Trefftz-DG Approximation for the Elasto-Acoustics

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DIP workshop

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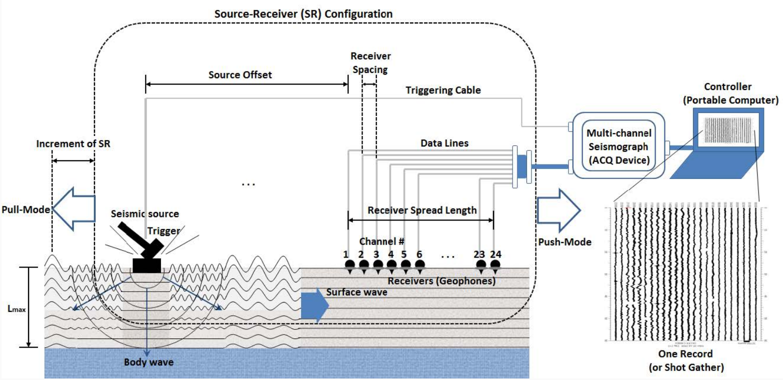
Abstract

Seismic survey

Figure 1: Schematic of overall field setup for a seismic survey[1].



[1] Park Seismic LLC. http://www.parkseismic.com. Internet resource.

Basic numerical methods

Table 1: Generic properties of the most widely used numerical methods[2].

Numerical method	Complex geometries	High-order accuracy and *hp*-adaptivity	Explicit semi-discrete form	Conservation laws	Elliptic problems
FDM	● (grey)	●	●	●	●
FVM	●	● (grey)	●	●	⊙
FEM	●	●	● (grey)	⊙	●
DG-FEM	●	●	●	●	⊙

[2] J.S.Hesthaven T.Warburton. Nodal discontinuous galerkin methods. Algorithms, analysis, and applications. Texts in Applied Mathematics, (54):1-370, 2007.

DG-FEM :

Adapted to the complex geometries ;

High-order accuracy and *hp*-adaptivity ;

Explicit semi-discrete form ;

Conservation laws ;

Higher number of degrees of freedom

comparing to the methods with continuous approximation.

Trefftz method [3] :

Given a region of an Euclidean space of some partitions of that region, a **Trefftz Method** is any procedure for solving boundary value problems of partial differential equations or systems of such equations, on such region, **using solutions of that differential equation or its adjoint**, defined in its subregions.

[3] I.Herrera. Trefftz method: a general theory. Instituto de Investigaciones en Matematicas Aplicadas y en Sistemas (IIMAS), pages 562-580, 2000.

Time-harmonic formulations :

O.Cassenat, B.Despres (1998);

C.Farhat, I.Harari, U.Hetmaniuk (2003); G.Gabard (2007);

T.Huttunen, P.Monk, J.P.Kaipo (2002);

R.Tezaur, C.Farhat (2006);

Time-domain formulations :

F.Kretzschmar, A.Moiola, I.Perugia (2015);

F.Kretzschmar, S.M.Schnepp, I.Tsukerman, T.Weiland(2014);

H.Egger, F.Kretzschmar, S.M.Schnepp, T.Weiland (2014).

Trefftz method

Expected advantages of Trefftz method :

Better orders of convergence ;

Flexibility in the choice of basis functions ;

Low dispersion ;

Incorporation of wave propagation directions in the discrete space ;

Adaptivity and local space-time mesh refinement.

[3] I.Herrera. Trefftz method: a general theory. Instituto de Investigaciones en Matematicas Aplicadas y en Sistemas (IIMAS), pages 562-580, 2000.

Mathematical formulation. Fluid case

Acoustic system. Problem equation

$\Omega_F \subset R^n$ - space domain ;

$I = (0, T)$ - time domain ;

$Q_F := \Omega_F \times I$;

$n_{Q_F} = (n^x_{Q_F}, n^t_{Q_F})$ - o.p. unit normal vector on ∂Q_F ;

Acoustic system

$$\frac{1}{c_f^2 \rho_f} \frac{\partial p}{\partial t} + \mathrm{div} v_f = f, \text{ in } Q_F;$$

$$\rho_f \frac{\partial v_f}{\partial t} + \nabla p = 0, \text{ in } Q_F;$$

$$v_f(\cdot, 0) = v_{f0}, \; p(\cdot, 0) = p_0, \text{ on } \Omega_F;$$

$$v_f = g_f^D, \text{ on } \partial\Omega_F \times I.$$

Acoustic system. Variational formulation

$K_F \subset Q_F$ (c_f, $\rho_f \equiv const.$ in K_F) ;

$n_{K_F} = (n^x_{K_F}, n^t_{K_F})$ - o.p. normal vector on ∂K_F ;

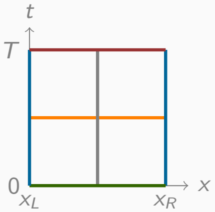
v_f, $p \in H^1(K_F)$;

ω_f, $q \in H^1(K_F)$;

Space-time integration

$$-\int_{K_F} (v_f \cdot (\rho_f \frac{\partial \omega_f}{\partial t} + \nabla q) + p(\frac{1}{c_f^2 \rho_f} \frac{\partial q}{\partial t} + \mathrm{div}\omega_f)) d\mathrm{v} +$$
$$+ \int_{\partial K_F} ((p\omega_f + v_f q) \cdot n^x_{K_F} + (\frac{1}{c_f^2 \rho_f} pq + \rho_f v_f \cdot \omega_f) n^t_{K_F}) d\mathrm{s} = \iint_{K_F} fq d\mathrm{v}.$$

Mesh and DG-notation



$\mathcal{T}_h$ - mesh on $Q_F := \Omega_F \times I$

$\mathcal{F}_h = \cup_{K_F \in \mathcal{T}_h} \partial K_F$ - mesh skeleton :

$\mathcal{F}_h^{\Omega_F}$:= union of the internal Ω-like faces, $(t \equiv const.)$;

$\mathcal{F}_h^{I_F}$:= union of the internal I-like faces, $(x \equiv const.)$;

$\mathcal{F}_h^{0_F} := \Omega_F \times \{0\}$;

$\mathcal{F}_h^{T_F} := \Omega_F \times \{T\}$;

$\mathcal{F}_h^{D_F} := \partial\Omega_F \times [0, T]$ - union of the Dirichlet boundary faces.

