Stable perfectly matched layers for a class of anisotropic dispersive models. Part II: Energy estimates

Maryna Kachanovska

STABLE PERFECTLY MATCHED LAYERS FOR A CLASS OF ANISOTROPIC DISPERSIVE MODELS. PART II: ENERGY ESTIMATES *

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Abstract. This article continues the stability analysis of the generalized perfectly matched layers for 2D anisotropic dispersive models studied in Part I of the work. We obtain explicit energy estimates for the PML system in the time domain, by making use of the ideas stemming from the analysis of the associated sesquilinear form in the Laplace domain. This analysis is based on the introduction of a particular set of auxiliary unknowns related to the PML, which simplifies the derivation of the energy estimates for the resulting system. For 2D dispersive systems, our analysis allows to demonstrate the stability of the PML system for a constant absorption parameter. For 1D dispersive systems, we show the stability of the PMLs with a non-constant absorption parameter.

1991 Mathematics Subject Classification. 65M12, 35Q60.

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Introduction

The method of the perfectly matched layers (PML), introduced by Bérenger [1, 2], is used ubiquitously in engineering and physics communities to compute a solution to a problem posed in an unbounded domain. However, it is well-known that for some classes of problems (e.g. the wave propagation in anisotropic and/or dispersive media) the classical PML method may result in instabilities [3,4]. For a class of anisotropic dispersive models PMLs were stabilized [5].

In this work we continue the analysis of the stable PMLs constructed in [5]. This class of systems describes the wave propagation in metamaterials and plasmas in 2D. In the frequency domain, they can be written in the form of an anisotropic dispersive wave equation:

$$\varepsilon_1(\omega)^{-1}\partial_x^2 u + \varepsilon_2(\omega)^{-1}\partial_y^2 u + \omega^2\mu(\omega)u = 0,\ \omega \in \mathbb{R},$$

where $\varepsilon_1(\omega)$, $\varepsilon_2(\omega)$ have a meaning of a dielectric permittivity and $\mu(\omega)$ is a magnetic permeability. They depend on frequency non-trivially and satisfy $\operatorname{Im}(\omega\varepsilon_j(\omega)) > 0$, $j = 1, 2$, $\operatorname{Im}(\omega\mu(\omega)) > 0$ for $\operatorname{Im}\omega > 0$, an assumption connected to the stability of the model, cf. [6].

In the time domain, such systems correspond to the Maxwell's equations with currents, which are coupled to the electromagnetic field through the ordinary differential equations.

Keywords and phrases: perfectly matched layers, Maxwell equations, Laplace transform, energy, Fourier analysis

* *This work was supported by a public grant as part of the Investissement d'avenir project, reference ANR-11-LABX-0056-LMH, LabEx LMH, as well as a co-financing program PRESTIGE*

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In [5] a method of construction of stable perfectly matched layers for this class of models is suggested. It is based on the ideas of Bécache et al. [4] for isotropic metamaterials, which, in turn, generalizes the work of Cummer [7] for the Drude model. However, the analysis in the aforementioned works does not cover the question of the decay of an energy associated with the newly constructed PML systems.

This questions was answered for the classical PML applied to isotropic, non-dispersive Maxwell equations by Bécache and Joly [8]. In a more general case, the derivation of energy estimates for PML systems does not seem trivial. For example, in [8] the authors have obtained energy estimates for the PML for isotropic Maxwell equations written in the Zhao-Cangellaris formulation [9]. The derivation of the energy estimates for this formulation is simpler compared to e.g. Bérenger's split formulation. Our approach is different from that of [8], where the PML system was given a priori. We suggest to look for a formulation that would be equivalent to the original system and would allow to obtain the energy in a simpler way. We make use of the stability analysis in the Laplace domain, as it was done in the work [5], which provides intuition on derivation of explicit (without appeal to the Plancherel's theorem) energy estimates.

An alternative way to derive energy estimates is offered by the use of the method of Hagstrom and Appelö [10]. It can be applied to a very general class of systems, however, it results in energy estimates on the spatial derivatives of unknowns [11], while most of the estimates we obtain involve the original unknowns. Moreover, the application of this approach to dispersive models does not seem trivial, due to the need to rewrite the original problem as a single equation, and then apply the method of [10] to the resulting problem.

This work is organized as follows. In Section 1 we state the problem under consideration. In Section 2 we show how to construct stable PMLs for so-called generalized Lorentz models and derive the energy estimates for the corresponding time-domain system. In Section 3 we extend these results to general passive materials. Finally, in Section 4 we demonstrate how the ideas of the previous sections can be used to prove the stability of the PMLs of [5] for 1D dispersive systems, for a non-constant absorption PML parameter.

1. Problem Setting

The wave propagation in 2D dispersive media is described by the Maxwell system:

$$\begin{aligned}\partial_t D_x &= \partial_y H_z,\\ \partial_t D_y &= -\partial_x H_z,\\ \partial_t B_z &= -\partial_x E_y + \partial_y E_x.\end{aligned}\tag{1}$$

In the above we use the scaling $c = \varepsilon_0 = \mu_0 = 1$. The relations between the electric field and the electric displacement field, as well as between the magnetic field and the magnetizing field are given in the Laplace domain. Denoting by s the Laplace variable, we set $\hat{u}(s) = (\mathcal{L}u)(s)$ the Laplace transform of $u(t)$ and introduce $\mathbb{C}_+ = \{s \in \mathbb{C} : \operatorname{Re} s > 0\}$. With this notation, the constitutive relations read $\hat{\mathbf{D}} = \underline{\underline{\varepsilon}}(s)\hat{\mathbf{E}}$, $\hat{H}_z = \mu^{-1}(s)\hat{B}_z$. We will assume that the dielectric permittivity is a diagonal matrix

$$\underline{\underline{\varepsilon}}(s) = \begin{pmatrix} \varepsilon_x(s) & 0 \\ 0 & \varepsilon_y(s) \end{pmatrix}.$$

The multiplication in the Laplace domain becomes a convolution in the time domain, which we formally denote

$$\mathbf{D} = \underline{\underline{\varepsilon}}(\partial_t)\mathbf{E}, \qquad B_z = \mu(\partial_t)H_z.$$

The system (1) in the Laplace domain can be rewritten as an anisotropic dispersive wave equation:

$$s^2\mu(s)\hat{H}_z - \varepsilon_y(s)^{-1}\partial_x^2\hat{H}_z - \varepsilon_x(s)^{-1}\partial_y^2\hat{H}_z = 0.\tag{2}$$

To explain the assumption on the coefficients of the above equation, let us introduce the following definition.

Definition 1.1. A function $c : \mathbb{C}_+ \to \mathbb{C}$ is passive if it is analytic in $\mathbb{C}_+$ and satisfies $\operatorname{Re}(sc(s)) > 0$ there.

In this work we will assume that $\varepsilon_x,\ \varepsilon_y,\ \mu$ are passive. One of the examples of passive models is offered by so-called generalized Lorentz materials [12]:

$$\begin{aligned}
\varepsilon_\alpha(s) &= 1 + \sum_{\ell=0}^{n_\alpha} \frac{\varepsilon_{\alpha\ell}}{s^2 + \omega_{\alpha\ell}^2}, \qquad \varepsilon_{\alpha\ell} > 0,\ \omega_{\alpha\ell} \in \mathbb{R}, \qquad \ell = 0, \ldots, n_\alpha, \qquad \alpha \in \{x, y\},\\
\mu(s) &= 1 + \sum_{\ell=0}^{n_\mu} \frac{\mu_\ell}{s^2 + \omega_{\mu\ell}^2}, \qquad \mu_\ell > 0,\ \omega_{\mu\ell} \in \mathbb{R}, \qquad \ell = 0, \ldots, n_\mu.
\end{aligned} \tag{3}$$

The generalized Lorentz materials include Drude materials [4] and a 2D uniaxial cold plasma [12, 13].

Remark 1.2. The passivity assumption covers nonhyperbolic models as well (e.g. the diffusion equation, with $\epsilon_x(s) = \epsilon_y(s) = 1,\ \mu(s) = s^{-1}$).

In [5] it was demonstrated that the well-posedness and stability of (1) follows from the passivity of coefficients of the sesquilinear form associated with (2). Introducing the scalar product

$$(u, v) = \int_{\mathbb{R}^2} u\bar{v}dx, \qquad u, v \in L^2,$$

we can define the following class of sesquilinear forms.

Definition 1.3. We will call a sesquilinear form

$$A(u, v) = a(s)(\partial_x u, \partial_x v) + b(s)(\partial_y u, \partial_y v) + s^2 c(s)(u, v), \qquad u, v \in H^1(\mathbb{R}^2),$$

passive if $a(s)^{-1}$, $b(s)^{-1}$, $c(s)$ are passive.

Informally,

$$\text{Passivity of a Sesquilinear Form} \implies \text{Time-Domain Stability of the Corresponding System.} \tag{4}$$

Instead of resorting to the Laplace-domain analysis of [5], one can prove stability results directly in the time domain, using energy techniques, cf. [14]. Extending these results to the PMLs is the subject of present work.

2. Stability of PMLs for Lorentz Materials

The analysis for the generalized Lorentz materials (3) will play a crucial role in this work. This is due to the fact that any passive material can be approximated by generalized Lorentz materials, see Section 3. Moreover, Lorentz materials are the only representatives of a fairly general class of passive materials. To demonstrate this, let us assume that $\varepsilon_x,\ \varepsilon_y,\ \mu$ satisfy the following assumptions, justified in [4, 12, 14].

Assumption 2.1. *A function $r(s)$ is a rational function $r(s) = 1 + \frac{p_r(s^2)}{q_r(s^2)}$, where the polynomials p_r and q_r have real coefficients, no common roots, and also $\deg p_r < \deg q_r$. All its zeros and poles lie in $i\mathbb{R}$.*

A class of passive functions satisfying the above condition is quite narrow, see [14] and references therein.

Theorem 2.2. *A function $r(s)$ satisfying Assumption 2.1 is passive if and only if $r(s) = 1 + \sum\limits_{\ell=0}^{n} \frac{r_\ell}{s^2+\omega_\ell^2}$, where $\omega_\ell \in \mathbb{R}$ and $r_\ell > 0$ for all $\ell = 0, \ldots, n$.*

Thus, passive functions that satisfy Assumption 2.1 correspond to Lorentz materials (3).

This section is organized as follows. In Section 2.1 we formulate the time-domain system corresponding to Lorentz materials without the PML. Next, in Section 2.2 we discuss the construction of stable PMLs for dispersive systems, concentrating in particular on Lorentz models. In Section 2.3 we show the main idea of the derivation of the PML system that would allow to obtain simplified energy estimates. Finally, in Sections 2.4 and 2.5 we derive the energy estimates for systems corresponding to different PMLs.

2.1. Time-Domain System without PMLs

We consider the Maxwell system (1) with Lorentz parameters (3). In the time domain, we obtain the Maxwell equations with currents, coupled to $\mathbf{E}$, H_z via ODEs, cf. [4]:

$$\partial_t D_x = \partial_y H_z, \tag{5a}$$
$$\partial_t D_y = -\partial_x H_z, \tag{5b}$$
$$\partial_t B_z = \partial_y E_x - \partial_x E_y, \tag{5c}$$
$$\partial_t D_\alpha = \partial_t E_\alpha + \sum_{\ell=0}^{n_\alpha} \varepsilon_{\alpha\ell}\lambda_{\alpha\ell}, \tag{5d}$$
$$\partial_t \lambda_{\alpha\ell} + \omega_{\alpha\ell}^2 p_{\alpha\ell} = E_\alpha, \qquad \partial_t p_{\alpha\ell} = \lambda_{\alpha\ell}, \qquad \ell = 0, \dots n_\alpha,\ \alpha \in \{x, y\}, \tag{5e}$$
$$\partial_t B_z = \partial_t H_z + \sum_{\ell=0}^{n_\mu} \mu_\ell \lambda_{\mu\ell}, \tag{5f}$$
$$\partial_t \lambda_{\mu\ell} + \omega_{\mu\ell}^2 p_{\mu\ell} = H_z, \qquad \partial_t p_{\mu\ell} = \lambda_{\mu\ell}, \qquad \ell = 0, \dots n_\mu. \tag{5g}$$

Let us clarify the time-domain realization of $\hat{B}_z = \mu(s)\hat{H}_z$ (the time-domain equivalent of $\mathbf{D} = \underline{\underline{\varepsilon}}(s)\mathbf{E}$ can be obtained similarly). The latter equivalently reads $s\hat{B}_z = s\mu(s)\hat{H}_z = \left(s + \sum\limits_{\ell=0}^{n_\mu} \frac{\mu_\ell}{s+\omega_{\mu\ell}^2 s^{-1}}\right)\hat{H}_z$. One defines

$$\hat{\lambda}_{\mu\ell} = \frac{1}{s + \omega_{\mu\ell}^2 s^{-1}}\hat{H}_z, \qquad \text{or, equivalently,} \qquad s\hat{\lambda}_{\mu\ell} + \omega_{\mu\ell}^2 \hat{p}_{\mu\ell} = \hat{H}_z, \qquad s\hat{p}_{\mu\ell} = \hat{\lambda}_{\mu\ell}.$$

Provided that $\lambda_{\mu\ell}|_{t=0} = 0 = p_{\mu\ell}|_{t=0}$, the above coincides with (5g) in the time domain. Similarly, for $B_z|_{t=0} = H_z|_{t=0}$, we verify that (5f) coincides in the Laplace domain with $\hat{B}_z = \mu(s)\hat{H}_z$. Let us formulate the following known result for the system (5a-5g), see also [4]. We provide its proof since it will be of use later.

