Stable broken H1 and H(div) polynomial extensions for polynomial-degree-robust potential and flux reconstruction in three space dimensions

Alexandre Ern, Martin Vohralík

Stable broken H^1 and $\boldsymbol{H}(\mathrm{div})$ polynomial extensions for polynomial-degree-robust potential and flux reconstruction in three space dimensions*

Alexandre Ern[†] Martin Vohralík[‡]

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Abstract

We study extensions of piecewise polynomial data prescribed on faces and possibly in elements of a patch of simplices sharing a vertex. In the H^1 setting, we look for functions whose jumps across the faces are prescribed, whereas in the $\boldsymbol{H}(\mathrm{div})$ setting, the normal component jumps and the piecewise divergence are prescribed. We show stability in the sense that the minimizers over piecewise polynomial spaces of the same degree as the data are subordinate in the broken energy norm to the minimizers over the whole broken H^1 and $\boldsymbol{H}(\mathrm{div})$ spaces. Our proofs are constructive and yield constants independent of the polynomial degree. One particular application of these results is in a posteriori error analysis, where the present results justify polynomial-degree-robust efficiency of potential and flux reconstructions.

Key words: polynomial extension operator, broken Sobolev space, potential reconstruction, flux reconstruction, a posteriori error estimate, robustness, polynomial degree, best approximation, patch of elements

1 Introduction

Braess *et al.* [1, Theorem 1] showed that equilibrated flux a posteriori error estimates lead to local efficiency and polynomial-degree robustness (in short, p-robustness). This means that the estimators upper-bounding the error also give local lower bounds for the error, up to a generic conctant independent of the polynomial degree of the approximate solution. These results apply to conforming finite element methods in two space dimensions. They are based on flux reconstructions obtained by solving, via the mixed finite element method, a homogeneous Neumann, hat-function-weighted residual problem on each vertex-centered patch of the mesh. The proof of the p-robustness in [1] relies on two key components: p-robust stability of the right inverse of the divergence operator shown in Costabel and McIntosh [9, Corollary 3.4] and p-robust stability of the right inverse of the normal trace shown in Demkowicz *et al.* [12, Theorem 7.1]. In our contribution [19, Theorem 3.17], we extended p-robustness of a posteriori error estimates to any numerical scheme satisfying a couple of clearly identified assumptions, including nonconforming, discontinuous Galerkin, and mixed finite elements, still in two space dimensions, while proceeding through similar stability arguments. A second type of local problem appears here, where one is led to solve a homogeneous Dirichlet, conforming finite element problem on each vertex-centered element patch, with a hat-function-weighted discontinuous datum, yielding a potential reconstruction.

The present work extends the results of [1] on flux reconstruction to three space dimensions and reformulates the methodology of [19] for potential reconstruction so that it can be applied in the same way in two and three space dimensions. In doing so, we adopt a different viewpoint leading to a larger abstract setting not necessarily linked to a posteriori error analysis. The two main results of this paper are Theorems 2.2 and 2.3. They concern a setting where one considers a shape-regular patch of simplicial mesh elements sharing a given vertex, say $\boldsymbol{a}$, together with a p-degree polynomial r_F associated with each face F of the

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[†]Université Paris-Est, CERMICS (ENPC), 77455 Marne-la-Vallée 2, France (alexandre.ern@enpc.fr).

[‡]INRIA Paris, 2 rue Simone Iff, 75589 Paris, France (martin.vohralik@inria.fr).

patch (H^1 potential reconstruction setting) or p-degree polynomials r_F and r_K associated with each face F and element K of the patch respectively ($\boldsymbol{H}(\mathrm{div})$ flux reconstruction setting). These data, satisfying appropriate compatibility conditions, are to be extended to functions defined over the patch, such that the jumps across the interior faces are prescribed by r_F (H^1 setting) or such that the normal component jumps and boundary values are prescribed by r_F and the piecewise divergence is prescribed by r_K ($\boldsymbol{H}(\mathrm{div})$ setting). Crucially, we prove that the extension into piecewise polynomials of degree p that minimizes the broken energy norm is, up to a constant only depending on the patch shape regularity, as good as the extension into the whole broken H^1 space with the same jump constraints. Similarly, our broken p-degree Raviart–Thomas–Nédélec extension is stable with respect to the broken $\boldsymbol{H}(\mathrm{div})$ one.

Section 3 reformulates equivalently the above theorems as Corollaries 3.1 and 3.3 to show that best-approximation of discontinuous or normal-trace discontinuous piecewise polynomial data by $H^1_0(\omega_{\boldsymbol{a}})$- or $\boldsymbol{H}_0(\mathrm{div},\omega_{\boldsymbol{a}})$-conforming piecewise polynomials (i.e., by continuous or normal-trace continuous piecewise polynomials) is, up to a p-independent constant, as good as by all $H^1_0(\omega_{\boldsymbol{a}})$ or $\boldsymbol{H}_0(\mathrm{div},\omega_{\boldsymbol{a}})$ Sobolev functions. This section also sheds more light on the continuous level, uncovering that three different equivalent formulations of our results can be devised using the equivalence principle of primal and dual energies. This, in particular, allows us to make a link with the previously obtained results in [1, 19] and to describe the application of our results to a posteriori error analysis in Section 4. In particular, p-robust local efficiency for flux reconstructions is stated in formula (4.9) and p-robust local efficiency for potential reconstructions is stated in formula (4.7).

The proofs of Theorems 2.2 and 2.3 are respectively presented in Sections 5 and 6. In contrast to [1], where the work with dual norms was essential, we design here a procedure only working in the (broken) energy norms. The proofs are constructive and therefore indicate a possible practical reconstruction of the potential and the flux avoiding the patchwise problem solves. These are replaced by a single explicit run through the patch, with possibly a solve on each element. The key ingredients on a single element are still the right inverse of the divergence [9, Corollary 3.4] and the right inverse of the normal trace [12, Theorem 7.1] in the $\boldsymbol{H}(\mathrm{div})$ setting, but this becomes the right inverse of the trace shown in Demkowicz *et al.* [10, Theorem 6.1] in the H^1 setting. We combine these building blocks into a stability result on one tetrahedron in Lemmas A.1 and A.3 in Appendix A. We find that they are of independent interest. Gluing the elemental contributions together at the patch level turns out to be another rather involved ingredient of the proofs in three space dimensions, and we collect some auxiliary results for that purpose in Appendix B. A first difficulty is that the two-dimensional argument of turning around a vertex can no longer be invoked. To achieve a suitable enumeration of the mesh cells composing the patch in three dimensions, we rely on the notion of shelling of polytopes, see Ziegler [21, Chap. 8], which we reformulate for the present purposes in Lemma B.1. Another difficulty is that we need to devise suitable functional transformations between different cells in the patch. This is done by introducing two- and three-coloring of some vertices lying on the boundary of the patch, possibly on a submesh of the original patch; how to achieve such colorings is described in Lemmas B.2 and B.3.

For the sake of clarity of our exposition, we focus on discussing in details patches completely surrounding the vertex $\boldsymbol{a}$, corresponding to an "interior" vertex when considering a mesh of some computational domain. Our technique, though, extends to the case where one considers a "boundary" vertex as well. Our main results in this context are Theorems 2.4 and 2.5, whereas the reformulations as best-approximation results on piecewise polynomial data can be found in Corollaries 3.7 and 3.8. The proofs of our main results concerning boundary vertices are given in Section 7, with the description of the enumeration of boundary patches presented in Appendix C. The aforementioned application to a posteriori error analysis (Section 4) then also covers general inhomogeneous Dirichlet and Neumann boundary conditions.

Our results combine and extend those of [9, Corollary 3.4], [12, Theorem 7.1], and [10, Theorem 6.1]. In particular, we obtain stable H^1 and $\boldsymbol{H}(\mathrm{div})$ polynomial extensions on an arbitrary tetrahedron in Lemmas A.1 and A.3 and on an arbitrary patch of tetrahedra in Theorems 2.2 and 2.3. In a further extension [17], we were recently able to employ them to construct p-robust $\boldsymbol{H}(\mathrm{div})$ liftings over arbitrary domains, not being patches of elements. A natural extension of [17] would be to obtain the same type of results in the H^1 setting. Another extension is to cover the $\boldsymbol{H}(\mathrm{curl})$ case, hinging on the single tetrahedron results of Demkowicz *et al.* [11, Theorem 7.2]. Finally, we mention that application of the present results to the construction of p-robust a posteriori error estimates for problems with arbitrarily jumping coefficients is detailed in [8], to eigenvalue problems in [5, 4], to the Stokes problem in [7], to linear elasticity in [16], and to the heat equation in [18].

2 Main results

This section presents our main results, once the setting and basic notation have been fixed.

2.1 Setting and basic notation

We call tetrahedron any non-degenerate (closed) simplex in $\mathbb{R}^3$, uniquely determined by four points in $\mathbb{R}^3$ not lying in a plane. Let $\boldsymbol{a}$ be a point in $\mathbb{R}^3$. We consider a patch of tetrahedra around $\boldsymbol{a}$, say $\mathcal{T}_{\boldsymbol{a}}$, i.e., a finite collection of tetrahedra having $\boldsymbol{a}$ as vertex, such that the intersection of any two distinct tetrahedra in $\mathcal{T}_{\boldsymbol{a}}$ is either $\boldsymbol{a}$, or a common edge, or a common face. A generic tetrahedron in $\mathcal{T}_{\boldsymbol{a}}$ is denoted by K and is also called an element or a cell. We let $\omega_{\boldsymbol{a}} \subset \mathbb{R}^3$ denote the interior of the subset $\cup_{K\in\mathcal{T}_{\boldsymbol{a}}} K$. For the time being, we focus on the case where $\omega_{\boldsymbol{a}}$ contains an open ball around $\boldsymbol{a}$. The main application we have in mind is when $\boldsymbol{a}$ is the interior vertex of a simplicial mesh $\mathcal{T}_h$ of some computational domain Ω. The case where $\boldsymbol{a}$ is a boundary vertex of the mesh entails some additional technicalities that we detail in Section 2.4.

All the faces of the elements in the patch $\mathcal{T}_{\boldsymbol{a}}$ are collected in the set $\mathcal{F}_{\boldsymbol{a}}$ which is split into $\mathcal{F}_{\boldsymbol{a}} = \mathcal{F}^{\mathrm{i}}_{\boldsymbol{a}} \cup \mathcal{F}^{\mathrm{b}}_{\boldsymbol{a}}$ with $\mathcal{F}^{\mathrm{i}}_{\boldsymbol{a}}$ collecting all the interior faces (containing the vertex $\boldsymbol{a}$ and shared by two distinct elements in $\mathcal{T}_{\boldsymbol{a}}$) and $\mathcal{F}^{\mathrm{b}}_{\boldsymbol{a}}$ collecting the faces located in $\partial\omega_{\boldsymbol{a}}$. For all faces $F \in \mathcal{F}_{\boldsymbol{a}}$, $\boldsymbol{n}_F$ denotes a unit normal vector to F whose orientation is arbitrary but fixed for all $F \in \mathcal{F}^{\mathrm{i}}_{\boldsymbol{a}}$ and coinciding with the unit outward normal to $\omega_{\boldsymbol{a}}$ for all $F \in \mathcal{F}^{\mathrm{b}}_{\boldsymbol{a}}$. We consider the jump operator $[\![\cdot]\!]_F$ for all $F \in \mathcal{F}^{\mathrm{i}}_{\boldsymbol{a}}$, yielding the difference (evaluated along $\boldsymbol{n}_F$) of the traces of the argument from the two elements that share the interior face F (the subscript F is omitted if there is no ambiguity). We also need to consider edges. Let $\mathcal{E}_{\boldsymbol{a}}$ collect all the edges in $\mathcal{T}_{\boldsymbol{a}}$ sharing the vertex $\boldsymbol{a}$; we refer to these edges as interior edges. Then, for each $e \in \mathcal{E}_{\boldsymbol{a}}$, the set $\mathcal{F}_e$ collects all the faces in $\mathcal{F}^{\mathrm{i}}_{\boldsymbol{a}}$ sharing e, and the set $\mathcal{T}_e$ collects all the cells in $\mathcal{T}_{\boldsymbol{a}}$ sharing e. For each $e \in \mathcal{E}_{\boldsymbol{a}}$, we fix one direction of rotation around e, and indicate by $\iota_{F,e}$ either equal to 1 or to -1 whether $\boldsymbol{n}_F$ complies with this direction or not for all $F \in \mathcal{F}_e$.

We define the broken H^1-space on the patch $\mathcal{T}_{\boldsymbol{a}}$ as

$$H^1(\mathcal{T}_{\boldsymbol{a}}) := \{v \in L^2(\omega_{\boldsymbol{a}});\, v|_K \in H^1(K),\ \forall K \in \mathcal{T}_{\boldsymbol{a}}\}, \tag{2.1}$$

and similarly the broken $\boldsymbol{H}(\mathrm{div})$-space on the patch $\mathcal{T}_{\boldsymbol{a}}$ as

$$\boldsymbol{H}(\mathrm{div}, \mathcal{T}_{\boldsymbol{a}}) := \{\boldsymbol{v} \in \boldsymbol{L}^2(\omega_{\boldsymbol{a}});\, \boldsymbol{v}|_K \in \boldsymbol{H}(\mathrm{div}, K),\ \forall K \in \mathcal{T}_{\boldsymbol{a}}\}. \tag{2.2}$$

For any $v \in H^1(\mathcal{T}_{\boldsymbol{a}})$, we can consider its piecewise (broken) gradient $\nabla_{\mathcal{T}} v$ defined as $(\nabla_{\mathcal{T}} v)|_K = \nabla(v|_K)$, and similarly for any $\boldsymbol{v} \in \boldsymbol{H}(\mathrm{div}, \mathcal{T}_{\boldsymbol{a}})$, we can consider its piecewise (broken) divergence $\nabla_{\mathcal{T}}\cdot\boldsymbol{v}$ defined as $(\nabla_{\mathcal{T}}\cdot\boldsymbol{v})|_K = \nabla\cdot(\boldsymbol{v}|_K)$, for all $K \in \mathcal{T}_{\boldsymbol{a}}$. For any $v \in H^1(\mathcal{T}_{\boldsymbol{a}})$, the jumps $[\![v]\!]_F$ across any face $F \in \mathcal{F}^{\mathrm{i}}_{\boldsymbol{a}}$ are well defined since the traces of v on F from the two cells sharing F are in $L^2(F)$; similarly, the traces $v|_{\mathcal{F}^{\mathrm{b}}_{\boldsymbol{a}}}$ are well-defined. We note that any smooth enough function $v \in H^1(\mathcal{T}_{\boldsymbol{a}})$ is such that

$$\sum_{F\in\mathcal{F}_e} \iota_{F,e}\, [\![v]\!]_F|_e = 0 \quad \text{for all interior edges } e \in \mathcal{E}_{\boldsymbol{a}}, \tag{2.3}$$

since the oriented sum of the jumps along a closed path around an interior edge is always zero. The definition of traces is a bit more subtle when one considers a field $\boldsymbol{v} \in \boldsymbol{H}(\mathrm{div}, \mathcal{T}_{\boldsymbol{a}})$. Let $r_F \in L^2(F)$ for all $F \in \mathcal{F}_{\boldsymbol{a}}$. Then we say that

$$\boldsymbol{v}\cdot\boldsymbol{n}_F = r_F \quad \forall F \in \mathcal{F}^{\mathrm{b}}_{\boldsymbol{a}}, \tag{2.4a}$$

$$[\![\boldsymbol{v}]\!]\cdot\boldsymbol{n}_F = r_F \quad \forall F \in \mathcal{F}^{\mathrm{i}}_{\boldsymbol{a}} \tag{2.4b}$$

for a function $\boldsymbol{v} \in \boldsymbol{H}(\mathrm{div}, \mathcal{T}_{\boldsymbol{a}})$ if and only if

$$(\nabla_{\mathcal{T}}\cdot\boldsymbol{v}, v)_{\omega_{\boldsymbol{a}}} + (\boldsymbol{v}, \nabla v)_{\omega_{\boldsymbol{a}}} = \sum_{F\in\mathcal{F}_{\boldsymbol{a}}} (r_F, v)_F \qquad \forall v \in H^1(\omega_{\boldsymbol{a}}). \tag{2.4c}$$