Acoustic system. Space-time DG formulation

Seek $(v_{f\,h}, p_h) \subset \mathbf{V}(\mathcal{T}_h) \subset H^1(\mathcal{T}_h)^2$, s.t. for all $K_F \in \mathcal{T}_h$, $(\omega_f, q) \subset \mathbf{V}(\mathcal{T}_h)$ it holds:

DG formulation

$$-\int_{K_F} \Big[p_h \Big(\frac{1}{c_f^2 \rho_f} \frac{\partial q}{\partial t} + \mathrm{div}\omega_f \Big) + v_{f\,h} \cdot \Big(\rho_f \frac{\partial \omega_f}{\partial t} + \nabla q \Big) \Big] d\mathrm{v}$$

$$+ \int_{\partial K_F} \Big[\Big(\frac{1}{c_f^2 \rho_f} \hat{p}_h q + \rho_f \hat{v_{f\,h}} \cdot \omega_f \Big) n^t_{K_F} + \big(\hat{p}_h \omega_f + \hat{v_{f\,h}} q \big) \cdot n^x_{K_F} \Big] d\mathrm{s} = \int_{K_F} f q \, d\mathrm{v}.$$

Acoustic system. Trefftz-DG formulation

Trefftz space

$$\mathbf{T}_F(\mathcal{T}_h) := \Big\{(\omega_f, q) \subset H^1(\mathcal{T}_h)^2,\ s.\,t.\ in\,all\ K_F \in \mathcal{T}_h$$

$$\rho_f \frac{\partial \omega_f}{\partial t} + \nabla q = 0,\ \frac{1}{c_f^2 \rho_f} \frac{\partial q}{\partial t} + \mathrm{div}\omega_f = 0\Big\}.$$

Acoustic system. Space-time Trefftz-DG formulation

Seek $(v_{f\,h}, p_h) \subset \mathbf{V}(\mathcal{T}_h) \subset H^1(\mathcal{T}_h)^2$, s.t. for all $K_F \in \mathcal{T}_h$, $(\omega_f, q) \subset \mathbf{V}(\mathcal{T}_h)$ it holds:

Trefftz-DG formulation ($f \equiv 0$)

$$\int_{\partial K_F} \Big[\big(\frac{1}{c_f^2 \rho_f} \hat{p_h} q + \rho_f \hat{v_{f\,h}} \cdot \omega_f \big) n^t_{K_F} + (\hat{p_h} \omega_f + \hat{v_{f\,h}} q) \cdot n^x_{K_F} \Big] ds = 0$$

Acoustic system. Space-time Trefftz-DG formulation

Numerical flux $\hat{v_f}_h$ and $\hat{p}_h$:

$$\text{on } \mathcal{F}_h^{I_F}: \quad \begin{pmatrix} \hat{v_f}_h \\ \hat{p}_h \end{pmatrix} := \begin{pmatrix} \{v_{f\,h}\} + \beta [\![p_h]\!]_x \\ \{p_h\} + \alpha [\![v_{f\,h}]\!]_x \end{pmatrix},$$

$$\text{on } \mathcal{F}_h^{\Omega_F}: \quad \begin{pmatrix} \hat{v_f}_h \\ \hat{p}_h \end{pmatrix} := \begin{pmatrix} {v_{f\,h}}^- \\ {p_h}^- \end{pmatrix},$$

$$\text{on } \mathcal{F}_h^{T_F}: \quad \begin{pmatrix} \hat{v_f}_h \\ \hat{p}_h \end{pmatrix} := \begin{pmatrix} v_{f\,h} \\ p_h \end{pmatrix},$$

$$\text{on } \mathcal{F}_h^{0_F}: \quad \begin{pmatrix} \hat{v_f}_h \\ \hat{p}_h \end{pmatrix} := \begin{pmatrix} v_{f0} \\ p_0 \end{pmatrix},$$

$$\text{on } \mathcal{F}_h^{D_F}: \quad \begin{pmatrix} \hat{v_f}_h \\ \hat{p}_h \end{pmatrix} := \begin{pmatrix} \hat{g_{D_F}} \\ p_h + \alpha (v_{f\,h} - g_{D_F}) \cdot n_{K_F}^x \end{pmatrix},$$

($\alpha \in L^\infty(\mathcal{F}_h^{I_F} \cup \mathcal{F}_h^{D_F})$, $\beta \in L^\infty(\mathcal{F}_h^{I_F})$ - positive flux parameters)

Trefftz-DG formulation :

$$
\begin{aligned}
&\int_{\mathcal{F}_h^{\Omega_F}} \left[\frac{1}{c_f^2\rho_f} p_h{}^- [\![q]\!]_t + \rho_f v_{f\,h}{}^- [\![\omega_f]\!]_t\right] d\mathrm{s} \\
&+ \int_{\mathcal{F}_h^{I_F}} \left[\{p_h\}[\![\omega_f]\!]_x + \{v_{f\,h}\}[\![q]\!]_x + \alpha[\![v_{f\,h}]\!]_x[\![\omega_f]\!]_x + \beta[\![p_h]\!]_x[\![q]\!]_x\right] d\mathrm{s} \\
&+ \int_{\mathcal{F}_h^{T_F}} \left[\frac{1}{c_f^2\rho_f} p_h q + \rho_f v_{f\,h}\cdot\omega_f\right] d\mathrm{s} - \frac{1}{2}\int_{\mathcal{F}_h^{0_F}} \left[\frac{1}{c_f^2\rho_f} p_h q + \rho_f v_{f\,h}\cdot\omega_f\right] d\mathrm{s} \\
&+ \int_{\mathcal{F}_h^{D_F}} \left[\omega_f\cdot(p n_{K_F}^x + \alpha v_{f\,h})\right] d\mathrm{s} = \\
&\frac{1}{2}\int_{\mathcal{F}_h^{0_F}} \left[\frac{1}{c_f^2\rho_f} p_h q + \rho_f v_{f\,h}\cdot\omega_f\right] d\mathrm{s} + \int_{\mathcal{F}_h^{D_F}} \left[\mathrm{g}_{D_F}(\alpha\omega_f - q\cdot n_{K_F}^x)\right] d\mathrm{s}.
\end{aligned}
$$

Acoustic system. Trefftz-DG formulation

Seek $(v_{f\,h}, p_h) \subset \mathbf{V}(\mathcal{T}_h) \subset \mathbf{T}_F(\mathcal{T}_h)$ s.t. for all $(\omega_f, q) \subset \mathbf{V}(\mathcal{T}_h)$ it holds:

Trefftz-DG formulation

$$\mathcal{A}_{TDG_F}((v_{f\,h}, p_h); (\omega_f, q)) = \ell_{TDG_F}(\omega_f, q).$$

Trefftz-DG formulation in terms of $L^2(\mathcal{T}_h)$-norms

$$
\begin{aligned}
&\mathcal{A}_{TDG_F}((\omega_f, q); (\omega_f, q)) := \\
&\qquad \frac{1}{2}\left\| \left(\frac{1}{c_f^2 \rho_f}\right)^{1/2} [\![q]\!]_t \right\|^2_{L^2(\mathcal{F}_h^{\Omega_F})} + \frac{1}{2}\left\| \rho_f^{1/2} [\![\omega_f]\!]_t \right\|^2_{L^2(\mathcal{F}_h^{\Omega_F})} \\
&\qquad + \left\| \alpha^{1/2} [\![\omega_f]\!]_x \right\|^2_{L^2(\mathcal{F}_h^{I_F})} + \left\| \beta^{1/2} [\![q]\!]_x \right\|^2_{L^2(\mathcal{F}_h^{I_F})} \\
&\qquad + \frac{1}{2}\left\| \left(\frac{1}{c_f^2 \rho_f}\right)^{1/2} q \right\|^2_{L^2(\mathcal{F}_h^{T_F})} + \frac{1}{2}\left\| \rho_f^{1/2} \omega_f \right\|^2_{L^2(\mathcal{F}_h^{T_F})} \\
&\qquad + \left\| \alpha^{1/2} \omega_f \right\|^2_{L^2(\mathcal{F}_h^{D_F})}.
\end{aligned}
$$