Theorem 2.3. *An energy $\mathcal{E}$ of (5a-5g), defined below, satisfies $\frac{d}{dt}\mathcal{E} = 0$. Here*

$$\mathcal{E} = \frac{1}{2}\left(\|E_x\|^2 + \|E_y\|^2 + \|H_z\|^2\right) + \mathcal{E}_x + \mathcal{E}_y + \mathcal{E}_z,$$
$$\mathcal{E}_\alpha = \frac{1}{2}\sum_{\ell=0}^{n_\alpha} \varepsilon_{\alpha\ell}\left(\|\lambda_{\alpha\ell}\|^2 + \omega_{\alpha\ell}^2\|p_{\alpha\ell}\|^2\right),\ \alpha \in \{x, y\}, \qquad \mathcal{E}_z = \frac{1}{2}\sum_{\ell=0}^{n_\mu} \mu_\ell\left(\|\lambda_{\mu\ell}\|^2 + \omega_{\mu\ell}^2\|p_{\mu\ell}\|^2\right).$$

Proof. Test the equation (5c) with H_z, to obtain

$$(\partial_t B_z, H_z) - (\partial_y E_x, H_z) + (\partial_x E_y, H_z) = (\partial_t B_z, H_z) + (E_x, \partial_y H_z) - (E_y, \partial_x H_z) = 0. \tag{6}$$

We consider the first term of the above expression; using (5f),

$$(\partial_t B_z, H_z) = \frac{1}{2}\frac{d}{dt}\|H_z\|^2 + \sum_{\ell=0}^{n_\mu} \mu_\ell \left(\lambda_{\mu\ell}, H_z\right) \overset{(5g)}{=} \frac{1}{2}\frac{d}{dt}\|H_z\|^2 + \sum_{\ell=0}^{n_\mu} \mu_\ell \left(\lambda_{\mu\ell}, \partial_t\lambda_{\mu\ell} + \omega_{\mu\ell}^2 p_{\mu\ell}\right)$$
$$\overset{(5g)}{=} \frac{1}{2}\frac{d}{dt}\|H_z\|^2 + \frac{1}{2}\frac{d}{dt}\sum_{\ell=0}^{n_\mu}\mu_\ell\|\lambda_{\mu\ell}\|^2 + \sum_{\ell=0}^{n_\mu}\mu_\ell\omega_{\mu\ell}^2(\partial_t p_{\mu\ell}, p_{\mu\ell}) = \frac{1}{2}\frac{d}{dt}\|H_z\|^2 + \frac{d}{dt}\mathcal{E}_z. \tag{7}$$

The term $(E_x, \partial_y H_z)$ in (6) can be rewritten exactly like in the previous case:

$$(E_x, \partial_y H_z) \overset{(5a)}{=} (E_x, \partial_t D_x) \overset{(5d)}{=} \frac{1}{2}\frac{d}{dt}\|E_x\|^2 + \frac{1}{2}\frac{d}{dt}\sum_{\ell=0}^{n_\mu}\varepsilon_\ell\left(\|\lambda_{x\ell}\|^2 + \omega_{x\ell}^2\|p_{x\ell}\|^2\right) = \frac{1}{2}\frac{d}{dt}\|E_x\|^2 + \frac{d}{dt}\mathcal{E}_x. \tag{8}$$

Similarly one shows that $-(E_y, \partial_x H_z) = \frac{1}{2}\frac{d}{dt}\|E_y\|^2 + \frac{d}{dt}\mathcal{E}_y$. □

2.2. Construction of Stable PMLs for Dispersive Models

Consider the system (1) written in the Laplace domain (2). To construct a PML in the region $x \geq 0$, one assumes that (2) is valid for $x = \tilde{x} \in \mathbb{C}$. The corresponding analytic continuation of $\hat{H}_z$, denoted by $\hat{\mathcal{H}}_z$, solves (2), with $x = \tilde{x}$. In [4] it is proposed to choose $\tilde{x}$ as follows, for some analytic function $\psi(s)$,

$$\tilde{x} = \begin{cases} x + s^{-1}\psi_x(s)\displaystyle\int_0^x \sigma_x(x')dx', & x \geq 0, \\ x, & x < 0, \end{cases} \qquad \sigma_x(x) = \begin{cases} \sigma(x) \geq 0, & x \geq 0, \\ 0, & x < 0. \end{cases} \tag{9}$$

The classical PML can be obtained taking $\psi_x(s) = 1$. For dispersive materials this choice may be unstable. We discuss how to choose $\psi_x(s)$ in further sections. With this change of variables, (2) transforms into

$$\varepsilon_y^{-1}\left(1 + \sigma_x s^{-1}\psi_x\right)^{-1}\partial_x\left(\left(1 + \sigma_x s^{-1}\psi_x\right)^{-1}\partial_x\hat{\mathcal{H}}_z\right) + \varepsilon_x^{-1}\partial_{yy}\hat{\mathcal{H}}_z - s^2\mu\hat{\mathcal{H}}_z = 0, \qquad x \in \mathbb{R}. \tag{10}$$

By analytic continuation, $\hat{\mathcal{H}}_z(x,y) = \hat{H}_z(x,y)$ for $x < 0$. The resulting system should be rewritten in the time domain. We postpone this question to future sections. The PML in other directions (e.g. for $y \geq 0$) can be constructed similarly. In corners the changes of both of the variables x and y is used.

Under (mild) assumptions on the coefficients of the sesquilinear form, one can show that the time-domain system corresponding to (10) is well-posed, for any $\sigma_x(x) \in L^\infty(\mathbb{R})$, see [15]. It seems significantly more difficult to show *the stability* of the corresponding problem for variable $\sigma_x(x)$, cf. [16]. Finer results can obtained by studying the sesquilinear form corresponding to (10) with $\sigma_x(x) = \text{const}$, $x \in \mathbb{R}$, when (9) becomes, cf. [5],

$$x \to x\left(1 + s^{-1}\psi_x\sigma_x\right), \qquad \sigma_x \equiv \text{const} \geq 0, \qquad x \in \mathbb{R}. \tag{11}$$

In this case the sesquilinear form corresponding to (10) becomes

$$A(u,v) = \frac{\varepsilon_y(s)^{-1}}{(1 + s^{-1}\psi_x(s)\sigma_x)^2}(\partial_x u, \partial_x v) + \varepsilon_x(s)^{-1}(\partial_y u, \partial_y v) + s^2\mu(s)(u,v), \qquad u, v \in H^1(\mathbb{R}^2). \tag{12}$$

As most of the results of this work are derived for (11), let us discuss the reasons for this choice and its place in the analysis of the stability of PMLs. To our knowledge, so far only the energy estimates for the classical PMLs for non-dispersive systems are available. These results are of either of two types:

(1) the energy estimates are derived for a PML system obtained from the change of variables (11), which, strictly speaking, is not what is used in practice. Moreover, the analysis is available only in 2D, for $\psi(s) = 1$ and non-dispersive models. Typically, such energy estimates result in the bound of the form $\|\mathbf{V}(t)\|_1 \leq C(t)\|\mathbf{V}(0)\|_2$, with $C(t)$ depending polynomially on t. They constitute one of the steps in the analysis of the stability of the PML models with $\sigma \neq \text{const}$. See [8,11,17] for this kind of analysis.

(2) the energy estimates are obtained for a PML system where the change of variables (9) is performed (again with $\psi(s) = 1$ and in non-dispersive case). This is indeed closer to the case of practical interest. For 1D problems these estimates are optimal and imply stability, see [18]. However, in higher dimensions they result in the bounds of the form $\|\mathbf{V}(t)\|_1 \leq C(t)\|\mathbf{V}(0)\|_2$, with $C(t)$ depending on t exponentially. Thus, such estimates are non-optimal, since they do not reflect the stability of the underlying PMLs (the latter fact suggested by numerous experiments). This kind of results was obtained e.g. in [8,19].

Our goal is to extend the results of the first type to 2D dispersive models, and show the stability of the PMLs for 1D dispersive problems for $\sigma_x \neq \text{const}$ (see Section 4). In this section we concentrate on the case $\sigma_x = \text{const}$.

2.2.1. *Construction of Stable PMLs for Lorentz Materials in One Direction*

The difficulty in the construction of the stable PML for dispersive models is the choice of the function $\psi_x(s)$ in (9). This question had been studied in [5], and for corners in particular in [20]. Let us provide several main results on the stability of PMLs, which are formulated as passivity of the corresponding sesquilinear forms, see (4). One of the choices of $\psi_x(s)$ resulting in stable PMLs is offered by the following statement.

Theorem 2.4 ([5]). *Let ε_x, ε_y, μ be passive and satisfy Assumption 2.1. Then the sesquilinear form (12) with $\psi_x(s) = \varepsilon_y(s)^{-1}$ is passive for any $\sigma_x \geq 0$.*

Remark 2.5. The passivity of the sesquilinear form (12), with $\psi(s) = \varepsilon_y(s)^{-1}$, implies its coercivity for all $s \in \mathbb{C}_+$. To see this, one tests (12) with su, to obtain

$$\begin{aligned}
\operatorname{Re}\left(A(u, su)\right) &= \operatorname{Re}\left(\frac{\bar{s}\varepsilon_y(s)^{-1}}{(1+s^{-1}\varepsilon_y(s)^{-1}\sigma_x)^2}\|\partial_x u\|^2 + \bar{s}\varepsilon_x(s)^{-1}\|\partial_y u\|^2 + s\mu(s)|s|^2\|u\|^2\right)\\
&= \operatorname{Re}\left(\frac{\bar{s}\varepsilon_y(s)^{-1}(1+\bar{s}^{-1}\bar{\varepsilon}_y(s)^{-1}\sigma_x)^2}{|1+s^{-1}\varepsilon_y(s)^{-1}\sigma_x|^4}\|\partial_x u\|^2 + \bar{s}\varepsilon_x(s)^{-1}\|\partial_y u\|^2 + s\mu(s)|s|^2\|u\|^2\right)\\
&= |1+s^{-1}\varepsilon_y(s)^{-1}\sigma_x|^{-4}\left(\operatorname{Re}(\bar{s}\varepsilon_y(s)^{-1}) + 2\sigma_x|\varepsilon_y(s)|^{-2} + \sigma_x^2|\varepsilon_y(s)|^{-2}\operatorname{Re}(\bar{s}^{-1}\bar{\varepsilon}_y(s)^{-1})\right)\|\partial_x u\|^2\\
&+ \operatorname{Re}(\bar{s}\varepsilon_x(s)^{-1})\|\partial_y u\|^2 + \operatorname{Re}(s\mu(s))\|u\|^2.
\end{aligned}$$

Notice that $\operatorname{Re}(\bar{s}\varepsilon_y^{-1}) = \operatorname{Re}\left(s\overline{\varepsilon_y}^{-1}\right) = \operatorname{Re}\left(s\varepsilon_y|\varepsilon_y|^{-2}\right) > 0$, for all $s \in \mathbb{C}_+$, thanks to passivity. For the same reason $\operatorname{Re}(\overline{(s\varepsilon_y)^{-1}}) > 0$, $\operatorname{Re}(\bar{s}\varepsilon_x^{-1})$ and $\operatorname{Re}(s\mu(s)) > 0$, $s \in \mathbb{C}_+$. From this the coercivity of $\operatorname{Re} A(u, sv)$ follows.

The coercivity alone is not sufficient for the well-posedness or stability of the corresponding time-domain system. In particular, one should show the analyticity of the Laplace-domain solution in $\mathbb{C}_+$, see [5].

This is indeed not the only stable PML change of variables. While the theorem below does not show how to construct $\psi_x(s)$, in [5] it is shown that its conditions can be ensured by choosing $\psi_x(s)$ that verifies certain sign conditions on the imaginary axis. We omit these details, since they are not important for the energy estimates.

Theorem 2.6 ([5]). *Let ε_x, ε_y, μ be passive and satisfy Assumption 2.1. Let additionally*

(1) $\psi_x(s)^{-1}$ satisfy Assumption 2.1 and be passive;
(2) $\varepsilon_x(s)\varepsilon_y(s)^{-1}\psi_x(s)^{-1}$ be passive;
(3) $\mu(s)\varepsilon_y(s)\psi_x(s)$ be passive.

Then the sesquilinear form

$$\tilde{A}(u, v) = \psi_x(s)\varepsilon_y(s)A(u, v), \qquad u, v \in H^1(\mathbb{R}^2),$$

with $A(u, v)$ given by (12), is passive for any $\sigma_x \geq 0$.

2.2.2. *Construction of Stable PMLs for Lorentz Materials in Corners*

Similar results can be formulated for the corner PMLs, which are obtained by the following change of variables:

$$x \to x\left(1+\sigma_x\psi_x(s)s^{-1}\right), \qquad y \to y\left(1+\sigma_y\psi_y(s)s^{-1}\right), \qquad \sigma_x,\ \sigma_y \geq 0. \tag{13}$$

The associated sesquilinear form reads

$$A_c(u,v) = \frac{\varepsilon_y(s)^{-1}}{(1+s^{-1}\psi_x(s)\sigma_x)^2}(\partial_x u, \partial_x v) + \frac{\varepsilon_x(s)^{-1}}{(1+s^{-1}\psi_y(s)\sigma_y)^2}(\partial_y u, \partial_y v) + s^2\mu(s)(u,v),\ u,v \in H^1(\mathbb{R}^2). \tag{14}$$

The corner PML analogue of Theorem 2.4 can be obtained by a change of variables with $\psi_x = \varepsilon_y^{-1}$ and $\psi_y = \varepsilon_x^{-1}$.

Theorem 2.7 ([5]). *Let ε_x, ε_y, μ be passive and satisfy Assumption 2.1. Then the sesquilinear form $A_c(u,v)$ of (14) with $\psi_x(s) = \varepsilon_y(s)^{-1}$ and $\psi_y(s) = \varepsilon_x(s)^{-1}$ is passive for any $\sigma_x \geq 0$ and $\sigma_y \geq 0$.*

Notice that the above result covers the stability of the PML in one direction (by setting $\sigma_x = 0$ or $\sigma_y = 0$) and in corners. A trivial analogue of Theorem 2.6 can be formulated as follows.

Theorem 2.8 ([20]). *Let ε_x, ε_y, μ be passive and satisfy Assumption 2.1. Let additionally*
(1) $\psi_x(s)^{-1}$, $\psi_y(s)^{-1}$ be passive and satisfy Assumption 2.1;
(2) $\psi_x\varepsilon_y = \psi_y\varepsilon_x$;
(3) $\mu(s)\varepsilon_y(s)\psi_x(s)$ be passive.
Then the sesquilinear form

$$\tilde{A}_c(u,v) = \psi_x(s)\varepsilon_y(s)A_c(u,v) = \psi_y(s)\varepsilon_x(s)A_c(u,v), \qquad u,v \in H^1(\mathbb{R}^2),$$

with $A_c(u,v)$ given by (14), is passive for any $\sigma_x \geq 0$ and $\sigma_y \geq 0$.

2.3. The Main Idea of the Energy Derivation for a PML System

There exists many ways to write a PML system for a given problem, see e.g. [8] and references therein, or [4]. To derive the energy conservation in an easy manner, it is crucial to introduce a very special set of auxiliary unknowns. One of such sets can be obtained by studying the coercivity of the corresponding sesquilinear form.