We will also need to prescribe the normal component of vector fields in a single cell $K \in \mathcal{T}_{\boldsymbol{a}}$ with unit outward normal $\boldsymbol{n}_K$. Consider a non-empty subset $\mathcal{F}^{\mathrm{N}}_K \subset \mathcal{F}_K$ where $\mathcal{F}_K$ collects the faces of K. Given functions $r_F \in L^2(F)$ for all $F \in \mathcal{F}^{\mathrm{N}}_K$, we say that $\boldsymbol{v}\cdot\boldsymbol{n}_K|_F = r_F$, $\forall F \in \mathcal{F}^{\mathrm{N}}_K$, for a function $\boldsymbol{v} \in \boldsymbol{H}(\mathrm{div}, K)$ if

$$(\nabla\cdot\boldsymbol{v}, \phi)_K + (\boldsymbol{v}, \nabla\phi)_K = \sum_{F\in\mathcal{F}^{\mathrm{N}}_K} (r_F, \phi)_F \qquad \forall \phi \in H^1(K) \text{ s.t. } \phi|_F = 0\ \forall F \in \mathcal{F}_K \setminus \mathcal{F}^{\mathrm{N}}_K. \tag{2.5}$$

Let $p \geq 0$ denote a polynomial degree. We use the notation $\mathbb{P}_p(K)$ for polynomials of order at most p in the element $K \in \mathcal{T}_{\boldsymbol{a}}$ and $\mathbb{P}_p(F)$ for polynomials of order at most p in the face $F \in \mathcal{F}_{\boldsymbol{a}}$. We denote by $\mathbb{P}_p(\mathcal{T}_{\boldsymbol{a}})$ the space composed of all functions supported on the patch $\mathcal{T}_{\boldsymbol{a}}$ whose restriction to any $K \in \mathcal{T}_{\boldsymbol{a}}$ is in $\mathbb{P}_p(K)$. Similarly, $\mathbb{P}_p(\mathcal{F}_{\boldsymbol{a}})$ stands for the space composed of all functions supported on all faces from $\mathcal{F}_{\boldsymbol{a}}$ whose restriction to any $F \in \mathcal{F}_{\boldsymbol{a}}$ is in $\mathbb{P}_p(F)$. Analogous notation is used for any subset of $\mathcal{F}_{\boldsymbol{a}}$. We denote by r_K the restriction of $r \in \mathbb{P}_p(\mathcal{T}_{\boldsymbol{a}})$ to $K \in \mathcal{T}_{\boldsymbol{a}}$ and similarly by r_F the restriction of $r \in \mathbb{P}_p(\mathcal{F}_{\boldsymbol{a}})$ to $F \in \mathcal{F}_{\boldsymbol{a}}$. Let $\boldsymbol{RTN}_p(K)$ be the Raviart–Thomas–Nédélec polynomial space of vector-valued functions of order p in the element $K \in \mathcal{T}_{\boldsymbol{a}}$, i.e., $\boldsymbol{RTN}_p(K) := [\mathbb{P}_p(K)]^3 + \mathbb{P}_p(K)\boldsymbol{x}$. Finally, $\boldsymbol{RTN}_p(\mathcal{T}_{\boldsymbol{a}})$ denotes the broken space composed of all functions whose restriction to any element $K \in \mathcal{T}_{\boldsymbol{a}}$ is in $\boldsymbol{RTN}_p(K)$.

For an element $K \in \mathcal{T}_{\boldsymbol{a}}$, its shape-regularity parameter γ_K is defined to be the ratio of its diameter to the diameter of the largest inscribed ball, and the shape-regularity parameter of the patch $\mathcal{T}_{\boldsymbol{a}}$ is then defined to be $\gamma_{\mathcal{T}_{\boldsymbol{a}}} := \max_{K \in \mathcal{T}_{\boldsymbol{a}}} \gamma_K$.

Remark 2.1 (Orientation)**.** *The orientation of $\boldsymbol{n}_F$ is irrelevant in (2.3). Indeed, changing the orientation of $\boldsymbol{n}_F$ changes the sign of the jumps (evaluated along $\boldsymbol{n}_F$) and at the same the sign of $\iota_{F,e}$. Similarly, the orientation of $\boldsymbol{n}_F$ is irrelevant also in the left-hand side of (2.4b).*

2.2 Broken H^1 polynomial extension

Our main result for broken scalar extensions is the following.

Theorem 2.2 (Stable broken H^1 polynomial extension)**.** *Let $p \geq 1$. Let the interface-based p-degree polynomial $r \in \mathbb{P}_p(\mathcal{F}^{\mathrm{i}}_{\boldsymbol{a}})$ satisfy the following compatibility conditions:*

$$r_F|_{F\cap\partial\omega_{\boldsymbol{a}}} = 0 \qquad \text{on all interior faces } F \in \mathcal{F}^{\mathrm{i}}_{\boldsymbol{a}}, \tag{2.6a}$$

$$\sum_{F\in\mathcal{F}_e} \iota_{F,e}\, r_F|_e = 0 \qquad \text{on all interior edges } e \in \mathcal{E}_{\boldsymbol{a}}. \tag{2.6b}$$

Then there exists a constant $C_{\mathrm{st}} > 0$ only depending on the patch shape-regularity parameter $\gamma_{\mathcal{T}_{\boldsymbol{a}}}$ such that

$$\min_{\substack{v_p\in\mathbb{P}_p(\mathcal{T}_{\boldsymbol{a}})\\ v_p|_F=0\ \forall F\in\mathcal{F}^{\mathrm{b}}_{\boldsymbol{a}},\\ [\![v_p]\!]_F=r_F\ \forall F\in\mathcal{F}^{\mathrm{i}}_{\boldsymbol{a}}}} \|\nabla_{\mathcal{T}} v_p\|_{\omega_{\boldsymbol{a}}} \leq C_{\mathrm{st}} \min_{\substack{v\in H^1(\mathcal{T}_{\boldsymbol{a}})\\ v|_F=0\ \forall F\in\mathcal{F}^{\mathrm{b}}_{\boldsymbol{a}},\\ [\![v]\!]_F=r_F\ \forall F\in\mathcal{F}^{\mathrm{i}}_{\boldsymbol{a}}}} \|\nabla_{\mathcal{T}} v\|_{\omega_{\boldsymbol{a}}}, \tag{2.7}$$

where the minimization sets are non-empty and both minimizers in (2.7) are unique.

The compatibility conditions (2.6) are natural since r_F is used to prescribe interface jumps. Indeed, these jumps necessarily vanish on the points of the interfaces located on $\partial\omega_{\boldsymbol{a}}$ since the considered functions vanish on $\partial\omega_{\boldsymbol{a}}$; moreover, (2.6b) follows from (2.3). The minimizers in (2.7) are respectively denoted by v_p^* and v^*, so that (2.7) becomes

$$\|\nabla_{\mathcal{T}} v_p^*\|_{\omega_{\boldsymbol{a}}} \leq C_{\mathrm{st}} \|\nabla_{\mathcal{T}} v^*\|_{\omega_{\boldsymbol{a}}}. \tag{2.8}$$

Note also that since the minimization sets are non-empty and the left one is a subset of the right one by definition, the inequality in the other direction, $\|\nabla_{\mathcal{T}} v^*\|_{\omega_{\boldsymbol{a}}} \leq \|\nabla_{\mathcal{T}} v_p^*\|_{\omega_{\boldsymbol{a}}}$, is trivial.

2.3 Broken $\boldsymbol{H}(\mathrm{div})$ polynomial extension

Our main result for broken vector extensions is the following.

Theorem 2.3 (Stable broken $\boldsymbol{H}(\mathrm{div})$ polynomial extension)**.** *Let $p \geq 0$. Let the element- and face-based p-degree polynomial $r \in \mathbb{P}_p(\mathcal{T}_{\boldsymbol{a}}) \times \mathbb{P}_p(\mathcal{F}_{\boldsymbol{a}})$ satisfy the following compatibility condition:*

$$\sum_{K\in\mathcal{T}_{\boldsymbol{a}}} (r_K, 1)_K - \sum_{F\in\mathcal{F}_{\boldsymbol{a}}} (r_F, 1)_F = 0. \tag{2.9}$$

Then there exists a constant $C_{\mathrm{st}} > 0$ only depending on the patch shape-regularity parameter $\gamma_{\mathcal{T}_{\boldsymbol{a}}}$ such that

$$\min_{\substack{\boldsymbol{v}_p\in\boldsymbol{RTN}_p(\mathcal{T}_{\boldsymbol{a}})\\ \boldsymbol{v}_p\cdot\boldsymbol{n}_F=r_F\ \forall F\in\mathcal{F}^{\mathrm{b}}_{\boldsymbol{a}}\\ [\![\boldsymbol{v}_p]\!]\cdot\boldsymbol{n}_F=r_F\ \forall F\in\mathcal{F}^{\mathrm{i}}_{\boldsymbol{a}}\\ \nabla_{\mathcal{T}}\cdot\boldsymbol{v}_p|_K=r_K\ \forall K\in\mathcal{T}_{\boldsymbol{a}}}} \|\boldsymbol{v}_p\|_{\omega_{\boldsymbol{a}}} \leq C_{\mathrm{st}} \min_{\substack{\boldsymbol{v}\in\boldsymbol{H}(\mathrm{div},\mathcal{T}_{\boldsymbol{a}})\\ \boldsymbol{v}\cdot\boldsymbol{n}_F=r_F\ \forall F\in\mathcal{F}^{\mathrm{b}}_{\boldsymbol{a}}\\ [\![\boldsymbol{v}]\!]\cdot\boldsymbol{n}_F=r_F\ \forall F\in\mathcal{F}^{\mathrm{i}}_{\boldsymbol{a}}\\ \nabla_{\mathcal{T}}\cdot\boldsymbol{v}|_K=r_K\ \forall K\in\mathcal{T}_{\boldsymbol{a}}}} \|\boldsymbol{v}\|_{\omega_{\boldsymbol{a}}}, \tag{2.10}$$

where the minimization sets are non-empty and both minimizers in (2.10) are unique.

The compatibility condition (2.9) is again natural here, since it follows from (2.4c) with the test function equal to 1 in $\omega_{\boldsymbol{a}}$. The minimizers in (2.10) are respectively denoted by $\boldsymbol{v}_p^*$ and $\boldsymbol{v}^*$, so that (2.10) becomes

$$\|\boldsymbol{v}_p^*\|_{\omega_{\boldsymbol{a}}} \leq C_{\mathrm{st}}\|\boldsymbol{v}^*\|_{\omega_{\boldsymbol{a}}}. \tag{2.11}$$

Since the minimization sets are non-empty and the left one is a subset of the right one by definition, the inequality in the other direction, $\|\boldsymbol{v}^*\|_{\omega_{\boldsymbol{a}}} \leq \|\boldsymbol{v}_p^*\|_{\omega_{\boldsymbol{a}}}$, again is trivial.

2.4 Boundary vertices

We consider in this section the case where the patch domain $\omega_{\boldsymbol{a}}$ does not contain an open ball around the point $\boldsymbol{a}$; typically, $\boldsymbol{a}$ is a mesh vertex lying on the boundary of some computational domain Ω. In this case, the patch domain $\omega_{\boldsymbol{a}}$ only contains an open ball around $\boldsymbol{a}$ minus some sector with solid angle $\theta_{\boldsymbol{a}} \in (0, 4\pi)$.

The set $\mathcal{F}_{\boldsymbol{a}}$ collecting all the faces of $\mathcal{T}_{\boldsymbol{a}}$ is now divided into four disjoint subsets: the set $\mathcal{F}_{\boldsymbol{a}}^{\mathrm{i}}$ collecting (as before) the interior faces containing the vertex $\boldsymbol{a}$ and shared by two distinct elements in $\mathcal{T}_{\boldsymbol{a}}$, the set $\mathcal{F}_{\boldsymbol{a}}^{\mathrm{b}}$ collecting the faces that are subsets of $\partial\omega_{\boldsymbol{a}}$ that do not contain $\boldsymbol{a}$, and the sets $\mathcal{F}_{\boldsymbol{a}}^{\mathrm{D}}$ and $\mathcal{F}_{\boldsymbol{a}}^{\mathrm{N}}$ collecting the faces that are subsets of $\partial\omega_{\boldsymbol{a}}$ that contain $\boldsymbol{a}$. Note that the faces in $\mathcal{F}_{\boldsymbol{a}}^{\mathrm{b}}$, $\mathcal{F}_{\boldsymbol{a}}^{\mathrm{D}}$, and $\mathcal{F}_{\boldsymbol{a}}^{\mathrm{N}}$ altogether cover $\partial\omega_{\boldsymbol{a}}$. Faces in these three sets are assigned a unit normal vector $\boldsymbol{n}_F$ pointing outward $\omega_{\boldsymbol{a}}$. The distinction between the two sets $\mathcal{F}_{\boldsymbol{a}}^{\mathrm{D}}$ and $\mathcal{F}_{\boldsymbol{a}}^{\mathrm{N}}$ is introduced so as to handle different types of boundary conditions in the context of a posteriori error estimates. We remark that $\mathcal{F}_{\boldsymbol{a}}^{\mathrm{i}}$ can be empty (if $\mathcal{T}_{\boldsymbol{a}}$ consists of a single tetrahedron), that $\mathcal{F}_{\boldsymbol{a}}^{\mathrm{b}}$ is always non-empty, and that either $\mathcal{F}_{\boldsymbol{a}}^{\mathrm{D}}$ or $\mathcal{F}_{\boldsymbol{a}}^{\mathrm{N}}$ can be empty, but not both at the same time. Let us set $\partial\omega_{\boldsymbol{a}}^{\mathrm{b}} = \cup_{F\in\mathcal{F}_{\boldsymbol{a}}^{\mathrm{b}}} F$. Finally, the set $\mathcal{E}_{\boldsymbol{a}}$ collects all the edges in $\mathcal{T}_{\boldsymbol{a}}$ sharing the vertex $\boldsymbol{a}$ (note that some of these edges are now located on $\partial\omega_{\boldsymbol{a}}$) and, for each edge $e \in \mathcal{E}_{\boldsymbol{a}}$, $\mathcal{F}_e$ collects all the faces in $\mathcal{F}_{\boldsymbol{a}}$ sharing e (note that $\mathcal{F}_e$ is now a subset of $\mathcal{F}_{\boldsymbol{a}}^{\mathrm{i}} \cup \mathcal{F}_{\boldsymbol{a}}^{\mathrm{D}} \cup \mathcal{F}_{\boldsymbol{a}}^{\mathrm{N}}$).

Our main results for boundary vertices are the following.

Theorem 2.4 (Stable broken H^1 polynomial extension)**.** *Let $p \geq 1$ and let Assumption C.1 hold. Let $r \in \mathbb{P}_p(\mathcal{F}_{\boldsymbol{a}}^{\mathrm{i}} \cup \mathcal{F}_{\boldsymbol{a}}^{\mathrm{D}})$ satisfy the following compatibility conditions:*

$$r_F|_{F\cap\partial\omega_{\boldsymbol{a}}^{\mathrm{b}}} = 0 \qquad \forall F \in \mathcal{F}_{\boldsymbol{a}}^{\mathrm{i}} \cup \mathcal{F}_{\boldsymbol{a}}^{\mathrm{D}}, \tag{2.12a}$$

$$\sum_{F\in\mathcal{F}_e} \iota_{F,e}\, r_F|_e = 0 \qquad \forall e \in \mathcal{E}_{\boldsymbol{a}} \text{ such that } \mathcal{F}_e \cap \mathcal{F}_{\boldsymbol{a}}^{\mathrm{N}} = \emptyset. \tag{2.12b}$$

Then there exists a constant $C_{\mathrm{st}} > 0$ only depending on the patch shape-regularity parameter $\gamma_{\mathcal{T}_{\boldsymbol{a}}}$ such that

$$\min_{\substack{v_p\in\mathbb{P}_p(\mathcal{T}_{\boldsymbol{a}})\\ v_p|_F=0\ \forall F\in\mathcal{F}_{\boldsymbol{a}}^{\mathrm{b}},\\ v_p|_F=r_F\ \forall F\in\mathcal{F}_{\boldsymbol{a}}^{\mathrm{D}},\\ [\![v_p]\!]_F=r_F\ \forall F\in\mathcal{F}_{\boldsymbol{a}}^{\mathrm{i}}}} \|\nabla_{\mathcal{T}} v_p\|_{\omega_{\boldsymbol{a}}} \leq C_{\mathrm{st}} \min_{\substack{v\in H^1(\mathcal{T}_{\boldsymbol{a}})\\ v|_F=0\ \forall F\in\mathcal{F}_{\boldsymbol{a}}^{\mathrm{b}},\\ v|_F=r_F\ \forall F\in\mathcal{F}_{\boldsymbol{a}}^{\mathrm{D}},\\ [\![v]\!]_F=r_F\ \forall F\in\mathcal{F}_{\boldsymbol{a}}^{\mathrm{i}}}} \|\nabla_{\mathcal{T}} v\|_{\omega_{\boldsymbol{a}}}, \tag{2.13}$$

where the minimization sets are non-empty and both minimizers in (2.13) are unique.