Acoustic system. Analysis of Trefftz-DG formulation

Norm $|||\cdot|||_{TDG_F}$ in $\mathbf{T}_F(\mathcal{T}_h)$

$$
\begin{aligned}
|||(\omega_f, q)|||^2_{TDG_F} :=& \frac{1}{2}\left\|\left(\frac{1}{c_f^2\rho_f}\right)^{1/2}[\![q]\!]_t\right\|^2_{L^2(\mathcal{F}_h^{\Omega_F})} + \frac{1}{2}\left\|\rho_f^{1/2}[\![\omega_f]\!]_t\right\|^2_{L^2(\mathcal{F}_h^{\Omega_F})} \\
&+ \left\|\alpha^{1/2}[\![\omega_f]\!]_x\right\|^2_{L^2(\mathcal{F}_h^{I_F})} + \left\|\beta^{1/2}[\![q]\!]_x\right\|^2_{L^2(\mathcal{F}_h^{I_F})} \\
&+ \frac{1}{2}\left\|\left(\frac{1}{c_f^2\rho_f}\right)^{1/2} q\right\|^2_{L^2(\mathcal{F}_h^{T_F})} + \frac{1}{2}\left\|\rho_f^{1/2}\omega_f\right\|^2_{L^2(\mathcal{F}_h^{T_F})} \\
&+ \left\|\alpha^{1/2}\omega_f\right\|^2_{L^2(\mathcal{F}_h^{D_F})}.
\end{aligned}
$$

Acoustic system. Analysis of Trefftz-DG formulation

Coercivity of Trefftz-DG formulation

$$\mathcal{A}_{TDG_F}((\omega_f, q); (\omega_f, q)) = |||(\omega_f, q)|||^2_{TDG_F}, \ \forall (\omega_f, q) \in \mathbf{T}_F(\mathcal{T}_h).$$

Acoustic system. Analysis of Trefftz-DG formulation

Add-on norm $|||\cdot|||_{TDG_F^*}$ in $\mathbf{T}_F(\mathcal{T}_h)$

$$
\begin{aligned}
|||(\omega_f, q)|||^2_{TDG_F^*} := & |||(\omega_f, q)|||^2_{TDG_F} \\
& + \|\rho_f^{1/2}\omega_f^-\|^2_{L^2(\mathcal{F}_h^{\Omega_F})} + \|(\frac{1}{c_f^2\rho_f})^{1/2} q^-\|^2_{L^2(\mathcal{F}_h^{\Omega_F})} \\
& + \|\beta^{-1/2}\{\omega_f\}\|^2_{L^2(\mathcal{F}_h^{I_F})} + \|\alpha^{-1/2}\{q\}\|^2_{L^2(\mathcal{F}_h^{I_F})} \\
& + \|\alpha^{-1/2}\omega_f\|^2_{L^2(\mathcal{F}_h^{D_F})}.
\end{aligned}
$$

Acoustic system. Analysis of Trefftz-DG formulation

Continuity of Trefftz-DG formulation

$$|\mathcal{A}_{TDG_F}((v_f, p); (\omega_f, q))| \leq 2\, |||(v_f, p)|||_{TDG_F^*}\, |||(\omega_f, q)|||_{TDG_F},$$

$$|\ell_{TDG_F}(\omega_f, q)| \leq \sqrt{2}\left[\|\rho_f^{1/2} v_{f0}\|^2_{L^2(\mathcal{F}_h^{0F})} + \|(\frac{1}{c_f^2 \rho_f})^{1/2} p_0\|^2_{L^2(\mathcal{F}_h^{0F})}\right]^{1/2}.$$

Acoustic case. Analysis of Trefftz-DG formulation

Well-posedness

$$|||(v_f - v_{f\,h}, p - p_h)|||_{TDG_F} \leq 3 \inf_{(\omega_f, q) \in \mathbf{V}(\mathcal{T}_h)} |||(v_f - \omega_f, p - q)|||_{TDG_F^*}.$$

Mathematical formulation. Solid case

Elastodynamic system. Problem equation

$\Omega_S \subset R^n$ - space domain ;

$I = (0, T)$ - time domain ;

$Q_S := \Omega_S \times I$;

$n_{Q_S} = (n^x_{Q_S}, n^t_{Q_S})$ - o.p. unit normal vector on ∂Q_S ;

Elastodynamic system

$$\frac{\partial \underline{\underline{\sigma}}}{\partial t} - \underline{\underline{C}}\,\underline{\underline{\varepsilon}}(v_s) = 0 \text{ in } Q_S;$$

$$\rho_s \frac{\partial v_s}{\partial t} - \mathrm{div}\underline{\underline{\sigma}} = 0 \text{ in } Q_S;$$

$$v_s(\cdot, 0) = v_{s0},\ \underline{\underline{\sigma}}(\cdot, 0) = \underline{\underline{\sigma}}_0 \text{ on } \Omega_S;$$

$$v_s = \mathrm{g}_{D_S} \text{ on } \partial\Omega_S \times I.$$

Elastodynamic system. Problem equation

$\Omega_S \subset R^n$ - space domain ;

$I = (0, T)$ - time domain ;

$Q_S := \Omega_S \times I$;

$n_{Q_S} = (n^x_{Q_S}, n^t_{Q_S})$ - o.p. unit normal vector on ∂Q_S ;

Elastodynamic system

$$\underline{\underline{A}} \frac{\partial \underline{\underline{\sigma}}}{\partial t} - \underline{\underline{\varepsilon}}(v_s) = 0 \text{ in } Q_S;$$
$$\rho_s \frac{\partial v_s}{\partial t} - \mathrm{div} \underline{\underline{\sigma}} = 0 \text{ in } Q_S;$$

$$v_s(\cdot, 0) = v_{s0}, \ \underline{\underline{\sigma}}(\cdot, 0) = \underline{\underline{\sigma}}_0 \text{ on } \Omega_S;$$
$$v_s = \mathrm{g}_{D_S} \text{ on } \partial\Omega_S \times I.$$

Elastodynamic system. Variational formulation

$K_S \subset Q_S$ ($\underline{\underline{A}}$, $\rho_s \equiv const.$ in K_S) ;

$n_{K_S} = (n^x_{K_S}, n^t_{K_S})$ - o.p. unit normal vector on ∂K_S ;

v_s, $\underline{\underline{\sigma}} \in H^1(K_S)$;

ω_s, $\underline{\underline{\xi}} \in H^1(K_S)$;

Space-time integration

$$-\int_{K_S} \left[\underline{\underline{\sigma}} : (\underline{\underline{A}} \frac{\partial \underline{\underline{\xi}}}{\partial t} - \underline{\underline{\varepsilon}}(\omega_s)) + v_s \cdot (\rho_s \frac{\partial \omega_s}{\partial t} - \mathrm{div} \underline{\underline{\xi}}) \right] dv$$

$$+ \int_{\partial K_S} \left[(\underline{\underline{A}}\,\underline{\underline{\sigma}} : \underline{\underline{\xi}} + \rho_s v_s \cdot w_s) \cdot n^t_{K_S} - (v_s \cdot \underline{\underline{\xi}} + \underline{\underline{\sigma}} \cdot \omega_s) \cdot n^x_{K_S} \right] ds = 0.$$