2.3.1. *A Toy Example: a 2D Isotropic Non-dispersive Wave Equation (Second-Order Formulation)*

Let us explain our idea based on the example of the isotropic non-dispersive 2D wave equation, to which the classical PMLs are applied in the x-direction. In the Laplace domain it reads (with $\hat{f}$ being its source term):

$$-\left(1+s^{-1}\sigma_x\right)^{-1}\partial_x\left(\left(1+s^{-1}\sigma_x\right)^{-1}\partial_x\hat{H}_z\right) - \partial_y^2\hat{H}_z + s^2\hat{H}_z = \hat{f}, \qquad s \in \mathbb{C}_+, \tag{15}$$

where we assumed $H_z|_{t=0} = 0$, $\partial_t H_z|_{t=0} = 0$. The corresponding variational formulation reads

$$A_{\sigma_x}(\hat{H}_z, v) = \left(1+s^{-1}\sigma_x\right)^{-2}(\partial_x\hat{H}_z, \partial_x v) + (\partial_y\hat{H}_z, \partial_y v) + s^2(\hat{H}_z, v) = (\hat{f}, v), \qquad \hat{H}_z, v \in H^1(\mathbb{R}^2).$$

The above sesquilinear form coincides with (12) for a particular case $\varepsilon_x = \varepsilon_y = \mu = 1$, where the change of variables of Theorem 2.4 is performed. To show the coercivity of the sesquilinear form for all $s \in \mathbb{C}_+$, one tests the above with $v = s\hat{H}_z$ (and then takes the real part), see Remark 2.5,

$$A_{\sigma_x}(\hat{H}_z, s\hat{H}_z) = \operatorname{Re}\left(\left(1+s^{-1}\sigma_x\right)^{-2}(\partial_x\hat{H}_z, s\partial_x\hat{H}_z) + (\partial_y\hat{H}_z, s\partial_y\hat{H}_z) + s(s\hat{H}_z, s\hat{H}_z)\right) = \operatorname{Re}\left(\hat{f}, s\hat{H}_z\right). \tag{16}$$

Testing with $v = s\hat{H}_z$ corresponds in the time domain to testing the corresponding equation with $\partial_t H_z$ (if $H_z|_{t=0} = 0$), which is exactly what one does in order to obtain the energy estimates for the problem without the PML. On the other hand, it is possible to obtain some energy identities directly from (16), using the Plancherel's equality. To proceed with this idea, let us rewrite the first term of (16) in a more convenient form:

$$\begin{aligned}\left(\left(1+s^{-1}\sigma_x\right)^{-2}\partial_x\hat{H}_z, s\partial_x\hat{H}_z\right) &= \left(\frac{\partial_x\hat{H}_z}{\left(1+s^{-1}\sigma_x\right)^2}, \frac{s\left(1+s^{-1}\sigma_x\right)^2}{\left(1+s^{-1}\sigma_x\right)^2}\partial_x\hat{H}_z\right) \qquad (17)\\ &= \left(\frac{\partial_x\hat{H}_z}{\left(1+s^{-1}\sigma_x\right)^2}, \frac{s+2\sigma_x+\sigma_x^2 s^{-1}}{\left(1+s^{-1}\sigma_x\right)^2}\partial_x\hat{H}_z\right)\\ &= \left(\bar{s}+2\sigma_x\right)\left\|\frac{\partial_x\hat{H}_z}{\left(1+s^{-1}\sigma_x\right)^2}\right\|^2 + \sigma_x^2 s\left\|\frac{\partial_x\hat{H}_z}{s\left(1+s^{-1}\sigma_x\right)^2}\right\|^2. \qquad (18)\end{aligned}$$

Thus, (16) can be rewritten as follows:

$$\operatorname{Re}\left(s+2\sigma_x\right)\left\|\frac{\partial_x\hat{H}_z}{\left(1+s^{-1}\sigma_x\right)^2}\right\|^2 + \sigma_x^2\operatorname{Re}s\left\|\frac{\partial_x\hat{H}_z}{s\left(1+s^{-1}\sigma_x\right)^2}\right\|^2 + \operatorname{Re}s\|\partial_y\hat{H}_z\|^2 + \operatorname{Re}s\|s\hat{H}_z\|^2 = \operatorname{Re}\left(\hat{f}, s\hat{H}_z\right). \quad (19)$$

Recall the Plancherel's identity (here $\eta > 0$) and its implication:

$$\int_0^{+\infty} \mathrm{e}^{-2\eta t}(H_z(t), v(t))dt = \frac{1}{2\pi i}\int_{\eta+i\mathbb{R}} (\hat{H}_z(s), \hat{v}(s))ds \implies \int_0^{+\infty} \mathrm{e}^{-2\eta t}\operatorname{Re}(H_z(t), v(t))dt = \frac{1}{2\pi i}\int_{\eta+i\mathbb{R}} \operatorname{Re}(\hat{H}_z(s), \hat{v}(s))ds.$$

Before applying the last identity of the above to (19), let us introduce two auxiliary unknowns:

$$\hat{\jmath} = \frac{\partial_x\hat{H}_z}{\left(1+s^{-1}\sigma_x\right)^2}, \qquad \hat{J} = \frac{\partial_x\hat{H}_z}{s\left(1+s^{-1}\sigma_x\right)^2}. \qquad (20)$$

This allows us to obtain a formal energy identity (assuming zero initial conditions for $j,\ J$):

$$\int_0^{+\infty} \mathrm{e}^{-2\eta t}\left((\eta+2\sigma_x)\|j\|^2 + \sigma_x^2\eta\|J\|^2 + \eta\|\partial_y H_z\|^2 + \eta\|\partial_t H_z\|^2\right)dt = \int_0^{+\infty} \mathrm{e}^{-2\eta t}(f, \partial_t H_z)dt.$$

Importantly, this time-domain equality involves newly introduced unknowns (20). This suggests that when writing the PML system corresponding to (15), it may be advantageous to introduce the unknowns (20). Let us rewrite (15) in time using this idea (now taking $f = 0$ but assuming non-zero initial conditions):

$$\partial_t^2 H_z - \partial_x j - \partial_y^2 H_z = 0, \qquad (21a)$$
$$\partial_t j + 2\sigma_x j + \sigma_x^2 J = \partial_x\partial_t H_z, \qquad (21b)$$
$$\partial_t J = j. \qquad (21c)$$

Testing (21a) with $\partial_t H_z$, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\|\partial_t H_z\|^2 + \|\partial_y H_z\|^2\right) + (j, \partial_x\partial_t H_z) = 0,\\ &(j, \partial_x\partial_t H_z) \overset{(21b)}{=} (j, \partial_t j + 2\sigma_x j + \sigma_x^2 J) \overset{(21c)}{=} \frac{1}{2}\frac{d}{dt}\|j\|^2 + 2\sigma_x\|j\|^2 + \sigma_x^2(\partial_t J, J)\\ &= \frac{1}{2}\frac{d}{dt}\left(\|j\|^2 + \|\sigma_x J\|^2\right) + 2\sigma_x\|j\|^2.\end{aligned}$$

Combining the above, we get the following energy decay law

$$\frac{d}{dt}\mathcal{E} = -2\sigma_x\|j\|^2, \qquad \mathcal{E} = \frac{1}{2}\left(\|\partial_t H_z\|^2 + \|\partial_y H_z\|^2 + \|j\|^2 + \sigma_x^2\|J\|^2\right). \tag{22}$$

Indeed, when $\sigma_x \neq \text{const}$, the formulation (21) is not perfectly matched. However, for $\sigma_x = \text{const}$, it is equivalent to another, perfectly matched formulation.

2.3.2. *Remaking the Toy Example: A First-Order Formulation*

Our goal is to derive the PML system in the first order formulation, like the original system (5a-5g). On one hand, we would like to introduce unknowns that would correspond to $s^{-1}\hat{\jmath}$, $s^{-1}\hat{J}$ (the energy of the first-order formulation for the wave equation coincides with the energy of the second-order formulation with u replaced by $\partial_t^{-1}u$). On the other hand, we would like to simplify the corresponding PML system of equations. First, the system of equations in the Laplace domain with the PML change of variables, and then comment on the introduction of the time-domain unknowns:

$$s\hat{H}_z = -\frac{\partial_x \hat{E}_y}{1+s^{-1}\sigma_x} + \partial_y \hat{E}_x, \tag{23a}$$

$$s\hat{E}_y = -\frac{\partial_x \hat{H}_z}{1+s^{-1}\sigma_x}, \tag{23b}$$

$$s\hat{E}_x = \partial_y \hat{H}_z. \tag{23c}$$

In this first-order formulation we suggest to introduce an auxiliary unknown E_y^* (it corresponds to $-s^{-1}\hat{J}$), s.t.

$$E_y^* = -\frac{\partial_x \hat{H}_z}{(s+\sigma_x)^2} \overset{(23b)}{=} \frac{\hat{E}_y}{s+\sigma_x}. \tag{24}$$

This choice is advantageous for two reasons:

- there is no need to introduce an unknown corresponding to $s^{-1}\hat{\jmath}$, since it can be represented as a linear combination of the existing unknowns

$$s^{-1}\hat{\jmath} = \frac{\partial_x \hat{H}_z}{(s+\sigma_x)(1+s^{-1}\sigma_x)} \overset{(23b)}{=} -\frac{\hat{E}_y}{1+s^{-1}\sigma_x} = -\left(1 - \frac{\sigma_x}{s+\sigma_x}\right)\hat{E}_y \overset{(24)}{=} -\hat{E}_y + \sigma_x \hat{E}_y^*.$$

- the above linear combination can replace the first term in the right-hand side of (23a), namely

$$\left(1+s^{-1}\sigma_x\right)^{-1}\partial_x \hat{E}_y$$

  by $\partial_x(\hat{E}_y - \sigma_x \hat{E}_y^*)$. Here we make use of the assumption $\sigma_x = \text{const}$.

Therefore, (23a-23c) can be rewritten in the time domain as follows:

$$\partial_t H_z = -\partial_x E_y + \sigma_x \partial_x E_y^* + \partial_y E_x, \tag{25a}$$

$$\partial_t E_y + \sigma_x E_y = -\partial_x H_z, \tag{25b}$$

$$\partial_t E_x = \partial_y H_z, \tag{25c}$$

$$\partial_t E_y^* + \sigma_x E_y^* = E_y. \tag{25d}$$

Let us show that the energy is consistent with (22). We test the first equation above with H_z, to obtain

$$(\partial_t H_z, H_z) + (\partial_x E_y - \sigma_x \partial_x E_y^*, H_z) - (\partial_y E_x, H_z) = 0. \tag{26}$$

The second term in the above

$$
\begin{aligned}
(\partial_x E_y - \sigma_x \partial_x E_y^*, H_z) &= -(E_y - \sigma_x E_y^*, \partial_x H_z) \\
&\overset{(25b)}{=} (E_y - \sigma_x E_y^*, \partial_t E_y + \sigma_x E_y) = (E_y - \sigma_x E_y^*, \partial_t (E_y - \sigma_x E_y^*) + \sigma_x \partial_t E_y^* + \sigma_x E_y) \\
&\overset{(25d)}{=} \frac{1}{2}\frac{d}{dt}\|E_y - \sigma_x E_y^*\|^2 + (E_y - \sigma_x E_y^*, \sigma_x \partial_t E_y^* + \sigma_x(\partial_t E_y^* + \sigma_x E_y^*)) \\
&= (E_y - \sigma_x E_y^*, 2\sigma_x \partial_t E_y^*) + (E_y - \sigma_x E_y^*, \sigma_x^2 E_y^*).
\end{aligned}
$$

Applying (25d) to each of the two terms above yields

$$
(E_y - \sigma_x E_y^*, 2\sigma_x \partial_t E_y^*) = 2\sigma_x \|E_y - \sigma_x E_y^*\|^2, \qquad (E_y - \sigma_x E_y^*, \sigma_x^2 E_y^*) = (\partial_t E_y^*, \sigma_x^2 E_y^*) = \frac{\sigma_x^2}{2}\frac{d}{dt}\|E_y^*\|^2.
$$

Finally, combining (26) and the latter expressions, we obtain:

$$
\frac{d}{dt}\mathcal{E} = -2\sigma_x \|E_y - \sigma_x E_y^*\|^2, \qquad \mathcal{E} = \|H_z\|^2 + \|E_x\|^2 + \|E_y - \sigma_x E_y^*\|^2 + \|\sigma_x E_y^*\|^2.
$$

As expected, the above corresponds to (22), with all the unknowns substituted by their primitives in time.

2.3.3. *Remarks on the Connection Between the Formulation (25a-25d) and the Bérenger's Split Formulation*

Recall the Bérenger's split formulation for the 2D Maxwell's equations:

$$
\begin{aligned}
&H_z = H_{zx} + H_{zy}, && (27a) \\
&\partial_t E_y + \sigma_x E_y = -\partial_x H_z, && (27b) \\
&\partial_t E_x = \partial_y E_x, && (27c) \\
&\partial_t H_{zx} + \sigma_x H_{zx} = -\partial_x E_y, && (27d) \\
&\partial_t H_{zy} = \partial_y E_x. && (27e)
\end{aligned}
$$

To see how from (25a-25d) one can obtain the above system, we apply ∂_x to (25d), and use $\sigma_x = \text{const}$,

$$
\partial_t \partial_x E_y^* + \sigma_x \partial_x E_y^* = \partial_x E_y.
$$

Then we rewrite (25a) as

$$
\partial_t H_z = -\partial_x E_y + \sigma_x \partial_x E_y^* + \partial_y E_x = -\partial_t \partial_x E_y^* + \partial_y E_x.
$$

Finally, we can introduce the additional unknown H_{zy} as in (27e). Thus, (25a-25d) can be rewritten as

$$
\begin{aligned}
&\partial_t H_z = -\partial_t \partial_x E_y^* + \partial_t H_{zy}, \\
&\partial_t E_y + \sigma_x E_y = -\partial_x H_z, \\
&\partial_t E_x = \partial_y H_z, \\
&\partial_t \partial_x E_y^* + \sigma_x \partial_x E_y^* = \partial_x E_y, \\
&\partial_t H_{zy} = \partial_y E_x.
\end{aligned}
$$

Then, choosing the initial conditions $\partial_x E_y^*\big|_{t=0} = -\left.H_{zx}\right|_{t=0}$, and $H_z|_{t=0} = H_{zx}|_{t=0} + H_{zy}|_{t=0}$, we can ensure that the solution of the above system coincides with the solution of (27a-27e), and, in particular, $\partial_x E_y^* = -H_{zx}$.

2.4. **Stability of the PML of Theorem 2.7, with $\psi_x = \varepsilon_y^{-1}$, $\psi_y = \varepsilon_x^{-1}$**

In this section we extend the ideas of the construction of the PML systems of Section 2.3 to the PMLs for problems with the Lorentz dispersion (3).

2.4.1. *Construction of a PML System Corresponding to Theorem 2.7, with $\psi_x = \varepsilon_y^{-1}$, $\psi_y = \varepsilon_x^{-1}$*

Let us write a PML system corresponding to (5a-5g) with the PML change of variables of Theorem 2.7. We perform the PML change of variables in both directions, however, assume that $\sigma_x,\ \sigma_y \geq 0$. Setting in the resulting system $\sigma_y = 0$ and discarding the auxiliary unknowns associated with the PML in the direction y, we will get the PML system in the direction x (similarly we can obtain the PML system in the direction y). Let us start with the first equation (5a), which in the Laplace domain with the PML change of variables of Theorem 2.7 and with the use of the constitutive relation $s\hat{D}_x = s\varepsilon_x \hat{E}_x$ reads

$$s\hat{D}_x = s\varepsilon_x \hat{E}_x = \left(1 + \frac{\sigma_y}{s\varepsilon_x}\right)^{-1} \partial_y \hat{H}_z.$$

In the time-domain this is equivalent to $\partial_t D_x + \sigma_y E_x = \partial_y H_z$.

Remark 2.9. When deriving the PML system, one ignores all the source terms and initial conditions, since they are necessarily supported outside of the domain where the PML is used.