Theorem 2.5 (Stable broken $\boldsymbol{H}(\mathrm{div})$ polynomial extension)**.** *Let $p \geq 0$. Let $r \in \mathbb{P}_p(\mathcal{T}_{\boldsymbol{a}}) \times \mathbb{P}_p(\mathcal{F}_{\boldsymbol{a}}^{\mathrm{i}} \cup \mathcal{F}_{\boldsymbol{a}}^{\mathrm{b}} \cup \mathcal{F}_{\boldsymbol{a}}^{\mathrm{N}})$ satisfy the following compatibility condition:*

$$\sum_{K\in\mathcal{T}_{\boldsymbol{a}}} (r_K, 1)_K - \sum_{F\in\mathcal{F}_{\boldsymbol{a}}^{\mathrm{i}}\cup\mathcal{F}_{\boldsymbol{a}}^{\mathrm{b}}\cup\mathcal{F}_{\boldsymbol{a}}^{\mathrm{N}}} (r_F, 1)_F = 0 \qquad \text{if } \mathcal{F}_{\boldsymbol{a}}^{\mathrm{D}} = \emptyset. \tag{2.14}$$

Then there exists a constant $C_{\mathrm{st}} > 0$ only depending on the patch shape-regularity parameter $\gamma_{\mathcal{T}_{\boldsymbol{a}}}$ such that

$$\min_{\substack{\boldsymbol{v}_p\in\boldsymbol{RTN}_p(\mathcal{T}_{\boldsymbol{a}})\\ \boldsymbol{v}_p\cdot\boldsymbol{n}_F=r_F\ \forall F\in\mathcal{F}_{\boldsymbol{a}}^{\mathrm{b}}\\ \boldsymbol{v}_p\cdot\boldsymbol{n}_F=r_F\ \forall F\in\mathcal{F}_{\boldsymbol{a}}^{\mathrm{N}}\\ [\![\boldsymbol{v}_p]\!]\cdot\boldsymbol{n}_F=r_F\ \forall F\in\mathcal{F}_{\boldsymbol{a}}^{\mathrm{i}}\\ \nabla_{\mathcal{T}}\cdot\boldsymbol{v}_p|_K=r_K\ \forall K\in\mathcal{T}_{\boldsymbol{a}}}} \|\boldsymbol{v}_p\|_{\omega_{\boldsymbol{a}}} \leq C_{\mathrm{st}} \min_{\substack{\boldsymbol{v}\in\boldsymbol{H}(\mathrm{div},\mathcal{T}_{\boldsymbol{a}})\\ \boldsymbol{v}\cdot\boldsymbol{n}_F=r_F\ \forall F\in\mathcal{F}_{\boldsymbol{a}}^{\mathrm{b}}\\ \boldsymbol{v}\cdot\boldsymbol{n}_F=r_F\ \forall F\in\mathcal{F}_{\boldsymbol{a}}^{\mathrm{N}}\\ [\![\boldsymbol{v}]\!]\cdot\boldsymbol{n}_F=r_F\ \forall F\in\mathcal{F}_{\boldsymbol{a}}^{\mathrm{i}}\\ \nabla_{\mathcal{T}}\cdot\boldsymbol{v}|_K=r_K\ \forall K\in\mathcal{T}_{\boldsymbol{a}}}} \|\boldsymbol{v}\|_{\omega_{\boldsymbol{a}}}, \tag{2.15}$$

where the minimization sets are non-empty and both minimizers in (2.15) are unique.

3 Equivalent reformulations

We reformulate in this section Theorems 2.2 and 2.3 in an equivalent way as best-approximation results of discontinuous piecewise polynomial data. This will in particular allow for a straightforward application to a posteriori error analysis in the forthcoming section. For further insight, as well as to make a link with previous contributions on the subject, we also give equivalent reformulations of the right-hand sides in (2.7) and (2.10).

3.1 Reformulation as best-approximation results

Let us set

$$H_0^1(\omega_{\boldsymbol{a}}) := \{v \in H^1(\omega_{\boldsymbol{a}});\ v|_{\partial\omega_{\boldsymbol{a}}} = 0\}, \tag{3.1a}$$

$$\boldsymbol{H}_0(\mathrm{div},\omega_{\boldsymbol{a}}) := \{\boldsymbol{v} \in \boldsymbol{H}(\mathrm{div},\omega_{\boldsymbol{a}});\ \boldsymbol{v}\cdot\boldsymbol{n}_{\partial\omega_{\boldsymbol{a}}} = 0\}. \tag{3.1b}$$

The result of Theorem 2.2 can be rephrased as follows.

Corollary 3.1 (H^1 best-approximation)**.** *Let the assumptions of Theorem 2.2 hold. Consider any $\tau_p \in \mathbb{P}_p(\mathcal{T}_{\boldsymbol{a}})$ so that $\tau_p|_F = 0\ \forall F \in \mathcal{F}_{\boldsymbol{a}}^{\mathrm{b}}$, and $[\![\tau_p]\!]_F = r_F\ \forall F \in \mathcal{F}_{\boldsymbol{a}}^{\mathrm{i}}$. Then the following holds:*

$$\min_{v_p\in\mathbb{P}_p(\mathcal{T}_{\boldsymbol{a}})\cap H_0^1(\omega_{\boldsymbol{a}})} \|\nabla_{\mathcal{T}}(\tau_p - v_p)\|_{\omega_{\boldsymbol{a}}} \le C_{\mathrm{st}} \min_{v\in H_0^1(\omega_{\boldsymbol{a}})} \|\nabla_{\mathcal{T}}(\tau_p - v)\|_{\omega_{\boldsymbol{a}}}. \tag{3.2}$$

Proof. Direct consequence of (2.7) upon shifting the minimization sets by τ_p. Note that the existence of τ_p follows from the non-emptiness of the discrete minimization set in (2.7). $\square$

Remark 3.2 (Minimizers)**.** *The unique minimizers in (3.2) are respectively $s_h^{\boldsymbol{a}} \in \mathbb{P}_p(\mathcal{T}_{\boldsymbol{a}}) \cap H_0^1(\omega_{\boldsymbol{a}})$ such that*

$$(\nabla s_h^{\boldsymbol{a}}, \nabla v_p)_{\omega_{\boldsymbol{a}}} = (\nabla_{\mathcal{T}}\tau_p, \nabla v_p)_{\omega_{\boldsymbol{a}}} \qquad \forall v_p \in \mathbb{P}_p(\mathcal{T}_{\boldsymbol{a}}) \cap H_0^1(\omega_{\boldsymbol{a}}), \tag{3.3}$$

and $s^{\boldsymbol{a}} \in H_0^1(\omega_{\boldsymbol{a}})$ such that

$$(\nabla s^{\boldsymbol{a}}, \nabla v)_{\omega_{\boldsymbol{a}}} = (\nabla_{\mathcal{T}}\tau_p, \nabla v)_{\omega_{\boldsymbol{a}}} \qquad \forall v \in H_0^1(\omega_{\boldsymbol{a}}). \tag{3.4}$$

The minimizers in (2.7) are such that $v_p^* = \tau_p - s_h^{\boldsymbol{a}}$ and $v^* = \tau_p - s^{\boldsymbol{a}}$.

Similarly, in the $\boldsymbol{H}(\mathrm{div})$-setting, Theorem 2.3 can be reformulated as follows.

Corollary 3.3 ($\boldsymbol{H}(\mathrm{div})$ best-approximation)**.** *Let the assumptions of Theorem 2.3 hold. Consider any $\boldsymbol{\tau}_p \in \boldsymbol{RTN}_p(\mathcal{T}_{\boldsymbol{a}})$ so that $\boldsymbol{\tau}_p\cdot\boldsymbol{n}_F = r_F\ \forall F \in \mathcal{F}_{\boldsymbol{a}}^{\mathrm{b}}$ and $[\![\boldsymbol{\tau}_p]\!]\cdot\boldsymbol{n}_F = r_F\ \forall F \in \mathcal{F}_{\boldsymbol{a}}^{\mathrm{i}}$. Then the following holds:*

$$\min_{\substack{\boldsymbol{v}_p\in\boldsymbol{RTN}_p(\mathcal{T}_{\boldsymbol{a}})\cap\boldsymbol{H}_0(\mathrm{div},\omega_{\boldsymbol{a}})\\ \nabla\cdot\boldsymbol{v}_p|_K = r_K - \nabla_{\mathcal{T}}\cdot\boldsymbol{\tau}_p|_K\ \forall K\in\mathcal{T}_{\boldsymbol{a}}}} \|\boldsymbol{\tau}_p + \boldsymbol{v}_p\|_{\omega_{\boldsymbol{a}}} \le C_{\mathrm{st}} \min_{\substack{\boldsymbol{v}\in\boldsymbol{H}_0(\mathrm{div},\omega_{\boldsymbol{a}})\\ \nabla\cdot\boldsymbol{v}|_K = r_K - \nabla_{\mathcal{T}}\cdot\boldsymbol{\tau}_p|_K\ \forall K\in\mathcal{T}_{\boldsymbol{a}}}} \|\boldsymbol{\tau}_p + \boldsymbol{v}\|_{\omega_{\boldsymbol{a}}}. \tag{3.5}$$

Proof. Direct consequence of (2.10) upon shifting the minimization sets by $\boldsymbol{\tau}_p$, the existence of $\boldsymbol{\tau}_p$ following from the non-emptiness of the discrete minimization set in (2.10). $\square$

Remark 3.4 (Minimizers)**.** *The unique minimizers in (3.5) are respectively $\boldsymbol{\sigma}_h^{\boldsymbol{a}} \in \boldsymbol{RTN}_p(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_0(\mathrm{div},\omega_{\boldsymbol{a}})$ with $\nabla\cdot\boldsymbol{\sigma}_h^{\boldsymbol{a}}|_K = r_K - \nabla_{\mathcal{T}}\cdot\boldsymbol{\tau}_p|_K$ for all $K \in \mathcal{T}_{\boldsymbol{a}}$ such that*

$$(\boldsymbol{\sigma}_h^{\boldsymbol{a}}, \boldsymbol{v}_p)_{\omega_{\boldsymbol{a}}} = -(\boldsymbol{\tau}_p, \boldsymbol{v}_p)_{\omega_{\boldsymbol{a}}} \qquad \forall \boldsymbol{v}_p \in \boldsymbol{RTN}_p(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_0(\mathrm{div},\omega_{\boldsymbol{a}}),\ \nabla\cdot\boldsymbol{v}_p = 0, \tag{3.6}$$

and $\boldsymbol{\sigma}^{\boldsymbol{a}} \in \boldsymbol{H}_0(\mathrm{div},\omega_{\boldsymbol{a}})$ with $\nabla\cdot\boldsymbol{\sigma}^{\boldsymbol{a}}|_K = r_K - \nabla_{\mathcal{T}}\cdot\boldsymbol{\tau}_p|_K$ for all $K \in \mathcal{T}_{\boldsymbol{a}}$ such that

$$(\boldsymbol{\sigma}^{\boldsymbol{a}}, \boldsymbol{v})_{\omega_{\boldsymbol{a}}} = -(\boldsymbol{\tau}_p, \boldsymbol{v})_{\omega_{\boldsymbol{a}}} \qquad \forall \boldsymbol{v} \in \boldsymbol{H}_0(\mathrm{div},\omega_{\boldsymbol{a}}),\ \nabla\cdot\boldsymbol{v} = 0. \tag{3.7}$$

The minimizers in (2.10) are such that $\boldsymbol{v}_p^* = \boldsymbol{\tau}_p + \boldsymbol{\sigma}_h^{\boldsymbol{a}}$ and $\boldsymbol{v}^* = \boldsymbol{\tau}_p + \boldsymbol{\sigma}^{\boldsymbol{a}}$.

3.2 Equivalent reformulations at the continuous level

We summarize here additional equivalence results on the continuous-level minimizations appearing in the right-hand sides of (3.2) and (3.5). Let us first set

$$H^1_*(\omega_{\boldsymbol{a}}) := \{v \in H^1(\omega_{\boldsymbol{a}});\ (v,1)_{\omega_{\boldsymbol{a}}} = 0\},$$

and let us define the following subspace of $\boldsymbol{H}(\mathrm{curl},\omega_{\boldsymbol{a}})$:

$$\boldsymbol{H}_*(\mathrm{curl},\omega_{\boldsymbol{a}}) := \{\boldsymbol{v} \in \boldsymbol{H}(\mathrm{curl},\omega_{\boldsymbol{a}});\ (\boldsymbol{v},\nabla\phi)_{\omega_{\boldsymbol{a}}} = 0,\ \forall \phi \in H^1_*(\omega_{\boldsymbol{a}})\}.$$

We first show that the $H^1_0(\omega_{\boldsymbol{a}})$-minimization of Corollary 3.1 is equivalent to evaluating a dual $\boldsymbol{H}(\mathrm{curl})$-norm of a suitable linear form defined from the data r_F, and consequently to evaluating the energy norm of its $\boldsymbol{H}_*(\mathrm{curl},\omega_{\boldsymbol{a}})$-lifting.

Corollary 3.5 ($\boldsymbol{H}(\mathrm{curl})$ form of the H^1-minimization)**.** *Let the assumptions of Corollary 3.1 hold. Let* $\boldsymbol{r}^{\boldsymbol{a}} \in \boldsymbol{H}_*(\mathrm{curl},\omega_{\boldsymbol{a}})$ *solve*

$$(\nabla\times\boldsymbol{r}^{\boldsymbol{a}}, \nabla\times\boldsymbol{v})_{\omega_{\boldsymbol{a}}} = -\sum_{F\in\mathcal{F}^{\mathrm{i}}_{\boldsymbol{a}}} (r_F\boldsymbol{n}_F, \nabla\times\boldsymbol{v})_F \qquad \forall \boldsymbol{v} \in \boldsymbol{H}_*(\mathrm{curl},\omega_{\boldsymbol{a}}). \tag{3.8}$$

Then

$$\min_{v\in H^1_0(\omega_{\boldsymbol{a}})} \|\nabla_{\mathcal{T}}(\tau_p - v)\|_{\omega_{\boldsymbol{a}}} = \|\nabla\times\boldsymbol{r}^{\boldsymbol{a}}\|_{\omega_{\boldsymbol{a}}} = \max_{\substack{\boldsymbol{v}\in\boldsymbol{H}(\mathrm{curl},\omega_{\boldsymbol{a}})\\ \|\nabla\times\boldsymbol{v}\|_{\omega_{\boldsymbol{a}}}=1}} \left\{\sum_{F\in\mathcal{F}^{\mathrm{i}}_{\boldsymbol{a}}} (r_F\boldsymbol{n}_F, \nabla\times\boldsymbol{v})_F\right\}. \tag{3.9}$$