Elastodynamic system. Space-time Trefftz-DG formulation

Trefftz space

$$\mathbf{T}_S(\mathcal{T}_h) := \Big\{(\omega_s, \underline{\underline{\xi}}) \subset H^1(\mathcal{T}_h)^2,\ s.\,t.\ \textit{in all}\ K_S \in \mathcal{T}_h$$
$$\rho_s \frac{\partial \omega_s}{\partial t} - \mathrm{div}\underline{\underline{\xi}} = 0,\ \underline{\underline{A}}\frac{\partial \underline{\underline{\xi}}}{\partial t} - \underline{\underline{\varepsilon}}(\omega_s) = 0\Big\}.$$

Elastodynamic system. Space-time Trefftz-DG formulation

Seek $(v_{s\,h}, \underline{\underline{\sigma}}_h) \subset \mathbf{V}(\mathcal{T}_h) \subset H^1(\mathcal{T}_h)^2$, s.t. for all $K_S \in \mathcal{T}_h$, $(\omega_s, \underline{\underline{\xi}}) \subset \mathbf{V}(\mathcal{T}_h)$ it holds:

Space-time Trefftz-DG formulation

$$\int_{\partial K_S} \left[(\underline{\underline{A}}\,\hat{\underline{\underline{\sigma}}}_h : \underline{\underline{\xi}} + \rho_s \hat{v_{s\,h}} \cdot w_s) \cdot n_K^t - (\hat{v_{s\,h}} \underline{\underline{\xi}} + \hat{\underline{\underline{\sigma}}}_h \omega_s) \cdot n_K^x \right] d\mathrm{s} = 0$$

Trefftz-DG formulation

$$\int_{\mathcal{F}_h^{\Omega_S}} \left[\underline{\underline{A}}\,\underline{\underline{\sigma}}_h^{-} : [\![\underline{\underline{\xi}}]\!]_t + \rho_s v_{s\,h}^{-}[\![\omega_s]\!]_t\right] d\mathrm{s}$$
$$- \int_{\mathcal{F}_h^{I_S}} \left[\{\underline{\underline{\sigma}}_h\}[\![\omega_s]\!]_x + \{v_{s\,h}\}[\![\underline{\underline{\xi}}]\!]_x - \gamma[\![v_{s\,h}]\!]_x[\![\omega_s]\!]_x - \delta[\![\underline{\underline{\sigma}}_h]\!]_x[\![\underline{\underline{\xi}}]\!]_x\right] d\mathrm{s}$$
$$+ \int_{\mathcal{F}_h^{T_S}} \left[\underline{\underline{A}}\,\underline{\underline{\sigma}}_h : \underline{\underline{\xi}} + \rho_s v_{s\,h} \cdot \omega_s\right] d\mathrm{s} - \frac{1}{2}\int_{\mathcal{F}_h^{0_S}} \left[\underline{\underline{A}}\,\underline{\underline{\sigma}}_h : \underline{\underline{\xi}} + \rho_s v_{s\,h} \cdot \omega_s\right] d\mathrm{s}$$
$$- \int_{\mathcal{F}_h^{D_S}} \left[\underline{\underline{\xi}}(v_{s\,h} \cdot n_{K_S}^x - \delta\underline{\underline{\sigma}}_h)\right] d\mathrm{s} =$$
$$\frac{1}{2}\int_{\mathcal{F}_h^{0_S}} \left[\underline{\underline{A}}\,\underline{\underline{\sigma}}_h : \underline{\underline{\xi}} + \rho_s v_{s\,h} \cdot \omega_s\right] d\mathrm{s} + \int_{\mathcal{F}_h^{D_S}} \left[\mathrm{g}_{D_S}(\omega_s \cdot n_{K_S}^x + \delta\underline{\underline{\xi}})\right] d\mathrm{s}.$$

Elastodynamic system. Space-time Trefftz-DG formulation

Seek $(v_{s\,h}, \underline{\underline{\sigma}}_h) \subset \mathbf{V}(\mathcal{T}_h) \subset \mathbf{T}_S(\mathcal{T}_h)$ s.t. for all $(\omega_s, \underline{\underline{\xi}}) \subset \mathbf{V}(\mathcal{T}_h)$ it holds:

Trefftz-DG formulation

$$\mathcal{A}_{TDG_S}((v_{f\,h}, \underline{\underline{\sigma}}_h); (\omega_s, \underline{\underline{\xi}})) = \ell_{TDG_S}(\omega_s, \underline{\underline{\xi}}).$$

Elastodynamic system. Analysis of Trefftz-DG formulation

Norms in $\mathbf{T}_S(\mathcal{T}_h)$

$$
\begin{aligned}
|||(\omega_s, \underline{\underline{\xi}})|||^2_{TDG_S} :=& \frac{1}{2}\left\|\left(\underline{\underline{A}}\right)^{1/2}[\![\underline{\underline{\xi}}]\!]_t\right\|^2_{L^2(\mathcal{F}_h^{\Omega_S})} + \frac{1}{2}\left\|\rho_s^{1/2}[\![\omega_s]\!]_t\right\|^2_{L^2(\mathcal{F}_h^{\Omega_S})} \\
&+ \left\|\gamma^{1/2}[\![\omega_s]\!]_x\right\|^2_{L^2(\mathcal{F}_h^{I_S})} + \left\|\delta^{1/2}[\![\underline{\underline{\xi}}]\!]_x\right\|^2_{L^2(\mathcal{F}_h^{I_S})} \\
&+ \frac{1}{2}\left\|\left(\underline{\underline{A}}\right)^{1/2}\underline{\underline{\xi}}\right\|^2_{L^2(\mathcal{F}_h^{T_S})} + \frac{1}{2}\left\|\rho_s^{1/2}\omega_s\right\|^2_{L^2(\mathcal{F}_h^{T_S})} \\
&+ \left\|\delta^{1/2}\underline{\underline{\xi}}\right\|^2_{L^2(\mathcal{F}_h^{D_S})}.
\end{aligned}
$$

Elastodynamic system. Analysis of Trefftz-DG formulation

Norms in $\mathbf{T}_S(\mathcal{T}_h)$

$$
\begin{aligned}
|||(\omega_s, \underline{\underline{\xi}})|||^2_{TDG_S^*} :=& |||(\omega_s, \underline{\underline{\xi}})|||^2_{TDG_S} \\
&+ \|\rho_s^{1/2}\omega_s^-\|^2_{L^2(\mathcal{F}_h^{\Omega_S})} + \|(\underline{\underline{A}})^{1/2}\underline{\underline{\xi}}^-\|^2_{L^2(\mathcal{F}_h^{\Omega_S})} \\
&+ \|\delta^{-1/2}\{\omega_s\}\|^2_{L^2(\mathcal{F}_h^{I_S})} + \|\gamma^{-1/2}\{\underline{\underline{\xi}}\}\|^2_{L^2(\mathcal{F}_h^{I_S})} \\
&+ \|\delta^{-1/2}\underline{\underline{\xi}}\|^2_{L^2(\mathcal{F}_h^{D_S})}.
\end{aligned}
$$

Elastodynamic system. Analysis of Trefftz-DG formulation

Coercivity of Trefftz-DG formulation

$$\mathcal{A}_{TDG_S}((\omega_s, \underline{\underline{\xi}}); (\omega_s, \underline{\underline{\xi}})) = |||(\omega_s, \underline{\underline{\xi}})|||^2_{TDG_S}, \ \forall (\omega_s, \underline{\underline{\xi}}) \in \mathbf{T}_S(\mathcal{T}_h).$$

