**Figure 3:** Element numbering.

# Implementation of the algorithm

Seek  $(v_{fh}, p_h) \in \mathbf{V}(\mathcal{T}_h) \subset \mathbf{T}_F(\mathcal{T}_h)$  s.t. for all  $(\omega_f, q) \in \mathbf{V}(\mathcal{T}_h)$  it holds:

## Treftz-DG formulation

$$\begin{aligned} & \int_{\mathcal{F}_h^{\Omega_F}} \left[ \frac{1}{c_f^2 \rho_f} p_h^- \llbracket q \rrbracket_t + \rho_f v_{fh}^- \llbracket \omega_f \rrbracket_t \right] ds \\ & + \int_{\mathcal{F}_h^{I_F}} \left[ \{p_h\} \llbracket \omega_f \rrbracket_x + \{v_{fh}\} \llbracket q \rrbracket_x + \alpha \llbracket v_{fh} \rrbracket_x \llbracket \omega_f \rrbracket_x + \beta \llbracket p_h \rrbracket_x \llbracket q \rrbracket_x \right] ds \\ & + \int_{\mathcal{F}_h^{T_F}} \left[ \frac{1}{c_f^2 \rho_f} p_h q + \rho_f v_{fh} \cdot \omega_f \right] ds = \int_{\mathcal{F}_h^{0_F}} \left[ \frac{1}{c_f^2 \rho_f} p_0 q + \rho_f v_{f0} \cdot \omega_f \right] ds. \end{aligned}$$

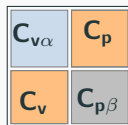
# Implementation of the algorithm

Treftz-DG formulation. 1<sup>st</sup> layer

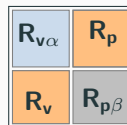
$$\begin{aligned} & [v_{K-1} \cdot \mathbf{L}_{v\alpha} + p_{K-1} \cdot \mathbf{L}_p] + [v_K \cdot \mathbf{C}_{v\alpha} + p_K \cdot \mathbf{C}_p] + [v_{K+1} \cdot \mathbf{R}_{v\alpha} + p_{K+1} \cdot \mathbf{R}_p] \\ & [v_{K-1} \cdot \mathbf{L}_v + p_{K-1} \cdot \mathbf{L}_{p\beta}] + [v_K \cdot \mathbf{C}_v + p_K \cdot \mathbf{C}_{p\beta}] + [v_{K+1} \cdot \mathbf{R}_v + p_{K+1} \cdot \mathbf{R}_{p\beta}] \end{aligned}$$



"Left" block



"Central" block



"Right" block

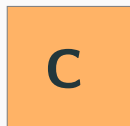
# Implementation of the algorithm

Treftz-DG formulation. 1<sup>st</sup> layer

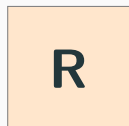
$$\begin{aligned} & [v_{K-1} \cdot \mathbf{L}_{v\alpha} + p_{K-1} \cdot \mathbf{L}_p] + [v_K \cdot \mathbf{C}_{v\alpha} + p_K \cdot \mathbf{C}_p] + [v_{K+1} \cdot \mathbf{R}_{v\alpha} + p_{K+1} \cdot \mathbf{R}_p] \\ & [v_{K-1} \cdot \mathbf{L}_v + p_{K-1} \cdot \mathbf{L}_{p\beta}] + [v_K \cdot \mathbf{C}_v + p_K \cdot \mathbf{C}_{p\beta}] + [v_{K+1} \cdot \mathbf{R}_v + p_{K+1} \cdot \mathbf{R}_{p\beta}] \end{aligned}$$



"Left" block



"Central" block



"Right" block



# Implementation of the algorithm

$$\begin{bmatrix} \boxed{C} & \boxed{R} & & \dots & & & \boxed{L} \\ \boxed{L} & \boxed{C} & \boxed{R} & & & & \\ & \boxed{L} & \boxed{C} & \boxed{R} & & & \\ \vdots & & & \ddots & & & \vdots \\ & & & & \boxed{L} & \boxed{C} & \boxed{R} \\ \boxed{R} & & \dots & & \boxed{L} & \boxed{C} \end{bmatrix} \cdot \begin{bmatrix} \boxed{U_1} \\ \boxed{U_2} \\ \boxed{U_3} \\ \vdots \\ \boxed{U_{N-2}} \\ \boxed{U_{N-1}} \end{bmatrix} = \begin{bmatrix} \boxed{U_1^0} \\ \boxed{U_2^0} \\ \boxed{U_3^0} \\ \vdots \\ \boxed{U_{N-2}^0} \\ \boxed{U_{N-1}^0} \end{bmatrix}$$