Proof. Since $(\nabla s^{\boldsymbol{a}} - \nabla_{\mathcal{T}}\tau_p, \nabla v)_{\omega_{\boldsymbol{a}}} = 0$ for all $v \in H^1_0(\omega_{\boldsymbol{a}})$, a distributional argument implies that the vector field $\nabla s^{\boldsymbol{a}} - \nabla_{\mathcal{T}}\tau_p$ is divergence-free in $\omega_{\boldsymbol{a}}$. The boundary $\partial\omega_{\boldsymbol{a}}$ being connected, we infer that there is $\boldsymbol{r}^{\boldsymbol{a}} \in \boldsymbol{H}(\mathrm{curl},\omega_{\boldsymbol{a}})$ such that $\nabla s^{\boldsymbol{a}} - \nabla_{\mathcal{T}}\tau_p = \nabla\times\boldsymbol{r}^{\boldsymbol{a}}$, and without loss of generality, we can take $\boldsymbol{r}^{\boldsymbol{a}} \in \boldsymbol{H}_*(\mathrm{curl},\omega_{\boldsymbol{a}})$ since $\boldsymbol{H}(\mathrm{curl},\omega_{\boldsymbol{a}}) = \boldsymbol{H}_*(\mathrm{curl},\omega_{\boldsymbol{a}}) \oplus \nabla H^1_*(\omega_{\boldsymbol{a}})$ (the sum being L^2-orthogonal) and fields in $\nabla H^1_*(\omega_{\boldsymbol{a}})$ are curl-free. We now observe that we have

$$(\nabla\times\boldsymbol{r}^{\boldsymbol{a}}, \nabla\times\boldsymbol{v})_{\omega_{\boldsymbol{a}}} = -(\nabla_{\mathcal{T}}\tau_p, \nabla\times\boldsymbol{v})_{\omega_{\boldsymbol{a}}} \qquad \forall \boldsymbol{v} \in \boldsymbol{H}_*(\mathrm{curl},\omega_{\boldsymbol{a}}). \tag{3.10}$$

Indeed, we have, for any $\boldsymbol{H}_*(\mathrm{curl},\omega_{\boldsymbol{a}})$,

$$(\nabla\times\boldsymbol{r}^{\boldsymbol{a}}, \nabla\times\boldsymbol{v})_{\omega_{\boldsymbol{a}}} = (\nabla s^{\boldsymbol{a}} - \nabla_{\mathcal{T}}\tau_p, \nabla\times\boldsymbol{v})_{\omega_{\boldsymbol{a}}} = -(\nabla_{\mathcal{T}}\tau_p, \nabla\times\boldsymbol{v})_{\omega_{\boldsymbol{a}}},$$

since $s^{\boldsymbol{a}} \in H^1_0(\omega_{\boldsymbol{a}})$. Now, elementwise Green formula using the definition of τ_p shows that $(\nabla_{\mathcal{T}}\tau_p, \nabla\times\boldsymbol{v})_{\omega_{\boldsymbol{a}}} = \sum_{F\in\mathcal{F}^{\mathrm{i}}_{\boldsymbol{a}}} (r_F\boldsymbol{n}_F, \nabla\times\boldsymbol{v})_F$, so that (3.8) follows. Finally,

$$\min_{v\in H^1_0(\omega_{\boldsymbol{a}})} \|\nabla_{\mathcal{T}}(\tau_p - v)\|_{\omega_{\boldsymbol{a}}} = \|\nabla_{\mathcal{T}}(\tau_p - s^{\boldsymbol{a}})\|_{\omega_{\boldsymbol{a}}} = \|\nabla\times\boldsymbol{r}^{\boldsymbol{a}}\|_{\omega_{\boldsymbol{a}}} = \max_{\substack{\boldsymbol{v}\in\boldsymbol{H}(\mathrm{curl},\omega_{\boldsymbol{a}})\\ \|\nabla\times\boldsymbol{v}\|_{\omega_{\boldsymbol{a}}}=1}} (\nabla_{\mathcal{T}}\tau_p, \nabla\times\boldsymbol{v})_{\omega_{\boldsymbol{a}}},$$

using (3.10) and noting that any function $\boldsymbol{v} \in \nabla H^1_*(\omega_{\boldsymbol{a}})$ is automatically excluded from the maximization set by the constraint $\|\nabla\times\boldsymbol{v}\|_{\omega_{\boldsymbol{a}}} = 1$. □

Let us now show that the constrained $\boldsymbol{H}_0(\mathrm{div},\omega_{\boldsymbol{a}})$-minimization of Corollary 3.3 is equivalent to evaluating a dual H^1-norm of a suitable linear form defined from the data r_K and r_F and consequently to evaluating the energy norm of its $H^1_*(\omega_{\boldsymbol{a}})$-lifting.

Corollary 3.6 (H^1 form of the $\boldsymbol{H}(\mathrm{div})$-minimization)**.** *Let the assumptions of Corollary 3.3 hold. Let* $r^{\boldsymbol{a}} \in H^1_*(\omega_{\boldsymbol{a}})$ *solve*

$$(\nabla r^{\boldsymbol{a}}, \nabla v)_{\omega_{\boldsymbol{a}}} = \sum_{K\in\mathcal{T}_{\boldsymbol{a}}} (r_K, v)_K - \sum_{F\in\mathcal{F}_{\boldsymbol{a}}} (r_F, v)_F \qquad \forall v \in H^1_*(\omega_{\boldsymbol{a}}). \tag{3.11}$$

Then

$$\min_{\substack{\boldsymbol{v}\in\boldsymbol{H}_0(\mathrm{div},\omega_{\boldsymbol{a}})\\ \nabla\cdot\boldsymbol{v}|_K = r_K - \nabla_{\mathcal{T}}\cdot\boldsymbol{\tau}_p|_K\ \forall K\in\mathcal{T}_{\boldsymbol{a}}}} \|\boldsymbol{\tau}_p + \boldsymbol{v}\|_{\omega_{\boldsymbol{a}}} = \|\nabla r^{\boldsymbol{a}}\|_{\omega_{\boldsymbol{a}}} = \max_{\substack{v\in H^1(\omega_{\boldsymbol{a}})\\ \|\nabla v\|_{\omega_{\boldsymbol{a}}}=1}} \left\{\sum_{K\in\mathcal{T}_{\boldsymbol{a}}} (r_K, v)_K - \sum_{F\in\mathcal{F}_{\boldsymbol{a}}} (r_F, v)_F\right\}. \tag{3.12}$$

Proof. Elementwise Green formula combined with the definition of $\boldsymbol{\tau}_p$ gives

$$\begin{aligned}\sum_{K\in\mathcal{T}_{\boldsymbol{a}}}(r_K,v)_K-\sum_{F\in\mathcal{F}_{\boldsymbol{a}}}(r_F,v)_F&=\sum_{K\in\mathcal{T}_{\boldsymbol{a}}}(r_K,v)_K-\sum_{K\in\mathcal{T}_{\boldsymbol{a}}}(\boldsymbol{\tau}_p{\cdot}\boldsymbol{n}_K,v)_{\partial K}\\&=\sum_{K\in\mathcal{T}_{\boldsymbol{a}}}(r_K-\nabla_{\mathcal{T}}{\cdot}\boldsymbol{\tau}_p,v)_K-\sum_{K\in\mathcal{T}_{\boldsymbol{a}}}(\boldsymbol{\tau}_p,\nabla v)_K,\end{aligned}$$

and we immediately see that (3.11) is the primal formulation of (3.7). As both formulations are equivalent, $\boldsymbol{\sigma}^{\boldsymbol{a}}=-\nabla r^{\boldsymbol{a}}-\boldsymbol{\tau}_p$, cf. [19, Remark 3.15]. The equality (3.12) follows immediately from (3.11), writing the maximum first for all $v\in H^1_*(\omega_{\boldsymbol{a}})$ with $\|\nabla v\|_{\omega_{\boldsymbol{a}}}=1$ and then noting that any function v constant on $\omega_{\boldsymbol{a}}$ is automatically excluded from the maximization set by the constraint $\|\nabla v\|_{\omega_{\boldsymbol{a}}}=1$. $\square$

Corollaries 3.5 and 3.6 allow us to draw insightful links with the literature. On the one hand, Corollary 3.5 explains how the right-hand side in (3.2) links to the continuous minimization used in [19, Lemma 3.13]. Therein, in two space dimensions, the field $\Re^{\perp}(\nabla_{\mathcal{T}}\tau_p)$ has been employed in the definition of the function $r_{\mathbf{a}}$ by formulas (3.19) and (3.32), where $\Re^{\perp}=\left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right)$ is the rotation by $\frac{\pi}{2}$; then $\|\nabla r_{\mathbf{a}}\|_{\omega_{\mathbf{a}}}$ of [19] equals the present $\min_{v\in H^1_0(\omega_{\boldsymbol{a}})}\|\nabla_{\mathcal{T}}(\tau_p-v)\|_{\omega_{\boldsymbol{a}}}$, and, in particular, we have

$$\min_{v\in H^1_0(\omega_{\boldsymbol{a}})}\|\nabla_{\mathcal{T}}(\tau_p-v)\|_{\omega_{\boldsymbol{a}}}=\max_{\substack{v\in H^1(\omega_{\boldsymbol{a}})\\\|\nabla v\|_{\omega_{\boldsymbol{a}}}=1}}\left\{-\left(\Re^{\perp}(\nabla_{\mathcal{T}}\tau_p),\nabla v\right)_{\omega_{\boldsymbol{a}}}\right\}.$$

On the other hand, the maximization form in Corollary 3.6 has been used previously in [1, Theorem 7] and [19, Lemma 3.12 and Corollary 3.16].

3.3 Boundary vertices

In this section, we reformulate Theorems 2.4 and 2.5 as best-approximation results on discontinuous piecewise polynomial data. The proofs are omitted since they are similar to the previous ones. In view of application to a posteriori error analysis of model problems with non-homogeneous boundary conditions, it is convenient to introduce some additional boundary data denoted by $u^{\mathrm{D}}_{\boldsymbol{a}}$ and $\sigma^{\mathrm{N}}_{\boldsymbol{a}}$ in the H^1 and $\boldsymbol{H}(\mathrm{div})$ settings, respectively.

Corollary 3.7 (H^1 best-approximation)**.** *Let the assumptions of Theorem 2.4 hold. Let $u^{\mathrm{D}}_{\boldsymbol{a}}\in\mathbb{P}_p(\mathcal{F}^{\mathrm{D}}_{\boldsymbol{a}})\cap C^0(\partial\omega^{\mathrm{D}}_{\boldsymbol{a}})$ with $\partial\omega^{\mathrm{D}}_{\boldsymbol{a}}=\cup_{F\in\mathcal{F}^{\mathrm{D}}_{\boldsymbol{a}}}F$. Consider any $\tau_p\in\mathbb{P}_p(\mathcal{T}_{\boldsymbol{a}})$ so that $\tau_p|_F=0$ $\forall F\in\mathcal{F}^{\mathrm{b}}_{\boldsymbol{a}}$, $\tau_p|_F-u^{\mathrm{D}}_{\boldsymbol{a}}|_F=r_F$ $\forall F\in\mathcal{F}^{\mathrm{D}}_{\boldsymbol{a}}$, and $[\![\tau_p]\!]_F=r_F$ $\forall F\in\mathcal{F}^{\mathrm{i}}_{\boldsymbol{a}}$. Then the following holds:*

$$\min_{\substack{v_p\in\mathbb{P}_p(\mathcal{T}_{\boldsymbol{a}})\cap H^1(\omega_{\boldsymbol{a}})\\v_p|_F=0\ \forall F\in\mathcal{F}^{\mathrm{b}}_{\boldsymbol{a}}\\v_p|_F=u^{\mathrm{D}}_{\boldsymbol{a}}|_F\ \forall F\in\mathcal{F}^{\mathrm{D}}_{\boldsymbol{a}}}}\|\nabla_{\mathcal{T}}(\tau_p-v_p)\|_{\omega_{\boldsymbol{a}}}\le C_{\mathrm{st}}\min_{\substack{v\in H^1(\omega_{\boldsymbol{a}})\\v|_F=0\ \forall F\in\mathcal{F}^{\mathrm{b}}_{\boldsymbol{a}}\\v|_F=u^{\mathrm{D}}_{\boldsymbol{a}}|_F\ \forall F\in\mathcal{F}^{\mathrm{D}}_{\boldsymbol{a}}}}\|\nabla_{\mathcal{T}}(\tau_p-v_p)\|_{\omega_{\boldsymbol{a}}}\tag{3.13}$$

Corollary 3.8 ($\boldsymbol{H}(\mathrm{div})$ best-approximation)**.** *Let the assumptions of Theorem 2.5 hold. Let $\sigma^{\mathrm{N}}_{\boldsymbol{a}}\in\mathbb{P}_p(\mathcal{F}^{\mathrm{N}}_{\boldsymbol{a}})$. Consider any $\boldsymbol{\tau}_p\in\boldsymbol{RTN}_p(\mathcal{T}_{\boldsymbol{a}})$ so that $\boldsymbol{\tau}_p{\cdot}\boldsymbol{n}_F=r_F$ $\forall F\in\mathcal{F}^{\mathrm{b}}_{\boldsymbol{a}}$, $\boldsymbol{\tau}_p{\cdot}\boldsymbol{n}_F+\sigma^{\mathrm{N}}_{\boldsymbol{a}}=r_F$ $\forall F\in\mathcal{F}^{\mathrm{N}}_{\boldsymbol{a}}$, and $[\![\boldsymbol{\tau}_p]\!]{\cdot}\boldsymbol{n}_F=r_F$ $\forall F\in\mathcal{F}^{\mathrm{i}}_{\boldsymbol{a}}$. Then the following holds:*

$$\min_{\substack{\boldsymbol{v}_p\in\boldsymbol{RTN}_p(\mathcal{T}_{\boldsymbol{a}})\cap\boldsymbol{H}(\mathrm{div},\omega_{\boldsymbol{a}})\\\boldsymbol{v}_p{\cdot}\boldsymbol{n}_F=0\ \forall F\in\mathcal{F}^{\mathrm{b}}_{\boldsymbol{a}}\\\boldsymbol{v}_p{\cdot}\boldsymbol{n}_F=\sigma^{\mathrm{N}}_{\boldsymbol{a}}|_F\ \forall F\in\mathcal{F}^{\mathrm{N}}_{\boldsymbol{a}}\\\nabla{\cdot}\boldsymbol{v}_p|_K=r_K-\nabla_{\mathcal{T}}{\cdot}\boldsymbol{\tau}_p|_K\ \forall K\in\mathcal{T}_{\boldsymbol{a}}}}\|\boldsymbol{\tau}_p+\boldsymbol{v}_p\|_{\omega_{\boldsymbol{a}}}\le C_{\mathrm{st}}\min_{\substack{\boldsymbol{v}\in\boldsymbol{H}(\mathrm{div},\omega_{\boldsymbol{a}})\\\boldsymbol{v}{\cdot}\boldsymbol{n}_F=0\ \forall F\in\mathcal{F}^{\mathrm{b}}_{\boldsymbol{a}}\\\boldsymbol{v}{\cdot}\boldsymbol{n}_F=\sigma^{\mathrm{N}}_{\boldsymbol{a}}|_F\ \forall F\in\mathcal{F}^{\mathrm{N}}_{\boldsymbol{a}}\\\nabla{\cdot}\boldsymbol{v}|_K=r_K-\nabla_{\mathcal{T}}{\cdot}\boldsymbol{\tau}_p|_K\ \forall K\in\mathcal{T}_{\boldsymbol{a}}}}\|\boldsymbol{\tau}_p+\boldsymbol{v}\|_{\omega_{\boldsymbol{a}}}.\tag{3.14}$$

4 Application to a posteriori error analysis

We show in this section the application of our results to a posteriori error analysis. For this purpose, let $\Omega\subset\mathbb{R}^3$ be a polyhedral Lipschitz domain (open, bounded, and connected set). Let $\mathcal{T}_h$ be a matching tetrahedral mesh of Ω, shape-regular with parameter $\gamma_{\mathcal{T}_h}>0$ that bounds the ratio of any element diameter to the diameter of its largest inscribed ball. All faces of the mesh are collected in the set $\mathcal{F}_h$, with faces lying on the boundary of Ω forming two disjoint sets $\mathcal{F}_h^{\mathrm{N}}$ and $\mathcal{F}_h^{\mathrm{D}}$ covering two subdomains Γ_{N} and Γ_{D} of $\partial\Omega$. Consider the Laplace problem

$$
\begin{aligned}
-\Delta u &= f && \text{in } \Omega, && (4.1\text{a})\\
u &= u_{\mathrm{D}} && \text{on } \Gamma_{\mathrm{D}}, && (4.1\text{b})\\
-\nabla u \cdot \boldsymbol{n}_\Omega &= \sigma_{\mathrm{N}} && \text{on } \Gamma_{\mathrm{N}}, && (4.1\text{c})
\end{aligned}
$$

where, for simplicity, $f \in \mathbb{P}_{p'-1}(\mathcal{T}_h)$, $u_{\mathrm{D}} \in \mathbb{P}_{p'}(\mathcal{F}_h^{\mathrm{D}}) \cap C^0(\Gamma_{\mathrm{D}})$, and $\sigma_{\mathrm{N}} \in \mathbb{P}_{p'-1}(\mathcal{F}_h^{\mathrm{N}})$, for a polynomial degree $p' \geq 1$. If $|\Gamma_{\mathrm{D}}| = 0$, we need to additionally suppose the Neumann compatibility condition $(f, 1)_\Omega = (\sigma_{\mathrm{N}}, 1)_{\partial\Omega}$. Weak solution of problem (4.1) is a function $u \in H^1(\Omega)$ such that $u|_{\Gamma_{\mathrm{D}}} = u_{\mathrm{D}}$ and such that

$$
(\nabla u, \nabla v)_\Omega = (f, v)_\Omega - (\sigma_{\mathrm{N}}, v)_{\Gamma_{\mathrm{N}}} \qquad \forall v \in H^1(\Omega) \text{ such that } v|_{\Gamma_{\mathrm{D}}} = 0. \tag{4.2}
$$

For more general data f, u_{D}, and σ_{N}, data oscillation terms arise in the a posteriori error analysis, see [15] and references therein for details.