Continuity of Trefftz-DG formulation

$$|\mathcal{A}_{TDG_S}((v_s, \underline{\underline{\sigma}}); (\omega_s, \underline{\underline{\xi}}))| \leq 2\, |||(v_s, \underline{\underline{\sigma}})|||_{TDG_S^*}\, |||(\omega_s, \underline{\underline{\xi}})|||_{TDG_S},$$

$$|\ell_{TDG_S}(\omega_s, \underline{\underline{\xi}})| \leq \sqrt{2}\left[\|\rho_s^{1/2} v_{s0}\|^2_{L^2(\mathcal{F}_h^{0S})} + \|\underline{\underline{A}}^{1/2}\underline{\underline{\sigma}}_0\|^2_{L^2(\mathcal{F}_h^{0S})}\right]^{1/2}.$$

Elastodynamic system. Analysis of Trefftz-DG formulation

Well-posedness

$$|||(v_s - v_{s\,h}, \underline{\underline{\sigma}} - \underline{\underline{\sigma}}_h)|||_{TDG_S} \leq 3 \inf_{(\omega_s, \underline{\underline{\xi}}) \in \mathbf{V}(\mathcal{T}_h)} |||(v_s - \omega_s, \underline{\underline{\sigma}} - \underline{\underline{\xi}})|||_{TDG_S^*}.$$

Mathematical formulation. Fluid-solid case

Representation of the **acoustic medium** as a limit case of an **elastic isotropic medium** with shear modulus μ tending or equal to 0 ?

Advantages of the numerical coupling:

- Computing 1 unknown scalar pressure instead of 6 components of stress tensor ;

  Avoiding presence of the numerical artifacts caused of the slow S-waves appearance.

Coupled system. Problem equation

Acoustic system

$$\frac{1}{c_f^2 \rho_f} \frac{\partial p}{\partial t} + \mathrm{div} v_f = f, \text{ in } Q_F;$$

$$\rho_f \frac{\partial v_f}{\partial t} + \nabla p = 0, \text{ in } Q_F;$$

$$v_f = \mathrm{g}_{D_F}, \text{ on } \partial\Omega_F \times I;$$

Elastodynamic system

$$\underline{\underline{A}} \frac{\partial \underline{\underline{\sigma}}}{\partial t} - \underline{\underline{\varepsilon}}(v_s) = 0 \text{ in } Q_S;$$

$$\rho_s \frac{\partial v_s}{\partial t} - \mathrm{div} \underline{\underline{\sigma}} = 0 \text{ in } Q_S;$$

$$v_s = \mathrm{g}_{D_S} \text{ on } \partial\Omega_S \times I;$$

Coupled system. Problem equation

Fluid-solid transmission conditions through the Γ_{FS}

$$v_f \cdot n_\Omega^x = v_s \cdot n_\Omega^x \text{ on } \Gamma_{FS};$$
$$\underline{\underline{\sigma}} n_\Omega^x = -p n_\Omega^x \text{ on } \Gamma_{FS}.$$

Coupled system. Space-time Trefftz-DG formulation

Trefftz space

$$\mathbf{T}(\mathcal{T}_h) := \Big\{(\omega_f, q, \omega_s, \underline{\underline{\xi}}) \subset H^1(\mathcal{T}_h)^4,\ s.t.$$

$$\rho_f \frac{\partial \omega_f}{\partial t} + \nabla q = 0, \quad \frac{1}{c_f^2 \rho_f} \frac{\partial q}{\partial t} + \mathrm{div}\omega_f = 0, \quad \forall\, K_F \in \mathcal{T}_h,$$

$$\rho_s \frac{\partial \omega_s}{\partial t} - \mathrm{div}\underline{\underline{\xi}} = 0, \quad \frac{\partial \underline{\underline{A}}\,\underline{\underline{\xi}}}{\partial t} - \underline{\underline{\varepsilon}}(\omega_s) = 0, \quad \forall\, K_S \in \mathcal{T}_h\Big\}.$$

Coupled system. Space-time Trefftz-DG formulation

Seek $(v_{f\,h}, p_h, v_{s\,h}, \underline{\underline{\sigma}}_h) \subset \mathbf{V}(\mathcal{T}_h) \subset \mathbf{T}(\mathcal{T}_h)$, s. t. for all $K_F, K_S \in \mathcal{T}_h$, $(\omega_f, q, \omega_s, \underline{\underline{\xi}}) \subset \mathbf{V}(\mathcal{T}_h)$ it holds :

Trefftz-DG formulation ($f \equiv 0$)

$$\int_{\partial K_F} \Big[(\frac{1}{c_f^2 \rho_f} \hat{p}_h q + \rho_f \hat{v_{f\,h}} \cdot \omega_f) n^t_{K_F} + (\hat{p}_h \omega_f + \hat{v_{f\,h}} q) \cdot n^x_{K_F} \Big] d\mathrm{s} = 0,$$

$$\int_{\partial K_S} \Big[(\underline{\underline{A}}\, \hat{\underline{\underline{\sigma}}}_h : \underline{\underline{\xi}} + \rho_s \hat{v_{s\,h}} \cdot w_s) \cdot n^t_K - (\hat{v_{s\,h}} \underline{\underline{\xi}} + \hat{\underline{\underline{\sigma}}}_h \omega_s) \cdot n^x_K \Big] d\mathrm{s} = 0$$

Coupled system. Space-time DG formulation

Numerical flux $\hat{v_f}_h$, $\hat{v_s}_h$, $\hat{p}_h$ and $\hat{\underline{\underline{\sigma}}}_h$ through $\mathcal{F}_h^{FS}$:

$$\text{on } \mathcal{F}_h^{FS}: \qquad \begin{pmatrix} \hat{v_f}_h \cdot n^x_{K_F} \\ \hat{p}_h \\ \hat{v_s}_h \\ \hat{\underline{\underline{\sigma}}}_h \cdot n^x_{K_S} \end{pmatrix} := \begin{pmatrix} v_{s\,h} \cdot n^x_{K_F} \\ p + \alpha(v_{f\,h} - v_{s\,h}) \cdot n^x_{K_F} \\ v_{s\,h} - \delta(\underline{\underline{\sigma}}_h - p_h) \cdot n^x_{K_S} \\ -p_h n^x_{K_S} \end{pmatrix}$$

Coupled system. Space-time Trefftz-DG formulation

Seek $(v_{f\,h}, p_h, v_{s\,h}, \underline{\underline{\sigma}}_h) \subset \mathbf{V}(\mathcal{T}_h) \subset \mathbf{T}(\mathcal{T}_h)$ s.t. $\forall\ (\omega_f, q, \omega_s, \underline{\underline{\xi}}) \subset \mathbf{V}(\mathcal{T}_h)$ it holds:

Trefftz-DG formulation

$$\mathcal{A}_{TDG}((v_{f\,h}, p_h, v_{s\,h}, \underline{\underline{\sigma}}_h); (\omega_f, q, \omega_s, \underline{\underline{\xi}})) = \ell_{TDG}(\omega_f, q, \omega_s, \underline{\underline{\xi}})$$