The equation (5b) can be treated similarly. To deal with (5c), we suggest to employ the splitting idea of Bérenger [1]. This is motivated by results of Section 2.3.3. Let us introduce split fields $\hat{H}_{zx}$ and $\hat{H}_{zy}$ so that they satisfy in the Laplace domain the following equations:

$$s\varepsilon_x(s)\hat{H}_{zy} = \partial_y \hat{E}_x, \qquad s\varepsilon_y(s)\hat{H}_{zx} = -\partial_x \hat{E}_y, \qquad s\hat{B}_z = \partial_y \hat{E}_x - \partial_x \hat{E}_y = s\varepsilon_x(s)\hat{H}_{zy} + s\varepsilon_y(s)\hat{H}_{zx}. \tag{28}$$

Applying the PML change of variables of Theorem 2.7 to the above equations, we obtain the following:

$$\begin{gathered} s\varepsilon_x \hat{H}_{zy} + \sigma_y \hat{H}_{zy} = \partial_y \hat{E}_x, \qquad s\varepsilon_y \hat{H}_{zx} + \sigma_x \hat{H}_{zx} = -\partial_x \hat{E}_y, \\ s\hat{B}_z = s\varepsilon_x(s)\hat{H}_{zy} + s\varepsilon_y(s)\hat{H}_{zx} = \partial_y \hat{E}_x - \partial_x \hat{E}_y - \sigma_y \hat{H}_{zy} - \sigma_x \hat{H}_{zx}. \end{gathered}$$

Finally, the time-domain expression of the operators $s\varepsilon_x(s),\ s\varepsilon_y(s)$ is done similarly to the system (5a-5g), via the introduction of the auxiliary unknowns. After the PML change of variables, (5a-5g) becomes:

$$\partial_t D_x + \sigma_y E_x = \partial_y H_z, \tag{29a}$$
$$\partial_t D_y + \sigma_x E_y = -\partial_x H_z, \tag{29b}$$
$$\partial_t B_z + \sigma_y H_{zy} + \sigma_x H_{zx} = \partial_y E_x - \partial_x E_y, \tag{29c}$$
$$\partial_t H_{zx} + \sum_{\ell=0}^{n_y} \varepsilon_{y\ell} h_{x\ell} + \sigma_x H_{zx} = -\partial_x E_y, \tag{29d}$$
$$\partial_t H_{zy} + \sum_{\ell=0}^{n_x} \varepsilon_{x\ell} h_{y\ell} + \sigma_y H_{zy} = \partial_y E_x, \tag{29e}$$
$$\partial_t h_{\alpha\ell} + \omega_{\beta\ell}^2 b_{\alpha\ell} = H_{z\alpha},\ \partial_t b_{\alpha\ell} = h_{\alpha\ell},\ \ell = 0, \ldots, n_\beta,\ (\alpha, \beta) \in \{(x, y), (y, x)\}, \tag{29f}$$

which we equip with the equations $(5d - 5g)$. In the derivation of the above system we did not use the assumption $\sigma_x,\ \sigma_y = \text{const}$, hence it is perfectly matched. However, even in the case $\sigma_x,\ \sigma_y = \text{const}$, obtaining the energy estimates for it directly requires to solve additional difficulties, besides the dispersive behaviour. We will report how to resolve them elsewhere. To simplify the discussion, we suggest to consider another formulation, equivalent to (29a-29f) for constant absorption parameters (see Remark 2.10 or Section 2.3.3). It differs from (29a-29f) only by treatment of the equation for B_z (5c). More precisely, we introduce $\hat{E}_x^*,\ \hat{E}_y^*$, s.t. $\hat{H}_{zy} = \partial_y \hat{E}_x^*,\ \hat{H}_{zx} = -\partial_x \hat{E}_y^*$, cf. (28). Then, formally, using $\sigma_x = \text{const}$, $\sigma_y = \text{const}$,

$$s\varepsilon_x \hat{E}_x^* + \sigma_y \hat{E}_x^* = \hat{E}_x, \qquad s\varepsilon_y \hat{E}_y^* + \sigma_x \hat{E}_y^* = \hat{E}_y, \qquad s\hat{B}_z = \partial_y \hat{E}_x - \partial_x \hat{E}_y - \sigma_y \partial_y \hat{E}_x^* + \sigma_x \partial_x \hat{E}_y^*. \tag{30}$$

The realization of the operators $s\varepsilon_x,\ s\varepsilon_y$ is done similarly to (5a-5g). We thus obtain the following PML system:

$$\partial_t D_x + \sigma_y E_x = \partial_y H_z, \tag{31a}$$
$$\partial_t D_y + \sigma_x E_y = -\partial_x H_z, \tag{31b}$$
$$\partial_t B_z + \sigma_y \partial_y E_x^* - \sigma_x \partial_x E_y^* = \partial_y E_x - \partial_x E_y, \tag{31c}$$
$$\partial_t E_x^* + \sum_{\ell=0}^{n_x} \varepsilon_{x\ell} \lambda_{x\ell}^* + \sigma_y E_x^* = E_x, \tag{31d}$$
$$\partial_t E_y^* + \sum_{\ell=0}^{n_y} \varepsilon_{y\ell} \lambda_{y\ell}^* + \sigma_x E_y^* = E_y, \tag{31e}$$
$$\partial_t \lambda_{\alpha\ell}^* + \omega_{\alpha\ell}^2 p_{\alpha\ell}^* = E_\alpha^*, \qquad \partial_t p_{\alpha\ell}^* = \lambda_{\alpha\ell}^*, \qquad \ell = 0, \dots, n_\alpha,\ \alpha \in \{x, y\}, \tag{31f}$$
$$\partial_t D_\alpha = \partial_t E_\alpha + \sum_{\ell=0}^{n_\alpha} \varepsilon_{\alpha\ell} \lambda_{\alpha\ell}, \tag{31g}$$
$$\partial_t \lambda_{\alpha\ell} + \omega_{\alpha\ell}^2 p_{\alpha\ell} = E_\alpha, \qquad \partial_t p_{\alpha\ell} = \lambda_{\alpha\ell}, \qquad \ell = 0, \dots, n_\alpha,\ \alpha \in \{x, y\}, \tag{31h}$$
$$\partial_t B_z = \partial_t H_z + \sum_{\ell=0}^{n_\mu} \mu_\ell \lambda_{\mu\ell}, \tag{31i}$$
$$\partial_t \lambda_{\mu\ell} + \omega_{\mu\ell}^2 p_{\mu\ell} = H_z, \qquad \partial_t p_{\mu\ell} = \lambda_{\mu\ell}, \qquad \ell = 0, \dots, n_\mu. \tag{31j}$$

In the above we introduced new PML unknowns E_x^*, $(\lambda_{x\ell}^*,\ p_{x\ell}^*)$, $\ell = 0, \dots, n_x$ and E_y^*, $(\lambda_{y\ell}^*,\ p_{y\ell}^*)$, $\ell = 0, \dots, n_y$.

Remark 2.10. The equivalence of (29a-29f) and (31a-31j) for constant σ_x, σ_y, i.e. that with properly chosen initial data, the solution $\mathbf{E}$, $\mathbf{D}$, B_z, H_z of (29a-29f) solves (31a-31j) and vice versa, can be shown as in [8].

Remark 2.11. In the system (31a-31j) the PML in the direction y is realized via the introduction of the auxiliary unknowns E_y^*, $\lambda_{y\ell}^*$, $p_{y\ell}^*$, $\ell = 0, \dots, n_y$. To obtain the PML in the direction x only, one sets $\sigma_y = 0$. In this case (31e) and (31f) for $\alpha = y$ are decoupled from the rest of the equations and can be discarded.

2.4.2. *Stability of the PML System (31a-31j)*

The energy non-growth result is summarized in the following theorem.

Theorem 2.12. *An energy associated with (31a-31j) defined as*

$$\mathcal{E} = \frac{1}{2}\left(\|E_x - \sigma_y E_x^*\|^2 + \|\sigma_y E_x^*\|^2 + \|E_y - \sigma_x E_y^*\|^2 + \|\sigma_x E_y^*\|^2 + \|H_z\|^2\right) + \mathcal{E}_x^* + \mathcal{E}_y^* + \mathcal{E}_z, \tag{32}$$
$$\mathcal{E}_x^* = \frac{1}{2}\sum_{\ell=0}^{n_x} \varepsilon_{x\ell}\left(\|\lambda_{x\ell} - \sigma_y \lambda_{x\ell}^*\|^2 + \omega_{x\ell}^2 \|p_{x\ell} - \sigma_y p_{x\ell}^*\|^2 + \|\sigma_y \lambda_{x\ell}^*\|^2 + \omega_{x\ell}^2 \|\sigma_y p_{x\ell}^*\|^2\right),$$
$$\mathcal{E}_y^* = \frac{1}{2}\sum_{\ell=0}^{n_y} \varepsilon_{y\ell}\left(\|\lambda_{y\ell} - \sigma_x \lambda_{y\ell}^*\|^2 + \omega_{y\ell}^2 \|p_{y\ell} - \sigma_x p_{y\ell}^*\|^2 + \|\sigma_x \lambda_{y\ell}^*\|^2 + \omega_{y\ell}^2 \|\sigma_x p_{y\ell}^*\|^2\right),$$
$$\mathcal{E}_z = \frac{1}{2}\sum_{\ell=0}^{n_\mu} \mu_\ell \left(\|\lambda_{\mu\ell}\|^2 + \omega_{\mu\ell}^2 \|p_{\mu\ell}\|^2\right),$$

does not grow, provided $\sigma_x,\ \sigma_y \geq 0$:

$$\frac{d}{dt}\mathcal{E} = -2\sigma_y\|E_x - \sigma_y E_x^*\|^2 - 2\sigma_x\|E_y - \sigma_x E_y^*\|^2.$$

Proof. First, we test (31c) with H_z to obtain

$$(\partial_t B_z, H_z) - (\partial_y E_x - \sigma_y \partial_y E_x^*, H_z) + (\partial_x E_y - \sigma_x \partial_x E_y^*, H_z) = 0. \tag{33}$$

Let us consider each term of the above expression separately. The first term is exactly as in (7):

$$(\partial_t B_z, H_z) \overset{(7)}{=} \frac{d}{dt}\left(\frac{1}{2}\|H_z\|^2 + \mathcal{E}_z\right).$$

The second term, after the integration by parts, becomes

$$\begin{aligned}
-(\partial_y E_x - \sigma_y \partial_y E_x^*, H_z) &= (E_x - \sigma_y E_x^*, \partial_y H_z) \overset{(31a)}{=} (E_x - \sigma_y E_x^*, \partial_t D_x + \sigma_y E_x) \\
&= (E_x - \sigma_y E_x^*, \partial_t D_x) + \sigma_y (E_x - \sigma_y E_x^*, E_x - \sigma_y E_x^* + \sigma_y E_x^*) \\
&= (E_x - \sigma_y E_x^*, \partial_t D_x) + \sigma_y \|E_x - \sigma_y E_x^*\|^2 + \sigma_y^2 (E_x - \sigma_y E_x^*, E_x^*) \\
&= \mathcal{T}_1 + \sigma_y \|E_x - \sigma_y E_x^*\|^2 + \mathcal{T}_2.
\end{aligned} \tag{34}$$

Let us first consider

$$\begin{aligned}
\mathcal{T}_1 &= (E_x - \sigma_y E_x^*, \partial_t D_x) \overset{(31g)}{=} \left(E_x - \sigma_y E_x^*, \partial_t E_x + \sum_{\ell=0}^{n_x} \varepsilon_{x\ell}\lambda_{x\ell}\right) \\
&= (E_x - \sigma_y E_x^*, \partial_t (E_x - \sigma_y E_x^*) + \sigma_y \partial_t E_x^*) + \left(E_x - \sigma_y E_x^*, \sum_{\ell=0}^{n_x} \varepsilon_{x\ell}\lambda_{x\ell}\right) \\
&\overset{(31d)}{=} \frac{1}{2}\frac{d}{dt}\|E_x - \sigma_y E_x^*\|^2 + \sigma_y \left(E_x - \sigma_y E_x^*, E_x - \sigma_y E_x^* - \sum_{\ell=0}^{n_x} \varepsilon_{x\ell}\lambda_{x\ell}^*\right) + \left(E_x - \sigma_y E_x^*, \sum_{\ell=0}^{n_x} \varepsilon_{x\ell}\lambda_{x\ell}\right) \\
&= \frac{1}{2}\frac{d}{dt}\|E_x - \sigma_y E_x^*\|^2 + \sigma_y \|E_x - \sigma_y E_x^*\|^2 + \sum_{\ell=0}^{n_x} \varepsilon_{x\ell} \left(E_x - \sigma_y E_x^*, \lambda_{x\ell} - \sigma_y \lambda_{x\ell}^*\right).
\end{aligned}$$

The last term in the above corresponds to a weighted sum of norms of a linear combination of auxiliary unknowns:

$$\begin{aligned}
\sum_{\ell=0}^{n_x} \varepsilon_{x\ell} \left(E_x - \sigma_y E_x^*, \lambda_{x\ell} - \sigma_y \lambda_{x\ell}^*\right) &\overset{(31h,31f)}{=} \sum_{\ell=0}^{n_x} \varepsilon_{x\ell} \left(\partial_t (\lambda_{x\ell} - \sigma_y \lambda_{x\ell}^*) + \omega_{x\ell}^2 (p_{x\ell} - \sigma_y p_{x\ell}^*), \lambda_{x\ell} - \sigma_y \lambda_{x\ell}^*\right) \\
&\overset{(31h,31f)}{=} \frac{1}{2}\frac{d}{dt}\sum_{\ell=0}^{n_x} \varepsilon_{x\ell} \|\lambda_{x\ell} - \sigma_y \lambda_{x\ell}^*\|^2 + \sum_{\ell=0}^{n_x} \varepsilon_{x\ell}\omega_{x\ell}^2 \left(p_{x\ell} - \sigma_y p_{x\ell}^*, \partial_t (p_{x\ell} - \sigma_y p_{x\ell}^*)\right) \\
&= \frac{1}{2}\frac{d}{dt}\sum_{\ell=0}^{n_x} \varepsilon_{x\ell} \left(\|\lambda_{x\ell} - \sigma_y \lambda_{x\ell}^*\|^2 + \omega_{x\ell}^2 \|p_{x\ell} - \sigma_y p_{x\ell}^*\|^2\right).
\end{aligned} \tag{35}$$

Summarizing the above,

$$\mathcal{T}_1 = \frac{1}{2}\frac{d}{dt}\|E_x - \sigma_y E_x^*\|^2 + \sigma_y \|E_x - \sigma_y E_x^*\|^2 + \frac{1}{2}\frac{d}{dt}\sum_{\ell=0}^{n_x} \varepsilon_{x\ell} \left(\|\lambda_{x\ell} - \sigma_y \lambda_{x\ell}^*\|^2 + \omega_{x\ell}^2 \|p_{x\ell} - \sigma_y p_{x\ell}^*\|^2\right).$$

It remains to obtain an explicit energy-like expression for the term $\mathcal{T}_2$ in (34):

$$\mathcal{T}_2 = \sigma_y^2(E_x - \sigma_y E_x^*, E_x^*) \overset{(31d)}{=} \sigma_y^2 \left(\partial_t E_x^* + \sum_{\ell=0}^{n_x} \varepsilon_{x\ell}\lambda_{x\ell}^*, E_x^*\right) \overset{(31f)}{=} \frac{\sigma_y^2}{2}\frac{d}{dt}\left(\|E_x^*\|^2 + \sum_{\ell=0}^{n_x} \varepsilon_{x\ell}\left(\|\lambda_{x\ell}^*\|^2 + \omega_{x\ell}^2\|p_{x\ell}^*\|^2\right)\right),$$

where the derivation of the latter identity is done similarly as e.g. in (7) or (35). Summarizing the above, we obtain the following expression for the second term in (33):

$$-(\partial_y E_x - \sigma_y \partial_y E_x^*, H_z) = \frac{1}{2}\frac{d}{dt}\left(\|E_x - \sigma_y E_x^*\|^2 + \|\sigma_y E_x^*\|^2\right) + 2\sigma_y\|E_x - \sigma_y E_x^*\|^2 + \frac{d}{dt}\mathcal{E}_x^*.$$

Repeating almost verbatim the above computations, one can show that the third term in (33) reads:

$$(\partial_x E_y - \sigma_x \partial_x E_y^*, H_z) = \frac{1}{2}\frac{d}{dt}\left(\|E_y - \sigma_x E_y^*\|^2 + \|\sigma_x E_y^*\|^2\right) + 2\sigma_x\|E_y - \sigma_x E_y^*\|^2 + \frac{d}{dt}\mathcal{E}_y^*.$$

Summing up all the terms of (33) we obtain the desired statement. □

Remark 2.13. The above result can be generalized to the case of Lorentz materials with dissipation, i.e.

$$\begin{aligned}
\varepsilon_\alpha(s) &= 1 + \sum_{\ell=0}^{n_\alpha} \frac{\varepsilon_{\alpha\ell}}{s^2 + 2\nu_{\alpha\ell}s + \omega_{\alpha\ell}^2}, && \varepsilon_{\alpha\ell} > 0,\ \nu_{\alpha\ell} \geq 0,\ \omega_{\alpha\ell} \geq 0, && \ell = 0, \dots, n_\alpha,\ \alpha \in \{x, y\},\\
\mu(s) &= 1 + \sum_{\ell=0}^{n_\mu} \frac{\mu_\ell}{s^2 + 2\nu_{\mu\ell}s + \omega_{\mu\ell}^2}, && \mu_\ell > 0,\ \nu_{\mu\ell} \geq 0,\ \omega_{\mu\ell} \geq 0, && \ell = 0, \dots, n_\mu.
\end{aligned}$$

In this case the choice of $\psi_x(s) = \varepsilon_y(s)^{-1}$, $\psi_y = \varepsilon_x(s)^{-1}$ would lead to a stable system.