Let $u_h \in \mathbb{P}_{p'}(\mathcal{T}_h)$ be an approximate solution to the problem (4.1); u_h can be primal-nonconforming in the sense that $u_h \notin H^1(\Omega)$ and $u_h|_{\Gamma_{\mathrm{D}}} \neq u_{\mathrm{D}}$, as well as dual-nonconforming in the sense that $-\nabla_{\mathcal{T}} u_h \notin \boldsymbol{H}(\mathrm{div}, \Omega)$, $\nabla \cdot (-\nabla_{\mathcal{T}} u_h) \neq f$, and $(-\nabla_{\mathcal{T}} u_h \cdot \boldsymbol{n}_\Omega)|_{\Gamma_{\mathrm{N}}} \neq \sigma_{\mathrm{N}}$. The results of this paper have a direct application to a posteriori error analysis since they allow us to construct two central objects leading to guaranteed reliability and p-robust local efficiency. The first is a so-called potential reconstruction $s_h \in \mathbb{P}_{p'+1}(\mathcal{T}_h) \cap H^1(\Omega)$, equal to u_{D} on Γ_{D}. The second one is a so-called equilibrated flux reconstruction $\boldsymbol{\sigma}_h \in \boldsymbol{RTN}_{p'}(\mathcal{T}_h) \cap \boldsymbol{H}(\mathrm{div}, \Omega)$, such that $\nabla \cdot \boldsymbol{\sigma}_h = f$ in Ω and $\boldsymbol{\sigma}_h \cdot \boldsymbol{n}|_{\Gamma_{\mathrm{N}}} = \sigma_{\mathrm{N}}$ on Γ_{N}. When $u_h \notin H^1(\Omega)$ or $u_h|_{\Gamma_{\mathrm{D}}} \neq u_{\mathrm{D}}$ so that the potential reconstruction s_h is necessary, we additionally need to suppose that Assumption C.1 below on boundary patches enumeration is satisfied; this may request some conditions on the structure of the boundary subsets Γ_{D} and Γ_{N}, like that they are locally connected (no checkerboard patters).

Collect all the mesh vertices in the set $\mathcal{V}_h$, and for any mesh vertex $\boldsymbol{a} \in \mathcal{V}_h$, let the patch $\mathcal{T}_{\boldsymbol{a}} \subset \mathcal{T}_h$ be given by the elements in $\mathcal{T}_h$ having $\boldsymbol{a}$ as vertex, whereas $\omega_{\boldsymbol{a}} \subset \Omega$ is the corresponding open subdomain of Ω. Let $\psi_{\boldsymbol{a}}$ be the "hat" function associated with the vertex $\boldsymbol{a}$: this is a continuous function, piecewise affine with respect to the mesh $\mathcal{T}_h$, which takes the value 1 at the vertex $\boldsymbol{a}$ and 0 at the other vertices. Its support is thus the patch subdomain $\omega_{\boldsymbol{a}}$. We also split the vertex set as $\mathcal{V}_h = \mathcal{V}_h^{\mathrm{i}} \cup \mathcal{V}_h^{\mathrm{b}}$, where $\mathcal{V}_h^{\mathrm{i}}$ contains all interior vertices and $\mathcal{V}_h^{\mathrm{b}}$ all boundary vertices. The faces of the elements in the interior patches $\mathcal{T}_{\boldsymbol{a}}$ (i.e., associated with an interior vertex $\boldsymbol{a}$) are collected in $\mathcal{F}_{\boldsymbol{a}} = \mathcal{F}_{\boldsymbol{a}}^{\mathrm{i}} \cup \mathcal{F}_{\boldsymbol{a}}^{\mathrm{b}}$, in conformity with Section 2.1. For a boundary vertex $\boldsymbol{a} \in \mathcal{V}_h^{\mathrm{b}}$, the split is $\mathcal{F}_{\boldsymbol{a}} = \mathcal{F}_{\boldsymbol{a}}^{\mathrm{i}} \cup \mathcal{F}_{\boldsymbol{a}}^{\mathrm{b}} \cup \mathcal{F}_{\boldsymbol{a}}^{\mathrm{D}} \cup \mathcal{F}_{\boldsymbol{a}}^{\mathrm{N}}$, as in Section 2.4, where $\mathcal{F}_{\boldsymbol{a}}^{\mathrm{i}}$ collects the interior faces containing the vertex $\boldsymbol{a}$ and shared by two distinct elements in $\mathcal{T}_{\boldsymbol{a}}$, $\mathcal{F}_{\boldsymbol{a}}^{\mathrm{b}}$ the patch boundary faces from $\partial\omega_{\boldsymbol{a}}$ but not sharing the point $\boldsymbol{a}$, $\mathcal{F}_{\boldsymbol{a}}^{\mathrm{D}}$ the Dirichlet boundary faces $\mathcal{F}_{\boldsymbol{a}}^{\mathrm{D}}$ from $\partial\omega_{\boldsymbol{a}} \cap \Gamma_{\mathrm{D}}$ and sharing the point $\boldsymbol{a}$, and $\mathcal{F}_{\boldsymbol{a}}^{\mathrm{N}}$ the Neumann boundary faces from $\partial\omega_{\boldsymbol{a}} \cap \Gamma_{\mathrm{N}}$ and sharing the point $\boldsymbol{a}$. To have a more unified formalism between interior and boundary vertices, we conventionally define $\mathcal{F}_{\boldsymbol{a}}^{\mathrm{D}}$ and $\mathcal{F}_{\boldsymbol{a}}^{\mathrm{N}}$ to be empty sets for all $\boldsymbol{a} \in \mathcal{V}_h^{\mathrm{i}}$.

We define the potential reconstruction following [6] and [19, Construction 3.8 and Remark 3.10] as $s_h := \sum_{\boldsymbol{a} \in \mathcal{V}_h} s_h^{\boldsymbol{a}}$, where $s_h^{\boldsymbol{a}}$ is the discrete minimizer of Corollary 3.1 given by (3.3) for interior vertices and similarly as the discrete minimizer of Corollary 3.7 for boundary vertices. We choose the polynomial degree p of our theory to be $p := p' + 1$ and we set for all $\boldsymbol{a} \in \mathcal{V}_h$,

$$
\begin{aligned}
\tau_p &:= \psi_{\boldsymbol{a}} u_h && \text{in } \omega_{\boldsymbol{a}}, && (4.3\text{a})\\
r_F &:= \psi_{\boldsymbol{a}} [\![u_h]\!]_F && \text{on all } F \in \mathcal{F}_{\boldsymbol{a}}^{\mathrm{i}}, && (4.3\text{b})\\
r_F &:= 0 && \text{on all } F \in \mathcal{F}_{\boldsymbol{a}}^{\mathrm{b}}, && (4.3\text{c})\\
r_F &:= \psi_{\boldsymbol{a}} (u_h - u_{\mathrm{D}}) && \text{on all } F \in \mathcal{F}_{\boldsymbol{a}}^{\mathrm{D}}, && (4.3\text{d})\\
u_{\boldsymbol{a}}^{\mathrm{D}} &:= \psi_{\boldsymbol{a}} u_{\mathrm{D}} && \text{on all } F \in \mathcal{F}_{\boldsymbol{a}}^{\mathrm{D}}. && (4.3\text{e})
\end{aligned}
$$

By construction, the polynomial data satisfy the compatibility conditions (2.6) and (2.12a). Similarly, following [13], [2, 1], and [19, Construction 3.4 and Remark 3.7], we define the equilibrated flux reconstruction as $\boldsymbol{\sigma}_h := \sum_{\boldsymbol{a} \in \mathcal{V}_h} \boldsymbol{\sigma}_h^{\boldsymbol{a}}$, where $\boldsymbol{\sigma}_h^{\boldsymbol{a}}$ is the discrete minimizer of Corollary 3.3 given by (3.6) for interior vertices and similarly the discrete minimizer of Corollary 3.8 for boundary vertices, with the polynomial degree

$p := p'$. Here we set, for all $\boldsymbol{a} \in \mathcal{V}_h$,

$$
\begin{aligned}
\boldsymbol{\tau}_p &:= \psi_{\boldsymbol{a}} \nabla_{\mathcal{T}} u_h && \text{in } \omega_{\boldsymbol{a}}, && \text{(4.4a)}\\
r_K &:= \psi_{\boldsymbol{a}}(f + \Delta_{\mathcal{T}} u_h) && \text{in all } K \in \mathcal{T}_{\boldsymbol{a}}, && \text{(4.4b)}\\
r_F &:= \psi_{\boldsymbol{a}} [\![\nabla_{\mathcal{T}} u_h]\!] \cdot \boldsymbol{n}_F && \text{on all } F \in \mathcal{F}_{\boldsymbol{a}}^{\mathrm{i}}, && \text{(4.4c)}\\
r_F &:= 0 && \text{on all } F \in \mathcal{F}_{\boldsymbol{a}}^{\mathrm{b}}, && \text{(4.4d)}\\
r_F &:= \psi_{\boldsymbol{a}}(\nabla_{\mathcal{T}} u_h \cdot \boldsymbol{n}_F + \sigma_{\mathrm{N}}) && \text{on all } F \in \mathcal{F}_{\boldsymbol{a}}^{\mathrm{N}}, && \text{(4.4e)}\\
\sigma_{\boldsymbol{a}}^{\mathrm{N}} &:= \psi_{\boldsymbol{a}} \sigma_{\mathrm{N}} && \text{on all } F \in \mathcal{F}_{\boldsymbol{a}}^{\mathrm{N}}. && \text{(4.4f)}
\end{aligned}
$$

Here, for all interior vertices and for those boundary vertices which are only shared by Neumann faces (i.e., $\mathcal{F}_{\boldsymbol{a}}^{\mathrm{D}} = \emptyset$), the hat-function orthogonality

$$
(\nabla_{\mathcal{T}} u_h, \nabla \psi_{\boldsymbol{a}})_{\omega_{\boldsymbol{a}}} = (f, \psi_{\boldsymbol{a}})_{\omega_{\boldsymbol{a}}} - (\sigma_{\mathrm{N}}, \psi_{\boldsymbol{a}})_{\partial\omega_{\boldsymbol{a}} \cap \Gamma_{\mathrm{N}}} \tag{4.5}
$$

is a necessary condition for the data compatibility conditions (2.9) and (2.14) to hold. This relation does not hold, for example, for certain discontinuous Galerkin methods; a simple use of the discrete gradient $\nabla_{\mathrm{d}} u_h$ from [14, Section 4.3] in place of the broken gradient $\nabla_{\mathcal{T}} u_h$ allows to fix this, see [19, 15].

The above potential and flux reconstructions lead to the guaranteed a posteriori error estimate

$$
\|\nabla_{\mathcal{T}}(u - u_h)\|_{\Omega}^2 \le \sum_{K \in \mathcal{T}_h} \left(\|\nabla_{\mathcal{T}} u_h + \boldsymbol{\sigma}_h\|_K^2 + \|\nabla_{\mathcal{T}}(u_h - s_h)\|_K^2 \right), \tag{4.6}
$$

see [19, Theorem 3.3] or [15, Theorem 3.3] and the references therein. The crucial use of our results is in the proof of the p-robust local efficiency. Corollary 3.1 for interior vertices and Corollary 3.7 for boundary vertices immediately give

$$
\|\nabla_{\mathcal{T}}(\psi_{\boldsymbol{a}} u_h - s_h^{\boldsymbol{a}})\|_{\omega_{\boldsymbol{a}}} \le C_{\mathrm{st}} \min_{\substack{v \in H^1(\omega_{\boldsymbol{a}}) \\ v|_F = 0 \ \forall F \in \mathcal{F}_{\boldsymbol{a}}^{\mathrm{b}} \\ v|_F = \psi_{\boldsymbol{a}} u_{\mathrm{D}} \ \forall F \in \mathcal{F}_{\boldsymbol{a}}^{\mathrm{D}}}} \|\nabla_{\mathcal{T}}(\psi_{\boldsymbol{a}} u_h - v)\|_{\omega_{\boldsymbol{a}}} \le C_{\mathrm{st}} \inf_{\substack{v \in H^1(\omega_{\boldsymbol{a}}) \\ v|_F = u_{\mathrm{D}} \ \forall F \in \mathcal{F}_{\boldsymbol{a}}^{\mathrm{D}}}} \|\nabla_{\mathcal{T}}(\psi_{\boldsymbol{a}}(u_h - v))\|_{\omega_{\boldsymbol{a}}}.
$$

Indeed, the right inequality follows immediately as any function $v \in H^1(\omega_{\boldsymbol{a}})$, equal to u_{D} on the faces from $\mathcal{F}_{\boldsymbol{a}}^{\mathrm{D}}$, belongs to the minimization set of the middle term above when multiplied by the hat function $\psi_{\boldsymbol{a}}$. This means that the discrete fully computable estimator $\|\nabla_{\mathcal{T}}(\psi_{\boldsymbol{a}} u_h - s_h^{\boldsymbol{a}})\|_{\omega_{\boldsymbol{a}}}$ is a local lower bound for a $\psi_{\boldsymbol{a}}$-weighted distance to the $H^1(\omega_{\boldsymbol{a}})$ space (or an affine subspace if $\mathcal{F}_{\boldsymbol{a}}^{\mathrm{D}} \neq \emptyset$). We now make the weak solution u of (4.1) appear in the bound. For $\boldsymbol{a} \in \mathcal{V}_h^{\mathrm{i}}$, let $\tilde{u} := u - c_{\boldsymbol{a}}$, where the constant $c_{\boldsymbol{a}}$ is chosen so that $(\tilde{u}, 1)_{\omega_{\boldsymbol{a}}} = (u_h, 1)_{\omega_{\boldsymbol{a}}}$. For a boundary vertex $\boldsymbol{a} \in \mathcal{V}_h^{\mathrm{b}}$ such that $\mathcal{F}_{\boldsymbol{a}}^{\mathrm{D}} = \emptyset$, let $\tilde{u} := u - c_{\boldsymbol{a}}$, where the constant $c_{\boldsymbol{a}}$ is chosen so that $\sum_{F \in \mathcal{F}_{\boldsymbol{a}}^{\mathrm{N}}} (\tilde{u}, 1)_F = \sum_{F \in \mathcal{F}_{\boldsymbol{a}}^{\mathrm{N}}} (u_h, 1)_F$. In the other situations, we let $\tilde{u} := u$. Then, proceeding as in [19, Lemma 3.13 and Section 4.3] while relying on the broken Poincaré–Friedrichs inequality [3], we obtain

$$
\begin{aligned}
\|\nabla_{\mathcal{T}}(\psi_{\boldsymbol{a}}(u_h - \tilde{u}))\|_{\omega_{\boldsymbol{a}}} &\le \|\nabla \psi_{\boldsymbol{a}}\|_{\infty,\omega_{\boldsymbol{a}}} \|u_h - \tilde{u}\|_{\omega_{\boldsymbol{a}}} + \|\psi_{\boldsymbol{a}}\|_{\infty,\omega_{\boldsymbol{a}}} \|\nabla_{\mathcal{T}}(u_h - \tilde{u})\|_{\omega_{\boldsymbol{a}}} \\
&\le (1 + C_{\mathrm{bPF},\omega_{\boldsymbol{a}}} h_{\omega_{\boldsymbol{a}}} \|\nabla \psi_{\boldsymbol{a}}\|_{\infty,\omega_{\boldsymbol{a}}}) \|\nabla_{\mathcal{T}}(u_h - u)\|_{\omega_{\boldsymbol{a}}} \\
&\quad + C_{\mathrm{bPF},\omega_{\boldsymbol{a}}} h_{\omega_{\boldsymbol{a}}} \|\nabla \psi_{\boldsymbol{a}}\|_{\infty,\omega_{\boldsymbol{a}}} \left\{ \sum_{F \in \mathcal{F}_{\boldsymbol{a}}^{\mathrm{i}}} h_F^{-1} \|\Pi_F^0 [\![u_h]\!]\|_F^2 + \sum_{F \in \mathcal{F}_{\boldsymbol{a}}^{\mathrm{D}}} h_F^{-1} \|\Pi_F^0 (u_h - u_{\mathrm{D}})\|_F^2 \right\}^{1/2}.
\end{aligned}
$$

In particular, if the mean values of the jumps in u_h are zero, i.e., $([\![u_h]\!], 1)_F = 0$ for all the faces $F \in \mathcal{F}_h^{\mathrm{i}}$ and $(u_h, 1)_F = (u_{\mathrm{D}}, 1)_F$ for all the Dirichlet faces $F \in \mathcal{F}_h^{\mathrm{D}}$, we infer that

$$
\|\nabla_{\mathcal{T}}(\psi_{\boldsymbol{a}} u_h - s_h^{\boldsymbol{a}})\|_{\omega_{\boldsymbol{a}}} \le C_{\mathrm{st}} C_{\mathrm{cont,bPF}} \|\nabla_{\mathcal{T}}(u_h - u)\|_{\omega_{\boldsymbol{a}}}, \tag{4.7}
$$

where the constant $C_{\mathrm{cont,bPF}} := \max_{\boldsymbol{a} \in \mathcal{V}_h} \{1 + C_{\mathrm{bPF},\omega_{\boldsymbol{a}}} h_{\omega_{\boldsymbol{a}}} \|\nabla \psi_{\boldsymbol{a}}\|_{\infty,\omega_{\boldsymbol{a}}}\}$ only depends on the shape-regularity parameter $\gamma_{\mathcal{T}_h}$.