## Implementation of the algorithm

$$\begin{bmatrix} \text{M} \\ 32N_x \times 32N_x \end{bmatrix} \cdot \begin{bmatrix} \text{U} \\ 1 \times 32N_x \end{bmatrix} = \begin{bmatrix} \text{U}^0 \\ 1 \times 32N_x \end{bmatrix}$$

# Implementation of the algorithm

## 1. Initialization

$$\begin{bmatrix} \mathbf{M} \end{bmatrix} ; \begin{bmatrix} \mathbf{invM} \end{bmatrix} ; \begin{bmatrix} \mathbf{U}^0 \end{bmatrix} ;$$

## 2. Propagation

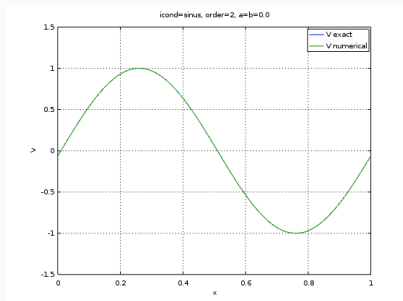
For  $t = 1 : N_t$

$$\begin{bmatrix} \mathbf{U} \end{bmatrix} = \begin{bmatrix} \mathbf{invM} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{U}^0 \end{bmatrix} ;$$

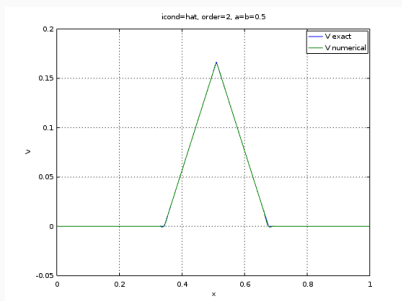
$$\begin{bmatrix} \mathbf{U}^0 \end{bmatrix} = \begin{bmatrix} \mathbf{U} \end{bmatrix} ;$$

end

# Implementation of the algorithm

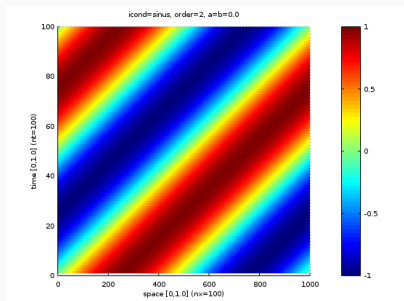


**Figure 4:** Exact and approximate solution for velocity  $v_f(x, t)$ ,  $t = \Delta t \times 1$ , on the first time-layer ( $\alpha = \beta = 0.5$ ).

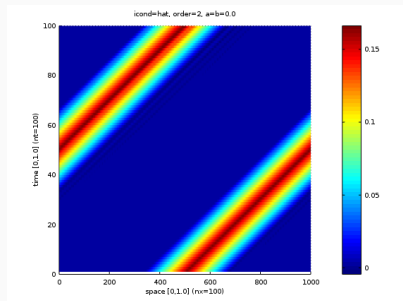


**Figure 5:** Exact and approximate solution for velocity  $v_f(x, t)$ ,  $t = \Delta t \times 1$ , on the first time-layer ( $\alpha = \beta = 0.5$ ).