Remark 2.14. With the help of the Young's inequality, one can show that the norm of the field $\mathbf{E}$ is controlled with the help of the stability result of Theorem 2.12. E.g. consider the first term in (32):

$$\begin{aligned}
\|E_x - \sigma_y E_x^*\|^2 + \|\sigma_y E_x^*\|^2 &\geq \|E_x\|^2 + 2\|\sigma_y E_x^*\|^2 - 2\|E_x\|\|\sigma_y E_x^*\| \geq \|E_x\|^2 + 2\|\sigma_y E_x^*\|^2 - \frac{2}{3}\|E_x\| - \frac{3}{2}\|\sigma_y E_x^*\|\\
&= \frac{1}{3}\|E_x\|^2 + \frac{1}{2}\|\sigma_y E_x^*\|^2.
\end{aligned}$$

2.5. Stability of the PML of Theorem 2.8, for General $\psi_x(s)$, $\psi_y(s)$

An exact expression of the corresponding to the PML with the change of variables as in Theorem 2.8 is slightly more complicated. It is easier to express the energy of this formulation in terms of new, 'effective' electric and magnetic fields, rather than the original unknowns. However, this requires the reformulation of the corresponding system without the PML. Similar ideas are used in [14].

2.5.1. *A Preliminary Reformulation of the System (5a-5g)*

Let us rewrite the Maxwell's equations in a more convenient form, equivalent to the original Lorentz system (5a-5g), which, however, would allow to handle easier the energy of the corresponding PML system. In particular, notice that the PML change of variables of Theorem 2.8 results in a passivity of a special scaled sesquilinear form. This form can be viewed as a sesquilinear form resulting from the PML of Theorem 2.7 applied to the problem with (passive) coefficients $\varepsilon_y^e = \psi_x^{-1}$, $\varepsilon_x^e = \varepsilon_x\psi_x^{-1}\varepsilon_y^{-1} = \psi_y^{-1}$ and $\mu^e = \varepsilon_y\psi_x\mu$.

Let us consider the solution $(\mathbf{E}, H_z)$ of the system (5a-5g) with the initial conditions $(\mathbf{E}, H_z)|_{t=0} = (\mathbf{E}_0,\ H_{z0})$. As discussed before the derivation of (5a-5g), we choose $\mathbf{D}_0 = \mathbf{E}_0$, $B_{z0} = H_{z0}$ and zero initial conditions for the rest of unknowns. Then, in the Laplace domain, the system (5a-5g) reads

$$s\varepsilon_x(s)\hat{E}_x - E_{x0} = \partial_y \hat{H}_z, \qquad s\varepsilon_y(s)\hat{E}_y - E_{y0} = -\partial_x \hat{H}_z, \qquad s\mu(s)\hat{H}_z - H_{z0} = \partial_y \hat{E}_x - \partial_x \hat{E}_y.$$

The above can be split into two systems:

$$s\varepsilon_x(s)\hat{E}_x^{(1)} = \partial_y \hat{H}_z^{(1)}, \qquad s\varepsilon_y(s)\hat{E}_y^{(1)} = -\partial_x \hat{H}_z^{(1)}, \qquad s\mu(s)\hat{H}_z^{(1)} - H_{z0} = \partial_y \hat{E}_x^{(1)} - \partial_x \hat{E}_y^{(1)}, \tag{36}$$

and

$$s\varepsilon_x(s)\hat{E}_x^{(2)} - E_{x0} = \partial_y \hat{H}_z^{(2)}, \qquad s\varepsilon_y(s)\hat{E}_y^{(2)} - E_{y0} = -\partial_x \hat{H}_z^{(2)}, \qquad s\mu(s)\hat{H}_z^{(2)} = \partial_y \hat{E}_x^{(2)} - \partial_x \hat{E}_y^{(2)}. \tag{37}$$

Due to linearity, $\mathbf{E} = \mathbf{E}^{(1)} + \mathbf{E}^{(2)}$, $H_z = H_z^{(1)} + H_z^{(2)}$. Let us consider the first equation of (36), which we multiply by $\psi_x(s)^{-1}\varepsilon_y(s)^{-1}$, to obtain

$$s\varepsilon_x(s)\psi_x(s)^{-1}\varepsilon_y(s)^{-1}\hat{E}_x^{(1)} = \partial_y \psi_x^{-1}(s)\varepsilon_y(s)^{-1}\hat{H}_z^{(1)} = \partial_y \hat{H}_z^e, \qquad \hat{H}_z^e = \psi_x^{-1}(s)\varepsilon_y(s)^{-1}\hat{H}_z^{(1)}.$$

Here the index 'e' stands for 'effective'. Setting $\varepsilon_x^e(s) = \varepsilon_x(s)\psi_x(s)^{-1}\varepsilon_y(s)^{-1}$, we rewrite the above as

$$s\varepsilon_x^e(s)\hat{E}_x^{(1)} = \partial_y \hat{H}_z^e. \tag{38}$$

Next, we repeat the procedure with the second equation of (36), defining $\varepsilon_y^e = \psi_x(s)^{-1}$:

$$s\varepsilon_y^e(s)\hat{E}_y^{(1)} = -\partial_x \hat{H}_z^e. \tag{39}$$

Finally, for the third equation of (36), it suffices to introduce $\mu^e(s) = \psi_x(s)\varepsilon_y(s)\mu(s)$ and rewrite

$$s\mu(s)\psi_x(s)\varepsilon_y(s)\psi_x(s)^{-1}\varepsilon_y(s)^{-1}\hat{H}_z^{(1)} - H_{z0} = s\mu^e(s)\hat{H}_z^e - H_{z0} = \partial_y \hat{E}_x^{(1)} - \partial_x \hat{E}_y^{(1)}. \tag{40}$$

Therefore, the system (36) is equivalent to the system of equations (38, 39, 40), provided that the initial conditions are chosen as $H_z^e(0) = H_{z0}$. Similarly the system (37) can be rewritten:

$$s\varepsilon_x^e(s)\hat{E}_x^e - E_{x0} = \partial_y \hat{H}_z^{(2)}, \qquad s\varepsilon_y^e \hat{E}_y^e - E_{y0} = -\partial_x \hat{H}_z^{(2)}, \qquad s\mu^e(s)\hat{H}_z^{(2)} = \partial_y \hat{E}_x^e - \partial_x \hat{E}_y^e,$$

with $\hat{E}_x^e = \varepsilon_y(s)\psi_x(s)\hat{E}_x^{(2)}$, $\hat{E}_y^e = \varepsilon_y(s)\psi_x(s)\hat{E}_y^{(2)}$. Since the above two systems have similar structures, from now on we will concentrated only on one of them, namely (38, 39, 40). Thanks to Theorem 2.8, the quantities ε_x^e, ε_y^e, μ^e are passive. Therefore, they are of the following form, see Theorem 2.2:

$$\varepsilon_\alpha^e(s) = 1 + \sum_{\ell=0}^{n_\alpha^e} \frac{\varepsilon_{\alpha\ell}^e}{s^2 + (\omega_{\alpha\ell}^e)^2}, \qquad \varepsilon_{\alpha\ell}^e > 0, \qquad \omega_{\alpha\ell}^e \in \mathbb{R}, \qquad \ell = 0, \ldots, n_\alpha^e, \qquad \alpha \in \{x, y\},$$

$$\mu(s) = 1 + \sum_{\ell=0}^{n_\mu^e} \frac{\mu_\ell}{s^2 + (\omega_{\mu\ell}^e)^2}, \qquad \mu_\ell^e > 0, \qquad \omega_{\mu\ell}^e \in \mathbb{R}, \qquad \ell = 0, \ldots, n_\mu^e.$$

The system without the PML mimics the Lorentz system (5a-5g):

$$
\begin{aligned}
&\partial_t D_x^e = \partial_y H_z^e, \qquad \partial_t D_y^e = -\partial_x H_z^e,\\
&\partial_t B_z^e = \partial_y E_x^{(1)} - \partial_x E_y^{(1)},\\
&\partial_t D_\alpha^e = \partial_t E_\alpha^{(1)} + \sum_{\ell=0}^{n_\alpha^e} \varepsilon_{\alpha\ell}^e \lambda_{\alpha\ell}^e,\\
&\partial_t \lambda_{\alpha\ell}^e + (\omega_{\alpha\ell}^e)^2 p_{\alpha\ell}^e = E_\alpha^{(1)}, \qquad \partial_t p_{\alpha\ell}^e = \lambda_{\alpha\ell}^e, \qquad \ell = 0,\dots n_\alpha^e,\ \alpha \in \{x,y\},\\
&\partial_t B_z^e = \partial_t H_z^e + \sum_{\ell=0}^{n_\mu^e} \mu_\ell^e \lambda_{\mu\ell}^e,\\
&\partial_t \lambda_{\mu\ell}^e + (\omega_{\mu\ell}^e)^2 p_{\mu\ell}^e = H_z^e, \qquad \partial_t p_{\mu\ell}^e = \lambda_{\mu\ell}^e, \qquad \ell = 0,\dots n_\mu^e.
\end{aligned}
\tag{41}
$$

Then the following result holds true.

Proposition 2.15. *Let* $(\mathbf{E}^{(1)},\ H_z^e,\ B_z^e)$ *solve the system (41) with the initial conditions chosen as* $H_z^e|_{t=0} = B_z^e|_{t=0} = H_{z0}$ *and as zero for the rest of unknowns. Let* $\mathbf{E}$, H_z *be the solution to the system (5a-5g) with the initial conditions chosen as* $H_z|_{t=0} = B_z|_{t=0} = H_{z0}$ *and as zero for the rest of unknowns. Then* $\mathbf{E}^{(1)}(t) = \mathbf{E}(t)$ *and* $B_z^e(t) = B_z(t)$ *for all* $t \geq 0$.

To recover $H_z^{(1)}$, we can use the identity $B_z = B_z^e$, or $s\mu(s)\hat{H}_z^{(1)} = s\mu^e(s)\hat{H}_z^e$, which gives in the time domain:

$$
\partial_t H_z^{(1)} + \sum_{\ell=0}^{n_\mu} \mu_\ell \lambda_{\mu\ell}^{(1)} = \partial_t H_z^e + \sum_{\ell=0}^{n_\mu} \mu_\ell \lambda_{\mu\ell}^e, \tag{42}
$$

$$
\partial_t \lambda_{\mu\ell}^{(1)} + \omega_{\mu\ell}^2 p_{\mu\ell}^{(1)} = H_z^{(1)}, \qquad \partial_t p_{\mu\ell}^{(1)} = \lambda_{\mu\ell}^{(1)}, \qquad \ell = 0,\dots n_\mu. \tag{43}
$$

With the system (41) we can associate a conservation of a certain energy.

Theorem 2.16. *An energy* $\mathcal{E}$ *of (41), defined below, satisfies* $\frac{d}{dt}\mathcal{E} = 0$. *Here*

$$
\begin{aligned}
&\mathcal{E} = \frac{1}{2}\left(\|E_x^{(1)}\|^2 + \|E_y^{(1)}\|^2 + \|H_z^e\|^2\right) + \mathcal{E}_x + \mathcal{E}_y + \mathcal{E}_z,\\
&\mathcal{E}_\alpha = \frac{1}{2}\sum_{\ell=0}^{n_\alpha} \varepsilon_{\alpha\ell}^e \left(\|\lambda_{\alpha\ell}^e\|^2 + (\omega_{\alpha\ell}^e)^2\|p_{\alpha\ell}^e\|^2\right),\ \alpha \in \{x,y\}, \qquad \mathcal{E}_z = \frac{1}{2}\sum_{\ell=0}^{n_\mu} \mu_\ell^e \left(\|\lambda_{\mu\ell}^e\|^2 + (\omega_{\mu\ell}^e)^2\|p_{\mu\ell}^e\|^2\right).
\end{aligned}
$$

As for the field $H_z^{(1)}$, we were not able to deduce the conservation of its norm from the equations (42-43). However, the following stability result which will be of use later.

Proposition 2.17. *The following bound holds for the solution of the system (42-43) coupled with (41):*

$$
\tilde{\mathcal{E}}_z(t) := \frac{1}{2}\left(\|H_z^{(1)}(t) - H_z^e(t)\|^2 + \sum_{\ell=0}^{n_\mu} \mu_\ell \left(\|\lambda_{\mu\ell}^{(1)}(t)\|^2 + \omega_{\mu\ell}^2\|p_{\mu\ell}^{(1)}(t)\|^2\right)\right) \leq 2\tilde{\mathcal{E}}_z(0) + C\mathcal{E}(0)t^2, \qquad t \geq 0,
$$

where $\mathcal{E}(t)$ *is defined in Theorem 2.16 and* $C > 0$ *is constant.*

Proof. See Appendix A. □

Remark 2.18. The result of (at most) linear growth of the field $H_z^{(1)}$ is non-optimal, at least when the initial conditions are chosen as in Proposition 2.15. This follows from the fact that $H_z^{(1)}$ solves the original Maxwell system, and the corresponding energy conservation result extends to $H_z^{(1)}$.