Let

$$
\begin{aligned}
H_*^1(\omega_{\boldsymbol{a}}) &:= \{v \in H^1(\omega_{\boldsymbol{a}});\ (v, 1)_{\omega_{\boldsymbol{a}}} = 0\}, && \boldsymbol{a} \in \mathcal{V}_h^{\mathrm{i}} \text{ or } \boldsymbol{a} \in \mathcal{V}_h^{\mathrm{b}} \text{ and } \mathcal{F}_{\boldsymbol{a}}^{\mathrm{D}} = \emptyset, && \text{(4.8a)}\\
H_*^1(\omega_{\boldsymbol{a}}) &:= \{v \in H^1(\omega_{\boldsymbol{a}});\ v = 0 \text{ on all } F \in \mathcal{F}_{\boldsymbol{a}}^{\mathrm{D}}\}, && \boldsymbol{a} \in \mathcal{V}_h^{\mathrm{b}} \text{ and } \mathcal{F}_{\boldsymbol{a}}^{\mathrm{D}} \neq \emptyset. && \text{(4.8b)}
\end{aligned}
$$

Then Corollary 3.3 for interior vertices and Corollary 3.8 for boundary vertices, in conjunction with Corollary 3.6, yield

$$\begin{aligned}
\|\psi_{\boldsymbol{a}}\nabla_{\mathcal{T}}u_h + \boldsymbol{\sigma}_h^{\boldsymbol{a}}\|_{\omega_{\boldsymbol{a}}} &\le C_{\mathrm{st}} \max_{\substack{v\in H^1_*(\omega_{\boldsymbol{a}})\\ \|\nabla v\|_{\omega_{\boldsymbol{a}}}=1}} \left\{ \sum_{K\in\mathcal{T}_{\boldsymbol{a}}} (r_K, v)_K - \sum_{F\in\mathcal{F}_{\boldsymbol{a}}} (r_F, v)_F \right\} \\
&= C_{\mathrm{st}} \max_{\substack{v\in H^1_*(\omega_{\boldsymbol{a}})\\ \|\nabla v\|_{\omega_{\boldsymbol{a}}}=1}} \left\{ (f, \psi_{\boldsymbol{a}} v)_{\omega_{\boldsymbol{a}}} - (\nabla_{\mathcal{T}} u_h, \nabla(\psi_{\boldsymbol{a}} v))_{\omega_{\boldsymbol{a}}} - \sum_{F\in\mathcal{F}_{\boldsymbol{a}}^{\mathrm{N}}} (\sigma_{\mathrm{N}}, \psi_{\boldsymbol{a}} v)_F \right\},
\end{aligned}$$

using the definition of r_K and r_F in (4.4) above and the Green theorem. Thus, the fully computable estimator $\|\psi_{\boldsymbol{a}}\nabla_{\mathcal{T}}u_h + \boldsymbol{\sigma}_h^{\boldsymbol{a}}\|_{\omega_{\boldsymbol{a}}}$ is a local lower bound for the local dual norm of the residual with $\psi_{\boldsymbol{a}}$-weighted test functions. We now note that $\psi_{\boldsymbol{a}} v$, extended by zero outside of the patch subdomain $\omega_{\boldsymbol{a}}$, is a function in $H^1(\Omega)$ which is zero on the Dirichlet part of the boundary Γ_{D}. Thus we can use the weak solution definition (4.2) to replace the right-hand side by $(\nabla_{\mathcal{T}}(u-u_h), \nabla(\psi_{\boldsymbol{a}} v))_{\omega_{\boldsymbol{a}}}$. Invoking the Cauchy–Schwarz inequality and the (continuous) Poincaré–Friedrichs inequality in $\omega_{\boldsymbol{a}}$, we infer that

$$\|\nabla(\psi_{\boldsymbol{a}} v)\|_{\omega_{\boldsymbol{a}}} \le (1 + C_{\mathrm{PF},\omega_{\boldsymbol{a}}} h_{\omega_{\boldsymbol{a}}} \|\nabla\psi_{\boldsymbol{a}}\|_{\infty,\omega_{\boldsymbol{a}}}) \|\nabla v\|_{\omega_{\boldsymbol{a}}},$$

see [1] or [19, Lemma 3.12]. This finally gives

$$\|\psi_{\boldsymbol{a}}\nabla_{\mathcal{T}}u_h + \boldsymbol{\sigma}_h^{\boldsymbol{a}}\|_{\omega_{\boldsymbol{a}}} \le C_{\mathrm{st}} C_{\mathrm{cont,PF}} \|\nabla_{\mathcal{T}}(u-u_h)\|_{\omega_{\boldsymbol{a}}}, \tag{4.9}$$

where $C_{\mathrm{cont,PF}} := \max_{\boldsymbol{a}\in\mathcal{V}_h}\{1 + C_{\mathrm{PF},\omega_{\boldsymbol{a}}} h_{\omega_{\boldsymbol{a}}} \|\nabla\psi_{\boldsymbol{a}}\|_{\infty,\omega_{\boldsymbol{a}}}\}$ only depends on the shape-regularity parameter $\gamma_{\mathcal{T}_h}$.

5 Proof for broken H^1 polynomial extensions

We prove here Theorem 2.2. In particular, we show in Section 5.1 the existence of the minimizers in (2.7), in Section 5.2 their uniqueness, and in Section 5.3 the stability bound (2.7). Let $p \ge 1$ and let $r \in \mathbb{P}_p(\mathcal{F}_{\boldsymbol{a}}^{\mathrm{i}})$ satisfy the compatibility conditions (2.6). Define

$$V_p(\mathcal{T}_{\boldsymbol{a}}) := \{v_p \in \mathbb{P}_p(\mathcal{T}_{\boldsymbol{a}});\ v_p|_F = 0\ \forall F \in \mathcal{F}_{\boldsymbol{a}}^{\mathrm{b}},\ [\![v_p]\!]_F = r_F\ \forall F \in \mathcal{F}_{\boldsymbol{a}}^{\mathrm{i}}\}, \tag{5.1a}$$

$$V(\mathcal{T}_{\boldsymbol{a}}) := \{v \in H^1(\mathcal{T}_{\boldsymbol{a}});\ v|_F = 0\ \forall F \in \mathcal{F}_{\boldsymbol{a}}^{\mathrm{b}},\ [\![v]\!]_F = r_F\ \forall F \in \mathcal{F}_{\boldsymbol{a}}^{\mathrm{i}}\}. \tag{5.1b}$$

Then the stability bound (2.7) becomes

$$\min_{v_p\in V_p(\mathcal{T}_{\boldsymbol{a}})} \|\nabla_{\mathcal{T}} v_p\|_{\omega_{\boldsymbol{a}}} \le C_{\mathrm{st}} \min_{v\in V(\mathcal{T}_{\boldsymbol{a}})} \|\nabla_{\mathcal{T}} v\|_{\omega_{\boldsymbol{a}}}. \tag{5.2}$$

To prove it, we crucially consider the enumeration of the cells in the patch $\mathcal{T}_{\boldsymbol{a}}$ from Lemma B.1 below in the form $K_1, \ldots, K_{|\mathcal{T}_{\boldsymbol{a}}|}$. Without loss of generality (see Remark 2.1), we orient all the interior faces $F = \partial K_i \cap \partial K_j \in \mathcal{F}_{\boldsymbol{a}}^{\mathrm{i}}$ so that $\boldsymbol{n}_F$ points from K_j to K_i with $j < i$.

In what follows, we abbreviate as $A \lesssim B$ the inequality $A \le cB$ with a generic constant c whose value can only depend on the patch regularity parameter $\gamma_{\mathcal{T}_{\boldsymbol{a}}}$; it is thus in particular independent of the polynomial degree p.

5.1 Existence of the minimizers

Let us first prove that the minimization sets $V_p(\mathcal{T}_{\boldsymbol{a}})$ and $V(\mathcal{T}_{\boldsymbol{a}})$ are non-empty; then the existence of the minimizers immediately follows. Since $V_p(\mathcal{T}_{\boldsymbol{a}}) \subset V(\mathcal{T}_{\boldsymbol{a}})$, we only consider $V_p(\mathcal{T}_{\boldsymbol{a}})$. The proof is constructive in that we build a function in $V_p(\mathcal{T}_{\boldsymbol{a}})$ by enumerating all the cells in $\mathcal{T}_{\boldsymbol{a}}$ while prescribing suitable Dirichlet data on some faces of each cell. For all $1 \le i \le |\mathcal{T}_{\boldsymbol{a}}|$, let us set $F_i^{\mathrm{b}} := \partial K_i \cap \partial\omega_{\boldsymbol{a}}$, i.e., F_i^{b} is the face of K_i lying on the patch subdomain boundary $\partial\omega_{\boldsymbol{a}}$. Consider a function $w_p \in \mathbb{P}_p(\mathcal{T}_{\boldsymbol{a}})$ such that its restrictions $w_p^i = w_p|_{K_i}$, for all $1 \le i \le |\mathcal{T}_{\boldsymbol{a}}|$, are defined by induction as follows:

(i) For $i = 1$, w_p^1 is any function in

$$V_p(K_1) := \{v_p \in \mathbb{P}_p(K_1);\ v_p|_{F_1^{\mathrm{b}}} = 0\}. \tag{5.3a}$$

(ii) For all $1 < i \le |\mathcal{T}_{\boldsymbol{a}}|$, w_p^i is any function in

$$V_p(K_i) := \{v_p \in \mathbb{P}_p(K_i);\ v_p|_{F_i^{\mathrm{b}}} = 0,\, v_p|_F = -r_F + w_p^j|_F\, \forall F \in \mathcal{F}_i^\sharp\}, \tag{5.3b}$$

where $j = j(i, F)$ is the index of the cell sharing F with K_i, i.e., $F = \partial K_i \cap \partial K_j$. Recall that by definition of the set of previously enumerated faces $\mathcal{F}_i^\sharp$ in Appendix B, we have $j < i$, so that w_p^j is already known from a previous step of the construction.

Lemma 5.1 below shows that the (affine) subspaces $V_p(K_i)$ are all non-empty, i.e., the above construction is meaningful. Then, it is easy to see that any function w_p constructed as above is in the discrete minimization set $V_p(\mathcal{T}_{\boldsymbol{a}})$; in particular, we note that the prescription (5.3b) on the faces in $\mathcal{F}_i^\sharp$ implies that $[\![w_p]\!]_F = w_p^j|_F - w_p^i|_F = r_F$.

Lemma 5.1 (Non-emptiness)**.** *For all $1 \le i \le |\mathcal{T}_{\boldsymbol{a}}|$, $V_p(K_i)$ is non-empty.*

Proof. The proof is carried out by induction.
(1) First, the linear space $V_p(K_1)$ is non-trivial. Its dimension is actually equal to the number of Lagrange nodes of order p in the tetrahedron K_1 that are not located on the face F_1^{b}.
(2) Let now $1 < i \le |\mathcal{T}_{\boldsymbol{a}}|$ and suppose that $V_p(K_{i-1})$ is non-empty. To prove that $V_p(K_i)$ is non-empty, we need to verify that the prescribed values on the faces of K_i are continuous across the edges they share. Once this is established, it will follow that $V_p(K_i)$ is an affine space whose tangent space has dimension equal to the number of Lagrange nodes of order p in K_i not located in the faces of K_i where a value is prescribed. We distinguish three cases.
(2.a) Case $1 < i < |\mathcal{T}_{\boldsymbol{a}}|$ and $|\mathcal{F}_i^\sharp| = 1$, say $\mathcal{F}_i^\sharp = \{F_i^1\}$. There is one edge to consider, namely $e = F_i^{\mathrm{b}} \cap F_i^1$. The compatibility condition (2.6a) implies that $r_F|_e = 0$, and we have by construction $w_p^j|_e = 0$ since $e \subset \partial\omega_{\boldsymbol{a}}$. Hence, the value we are prescribing on the face F_i^1 restricted to the edge e is zero.
(2.b) Case $1 < i < |\mathcal{T}_{\boldsymbol{a}}|$ and $|\mathcal{F}_i^\sharp| = 2$, say $\mathcal{F}_i^\sharp = \{F_i^1, F_i^2\}$ (note that $|\mathcal{F}_i^\sharp| < 3$ if $i < |\mathcal{T}_{\boldsymbol{a}}|$ owing to Lemma B.1(ii), although we do not need to make use of this property here). There are three edges to consider, namely $e^1 = F_i^{\mathrm{b}} \cap F_i^1$, $e^2 = F_i^{\mathrm{b}} \cap F_i^2$, and $e^{12} = F_i^1 \cap F_i^2$. For e^1 and e^2, the reasoning is the same as above. For $e := e^{12}$, we first use Lemma B.1(i), giving that K_i is the last cell to be enumerated in the rotational path of cells around e. Thus w_p has been already defined in the previous elements by the induction argument and the algebraic properties of the jump operator (see (2.3)) give

$$\sum_{F \in \mathcal{F}_e \setminus \{F_i^1, F_i^2\}} \iota_{F,e} [\![w_p]\!] = w_p^{j_1} - w_p^{j_2}, \tag{5.4}$$

where $F_i^1 = \partial K_i \cap \partial K_{j_1}$ and $F_i^2 = \partial K_i \cap \partial K_{j_2}$, with K_{j_1} following K_i in the rotation direction around the edge e. Second, employing in (5.4) $[\![w_p]\!]_F = r_F$ for all $F \in \mathcal{F}_e \setminus \{F_i^1, F_i^2\}$ and the compatibility condition (2.6b), we conclude that

$$w_p^{j_1} - w_p^{j_2} = -\iota_{F_i^1,e} r_{F_i^1} - \iota_{F_i^2,e} r_{F_i^2} = r_{F_i^1} - r_{F_i^2}$$

on the edge e; the last inequality also employs that $\iota_{F_i^1,e} = -1$ and $\iota_{F_i^2,e} = 1$ in the chosen notation (recall that both normal vectors $\boldsymbol{n}_{F_i^1}$ and $\boldsymbol{n}_{F_i^2}$ point inward K_i as $j_1, j_2 < i$). This provides the desired continuity property on e.
(2.c) Case $i = |\mathcal{T}_{\boldsymbol{a}}|$. Then $\mathcal{F}_i^\sharp$ contains three faces, and the six edges of $K_{|\mathcal{T}_{\boldsymbol{a}}|}$ need to be considered. The reasoning is the same as above for the three edges located on $\partial\omega_{\boldsymbol{a}}$ and the three edges inside $\omega_{\boldsymbol{a}}$ (using again Lemma B.1(i) and the compatibility condition (2.6b)). ☐

5.2 Uniqueness of the minimizers

The uniqueness of the minimizers in (2.7) results from the fact that $V_p(\mathcal{T}_{\boldsymbol{a}})$ and $V(\mathcal{T}_{\boldsymbol{a}})$ are convex sets (being affine spaces) and that the functional we are minimizing is strictly convex on the tangent spaces of $V_p(\mathcal{T}_{\boldsymbol{a}})$ and $V(\mathcal{T}_{\boldsymbol{a}})$ (both tangent spaces are composed of functions vanishing on $\partial\omega_{\boldsymbol{a}}$, where the H^1-seminorm defines a strictly convex functional).