# Implementation of the algorithm

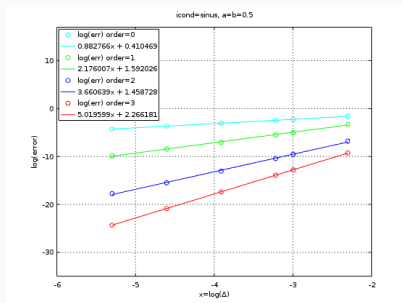


**Figure 6:** Time propagation of the approximate solution for velocity  $v_f(x, t)$  ( $\alpha = \beta = 0.5$ ).

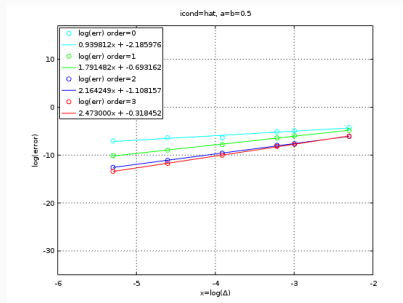


**Figure 7:** Time propagation of the approximate solution for velocity  $v_f(x, t)$  ( $\alpha = \beta = 0.5$ ).

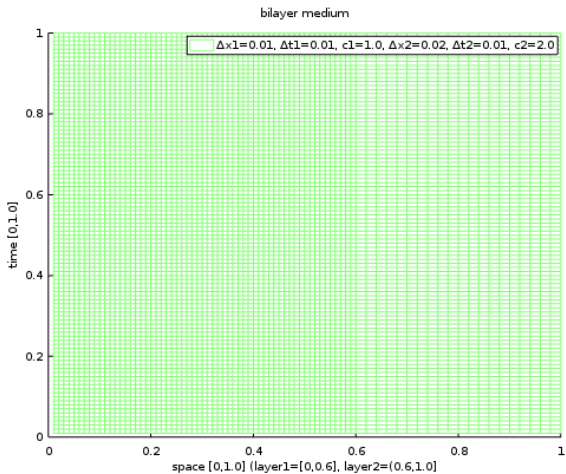
# Implementation of the algorithm



**Figure 8:** Convergence of numerical solution for velocity  $v_f(x, t)$  in logarithmic scale ( $\alpha = \beta = 0.5$ ).



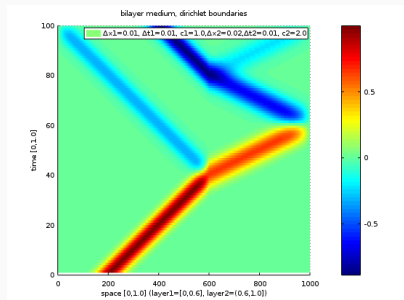
**Figure 9:** Convergence of numerical solution for velocity  $v_f(x, t)$  in logarithmic scale ( $\alpha = \beta = 0.5$ ).



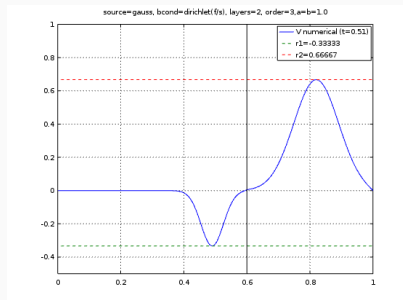
**Figure 10:** Mesh for heterogeneous case (bilayer medium).

$\Delta x_1 = 0.01, \Delta t_1 = 0.01, c_1 = 1.0, \Delta x_2 = 0.02, \Delta t_2 = 0.01, c_1 = 2.0.$

# Implementation of the algorithm



**Figure 11:** Approximate solution for velocity  $v(x, t)$ .



**Figure 12:** Approximate solution for velocity  $v(x, t)$ ,  $t = \Delta t \times 51$ .



## Conclusion

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## On-going work :



Numerical implementation of the method for 1D AS ;  
Different choice of the basis functions ;  
Analysis of efficiency of the method / algorithm / code.

## Perspectives :



- Numerical implementation of the method for ES 1D,2D,3D ;
- Numerical coupling 1D,2D,3D ;
- Coupling Trefftz-DG in solid with FVM in fluid,  
integrating of the transmission layer.

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**Thank you for your attention!**

**Questions?**

# Implementation of the algorithm

## Polynomial waves basis functions

- Generating functions  $g^V(x, t)$  and  $g^P(x, t)$  - solution of wave equation if  $a^2 = b^2$

$$g^V(a, b, x, t) = e^{i(ax+bc_f t)}, \quad g^P(a, b, x, t) = -c_f e^{i(bx+ac_f t)}.$$

- Taylor expansions for  $g^V(x, t)$ ,  $g^P(x, t)$  ( $a^2 = b^2$ ):

$$e^{i(ax+bc_f t)} = \sum_{n=0}^{\infty} \sum_{k=0, k < 2}^n Q_{n-k, k}^V(x, t) a^{n-k} b^k,$$
$$-c_f e^{i(bx+ac_f t)} = \sum_{n=0}^{\infty} \sum_{k=0, k < 2}^n Q_{n-k, k}^P(x, t) a^{n-k} b^k.$$

$$R_{n-k, k}^V(x, t) = \Re(Q_{n-k, k}^V(x, t)), \quad I_{n-k, k}^V(x, t) = \Im(Q_{n-k, k}^V(x, t))$$
$$R_{n-k, k}^P(x, t) = \Re(Q_{n-k, k}^P(x, t)), \quad I_{n-k, k}^P(x, t) = \Im(Q_{n-k, k}^P(x, t)).$$