2.5.2. The PML System and Its Stability

The application of the PML of Theorem 2.8 to (41) results in the system that has the same structure as (31a-31j). Indeed, $\psi_x = (\varepsilon_y^e)^{-1}$ and $\psi_y = (\varepsilon_x^e)^{-1}$. We can immediately write the corresponding PML system:

$$
\begin{aligned}
&\partial_t D_x^e + \sigma_y E_x^{(1)} = \partial_y H_z^e, \qquad \partial_t D_y^e + \sigma_x E_y^{(1)} = -\partial_x H_z^e,\\
&\partial_t B_z^e = \partial_y E_x^{(1)} - \sigma_y \partial_y E_x^* - \partial_x E_y^{(1)} + \sigma_x \partial_x E_y^*,\\
&\partial_t E_x^* + \sum_{\ell=0}^{n_x^e} \varepsilon_{x\ell}^e \lambda_{x\ell}^* + \sigma_y E_x^* = E_x^{(1)}, \qquad \partial_t E_y^* + \sum_{\ell=0}^{n_y^e} \varepsilon_{y\ell}^e \lambda_{y\ell}^* + \sigma_x E_y^* = E_y^{(1)},\\
&\partial_t \lambda_{\alpha\ell}^* + (\omega_{\alpha\ell}^e)^2 p_{\alpha\ell}^e = E_\alpha^{(1)}, \qquad \partial_t p_{\alpha\ell}^* = \lambda_{\alpha\ell}^*, \qquad \ell = 0, \dots, n_\alpha, \qquad \alpha \in \{x, y\},\\
&\partial_t D_\alpha^e = \partial_t E_\alpha + \sum_{\ell=0}^{n_\alpha^e} \varepsilon_{\alpha\ell}^e \lambda_{\alpha\ell}^e,\\
&\partial_t \lambda_{\alpha\ell}^e + (\omega_{\alpha\ell}^e)^2 p_{\alpha\ell}^e = E_\alpha^{(1)}, \qquad \partial_t p_{\alpha\ell}^e = \lambda_{\alpha\ell}^e, \qquad \ell = 0, \dots n_\alpha^e,\ \alpha \in \{x, y\},\\
&\partial_t B_z^e = \partial_t H_z^e + \sum_{\ell=0}^{n_\mu^e} \mu_\ell^e \lambda_{\mu\ell}^e,\\
&\partial_t \lambda_{\mu\ell}^e + (\omega_{\mu\ell}^e)^2 p_{\mu\ell}^e = H_z^e, \qquad \partial_t p_{\mu\ell}^e = \lambda_{\mu\ell}^e, \qquad \ell = 0, \dots n_\mu^e.
\end{aligned}
\tag{44}
$$

As before, the physical field $H_z^{(1)}$ can be recovered using (42-43). An energy associated with the above system can be derived as in Theorem 2.12, and the bound on $H_z^{(1)}$ as in Proposition 2.17.

Theorem 2.19. *An energy associated with (44) defined as*

$$
\begin{aligned}
\mathcal{E} &= \frac{1}{2}\left(\|E_x^{(1)} - \sigma_y E_x^*\|^2 + \|\sigma_y E_x^*\|^2 + \|E_y^{(1)} - \sigma_x E_y^*\|^2 + \|\sigma_x E_y^*\|^2 + \|H_z^e\|^2\right) + \mathcal{E}_x^{e,*} + \mathcal{E}_y^{e,*} + \mathcal{E}_z^e, \qquad (45)\\
\mathcal{E}_x^{e,*} &= \frac{1}{2}\sum_{\ell=0}^{n_x} \varepsilon_{x\ell}^e \left(\|\lambda_{x\ell}^e - \sigma_y \lambda_{x\ell}^*\|^2 + (\omega_{x\ell}^e)^2 \|p_{x\ell}^e - \sigma_y p_{x\ell}^*\|^2 + \|\sigma_y \lambda_{x\ell}^*\|^2 + (\omega_{x\ell}^e)^2 \|\sigma_y p_{x\ell}^*\|^2\right),\\
\mathcal{E}_y^{e,*} &= \frac{1}{2}\sum_{\ell=0}^{n_y} \varepsilon_{y\ell}^e \left(\|\lambda_{y\ell}^e - \sigma_x \lambda_{y\ell}^*\|^2 + (\omega_{y\ell}^e)^2 \|p_{y\ell}^e - \sigma_x p_{y\ell}^*\|^2 + \|\sigma_x \lambda_{y\ell}^*\|^2 + (\omega_{y\ell}^e)^2 \|\sigma_x p_{y\ell}^*\|^2\right),\\
\mathcal{E}_z^e &= \frac{1}{2}\sum_{\ell=0}^{n_\mu} \mu_\ell^e \left(\|\lambda_{\mu\ell}^e\|^2 + (\omega_{\mu\ell}^e)^2 \|p_{\mu\ell}^e\|^2\right)
\end{aligned}
$$

does not grow, provided $\sigma_x,\ \sigma_y \geq 0$:

$$
\frac{d}{dt}\mathcal{E} = -2\sigma_y \|E_x^{(1)} - \sigma_y E_x^*\|^2 - 2\sigma_x \|E_y^{(1)} - \sigma_x E_y^*\|^2.
$$

Moreover, for the solution of (42-43) coupled with (44), it holds

$$
\tilde{\mathcal{E}}_z(t) := \frac{1}{2}\left(\|H_z^{(1)}(t) - H_z^e(t)\|^2 + \sum_{\ell=0}^{n_\mu} \mu_\ell \left(\|\lambda_{\mu\ell}^{(1)}(t)\|^2 + \omega_{\mu\ell}^2 \|p_{\mu\ell}^{(1)}(t)\|^2\right)\right) \leq 2\tilde{\mathcal{E}}_z(0) + C\mathcal{E}(0)t^2, \qquad t \geq 0,
$$

where $C > 0$ is constant.

3. General Passive Materials: Stability with Energy Techniques

In this section we summarize some results on the materials whose dielectric permittivity and magnetic permeability do not satisfy the conditions of Section 2. One of the examples is given by Warburg's law, when the dielectric permittivity $\varepsilon(s) = 1 + \frac{1+a\sqrt{s}}{s\left(1+b\sqrt{s}\right)}$, $a, b > 0$, see e.g. [21,22] for the corresponding numerical method.

The construction of the PML for this types of systems is done as before, via a special change of variables (9). In this section we will show the stability of the following PML.

Theorem 3.1 ([5]). *Let ε_x, ε_y, μ be passive. Then the sesquilinear form (14) with $\psi_x(s) = \varepsilon_y(s)^{-1}$ and $\psi_y(s) = \varepsilon_x(s)^{-1}$ is passive for any $\sigma_x \geq 0$ and $\sigma_y \geq 0$.*

The stability of the corresponding system without the PML was shown in [14] with the help of a Herglotz-Nevanlinna representation formula. This integral representation allows to write any passive system (even dissipative) in the form (5a-5g), and thus, associate with any system an energy that is conserved (while at the same time, there may exist another energy that decays). A conservative extension for a general class of dissipative systems had been constructed, though in a slightly different form, in the seminal work by Figotin and Schenker in [23]. Such an approximation of an arbitrary material with the help of Lorentz materials will be used to construct a PML system similar to (31a-31j), see Section 3.2.

3.1. Time-Domain System without PMLs

The results of this section are due to [14], where it is suggested that passive functions corresponding to the Laplace transform of real-valued in the time domain distributions (i.e., in the Laplace domain, $\overline{\hat{f}(s)} = \hat{f}(\overline{s})$, $s \in \mathbb{C}_+$) satisfy the property formulated below. The following lemma is nothing more but a version of the Herglotz-Nevanlinna representation for such functions.

Lemma 3.2 (Herglotz-Nevanlinna Representation). *The function $f(s)$, which satisfies $\overline{f(s)} = f(\overline{s})$, with $s \in \mathbb{C}_+$, is passive if and only if there exists a non-decreasing function of a bounded variation $\nu_f(\xi)$, for which the Stiltjes integral $\int\limits_{-\infty}^{\infty} \frac{d\nu_f(\xi)}{1+\xi^2}$ is finite and $f(s)$ is represented as the following Stiltjes integral:*

$$f(s) = a + \int\limits_{-\infty}^{\infty} \frac{d\nu_f(\xi)}{s^2 + \xi^2}, \qquad a \geq 0. \tag{46}$$

Here $a = \lim\limits_{s_r \to +\infty} f(s_r)$, and the measure ν_f satisfies $\int\limits_{-\infty}^{+\infty} \frac{d\nu_f(\xi)}{1+\xi^2} < \infty$.

Proof. Notice that $f(s)$ is passive if and only if $\operatorname{Re}\left(-i\omega f(-i\omega)\right) > 0$ for $\operatorname{Im}\omega > 0$, or $\operatorname{Im}(\omega f(-i\omega)) > 0$ for $\operatorname{Im}\omega > 0$. Hence, $\omega f(-i\omega)$ is Herglotz, and one can apply the results of [14, Section 4.1] to $f(-i\omega)$. ☐

For instance, to obtain the Lorentz magnetic permeability (3), we can take a sum of δ-measures $\nu_\mu(\xi) = \sum\limits_{\ell=0}^{n_\mu} \mu_\ell \delta(\xi - \omega_{\mu\ell})$. Without loss of generality, let us assume that [6,14]

$$\varepsilon_x(s) \to 1, \qquad \varepsilon_y(s) \to 1, \qquad \mu(s) \to 1, \qquad \text{as } \operatorname{Re} s \to +\infty.$$

Then Lemma 3.2 yields (where we take into account $\varepsilon_x(s_r)$, $\varepsilon_y(s_r)$, $\mu(s_r) \to 1$ as $s_r \to +\infty$)

$$\varepsilon_\alpha(s) = 1 + \int\limits_{-\infty}^{+\infty} \frac{d\nu_\alpha}{s^2 + \xi^2}, \qquad \alpha \in \{x, y\}, \qquad \mu(s) = 1 + \int\limits_{-\infty}^{+\infty} \frac{d\nu_\mu}{s^2 + \xi^2}, \qquad s \in \mathbb{C}_+.$$

These identities generalize the corresponding definitions for the Lorentz materials (3), with the exception that the weighted sums are substituted by integrals with Borel measures. Our goal is to write the system (1) in the time domain, using the above representation of $\underline{\underline{\varepsilon}}$, μ. Let us recall from [14] how the equation $s\hat{D}_x = s\varepsilon_x(s)\hat{E}_x$ can be written in the time domain. Let us introduce $\hat{\lambda}_x(\xi)$ and $\hat{p}_x(\xi)$, which map $\xi \in \mathbb{R}$ into a set of analytic functions from $s \in \mathbb{C}_+$ into $L^2(\mathbb{R}^2)$, so that they satisfy the following identities:

$$s\hat{\lambda}_x(\xi) + \xi^2\hat{p}_x(\xi) = \hat{E}_x, \qquad s\hat{p}_x(\xi) = \hat{\lambda}_x(\xi),\ \xi \in \mathbb{R}.$$

Then, in the time domain, we can rewrite (formally) $s\hat{D}_x = s\varepsilon\hat{E}_x$ as:

$$\begin{aligned}
&\partial_t D_x = \partial_t E_x + \int_{-\infty}^{\infty} \lambda_x(\xi) d\nu_x(\xi),\\
&\partial_t \lambda_x(\xi) + \xi^2 p_x(\xi) = E_x, \qquad \partial_t p_x(\xi) = \lambda_x(\xi), \qquad \xi \in \mathbb{R}.
\end{aligned} \tag{47}$$

If the measure ν_x has a density f_{ν_x}, we can define the quantities $\lambda_x(\xi),\ p_x(\xi)$ only for $\xi \in \operatorname{supp} f_{\nu_x}(\xi)$ (this is the case for the Lorentz materials, where the density has a discrete support). Thus, extending the above idea to the rest of the constitutive relations, we write (1) in the form (5a-5g):

$$\begin{aligned}
&\partial_t D_x = \partial_y H_z, \qquad \partial_t D_y = -\partial_x H_z,\\
&\partial_t B_z = \partial_y E_x - \partial_x E_y,\\
&\partial_t D_\alpha = \partial_t E_\alpha + \int_{-\infty}^{\infty} \lambda_\alpha(\xi) d\nu_\alpha(\xi), \qquad \partial_t\lambda_\alpha + \xi^2 p_\alpha = E_\alpha,\ \partial_t p_\alpha = \lambda_\alpha,\ \xi \in \mathbb{R}, \alpha \in \{x, y\},\\
&\partial_t B_z = \partial_t H_z + \int_{-\infty}^{\infty} \lambda_\mu(\xi) d\nu_\mu(\xi), \qquad \partial_t\lambda_\mu + \xi^2 p_\mu = E_\mu,\ \partial_t p_\mu = \lambda_\mu,\ \xi \in \mathbb{R}.
\end{aligned} \tag{48}$$

Here $\lambda_\alpha(\xi),\ p_\alpha(\xi)$, $\alpha \in \{x, y, \mu\}$, are functions from $\mathbb{R}$ to $C\left(\mathbb{R}; L^2(\mathbb{R}^2)\right)$. An associated energy is conserved [14].

Proposition 3.3. *An energy $\mathcal{E}$ associated with (48), defined below, satisfies $\frac{d}{dt}\mathcal{E} = 0$. Here*

$$\begin{aligned}
&\mathcal{E} = \frac{1}{2}\left(\|E_x\|^2 + \|E_y\|^2 + \|H_z\|^2\right) + \mathcal{E}_x + \mathcal{E}_y + \mathcal{E}_z,\\
&\mathcal{E}_\alpha = \frac{1}{2}\int_{-\infty}^{\infty} \left(\|\lambda_\alpha\|^2 + \xi^2\|p_\alpha\|^2\right) d\nu_\alpha(\xi),\ \alpha \in \{x, y\}, \qquad \mathcal{E}_z = \frac{1}{2}\int_{-\infty}^{\infty} \left(\|\lambda_\mu\|^2 + \xi^2\|p_\mu\|^2\right) d\nu_\mu(\xi).
\end{aligned}$$

3.2. The PML System and Its Stability

Let us perform the PML change of variables for the system (48) as per Theorem 3.1. This results in the following analogue of the system (31a-31j) (where again the difference is that the sums are substituted by integrals). The derivation of this system follows the derivation of (31a-31j), combined with the use of (47).

$$
\begin{aligned}
&\partial_t D_x + \sigma_y E_x = \partial_y H_z, \qquad \partial_t D_y + \sigma_x E_y = -\partial_x H_z,\\
&\partial_t B_z + \sigma_y \partial_y E_x^* - \sigma_x \partial_x E_y^* = \partial_y E_x - \partial_x E_y,\\
&\partial_t D_\alpha = \partial_t E_\alpha + \int_{-\infty}^{\infty} \lambda_\alpha(\xi) d\nu_\alpha(\xi), \qquad \partial_t \lambda_\alpha + \xi^2 p_\alpha = E_\alpha, \qquad \partial_t p_\alpha = \lambda_\alpha,\ \xi \in \mathbb{R},\ \alpha \in \{x, y\},\\
&\partial_t E_x^* + \int_{-\infty}^{\infty} \lambda_x^*(\xi) d\nu_x(\xi) + \sigma_y E_x^* = E_x, \qquad \partial_t E_y^* + \int_{-\infty}^{\infty} \lambda_y^*(\xi) d\nu_y(\xi) + \sigma_x E_y^* = E_y,\\
&\partial_t \lambda_\alpha^* + \xi^2 p_\alpha^* = E_\alpha^*, \qquad \partial_t p_\alpha^* = \lambda_\alpha^*, \qquad \xi \in \mathbb{R},\ \alpha \in \{x, y\},\\
&\partial_t B_z = \partial_t H_z + \int_{-\infty}^{\infty} \lambda_\mu(\xi) d\nu_\mu(\xi), \qquad \partial_t \lambda_\mu + \xi^2 p_\mu = E_\mu,\ \partial_t p_\mu = \lambda_\mu,\ \xi \in \mathbb{R}.
\end{aligned}
\tag{49}
$$

We can formulate the following analogue of Theorem 2.12, with the proof almost verbatim the same.