5.3 Proof of the stability bound (2.7)

We now construct two functions $\zeta_p \in \mathbb{P}_p(\mathcal{T}_{\boldsymbol{a}})$ and $\zeta \in H^1(\mathcal{T}_{\boldsymbol{a}})$ such that their restrictions $\zeta_p^i = \zeta_p|_{K_i}$ and $\zeta^i = \zeta|_{K_i}$, for all $1 \le i \le |\mathcal{T}_{\boldsymbol{a}}|$, are defined by induction as follows:

(i) For $i = 1$, we define the spaces

$$V_p(K_1) := \{v_p \in \mathbb{P}_p(K_1);\ v_p|_{F_1^{\mathrm{b}}} = 0\}, \tag{5.5a}$$
$$V(K_1) := \{v \in H^1(K_1);\ v|_{F_1^{\mathrm{b}}} = 0\}, \tag{5.5b}$$

and consider the following unique minimizers:

$$\zeta_p^1 := \underset{v_p \in V_p(K_1)}{\arg\min} \|\nabla v_p\|_{K_1}, \qquad \zeta^1 := \underset{v \in V(K_1)}{\arg\min} \|\nabla v\|_{K_1}. \tag{5.6}$$

Note that these problems are actually trivial, so that we have $\zeta_p^1 = \zeta^1 = 0$ in K_1.

(ii) For all $1 < i \le |\mathcal{T}_{\boldsymbol{a}}|$, we define the spaces

$$V_p(K_i) := \{v_p \in \mathbb{P}_p(K_i);\ v_p|_{F_i^{\mathrm{b}}} = 0,\ v_p|_F = -r_F + \zeta_p^j|_F\ \forall F \in \mathcal{F}_i^\sharp\}, \tag{5.7a}$$
$$V(K_i) := \{v \in H^1(K_i);\ v|_{F_i^{\mathrm{b}}} = 0,\ v|_F = -r_F + \zeta_p^j|_F\ \forall F \in \mathcal{F}_i^\sharp\}, \tag{5.7b}$$

where $j = j(i, F)$ is the index of the cell sharing F with K_i, i.e., $F = \partial K_i \cap \partial K_j$. Note that both spaces are defined using the same Dirichlet data which are piecewise polynomials of degree p. Consider the following unique minimizers:

$$\zeta_p^i := \underset{v_p \in V_p(K_i)}{\arg\min} \|\nabla v_p\|_{K_i}, \qquad \zeta^i := \underset{v \in V(K_i)}{\arg\min} \|\nabla v\|_{K_i}. \tag{5.8}$$

Note that the above minimization problems are well-posed since the minimization sets are non-empty. Indeed, the Dirichlet condition is a continuous, piecewise polynomial, as established in Section 5.1. Moreover, the set of faces where a Dirichlet condition is prescribed is always non-empty, so that the minimized functional is strictly convex. We can also observe that the continuous minimizer $\zeta^i \in H^1(K_i)$ is the weak solution of the problem

$$-\Delta \zeta^i = 0 \qquad \text{in } K_i, \tag{5.9a}$$
$$\zeta^i|_F = -r_F + \zeta_p^j|_F \qquad \text{on all } F \in \mathcal{F}_i^\sharp, \tag{5.9b}$$
$$\zeta^i|_F = 0 \qquad \text{on } F_i^{\mathrm{b}}, \tag{5.9c}$$
$$-\nabla \zeta^i \cdot \boldsymbol{n}_{K_i}|_F = 0 \qquad \text{on all } F \in \mathcal{F}_i^\flat, \tag{5.9d}$$

whereas ζ_p^i is its (spectral) finite element approximation.

We will show the following two statements for all $1 < i \le |\mathcal{T}_{\boldsymbol{a}}|$ (the statements being trivial for $i = 1$):

$$\|\nabla \zeta_p^i\|_{K_i} \lesssim \|\nabla \zeta^i\|_{K_i}, \tag{5.10a}$$
$$\|\nabla \zeta^i\|_{K_i} \lesssim \|\nabla_{\mathcal{T}} v^*\|_{\omega_{\boldsymbol{a}}} + \sum_{j<i} \|\nabla \zeta_p^j\|_{K_j}, \tag{5.10b}$$

where $v^* \in V(\mathcal{T}_{\boldsymbol{a}})$ is the global continuous minimizer in (2.7). Since the above inductive construction implies that $\zeta_p \in V_p(\mathcal{T}_{\boldsymbol{a}})$, the global discrete minimizer $v_p^* \in V_p(\mathcal{T}_{\boldsymbol{a}})$ in (2.7) is such that

$$\|\nabla_{\mathcal{T}} v_p^*\|_{\omega_{\boldsymbol{a}}} \le \|\nabla_{\mathcal{T}} \zeta_p\|_{\omega_{\boldsymbol{a}}}.$$

Moreover, combining (5.10a) with (5.10b) proves by induction that $\|\nabla \zeta_p^i\|_{K_i} \lesssim \|\nabla_{\mathcal{T}} v^*\|_{\omega_{\boldsymbol{a}}}$ for all $1 \le i \le |\mathcal{T}_{\boldsymbol{a}}|$, so that $\|\nabla_{\mathcal{T}} \zeta_p\|_{\omega_{\boldsymbol{a}}} \lesssim \|\nabla_{\mathcal{T}} v^*\|_{\omega_{\boldsymbol{a}}}$. Hence, $\|\nabla_{\mathcal{T}} v_p^*\|_{\omega_{\boldsymbol{a}}} \lesssim \|\nabla_{\mathcal{T}} v^*\|_{\omega_{\boldsymbol{a}}}$, and this concludes the proof of (2.7).

Proof of (5.10a). We apply Lemma A.1 on $K = K_i$ with $\mathcal{F}_K^{\mathrm{D}} = \{F_i^{\mathrm{b}}\} \cup \mathcal{F}_i^\sharp$ and the p-degree polynomials given by 0 for $F = F_i^{\mathrm{b}}$ and $-r_F + \zeta_p^j|_F$ for all $F \in \mathcal{F}_i^\sharp$. The proof that these polynomials are continuous over the edges shared by two faces in $\mathcal{F}_K^{\mathrm{D}}$ has been given in Section 5.1. $\square$

Proof of (5.10b). We distinguish three cases.
(1) Case $1 < i < |\mathcal{T}_{\boldsymbol{a}}|$ and $|\mathcal{F}_i^\sharp| = 1$, say $\mathcal{F}_i^\sharp = \{F\}$. Let $K_j \in \mathcal{T}_{\boldsymbol{a}}$ be the cell such that $F = \partial K_i \cap \partial K_j$; the definition of $\mathcal{F}_i^\sharp$ implies that $j < i$. Let $\boldsymbol{T} : K_j \to K_i$ be the (unique) bijective affine map leaving F pointwise invariant. Consider in the cell K_i the function

$$v := v^*|_{K_i} - (v^*|_{K_j} - \zeta_p^j) \circ \boldsymbol{T}^{-1}.$$

The crucial observation is that $v \in V(K_i)$. Indeed, the properties $v \in H^1(K_i)$ and $v|_{F_i^{\mathrm{b}}} = 0$ are straightforward to verify. Moreover, since $j < i$, we have $v^*|_{K_j} - v^*|_{K_i} = [\![v^*]\!]_F = r_F$, so that indeed $v|_F = -r_F + \zeta_p^j|_F$. Using the definition (5.8) of ζ^i together with the properties of the map $\boldsymbol{T}$ which follow from mesh regularity, we infer that

$$\begin{aligned}\|\nabla \zeta^i\|_{K_i} &\le \|\nabla v\|_{K_i} \le \|\nabla v^*\|_{K_i} + \|\nabla((v^*|_{K_j} - \zeta_p^j) \circ \boldsymbol{T}^{-1})\|_{K_i} \\ &\lesssim \|\nabla v^*\|_{K_i} + \|\nabla(v^*|_{K_j} - \zeta_p^j)\|_{K_j} \le \|\nabla v^*\|_{K_i} + \|\nabla v^*\|_{K_j} + \|\nabla \zeta_p^j\|_{K_j},\end{aligned}$$

so that (5.10b) holds (recall that $j < i$).
(2) Case $1 < i < |\mathcal{T}_{\boldsymbol{a}}|$ and $|\mathcal{F}_i^\sharp| = 2$, say $\mathcal{F}_i^\sharp = \{F^1, F^2\}$ with $e = F^1 \cap F^2$. We consider the conforming refinement $\mathcal{T}_e'$ of $\mathcal{T}_e$ from Lemma B.2 applied with $K_* = K_i$. Recall that $\mathcal{T}_e'$ contains K_i, that all the tetrahedra in $\mathcal{T}_e'$ have e as edge with their two other vertices lying on $\partial\omega_{\boldsymbol{a}}$, and that, collecting all the vertices of $\mathcal{T}_e'$ that are not endpoints of e in the set $\mathcal{V}_e'$, there is a two-color map $\texttt{col} : \mathcal{V}_e' \to \{1, 2\}$ so that for all $\kappa \in \mathcal{T}_e'$, the two vertices of κ that are not endpoints of e, say $\{\boldsymbol{a}_\kappa^n\}_{1 \le n \le 2}$, satisfy $\texttt{col}(\boldsymbol{a}_\kappa^n) = n$. We use the two-color map to define, for all $\kappa \in \mathcal{T}_e'$, the (unique) bijective affine map $\boldsymbol{T}_\kappa : \kappa \to K_i$ leaving e pointwise invariant and preserving the color of the two other vertices of κ, i.e., $\texttt{col}(\boldsymbol{T}_\kappa(\boldsymbol{a}_\kappa^n)) = n$ for each $n \in \{1, 2\}$. Consider in the cell K_i a function v defined from the global continuous minimizer v^* in K_i and from its difference with the piecewise polynomial ζ_p on the previously enumerated elements by

$$v := v^*|_{K_i} + \sum_{\kappa \in \mathcal{T}_e' \setminus \{K_i\}} \frac{\epsilon_\kappa}{\epsilon_{K_i}} (v^* - \zeta_p)|_\kappa \circ \boldsymbol{T}_\kappa^{-1}, \tag{5.11}$$

where $\epsilon_\kappa = 1$ if the orientation of the vector $\boldsymbol{a}_\kappa^1 \boldsymbol{a}_\kappa^2$ is compatible with the fixed orientation of rotation around the edge e and $\epsilon_\kappa = -1$ otherwise. Note that the binary coloring implies that $\epsilon_\kappa + \epsilon_{\kappa'} = 0$ if the two cells κ and κ' share an interior face. Let us verify that $v \in V(K_i)$. The properties $v \in H^1(K_i)$ and $v|_{F_i^{\mathrm{b}}} = 0$ are again straightforward to verify. It remains to show that $v|_F = -r_F + \zeta_p^j|_F$ for all $F \in \mathcal{F}_i^\sharp$. We will do the proof for the face F^1; the proof for the face F^2 is similar. Let $\boldsymbol{x} \in F^1$, and assume without loss of generality that the vertex of F^1 that is not in e has color 1. Let $\mathcal{F}_e^1$ collect all the interior faces in $\mathcal{T}_e'$ whose vertex that is not in e has color 1. Then, $F^1 \in \mathcal{F}_e^1$ and $F^2 \notin \mathcal{F}_e^1$. Moreover, the definition of $\boldsymbol{T}_\kappa$ implies that, for all $\kappa \in \mathcal{T}_e'$, $\kappa \neq K_i$, the point $\boldsymbol{T}_\kappa^{-1}(\boldsymbol{x})$ belongs to the face of κ in the set $\mathcal{F}_e^1$. Recall that ϵ_κ has opposite sign on the two cells sharing each interior face. As a result, we find that the function v in (5.11) satisfies

$$v|_{F^1} = v^*|_{K_i}|_{F^1} - (v^*|_{K_{j_1}} - \zeta_p^{j_1})|_{F^1} \circ \boldsymbol{T}_{K_{j_1}}^{-1} + \sum_{F \in \mathcal{F}_e^1 \setminus \{F^1\}} \epsilon_F ([\![v^*]\!]_F - [\![\zeta_p]\!]_F)|_F \circ \boldsymbol{T}_{\kappa_F}^{-1}, \tag{5.12}$$

where K_{j_1} is the cell sharing F^1 with K_i, i.e., $F^1 = \partial K_i \cap \partial K_{j_1}$, whereas $\epsilon_F = \epsilon_{\kappa_F}/\epsilon_{K_i} = \pm 1$ where κ_F is the element sharing F having the lowest enumeration index. Since $[\![v^*]\!]_F = [\![\zeta_p]\!]_F$ for all $F \in \mathcal{F}_e^1$ (the common value being r_F if F is already a face in $\mathcal{T}_e$ or zero if F is a newly created face in $\mathcal{T}_e'$) and since $j_1 < i$, this yields

$$v|_{F^1} = v^*|_{K_i}|_{F^1} - (v^*|_{K_{j_1}} - \zeta_p^{j_1})|_{F^1} = -r_{F^1}|_{F^1} + \zeta_p^{j_1}|_{F^1}.$$

Hence, $v \in V(K_i)$. In view of (5.11) and (5.8), we conclude that

$$\|\nabla \zeta^i\|_{K_i} \le \|\nabla v\|_{K_i} \lesssim \|\nabla v^*\|_{K_i} + \sum_{\kappa \in \mathcal{T}_e' \setminus \{K_i\}} \{\|\nabla v^*\|_\kappa + \|\nabla \zeta_p\|_\kappa\} \le |\mathcal{T}_e'|^{1/2} \|\nabla_{\mathcal{T}} v^*\|_{\omega_{\boldsymbol{a}}} + 2^{1/2} \sum_{j<i} \|\nabla \zeta_p\|_{K_j};$$

observe that the partition $\mathcal{T}_e'$ may add one element with respect to $\mathcal{T}_e$.