Coupled system. Analysis of Trefftz-DG formulation

Norms in $\mathbf{T}(\mathcal{T}_h)$

$$|||(\omega_f, q, \omega_s, \underline{\underline{\xi}}))|||^2_{TDG} := |||(\omega_f, q)|||^2_{TDG_F} + |||(\omega_s, \underline{\underline{\xi}})|||^2_{TDG_S} + 2\|\delta^{1/2}\underline{\underline{\xi}}\|^2_{L^2(\mathcal{F}_h^{FS})}$$

$$|||(\omega_f, q, \omega_s, \underline{\underline{\xi}})|||^2_{TDG^*} := |||(\omega_f, q)|||^2_{TDG_F^*} + |||(\omega_s, \underline{\underline{\xi}})|||^2_{TDG_S^*} + \frac{1}{2}\|\delta^{-1/2}\underline{\underline{\xi}}\|^2_{L^2(\mathcal{F}_h^{FS})}$$

Coupled system. Analysis of Trefftz-DG formulation

Coercivity of Trefftz-DG formulation

$$\mathcal{A}_{TDG}((\omega_f, q, \omega_s, \underline{\underline{\xi}}); (\omega_f, q, \omega_s, \underline{\underline{\xi}})) = |||(\omega_f, q, \omega_s, \underline{\underline{\xi}})|||^2_{TDG}$$

Continuity of Trefftz-DG formulation

$$|\mathcal{A}_{TDG}((v_f, p, v_s, \underline{\underline{\sigma}}); (\omega_f, q, \omega_s, \underline{\underline{\xi}}))| \leq 2\, |||(v_f, p, v_s, \underline{\underline{\sigma}})|||_{TDG^*}\, |||(\omega_f, q\omega_s, \underline{\underline{\xi}})|||_{TDG},$$

$$|\ell_{TDG}(\omega_f, q, \omega_s, \underline{\underline{\xi}})| \leq \sqrt{2} \Big[\|\rho_f^{1/2} v_{f0}\|^2_{L^2(\mathcal{F}_h^{0F})} + \|(\frac{1}{c_f^2 \rho_f})^{1/2} p_0\|^2_{L^2(\mathcal{F}_h^{0F})} + \|\rho_s^{1/2} v_{s0}\|^2_{L^2(\mathcal{F}_h^{0S})} + \|\underline{\underline{A}}^{1/2} \underline{\underline{\sigma}}_0\|^2_{L^2(\mathcal{F}_h^{0S})} \Big]^{1/2}.$$

Well-posedness

$$|||(v_f - v_{f\,h}, p - p_h, v_s - v_{s\,h}, \underline{\underline{\sigma}} - \underline{\underline{\sigma}}_h)|||_{TDG}$$

$$\leq 3 \inf_{(\omega_f, q, \omega_s, \underline{\underline{\xi}}) \in \mathbf{V}(\mathcal{T}_h)} |||(v_f - \omega_f, p - q, v_s - \omega_s, \underline{\underline{\sigma}} - \underline{\underline{\xi}})|||_{TDG^*}.$$

Implementation of the algorithm. Acoustic system

Implementation of the algorithm

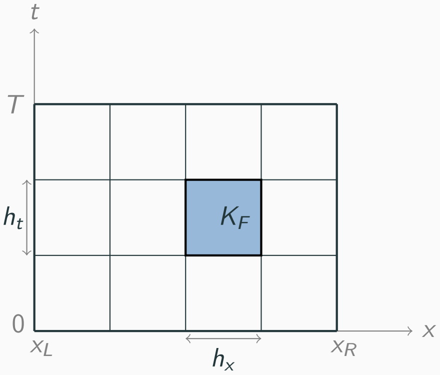
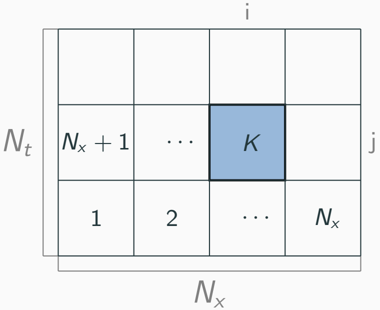


Figure 2: Rectangular mesh on $Q_F := [x_L, x_R] \times [0, T]$.

Implementation of the algorithm



$$K = (j-1)N_x + i$$

Figure 3: Element numbering.

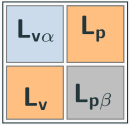
Seek $(v_{f\,h}, p_h) \subset \mathbf{V}(\mathcal{T}_h) \subset \mathbf{T}_F(\mathcal{T}_h)$ s.t. for all $(\omega_f, q) \subset \mathbf{V}(\mathcal{T}_h)$ it holds:

Treftz-DG formulation

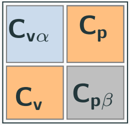
$$
\begin{aligned}
&\int_{\mathcal{F}_h^{\Omega_F}} \left[\frac{1}{c_f^2\rho_f} {p_h}^- [\![q]\!]_t + \rho_f {v_{f\,h}}^- [\![\omega_f]\!]_t\right] d\mathrm{s} \\
&+ \int_{\mathcal{F}_h^{I_F}} \left[\{\!\{p_h\}\!\}[\![\omega_f]\!]_x + \{\!\{v_{f\,h}\}\!\}[\![q]\!]_x + \alpha[\![v_{f\,h}]\!]_x[\![\omega_f]\!]_x + \beta[\![p_h]\!]_x[\![q]\!]_x\right] d\mathrm{s} \\
&+ \int_{\mathcal{F}_h^{T_F}} \left[\frac{1}{c_f^2\rho_f} p_h q + \rho_f v_{f\,h} \cdot \omega_f\right] d\mathrm{s} = \int_{\mathcal{F}_h^{0_F}} \left[\frac{1}{c_f^2\rho_f} p_0 q + \rho_f v_{f0} \cdot \omega_f\right] d\mathrm{s}.
\end{aligned}
$$

Treftz-DG formulation. 1st layer

$$\left[v_{K-1}\cdot\mathbf{L}_{\mathbf{v}\alpha}+p_{K-1}\cdot\mathbf{L}_{\mathbf{p}}\right]+\left[v_{K}\cdot\mathbf{C}_{\mathbf{v}\alpha}+p_{K}\cdot\mathbf{C}_{\mathbf{p}}\right]+\left[v_{K+1}\cdot\mathbf{R}_{\mathbf{v}\alpha}+p_{K+1}\cdot\mathbf{R}_{\mathbf{p}}\right]$$
$$\left[v_{K-1}\cdot\mathbf{L}_{\mathbf{v}}+p_{K-1}\cdot\mathbf{L}_{\mathbf{p}\beta}\right]+\left[v_{K}\cdot\mathbf{C}_{\mathbf{v}}+p_{K}\cdot\mathbf{C}_{\mathbf{p}\beta}\right]+\left[v_{K+1}\cdot\mathbf{R}_{\mathbf{v}}+p_{K+1}\cdot\mathbf{R}_{\mathbf{p}\beta}\right]$$