Theorem 3.4. *An energy $\mathcal{E}$ associated with (49), defined as*

$$
\begin{aligned}
\mathcal{E} &= \frac{1}{2}\left(\|E_x - \sigma_y E_x^*\|^2 + \|\sigma_y E_x^*\|^2 + \|E_y - \sigma_x E_y^*\|^2 + \|\sigma_x E_y^*\|^2 + \|H_z\|^2\right) + \mathcal{E}_x + \mathcal{E}_y + \mathcal{E}_z,\\
\mathcal{E}_x &= \frac{1}{2}\left(\int_{-\infty}^{+\infty} \left(\|\lambda_x - \sigma_y \lambda_x^*\|^2 + \xi^2 \|p_x - \sigma_y p_x^*\|^2 + \|\sigma_y \lambda_x^*\|^2 + \xi^2 \|\sigma_y p_x^*\|^2\right) d\nu_x\right),\\
\mathcal{E}_y &= \frac{1}{2}\left(\int_{-\infty}^{+\infty} \left(\|\lambda_y - \sigma_x \lambda_y^*\|^2 + \xi^2 \|p_y - \sigma_x p_y^*\|^2 + \|\sigma_x \lambda_y^*\|^2 + \xi^2 \|\sigma_x p_y^*\|^2\right) d\nu_y\right),\\
\mathcal{E}_z &= \frac{1}{2}\int_{-\infty}^{+\infty} \left(\|\lambda_\mu\|^2 + \xi^2 \|p_\mu\|^2\right) d\nu_\mu,
\end{aligned}
$$

does not grow, provided any $\sigma_x,\ \sigma_y \geq 0$:

$$
\frac{d}{dt}\mathcal{E} = -2\sigma_x \|E_y - \sigma_x E_y^*\|^2 - 2\sigma_y \|E_x - \sigma_y E_x^*\|^2.
$$

4. Stability of PMLs for a Non-constant Absorption Parameter for the 1D Dispersive System

While we have proven the stability of the new PMLs under the assumption $\sigma_x = \text{const}$, obtaining the energy estimates for $\sigma_x \neq \text{const}$ remains an open question. It seems that showing the stability for varying σ_x is not too trivial even for 1D dispersive models (being almost obvious for classical PMLs for a 1D non-dispersive wave equation). The goal of this section is to prove the stability of the PMLs, that were proposed and analyzed in [5] and in this work for $\sigma_x \equiv \text{const}$, in the case when $\sigma_x \neq \text{const}$ for a 1D dispersive system of equations.

All over this section, without loss of generality, we will assume that the permittivity and the permeability are Lorentz (3). Moreover, we will consider only the PMLs suggested in Theorem 2.7. However, all the results can be extended to more general $\varepsilon_x,\ \varepsilon_y,\ \mu$, and other PMLs (e.g. given in Theorem 3.1).

The 1D analogue of the dispersive system of equations (5a-5g) in the Laplace domain reads:

$$s\varepsilon_y(s)\hat{E}_y = -\partial_x \hat{H}_z, \qquad s\mu(s)\hat{H}_z = -\partial_x \hat{E}_y. \tag{50}$$

The time-domain equivalent of the above is given by the system of equations

$$\begin{aligned}
&\partial_t D_y = -\partial_x H_z, \qquad \partial_t B_z = -\partial_x E_y,\\
&\partial_t D_y = \partial_t E_y + \sum_{\ell=0}^{n_y} \varepsilon_{y\ell}\lambda_{y\ell}, \qquad \partial_t \lambda_{y\ell} + \omega_{y\ell}^2 p_{y\ell} = E_y, \qquad \partial_t p_{y\ell} = \lambda_{y\ell}, \qquad \ell = 0,\dots,n_y,\\
&\partial_t B_z = \partial_t H_z + \sum_{\ell=0}^{n_y} \mu_\ell \lambda_{\mu\ell}, \qquad \partial_t \lambda_{\mu\ell} + \omega_{\mu\ell}^2 p_{\mu\ell} = H_z, \qquad \partial_t p_{\mu\ell} = \lambda_{\mu\ell}, \qquad \ell = 0,\dots,n_\mu.
\end{aligned} \tag{51}$$

This section is organized as follows. First, we consider a simplified case $\varepsilon_y = \mu$, for which the derivation of the energy estimates can be done as for the 1D non-dispersive wave equation. Next, we will concentrate on a general case $\varepsilon_y \neq \mu$, and show how to deal with it by rewriting the PML system of equations in a different form and deriving the energy estimates for this new system.

4.1. Stability of PMLs with $\psi_x = \varepsilon_y^{-1}$ for a Special Case $\varepsilon_y = \mu$

Let us assume that $\varepsilon_y(s) = \mu(s)$, and apply the PML change of variables (9) with $\psi_x(s) = \varepsilon_y(s)^{-1}$ to (50):

$$s\varepsilon_y(s)\hat{E}_y = -\left(1 + \frac{\sigma_x}{s\varepsilon_y(s)}\right)^{-1} \partial_x \hat{H}_z, \qquad s\varepsilon_y(s)\hat{H}_z = -\left(1 + \frac{\sigma_x}{s\varepsilon_y(s)}\right)^{-1} \partial_x \hat{E}_y.$$

This results in the following system:

$$s\varepsilon_y(s)\hat{E}_y + \sigma_x \hat{E}_y = -\partial_x \hat{H}_z, \qquad s\varepsilon_y(s)\hat{H}_z + \sigma_x \hat{H}_z = -\partial_x \hat{E}_y, \qquad x \in \mathbb{R}.$$

In the time domain, the above reads:

$$\partial_t D_y + \sigma_x E_y = -\partial_x H_z, \qquad \partial_t B_z + \sigma_x H_z = -\partial_x E_y, \tag{52a}$$

$$\partial_t D_y = \partial_t E_y + \sum_{\ell=0}^{n_y} \varepsilon_{y\ell}\lambda_{y\ell}, \qquad \partial_t \lambda_{y\ell} + \omega_{y\ell}^2 p_{y\ell} = E_y, \qquad \partial_t p_{y\ell} = \lambda_{y\ell}, \tag{52b}$$

$$\partial_t B_z = \partial_t H_z + \sum_{\ell=0}^{n_y} \varepsilon_{y\ell}\lambda_{z\ell}, \qquad \partial_t \lambda_{z\ell} + \omega_{y\ell}^2 p_{z\ell} = H_z, \qquad \partial_t p_{z\ell} = \lambda_{z\ell}, \qquad \ell = 0,\dots,n_y. \tag{52c}$$

The energy nongrowth result follows trivially.

Proposition 4.1. *An energy $\mathcal{E}$ associated with the system (52a-52c) defined as*

$$\mathcal{E} = \|E_y\|^2 + \|H_z\|^2 + \mathcal{E}_y + \mathcal{E}_z, \qquad \mathcal{E}_y = \sum_{\ell=0}^{n_y} \varepsilon_{y\ell}\left(\|\lambda_{y\ell}\|^2 + \omega_{y\ell}^2\|p_{y\ell}\|^2\right), \qquad \mathcal{E}_z = \sum_{\ell=0}^{n_y} \varepsilon_{y\ell}\left(\|\lambda_{z\ell}\|^2 + \omega_{y\ell}^2\|p_{z\ell}\|^2\right),$$

does not grow, provided any non-negative $\sigma_x \in L^\infty(\mathbb{R})$:

$$\frac{d}{dt}\mathcal{E} = -\int_0^{+\infty} \left(|E_y|^2 + |H_z|^2\right)\sigma_x(x)dx.$$

Proof. The energy identity is deduced by testing the first equation of (52a) with E_y, and the second equation of (52a) with H_z, and summing up the obtained identities. The details are left to the reader. $\square$

4.2. Stability of PMLs with $\psi_x = \varepsilon_y^{-1}$ for a General Case when $\varepsilon_y \neq \mu$

The case $\varepsilon_y \neq \mu$ is less trivial. Applying the PML change of variables (9) with $\psi_x = \varepsilon_y^{-1}$ to (50), we can rewrite the corresponding system in the second order formulation, cf. (10),

$$\partial_x \left(\varepsilon_y(s)^{-1} \left(1 + \frac{\sigma_x(x)}{s\varepsilon_y(s)}\right)^{-1} \partial_x \hat{H}_z \right) - s^2 \mu(s) \left(1 + \frac{\sigma_x(x)}{s\varepsilon_y(s)}\right) \hat{H}_z = 0, \qquad x \in \mathbb{R},$$

with $\sigma(x) \equiv 0$ for $x < 0$. Following the ideas of Section 2.3, we will show the coercivity of the corresponding sesquilinear form for all $s \in \mathbb{C}_+$, namely

$$A(\hat{H}_z, v) = \left(\frac{\varepsilon_y(s)^{-1}}{1 + \sigma_x(x)(s\varepsilon_y(s))^{-1}} \partial_x \hat{H}_z, \partial_x \hat{H}_z \right) + s^2 \mu(s) \left(\left(1 + \sigma_x(x)(s\varepsilon_y(s))^{-1}\right) \hat{H}_z, v \right), \; \hat{H}_z, v \in H^1(\mathbb{R}). \quad (53)$$

So far, we were able to show only a very special estimate, derived by testing (53) with $v = s\hat{H}_z$ and using a proper scaling. Defining the branch cut of the square root as $\mathbb{R}_{\leq 0}$, we multiply the above with $\sqrt{\frac{s\varepsilon_y(s)}{s\mu(s)}}$ and test the result with $s\hat{H}_z$. This results in the following:

$$\begin{aligned} \operatorname{Re} \left(\sqrt{\frac{s\varepsilon_y}{s\mu}} A\left(\hat{H}_z, s\hat{H}_z\right) \right) &= \operatorname{Re} \left(\frac{\bar{s}\sqrt{\frac{s\varepsilon_y}{s\mu}} \varepsilon_y^{-1} \left(1 + \sigma_x(x)\overline{(s\varepsilon_y)^{-1}}\right)}{|1 + \sigma(x)(s\varepsilon_y)^{-1}|^2} \partial_x \hat{H}_z, \partial_x \hat{H}_z \right) \\ &+ \operatorname{Re} \left(\bar{s}\sqrt{\frac{s\varepsilon_y}{s\mu}} s^2 \mu \left(\left(1 + \frac{\sigma_x(x)}{s\varepsilon_y}\right) \hat{H}_z, \hat{H}_z \right) \right). \end{aligned}$$

We use the notation $\sqrt{\frac{s\varepsilon_y}{s\mu}}$ to underline that the real parts of the numerator and denominator are positive in $\mathbb{C}_+$, and hence $\frac{s\varepsilon_y}{s\mu}$ does not cross the branch cut of the square root when $s \in \mathbb{C}_+$. In this case we can use the identity $\sqrt{z_1 z_2} = \sqrt{z_1}\sqrt{z_2}$ for $z_1,\ z_2 \in \mathbb{C}_+$, and rewrite the above as follows:

$$\begin{aligned} \operatorname{Re} \left(\sqrt{\frac{s\varepsilon_y}{s\mu}} A\left(\hat{H}_z, s\hat{H}_z\right) \right) &= \operatorname{Re} \left(|s|^2 ((s\mu)(s\varepsilon_y))^{-\frac{1}{2}} \left\| \frac{\partial_x \hat{H}_z}{1 + \sigma_x(x)(s\varepsilon_y)^{-1}} \right\|^2 \right) \\ &+ \operatorname{Re} \left(|\varepsilon_y|^{-2} ((s\mu)^{-1}(s\varepsilon_y))^{\frac{1}{2}} \int_{\mathbb{R}} \sigma_x(x) \left| \frac{\hat{H}_z}{1 + \sigma_x(x)(s\varepsilon_y)^{-1}} \right|^2 dx \right) \\ &+ \operatorname{Re} \left(|s|^2 \left((s\varepsilon_y)(s\mu)\right)^{\frac{1}{2}} \left\| \hat{H}_z \right\|^2 + |s|^2 ((s\varepsilon_y)^{-1}(s\mu))^{\frac{1}{2}} \int_{\mathbb{R}} \sigma_x(x) \left| \hat{H}_z \right|^2 dx \right). \end{aligned} \quad (54)$$

Thanks to $\operatorname{Re} \sqrt{(s\mu)(s\varepsilon_y)} > 0$ and $\operatorname{Re} \sqrt{(s\mu)^{-1}(s\varepsilon_y)} > 0$ for $s \in \mathbb{C}_+$ we can see that the real part of every term in the above is positive. This implies the coercivity of the scaled sesquilinear form $A(\hat{H}_z, v)$. Moreover, the apparatus of the work [5] can be used to show the well-posedness and the stability of the resulting PML system in the time domain. However, to perform the energy analysis, it is necessary to rewrite the original system of 1D equations in a very special form. This is the subject of the following section.

4.2.1. *A Preliminary Reformulation of the System (51)*

For constant σ_x = const, the estimate (54) would imply the passivity of the scaled sesquilinear form $\sqrt{\frac{s\varepsilon_y}{s\mu}}A(\hat{H}_z, v)$, cf. Theorem 2.8. This suggests that to write the time-domain PML system for which the energy estimates are derived in an easy manner, it may be first necessary to reformulate the original system without the PML (51) in an equivalent form, making use of the ideas of Section 2.5.1.