## Polynomial waves basis functions 1D

$$\begin{array}{llll} \phi_1^v = 0 & \phi_2^v = 1 & \phi_3^v = x & \phi_4^v = c_f t \\ \phi_5^v = -\frac{x^2}{2} - \frac{c_f^2 t^2}{2} & \phi_6^v = -c_f x t & \phi_7^v = -\frac{x^3}{6} - \frac{x c_f^2 t^2}{2} & \phi_8^v = -\frac{c_f^3 t^3}{6} - \frac{x^2 c_f t}{2} \\ \\ \phi_1^p = -c_f & \phi_2^p = 0 & \phi_3^p = -c_f^2 t & \phi_4^p = -c_f x \\ \phi_5^p = c_f^2 x t & \phi_6^p = c_f \left( \frac{x^2}{2} + \frac{c_f^2 t^2}{2} \right) & \phi_7^p = c_f \left( \frac{c_f^3 t^3}{6} + \frac{x^2 c_f t}{2} \right) & \phi_8^p = c_f \left( \frac{x^3}{6} + \frac{x c_f^2 t^2}{2} \right). \end{array}$$



# Implementation of the algorithm

## Polynomial waves basis functions 2D

<b>p = 0</b>		
$\phi_1^{vx} = 0$ $\phi_2^{vx} = 1$	$\phi_1^{vy} = 0$ $\phi_2^{vy} = 1$	$\phi_1^p = -c_f$ $\phi_2^p = 0$
<b>p = 1</b>		
$\phi_3^{vx} = x$ $\phi_4^{vx} = c_f t$ $\phi_5^{vx} = y$	$\phi_3^{vy} = y$ $\phi_4^{vy} = x$ $\phi_5^{vy} = c_f t$	$\phi_3^p = -c_f(c_f t)$ $\phi_4^p = -c_f x$ $\phi_5^p = -c_f y$
<b>p = 2</b>		
$\phi_7^{vx} = -\frac{x^2}{2} - \frac{y^2}{2} - \frac{c_f^2 t^2}{2}$ $\phi_8^{vx} = -xy$ $\phi_9^{vx} = -c_f t y$ $\phi_{10}^{vx} = -c_f t x$	$\phi_7^{vy} = -xy$ $\phi_8^{vy} = -\frac{x^2}{2} - \frac{y^2}{2} - \frac{c_f^2 t^2}{2}$ $\phi_9^{vy} = -c_f t x$ $\phi_{10}^{vy} = -c_f t y$	$\phi_7^p = c_f(c_f t x)$ $\phi_8^p = c_f(c_f t y)$ $\phi_9^p = c_f(x y)$ $\phi_{10}^p = c_f\left(\frac{x^2}{2} + \frac{y^2}{2} + \frac{c_f^2 t^2}{2}\right)$
<b>p = 3</b>		
$\phi_{11}^{vx} = -\frac{x^2 y}{2} - \frac{c_f^2 t^2 y}{2} - \frac{y^3}{6}$ $\phi_{12}^{vx} = -c_f t x y$ $\phi_{13}^{vx} = -\frac{c_f^3 t^3}{6} - \frac{c_f t x^2}{2} - \frac{c_f t y^2}{2}$ $\phi_{14}^{vx} = -\frac{x y^2}{2} - \frac{c_f^2 t^2 x}{2} - \frac{x^3}{6}$	$\phi_{11}^{vy} = -\frac{x y^2}{2} - \frac{c_f^2 t^2 x}{2} - \frac{x^3}{6}$ $\phi_{12}^{vy} = -\frac{c_f^3 t^3}{6} - \frac{c_f t x^2}{2} - \frac{c_f t y^2}{2}$ $\phi_{13}^{vy} = -c_f t y$ $\phi_{14}^{vy} = -\frac{x^2 y}{2} - \frac{c_f^2 t^2 y}{2} - \frac{y^3}{6}$	$\phi_{11}^p = c_f(c_f t x y)$ $\phi_{12}^p = c_f\left(\frac{x^2 y}{2} + \frac{c_f^2 t^2 y}{2} + \frac{y^3}{6}\right)$ $\phi_{13}^p = c_f\left(\frac{x y^2}{2} + \frac{c_f^2 t^2 x}{2} + \frac{x^3}{6}\right)$ $\phi_{14}^p = c_f\left(\frac{c_f^3 t^3}{6} + \frac{c_f t x^2}{2} + \frac{c_f t y^2}{2}\right)$