"Left" block



"Central" block

"Right" block

Implementation of the algorithm

Treftz-DG formulation. 1^{st} layer

$$\left[v_{K-1}\cdot\mathbf{L}_{\mathbf{v}\alpha}+p_{K-1}\cdot\mathbf{L}_{\mathbf{p}}\right]+\left[v_{K}\cdot\mathbf{C}_{\mathbf{v}\alpha}+p_{K}\cdot\mathbf{C}_{\mathbf{p}}\right]+\left[v_{K+1}\cdot\mathbf{R}_{\mathbf{v}\alpha}+p_{K+1}\cdot\mathbf{R}_{\mathbf{p}}\right]$$

$$\left[v_{K-1}\cdot\mathbf{L}_{\mathbf{v}}+p_{K-1}\cdot\mathbf{L}_{\mathbf{p}\beta}\right]+\left[v_{K}\cdot\mathbf{C}_{\mathbf{v}}+p_{K}\cdot\mathbf{C}_{\mathbf{p}\beta}\right]+\left[v_{K+1}\cdot\mathbf{R}_{\mathbf{v}}+p_{K+1}\cdot\mathbf{R}_{\mathbf{p}\beta}\right]$$

L	C	R
"Left" block	**"Central" block**	**"Right" block**

Implementation of the algorithm

$$
\begin{bmatrix}
C & R & & \cdots & & L \\
L & C & R & & & \\
& L & C & R & & \\
\vdots & & & \ddots & & \vdots \\
& & & L & C & R \\
R & & \cdots & & L & C
\end{bmatrix}
\cdot
\begin{bmatrix}
U_1 \\ U_2 \\ U_3 \\ \vdots \\ U_{N-2} \\ U_{N-1}
\end{bmatrix}
=
\begin{bmatrix}
U_1^0 \\ U_2^0 \\ U_3^0 \\ \vdots \\ U_{N-2}^0 \\ U_{N-1}^0
\end{bmatrix}
$$

Implementation of the algorithm

$$\underbrace{\left[\mathbf{M}\right]}_{32N_x \times 32N_x} \cdot \underbrace{\left[\mathbf{U}\right]}_{1 \times 32N_x} = \underbrace{\left[\mathbf{U}^0\right]}_{1 \times 32N_x}$$

Implementation of the algorithm

1. Initialization

$$\begin{bmatrix} \mathbf{M} \end{bmatrix} ; \begin{bmatrix} \mathbf{invM} \end{bmatrix} ; \begin{bmatrix} \mathbf{U}^0 \end{bmatrix} ;$$

2. Propagation

`For t = 1 : N_t`

$$\begin{bmatrix} \mathbf{U} \end{bmatrix} = \begin{bmatrix} \mathbf{invM} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{U}^0 \end{bmatrix} ;$$

$$\begin{bmatrix} \mathbf{U}^0 \end{bmatrix} = \begin{bmatrix} \mathbf{U} \end{bmatrix} ;$$

`end`

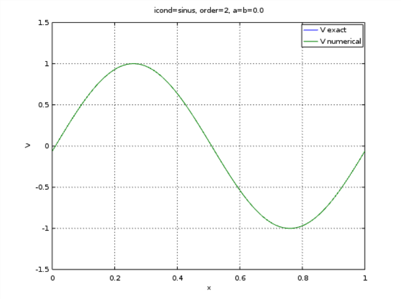


Figure 4: Exact and approximate solution for velocity $v_f(x,t)$, $t = \Delta t \times 1$, on the first time-layer ($\alpha = \beta = 0.5$).

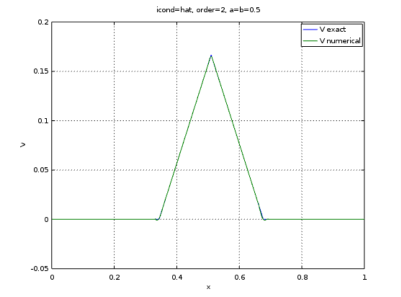


Figure 5: Exact and approximate solution for velocity $v_f(x,t)$, $t = \Delta t \times 1$, on the first time-layer ($\alpha = \beta = 0.5$).

Implementation of the algorithm

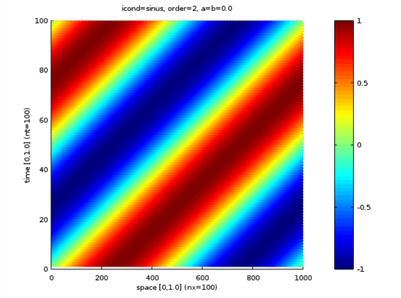


Figure 6: Time propagation of the approximate solution for velocity $v_f(x,t)$ ($\alpha = \beta = 0.5$).

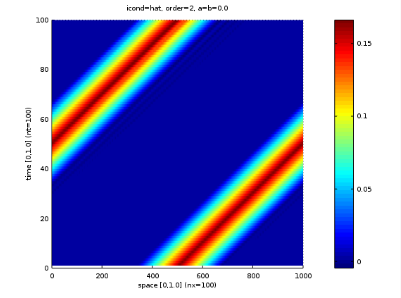


Figure 7: Time propagation of the approximate solution for velocity $v_f(x,t)$ ($\alpha = \beta = 0.5$).

Implementation of the algorithm

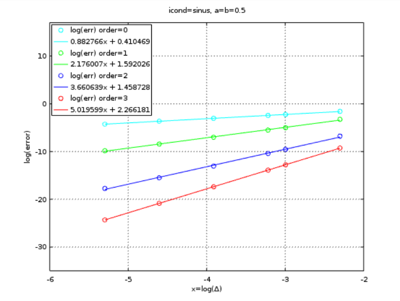


Figure 8: Convergence of numerical solution for velocity $v_f(x,t)$ in logarithmic scale ($\alpha = \beta = 0.5$).

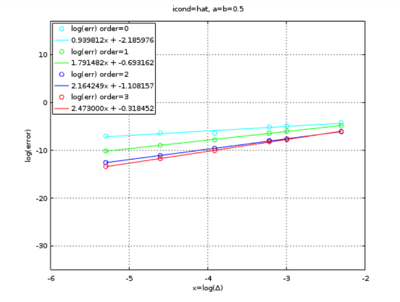


Figure 9: Convergence of numerical solution for velocity $v_f(x,t)$ in logarithmic scale ($\alpha = \beta = 0.5$).

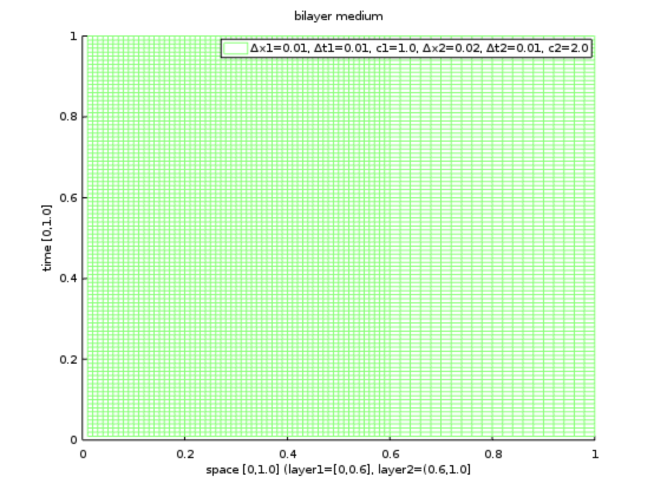


Figure 10: Mesh for heterogeneous case (bilayer medium).
$\Delta x_1 = 0.01$, $\Delta t_1 = 0.01$, $c_1 = 1.0$, $\Delta x_2 = 0.02$, $\Delta t_2 = 0.01$, $c_1 = 2.0$.