As in Section 2.5.1, we will make use of the linearity, and rewrite the corresponding system in the time domain, assuming the initial conditions $(E_y,\ H_z)|_{t=0} = (0, H_{z0})$. The case $(E_{y0}, 0)$ can be treated similarly. We thus start with (50) with the initial conditions $(0, H_{z0})$, using the same notation as in Section 2.5.1:

$$s\varepsilon_y \hat{E}_y^{(1)} = -\partial_x \hat{H}_z^{(1)}, \qquad s\mu \hat{H}_z^{(1)} - H_{z0} = -\partial_x \hat{E}_y,$$

and then define $\hat{H}_z^e = \sqrt{(s\varepsilon_y)^{-1}(s\mu)}\hat{H}_z^{(1)}$. Then

$$\sqrt{(s\mu)(s\varepsilon_y)}\hat{E}_y^{(1)} = -\partial_x \hat{H}_z^e, \qquad \sqrt{(s\mu)(s\varepsilon_y)}\hat{H}_z^e - H_{z0} = -\partial_x \hat{E}_y^{(1)}. \tag{55}$$

Since for $\mu \neq \varepsilon_y$, the function $\sqrt{(s\mu)(s\varepsilon_y)}$ is not rational, to rewrite the above system in the time domain, we will make use of the ideas of Section 3. In particular, notice that $\mathrm{Re}\sqrt{(s\varepsilon_y)(s\mu)} > 0$ for $s \in \mathbb{C}_+$. Lemma 3.2 applied to $s^{-1}\sqrt{(s\varepsilon_y)(s\mu)}$, with the use of $\lim\limits_{s_r \to +\infty}\left(s^{-1}\sqrt{(s\varepsilon_y)(s\mu)}\right) = 1$, yields

$$s^{-1}\sqrt{(s\varepsilon_y)(s\mu)} = 1 + \int\limits_{-\infty}^{+\infty} \frac{d\nu(\xi)}{s^2 + \xi^2}.$$

Then the system (55) in the time domain reads

$$\begin{aligned}
&\partial_t D_y^e = -\partial_x H_z^e, \qquad \partial_t B_z^e = -\partial_x E_y^{(1)},\\
&\partial_t D_y^e = \partial_t E_y^{(1)} + \int\limits_{-\infty}^{+\infty} \lambda_y^e(\xi) d\nu(\xi), \qquad \partial_t \lambda_y^e + \xi^2 p_y^e = E_y^{(1)}, \qquad \partial_t p_y^e = \lambda_y^e,\\
&\partial_t B_z^e = \partial_t H_z^e + \int\limits_{-\infty}^{+\infty} \lambda_z^e(\xi) d\nu(\xi), \qquad \partial_t \lambda_z^e + \xi^2 p_z^e = H_z^e, \qquad \partial_t p_z^e = \lambda_z^e.
\end{aligned} \tag{56}$$

Let us remark that here we set $H_z^e|_{t=0} = H_{z0}$ and $B_z^e|_{t=0} = H_{z0}$. As before, to recover $H_z^{(1)}$, we can make use of the identities similar to (42-43), more precisely

$$\begin{aligned}
&\partial_t H_z^{(1)} + \sum_{\ell=0}^{n_\mu} \mu_\ell \lambda_{\mu\ell} = \partial_t H_z^e + \int\limits_{-\infty}^{+\infty} \lambda_z^e(\xi) d\nu(\xi),\\
&\partial_t \lambda_{\mu\ell} + \omega_{\mu\ell}^2 p_{\mu\ell} = H_z^{(1)}, \qquad \partial_t p_{\mu\ell} = \lambda_{\mu\ell}, \qquad \ell = 0, \ldots, n_\mu.
\end{aligned} \tag{57}$$

The equivalence result of this system to the system (51) mimics the result of Proposition 2.15.

Proposition 4.2. *Let $(E_y^{(1)},\ H_z^e)$ solve the system (56) with the initial conditions chosen as $H_z^e|_{t=0} = B_z|_{t=0} = H_{z0}$ and as zero for the rest of unknowns. Let E_y, H_z be the solution to the system (51) with the initial conditions chosen as $H_z|_{t=0} = B_z|_{t=0} = H_{z0}$ and as zero for the rest of unknowns. Then $E_y^{(1)}(t) = E_y(t)$ and $B_z^e(t) = B_z(t)$ for all $t \geq 0$.*

Similarly, the energy conservation is a corollary of Proposition 3.3, since (56) is a particular case of (48).

Proposition 4.3. *An energy $\mathcal{E}$ associated with (56), defined below, satisfies $\frac{d}{dt}\mathcal{E} = 0$. Here*

$$\mathcal{E} = \frac{1}{2}\left(\|E_y^{(1)}\|^2 + \|H_z^e\|^2\right) + \mathcal{E}_y^e + \mathcal{E}_z^e, \qquad \mathcal{E}_\alpha^e = \frac{1}{2}\int\limits_{-\infty}^{\infty}\left(\|\lambda_\alpha^e\|^2 + \xi^2\|p_\alpha^e\|^2\right)d\nu(\xi), \qquad \alpha \in \{x, y\}.$$

4.2.2. *The PML System and Its Stability*

Let us now apply the PMLs to the time-domain system (56), or, in the Laplace domain (55). Performing the change of variables (9) with $\psi_x = \varepsilon_y^{-1}$ results in the following system (recall that H_{z0} is supported outside of the perfectly matched layer) inside the perfectly matched layer:

$$\sqrt{(s\mu)(s\varepsilon_y)}\hat{E}_y^{(1)}\left(1 + \frac{\sigma_x(x)}{s\varepsilon_y}\right) = -\partial_x \hat{H}_z^e, \qquad \sqrt{(s\mu)(s\varepsilon_y)}\hat{H}_z^e\left(1 + \frac{\sigma_x(x)}{s\varepsilon_y}\right) = -\partial_x \hat{E}_y^{(1)},$$

or, alternatively, in the whole domain:

$$\begin{aligned} &\sqrt{(s\mu)(s\varepsilon_y)}\hat{E}_y^{(1)} + \sigma_x(x)\sqrt{(s\mu)(s\varepsilon_y)^{-1}}\hat{E}_y^{(1)} = -\partial_x \hat{H}_z^e, \\ &\sqrt{(s\mu)(s\varepsilon_y)}\hat{H}_z^e + \sigma_x(x)\sqrt{(s\mu)(s\varepsilon_y)^{-1}}\hat{H}_z^e - H_{z0} = -\partial_x \hat{E}_y^{(1)}. \end{aligned} \tag{58}$$

To write the above in the time domain, we again apply Lemma 3.2 to the passive function $s^{-1}\sqrt{(s\varepsilon_y)^{-1}(s\mu)}$, with the use of $\lim\limits_{s_r \to +\infty} s^{-1}\sqrt{(s\varepsilon_y)^{-1}(s\mu)} = 0$. This results in the following representation:

$$s^{-1}\sqrt{(s\varepsilon_y)^{-1}(s\mu)} = \int\limits_{-\infty}^{+\infty}\frac{d\nu^*(\xi)}{s^2 + \xi^2}. \tag{59}$$

With the help of the above we can write the system (58) in the time domain. We make use of $s(s^2+\xi^2)^{-1}\hat{E}_y^{(1)} = \hat{\lambda}_y^e$ and a similar identity for $\hat{\lambda}_z^e$, $\hat{H}_z^e$:

$$\begin{aligned} &\partial_t D_y^e + \sigma_x(x)\int\limits_{-\infty}^{+\infty}\lambda_y^e(\xi)d\nu^*(\xi) = -\partial_x H_z^e, \qquad \partial_t B_z^e + \sigma_x(x)\int\limits_{-\infty}^{+\infty}\lambda_z^e(\xi)d\nu^*(\xi) = -\partial_x E_y^{(1)}, \\ &\partial_t D_y^e = \partial_t E_y^{(1)} + \int\limits_{-\infty}^{+\infty}\lambda_y^e(\xi)d\nu(\xi), \qquad \partial_t\lambda_y^e + \xi^2 p_y^e = E_y^{(1)}, \qquad \partial_t p_y^e = \lambda_y^e, \\ &\partial_t B_z^e = \partial_t H_z^e + \int\limits_{-\infty}^{+\infty}\lambda_z^e(\xi)d\nu(\xi), \qquad \partial_t\lambda_z^e + \xi^2 p_z^e = H_z^e, \qquad \partial_t p_z^e = \lambda_z^e. \end{aligned} \tag{60}$$

In order to recover $H_z^{(1)}$, we use (57). The following stability result holds for the above system.

Theorem 4.4. *Let $\sigma_x(x) \in L^\infty(\mathbb{R})$. An energy of the PML system (60) defined by*

$$\mathcal{E} = \|E_y^{(1)}\|^2 + \|H_z^e\|^2 + \mathcal{E}_y + \mathcal{E}_z + \mathcal{E}_y^* + \mathcal{E}_z^*,$$

$$\mathcal{E}_\alpha^e = \int_{-\infty}^{+\infty} \left(\|\lambda_\alpha^e\| + \xi^2 \|p_\alpha^e\|^2\right) d\nu(\xi), \qquad \mathcal{E}_\alpha^* = \int_{-\infty}^{+\infty} \int_{\mathbb{R}} \left(|\lambda_\alpha^e|^2 + \xi^2 |p_\alpha^e|^2\right) \sigma_x(x) dx d\nu^*(\xi), \qquad \alpha \in \{x, z\},$$

remains constant: $\frac{d}{dt}\mathcal{E} = 0$.

Proof. The result is obtained by testing the first equation of (60) with $E_y^{(1)}$, the second equation with H_z^e and summing up the two obtained identities. The details are left to the reader. $\square$

First of all, in the above we can clearly see the role of the condition $\sigma_x(x) \geq 0$: it is required for the stability of the PML system. Unlike the results of previous sections for $\sigma_x \equiv \text{const}$, the above result shows the conservation of a certain energy, rather than its decay. In practice this may appear non-optimal.

The non-optimality becomes clear when one considers the case $\mu(s) = \varepsilon_y(s)$, for which one can obtain a finer energy decay result in Section 4.1, but for which the results of Theorem 4.4 still remain valid. For instance, in this case, (59) degenerates to a very special representation of the symbol s^{-1}, namely $s^{-1} = \pi^{-1} \int_{-\infty}^{+\infty} \frac{d\xi}{s^2+\xi^2}$, $s \in \mathbb{C}_+$. In this case, on one hand $\sqrt{(s\varepsilon_y)^{-1}(s\mu)}\hat{E}_y = \hat{E}_y$ in the time domain corresponds to E_y, and on the other hand it can be represented as $\pi^{-1} \int_{-\infty}^{+\infty} \lambda_y^e d\xi$, see (60).

Nonetheless, the energy conservation result obtained in this section is not surprising, and is a consequence of the fact that the system (60) can be viewed as a conservative extension of an equivalent PML system, see [14].

As for the original unknown $H_z^{(1)}$, the corresponding stability result mimics Proposition 2.17.

Proposition 4.5. *For the solution of the system (60) coupled with (57), the following holds true:*

$$\tilde{\mathcal{E}}_z(t) := \frac{1}{2}\left(\|H_z^{(1)}(t) - H_z^e(t)\|^2 + \sum_{\ell=0}^{n_\mu} \mu_\ell \left(\|\lambda_{\mu\ell}^{(1)}(t)\|^2 + \omega_{\mu\ell}^2 \|p_{\mu\ell}^{(1)}(t)\|^2\right)\right) \leq 2\tilde{\mathcal{E}}_z(0) + C\mathcal{E}(0)t^2,$$

where $\mathcal{E}(t)$ is defined in Theorem 4.4.

Proof. See the proof of Proposition 2.17 in Appendix A. $\square$

The above result indicates a possible linear growth of the field $H_z^{(1)}$. At the moment it is not clear whether this estimate on the behaviour of the field $H_z^{(1)}$ is optimal, cf. Remark 2.18.

5. Conclusions and Open Questions

In this work we have examined the stability of the PMLs of [5] for dispersive models with the help of the energy techniques. We exploit the coercivity of the corresponding sesquilinear form in the Laplace domain, and based on its analysis, we show how a set of auxiliary PML unknowns can be introduced to simplify the derivation of the energy estimates. This technique allows to obtain the stability result for a constant absorption parameter in 2D, and can be extended to the analysis of the PML with a non-constant absorption parameter in 1D.

There are many open questions remaining, among them the derivation of the energy estimates for a non-constant absorption parameter in two dimensions, even for classical PMLs and non-dispersive case. Another interesting question is whether the technique of this work can be extended to analyze the stability of classical PMLs in 3D. To our knowledge, no energy estimates are known in this case even for a constant absorption parameter. This constitutes a subject of a future research.

Acknowledgements

I am grateful to Patrick Joly for many useful suggestions and remarks on this work.

Appendix A. Proof of Proposition 2.17

Proof. Testing (42) with $H_z^{(1)} - H_z^e$, we obtain

$$\left(\partial_t(H_z^{(1)} - H_z^e), H_z^{(1)} - H_z^e\right) + \sum_{\ell=0}^{n_\mu} \mu_\ell \left(\lambda_{\mu\ell}^{(1)}, H_z^{(1)}\right) = \sum_{\ell=0}^{n_\mu} \mu_\ell \left(\lambda_{\mu\ell}^{(1)}, H_z^e\right) + \sum_{\ell=0}^{n_\mu^e} \mu^e \left(\lambda_{\mu\ell}^e, H_z^{(1)} - H_z^e\right). \tag{61}$$

The left-hand side of the above is $\frac{d}{dt}\tilde{\mathcal{E}}_z$, see also (7). As for the right-hand side, let us consider each term separately. The first term, with the use of the Cauchy-Schwartz formula, can be bounded

$$\left|\sum_{\ell=0}^{n_\mu} \mu_\ell \left(\lambda_{\mu\ell}^{(1)}, H_z^e\right)\right| \leq \|H_z^e\| \sum_{\ell=0}^{n_\mu} \mu_\ell \left\|\lambda_{\mu\ell}^{(1)}\right\| \leq C\|H_z^e\| \left(\sum_{\ell=0}^{n_\mu} \mu_\ell \|\lambda_{\mu\ell}^{(1)}\|^2\right)^{\frac{1}{2}} \leq C_0\sqrt{\mathcal{E}(0)\tilde{\mathcal{E}}_z}, \qquad C_0 > 0,$$

where we used the energy identity of Theorem 2.19 and the definition of $\tilde{\mathcal{E}}_z$. Similarly, for some $C' > 0$, the second term of the right-hand side of (61) satisfies

$$\left|\sum_{\ell=0}^{n_\mu^e} \mu^e \left(\lambda_{\mu\ell}^e, H_z^{(1)} - H_z^e\right)\right| \leq C'\|H_z^{(1)} - H_z^e\| \sum_{\ell=0}^{n_\mu^e} \mu^e \|\lambda_{\mu\ell}^e\|.$$

To bound the above, we notice that Theorem 2.19 implies that $\|\lambda_{\mu\ell}^e(t)\| \leq C^*\mathcal{E}(0)$, with $t \geq 0$ and C^* independent of t, see also Remark 2.14. Thus, for some constant $C_1 > 0$,

$$\left|\sum_{\ell=0}^{n_\mu^e} \mu^e \left(\lambda_{\mu\ell}^e, H_z^{(1)} - H_z^e\right)\right| \leq C_1\sqrt{\mathcal{E}(0)\tilde{\mathcal{E}}_z}.$$

Thus we get from (61) the following differential inequality:

$$\frac{d}{dt}\tilde{\mathcal{E}}_z \leq C_2\sqrt{\mathcal{E}(0)\tilde{\mathcal{E}}_z}, \qquad C_2 > 0,$$

or

$$\frac{1}{2}\frac{d}{dt}\sqrt{\tilde{\mathcal{E}}_z} \leq C_2\sqrt{\mathcal{E}(0)}.$$

From the above we obtain

$$\tilde{\mathcal{E}}_z(t) \leq \left(2C_2\sqrt{\mathcal{E}(0)}t + \sqrt{\tilde{\mathcal{E}}_z(0)}\right)^2 \leq 2\left(4C_2^2\mathcal{E}(0)t^2 + \tilde{\mathcal{E}}_z(0)\right).$$

$\square$

The above result shows that $\|H_z^{(1)}(t) - H_z^e(t)\|$ grows at most linearly, and since $\|H_z^e(t)\|$ is uniformly bounded in t, the growth of $\|H_z^{(1)}(t)\|$ is at most linear.

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