Implementation of the algorithm

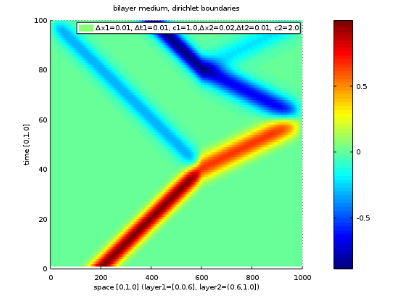


Figure 11: Approximate solution for velosity $v(x,t)$.

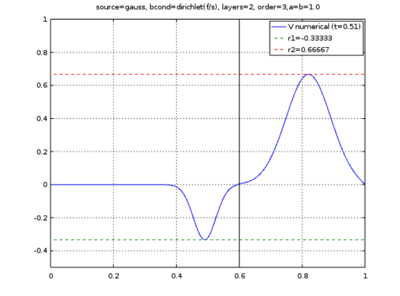


Figure 12: Approximate solution for velocity $v(x,t)$, $t = \Delta t \times 51$.

Conclusion

On-going work :

- Numerical implementation of the method for 1D AS ;
- Different choice of the basis functions ;
- Analysis of efficiency of the method / algorithm / code.

Perspectives :

- Numerical implementation of the method for ES 1D,2D,3D ;
- Numerical coupling 1D,2D,3D ;
- Coupling Trefftz-DG in solid with FVM in fluid, integrating of the transmission layer.

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Thank you for your attention!

Questions?

Implementation of the algorithm

Polynomial waves basis functions

- Generating functions $g^v(x,t)$ and $g^p(x,t)$ - solution of wave equation if $a^2 = b^2$

$$g^v(a,b,x,t) = e^{i(ax+bc_f t)}, \;\; g^p(a,b,x,t) = -c_f e^{i(bx+ac_f t)}.$$

- Taylor expansions for $g^v(x,t)$, $g^p(x,t)$ ($a^2 = b^2$):

$$e^{i(ax+bc_f t)} = \sum_{n=0}^{\infty} \sum_{k=0,k<2}^{n} Q^v_{n-k,k}(x,t) a^{n-k} b^k,$$

$$-c_f e^{i(bx+ac_f t)} = \sum_{n=0}^{\infty} \sum_{k=0,k<2}^{n} Q^p_{n-k,k}(x,t) a^{n-k} b^k.$$

$$R^v_{n-k,k}(x,t) = \Re(Q^v_{n-k,k}(x,t)), \quad I^v_{n-k,k}(x,t) = \Im(Q^v_{n-k,k}(x,t))$$
$$R^p_{n-k,k}(x,t) = \Re(Q^p_{n-k,k}(x,t)), \quad I^p_{n-k,k}(x,t) = \Im(Q^p_{n-k,k}(x,t)).$$

Polynomial waves basis functions 1D

$$\phi_1^v = 0 \qquad \phi_2^v = 1 \qquad \phi_3^v = x \qquad \phi_4^v = c_f t$$
$$\phi_5^v = -\frac{x^2}{2} - \frac{c_f^2 t^2}{2} \qquad \phi_6^v = -c_f x t \qquad \phi_7^v = -\frac{x^3}{6} - \frac{x c_f^2 t^2}{2} \qquad \phi_8^v = -\frac{c_f^3 t^3}{6} - \frac{x^2 c_f t}{2}$$

$$\phi_1^p = -c_f \qquad \phi_2^p = 0 \qquad \phi_3^p = -c_f^2 t \qquad \phi_4^p = -c_f x$$
$$\phi_5^p = c_f^2 x t \qquad \phi_6^p = c_f\left(\frac{x^2}{2} + \frac{c_f^2 t^2}{2}\right) \qquad \phi_5^p = c_f\left(\frac{c_f^3 t^3}{6} + \frac{x^2 c_f t}{2}\right) \qquad \phi_6^p = c_f\left(\frac{x^3}{6} + \frac{x c_f^2 t^2}{2}\right).$$

Implementation of the algorithm

Polynomial waves basis functions 2D

p = 0		
$\phi_1^{vx} = 0$ $\phi_2^{vx} = 1$	$\phi_1^{vy} = 0$ $\phi_2^{vy} = 1$	$\phi_1^{p} = -c_f$ $\phi_2^{p} = 0$
p = 1		
$\phi_3^{vx} = x$ $\phi_4^{vx} = c_f t$ $\phi_5^{vx} = y$	$\phi_3^{vy} = y$ $\phi_4^{vy} = x$ $\phi_5^{vy} = c_f t$	$\phi_3^{p} = -c_f(c_f t)$ $\phi_4^{p} = -c_f x$ $\phi_5^{p} = -c_f y$
p = 2		
$\phi_7^{vx} = -\frac{x^2}{2} - \frac{y^2}{2} - \frac{c_f^2 t^2}{2}$ $\phi_8^{vx} = -xy$ $\phi_9^{vx} = -c_f ty$ $\phi_{10}^{vx} = -c_f tx$	$\phi_7^{vy} = -xy$ $\phi_8^{vy} = -\frac{x^2}{2} - \frac{y^2}{2} - \frac{c_f^2 t^2}{2}$ $\phi_9^{vy} = -c_f tx$ $\phi_{10}^{vy} = -c_f ty$	$\phi_7^{p} = c_f(c_f tx)$ $\phi_8^{p} = c_f(c_f ty)$ $\phi_9^{p} = c_f(xy)$ $\phi_{10}^{p} = c_f(\frac{x^2}{2} + \frac{y^2}{2} + \frac{c_f^2 t^2}{2})$
p = 3		
$\phi_{11}^{vx} = -\frac{x^2 y}{2} - \frac{c_f^2 t^2 y}{2} - \frac{y^3}{6}$ $\phi_{12}^{vx} = -c_f txy$ $\phi_{13}^{vx} = -\frac{c_f^3 t^3}{6} - \frac{c_f tx^2}{2} - \frac{c_f ty^2}{2}$ $\phi_{14}^{vx} = -\frac{xy^2}{2} - \frac{c_f^2 t^2 x}{2} - \frac{x^3}{6}$	$\phi_{11}^{vy} = -\frac{xy^2}{2} - \frac{c_f^2 t^2 x}{2} - \frac{x^3}{6}$ $\phi_{12}^{vy} = -\frac{c_f^3 t^3}{6} - \frac{c_f tx^2}{2} - \frac{c_f ty^2}{2}$ $\phi_{13}^{vy} = -c_f ty$ $\phi_{14}^{vy} = -\frac{x^2 y}{2} - \frac{c_f^2 t^2 y}{2} - \frac{y^3}{6}$	$\phi_{11}^{p} = c_f(c_f txy)$ $\phi_{12}^{p} = c_f(\frac{x^2 y}{2} + \frac{c_f^2 t^2 y}{2} + \frac{y^3}{6})$ $\phi_{13}^{p} = c_f(\frac{xy^2}{2} + \frac{c_f^2 t^2 x}{2} + \frac{x^3}{6})$ $\phi_{14}^{p} = c_f(\frac{c_f^3 t^3}{6} + \frac{c_f tx^2}{2} + \frac{c_f ty^2}{2})$