(3) Case $i = |\mathcal{T}_{\boldsymbol{a}}|$. Note that owing to Lemma B.1(ii), this is the only case where $|\mathcal{F}_i^\sharp| = 3$ happens, so that we need to work with all the three interior faces of the element K_i. For this purpose, we apply

Lemma B.3 with $K_* = K_i$. Recall that $\mathcal{T}'_{\boldsymbol{a}}$ contains K_i, that all the tetrahedra in $\mathcal{T}'_{\boldsymbol{a}}$ have $\boldsymbol{a}$ as vertex with their three other vertices lying on $\partial\omega_{\boldsymbol{a}}$, and that, collecting all the vertices of $\mathcal{T}'_{\boldsymbol{a}}$ that lie on $\partial\omega_{\boldsymbol{a}}$ in the set $\mathcal{V}'_{\boldsymbol{a}}$, there is a three-color map $\texttt{col} : \mathcal{V}'_{\boldsymbol{a}} \to \{1,2,3\}$ so that for all $\kappa \in \mathcal{T}'_{\boldsymbol{a}}$, the three vertices of κ that are not $\boldsymbol{a}$, say $\{\boldsymbol{a}^n_\kappa\}_{1\le n\le 3}$, satisfy $\texttt{col}(\boldsymbol{a}^n_\kappa) = n$. We use the three-color map to define, for all $\kappa \in \mathcal{T}'_{\boldsymbol{a}}$, the (unique) bijective affine map $\boldsymbol{T}_\kappa : \kappa \to K_i$ leaving $\boldsymbol{a}$ invariant and preserving the color of the three other vertices of κ. Consider in the cell K_i the function

$$v := v^*|_{K_i} + \sum_{\kappa\in\mathcal{T}'_{\boldsymbol{a}}\setminus\{K_i\}} \frac{\epsilon_\kappa}{\epsilon_{K_i}}(v^* - \zeta_p)|_\kappa \circ \boldsymbol{T}_\kappa^{-1}, \tag{5.13}$$

where, for all $\kappa \in \mathcal{T}'_{\boldsymbol{a}}$, $\epsilon_\kappa = 1$ if the vector $\boldsymbol{a}^1_\kappa\boldsymbol{a}^2_\kappa \times \boldsymbol{a}^1_\kappa\boldsymbol{a}^3_\kappa$ points outward $\omega_{\boldsymbol{a}}$ and $\epsilon_\kappa = -1$ otherwise. Let us verify that $v \in V(K_i)$. The properties $v \in H^1(K_i)$ and $v|_{F^{\mathrm{b}}_{|\mathcal{T}_{\boldsymbol{a}}|}} = 0$ are straightforward. It remains to verify that v satisfies the appropriate boundary condition on the three faces in $F^\sharp_i$, i.e., on the three faces of K_i sharing the vertex $\boldsymbol{a}$. We can call these faces F^{12}, F^{13}, and F^{23}, where the superscripts refer to the two colors of the two vertices of the face that are not $\boldsymbol{a}$. Let us verify the boundary condition on F^{12}; the proof for the two other faces is similar. Let $\mathcal{F}^{12}$ collect all the interior faces in $\mathcal{T}'_{\boldsymbol{a}}$ such that their two vertices which are not $\boldsymbol{a}$ have colors 1 and 2. Since any interior face in $\mathcal{F}^{12}$ is shared by two cells in $\mathcal{T}'_{\boldsymbol{a}}$ having opposite number ϵ_κ and since any cell in $\mathcal{T}'_{\boldsymbol{a}}$ has one interior face in $\mathcal{F}^{12}$, we infer that

$$v|_{F^{12}} = v^*|_{K_i} - (v^*|_{K_{j_{12}}} - \zeta_p^{j_{12}})|_{F^{12}} \circ \boldsymbol{T}^{-1}_{K_{j_{12}}} + \sum_{F\in\mathcal{F}^{12}\setminus\{F^{12}\}} \epsilon_F([\![v^*]\!]_F - [\![\zeta_p]\!]_F)|_F \circ \boldsymbol{T}^{-1}_{\kappa_F},$$

with ϵ_F and κ_F defined as above and where j_{12} is the index of the cell sharing F^{12} with K_i. Since $[\![v^*]\!]_F = [\![\zeta_p]\!]_F$ (the common value being either r_F or 0) and since $j_{12} < i$, we conclude that $v|_{F^{12}} = -r_{F^{12}}|_{F^{12}} + \zeta_p^{j_{12}}|_{F^{12}}$. Hence, $v \in V(K_i)$. Finally, we can bound $\|\nabla v\|_{K_i}$ as above, and this completes the proof. $\square$

6 Proof for broken $\boldsymbol{H}(\mathrm{div})$ polynomial extensions

We prove here Theorem 2.3. In particular, we show in Section 6.1 the existence of the minimizers in (2.10), in Section 6.2 their uniqueness, and in Section 6.3 the stability bound (2.10). Let $p \ge 0$. Let $r \in \mathbb{P}_p(\mathcal{T}_{\boldsymbol{a}}) \times \mathbb{P}_p(\mathcal{F}_{\boldsymbol{a}})$ satisfy the compatibility condition (2.9). Let us set

$$\begin{aligned} \boldsymbol{V}_p(\mathcal{T}_{\boldsymbol{a}}) &:= \{\boldsymbol{v}_p \in \boldsymbol{RTN}_p(\mathcal{T}_{\boldsymbol{a}});\ \boldsymbol{v}_p{\cdot}\boldsymbol{n}_F = r_F\ \forall F \in \mathcal{F}^{\mathrm{b}}_{\boldsymbol{a}},\ \nabla_{\mathcal{T}}{\cdot}\boldsymbol{v}_p|_K = r_K\ \forall K \in \mathcal{T}_{\boldsymbol{a}}, \\ &\qquad [\![\boldsymbol{v}_p]\!]{\cdot}\boldsymbol{n}_F = r_F\ \forall F \in \mathcal{F}^{\mathrm{i}}_{\boldsymbol{a}}\}, \qquad (6.1\mathrm{a}) \\ \boldsymbol{V}(\mathcal{T}_{\boldsymbol{a}}) &:= \{\boldsymbol{v} \in \boldsymbol{H}(\mathrm{div}, \mathcal{T}_{\boldsymbol{a}});\ \boldsymbol{v}{\cdot}\boldsymbol{n}_F = r_F\ \forall F \in \mathcal{F}^{\mathrm{b}}_{\boldsymbol{a}},\ \nabla_{\mathcal{T}}{\cdot}\boldsymbol{v}|_K = r_K\ \forall K \in \mathcal{T}_{\boldsymbol{a}}, \\ &\qquad [\![\boldsymbol{v}]\!]{\cdot}\boldsymbol{n}_F = r_F\ \forall F \in \mathcal{F}^{\mathrm{i}}_{\boldsymbol{a}}\}. \qquad (6.1\mathrm{b}) \end{aligned}$$

Then the stability bound (2.10) becomes

$$\min_{\boldsymbol{v}_p\in\boldsymbol{V}_p(\mathcal{T}_{\boldsymbol{a}})} \|\boldsymbol{v}_p\|_{\omega_{\boldsymbol{a}}} \le C_{\mathrm{st}} \min_{\boldsymbol{v}\in\boldsymbol{V}(\mathcal{T}_{\boldsymbol{a}})} \|\boldsymbol{v}\|_{\omega_{\boldsymbol{a}}}. \tag{6.2}$$

As in Section 5, we consider the enumeration of the cells in $\mathcal{T}_{\boldsymbol{a}}$ from Lemma B.1 in the form $K_1, \dots, K_{|\mathcal{T}_{\boldsymbol{a}}|}$. Without loss of generality (see Remark 2.1), we orient all the interior faces $F = \partial K_i \cap \partial K_j \in \mathcal{F}^{\mathrm{i}}_{\boldsymbol{a}}$ so that $\boldsymbol{n}_F$ points from K_j to K_i with $j < i$.

6.1 Existence of the minimizers

Let us first prove that the minimization sets $\boldsymbol{V}_p(\mathcal{T}_{\boldsymbol{a}})$ and $\boldsymbol{V}(\mathcal{T}_{\boldsymbol{a}})$ are non-empty, yielding the existence of the minimizers. Since $\boldsymbol{V}_p(\mathcal{T}_{\boldsymbol{a}}) \subset \boldsymbol{V}(\mathcal{T}_{\boldsymbol{a}})$, we only consider $\boldsymbol{V}_p(\mathcal{T}_{\boldsymbol{a}})$. Recall that, for all $1 \le i \le |\mathcal{T}_{\boldsymbol{a}}|$, $F^{\mathrm{b}}_i = \partial K_i \cap \partial\omega_{\boldsymbol{a}}$ is the face of the element K_i lying on the patch boundary $\partial\omega_{\boldsymbol{a}}$. Consider a function $\boldsymbol{w}_p \in \boldsymbol{RTN}_p(\mathcal{T}_{\boldsymbol{a}})$ such that its restrictions $\boldsymbol{w}^i_p = \boldsymbol{w}_p|_{K_i}$, for all $1 \le i \le |\mathcal{T}_{\boldsymbol{a}}|$, are defined by induction as follows:

(i) For $i = 1$, $\boldsymbol{w}_p^1$ is any function in

$$\boldsymbol{V}_p(K_1) := \{\boldsymbol{v}_p \in \boldsymbol{RTN}_p(K_1);\ \boldsymbol{v}_p{\cdot}\boldsymbol{n}_{F_1^{\mathrm{b}}} = r_{F_1^{\mathrm{b}}},\ \nabla{\cdot}\boldsymbol{v}_p = r_{K_1}\}. \tag{6.3a}$$

(ii) For all $1 < i \le |\mathcal{T}_{\boldsymbol{a}}|$, $\boldsymbol{w}_p^i$ is any function in

$$\boldsymbol{V}_p(K_i) := \{\boldsymbol{v}_p \in \boldsymbol{RTN}_p(K_i);\ \boldsymbol{v}_p{\cdot}\boldsymbol{n}_{F_i^{\mathrm{b}}} = r_{F_i^{\mathrm{b}}},\ \nabla{\cdot}\boldsymbol{v}_p = r_{K_i},\ \boldsymbol{v}_p{\cdot}\boldsymbol{n}_F = -r_F + \boldsymbol{w}_p^j{\cdot}\boldsymbol{n}_F\ \forall F \in \mathcal{F}_i^\sharp\}, \tag{6.3b}$$

where $j = j(i, F)$ is the index of the cell sharing F with K_i, i.e., $F = \partial K_i \cap \partial K_j$. Recall that by definition of $\mathcal{F}_i^\sharp$, we have $j < i$, so that $\boldsymbol{w}_p^j$ is already known from a previous step.

Lemma 6.1 below shows that the (affine) subspaces $\boldsymbol{V}_p(K_i)$ are all non-empty, i.e., the above construction is meaningful. Then, it is easy to see that any function $\boldsymbol{w}_p$ constructed as above is in the discrete minimization set $\boldsymbol{V}_p(\mathcal{T}_{\boldsymbol{a}})$; in particular, we note that the prescription (6.3b) on the faces in $\mathcal{F}_i^\sharp$ implies that $[\![\boldsymbol{w}_p]\!]{\cdot}\boldsymbol{n}_F = (\boldsymbol{w}_p^j - \boldsymbol{w}_p^i){\cdot}\boldsymbol{n}_F = r_F$.

Lemma 6.1 (Non-emptiness)**.** *For all $1 \le i \le |\mathcal{T}_{\boldsymbol{a}}|$, $\boldsymbol{V}_p(K_i)$ is non-empty.*

Proof. The proof is actually simpler than that of Lemma 5.1 in the H^1-setting in Section 5.1, as Neumann boundary data in (6.3) do not request any condition of continuity on edges between faces of each K_i. Congruently, property (i) of Lemma B.1 is not used here. The only non-trivial property to verify is the compatibility between the prescriptions of the normal component and the divergence whenever $\mathcal{F}_i^\sharp \cup \{F_i^{\mathrm{b}}\} = \mathcal{F}_{K_i}$, i.e., whenever the normal component is prescribed over the whole boundary of K_i. Here, it is important that this situation only happens when $i = |\mathcal{T}_{\boldsymbol{a}}|$, i.e., for the last cell in the enumeration; this is indeed the case owing to Lemma B.1(ii). Then the non-emptiness of $\boldsymbol{V}_p(K_i)$ follows from classical properties of Raviart–Thomas–Nédélec finite elements. Let thus $i = |\mathcal{T}_{\boldsymbol{a}}|$. We need to check the Neumann compatibility condition

$$(r_{K_i}, 1)_{K_i} = (r_{F_i^{\mathrm{b}}}, 1)_{F_i^{\mathrm{b}}} + \sum_{F \in \mathcal{F}_i^\sharp} (r_F - \boldsymbol{w}_p^j{\cdot}\boldsymbol{n}_F, 1)_F. \tag{6.4}$$

(Note that $\boldsymbol{n}_F$ points inward K_i for all $F \in \mathcal{F}_i^\sharp$ since $i = |\mathcal{T}_{\boldsymbol{a}}|$.) Using the divergence theorem in each cell K_j, $1 \le j < |\mathcal{T}_{\boldsymbol{a}}|$, we write

$$\begin{aligned}
\sum_{1 \le j < |\mathcal{T}_{\boldsymbol{a}}|} (r_{K_j}, 1)_{K_j} &= \sum_{1 \le j < |\mathcal{T}_{\boldsymbol{a}}|} (\nabla{\cdot}\boldsymbol{w}_p^j, 1)_{K_j} = \sum_{1 \le j < |\mathcal{T}_{\boldsymbol{a}}|} \sum_{F \in \mathcal{F}_{K_j}} (\boldsymbol{w}_p^j{\cdot}\boldsymbol{n}_{K_j}, 1)_F \\
&= \sum_{1 \le j < |\mathcal{T}_{\boldsymbol{a}}|} (r_{F_j^{\mathrm{b}}}, 1)_{F_j^{\mathrm{b}}} + \sum_{F \in \mathcal{F}_{\boldsymbol{a}}^{\mathrm{i}} \setminus \mathcal{F}_i^\sharp} ([\![\boldsymbol{w}_p]\!]{\cdot}\boldsymbol{n}_F, 1)_F + \sum_{F \in \mathcal{F}_i^\sharp} (\boldsymbol{w}_p^j{\cdot}\boldsymbol{n}_F, 1)_F \\
&= \sum_{1 \le j < |\mathcal{T}_{\boldsymbol{a}}|} (r_{F_j^{\mathrm{b}}}, 1)_{F_j^{\mathrm{b}}} + \sum_{F \in \mathcal{F}_{\boldsymbol{a}}^{\mathrm{i}} \setminus \mathcal{F}_i^\sharp} (r_F, 1)_F + \sum_{F \in \mathcal{F}_i^\sharp} (\boldsymbol{w}_p^j{\cdot}\boldsymbol{n}_F, 1)_F,
\end{aligned}$$

where $\boldsymbol{n}_{K_j}$ is the unit normal pointing outward K_j. Then (6.4) follows by combining the above relation with the compatibility condition (2.9). $\square$

6.2 Uniqueness of the minimizers

As the affine subspaces $\boldsymbol{V}_p(\mathcal{T}_{\boldsymbol{a}})$ and $\boldsymbol{V}(\mathcal{T}_{\boldsymbol{a}})$ are non-empty convex sets, the uniqueness of the minimizers in (2.10) follows from the fact that the functional we are minimizing is strictly convex on the tangent spaces (both tangent spaces are composed of divergence-free functions, so that the $\|{\cdot}\|_{\omega_{\boldsymbol{a}}} = \|{\cdot}\|_{\boldsymbol{H}(\mathrm{div},\mathcal{T}_{\boldsymbol{a}})}$ for such functions).

6.3 Proof of the stability bound (2.10)

We now construct two functions $\boldsymbol{\zeta}_p \in \boldsymbol{RTN}_p(\mathcal{T}_{\boldsymbol{a}})$ and $\boldsymbol{\zeta} \in \boldsymbol{H}(\mathrm{div}, \mathcal{T}_{\boldsymbol{a}})$ such that their restrictions $\boldsymbol{\zeta}_p^i = \boldsymbol{\zeta}_p|_{K_i}$ and $\boldsymbol{\zeta}^i = \boldsymbol{\zeta}|_{K_i}$, for all $1 \le i \le |\mathcal{T}_{\boldsymbol{a}}|$, are defined by induction as follows:

(i) For $i = 1$, we define the spaces

$$\boldsymbol{V}_p(K_1) := \{\boldsymbol{v}_p \in \boldsymbol{RTN}_p(K_1);\ \boldsymbol{v}_p \cdot \boldsymbol{n}_{F_1^{\mathrm{b}}} = r_{F_1^{\mathrm{b}}},\ \nabla \cdot \boldsymbol{v}_p = r_{K_1}\}, \tag{6.5a}$$
$$\boldsymbol{V}(K_1) := \{\boldsymbol{v} \in \boldsymbol{H}(\mathrm{div}, K_1);\ \boldsymbol{v} \cdot \boldsymbol{n}_{F_1^{\mathrm{b}}} = r_{F_1^{\mathrm{b}}},\ \nabla \cdot \boldsymbol{v} = r_{K_1}\}, \tag{6.5b}$$

and consider the following unique minimizers:

$$\boldsymbol{\zeta}_p^1 := \underset{\boldsymbol{v}_p \in \boldsymbol{V}_p(K_1)}{\arg\min} \|\boldsymbol{v}_p\|_{K_1}, \qquad \boldsymbol{\zeta}^1 := \underset{\boldsymbol{v} \in \boldsymbol{V}(K_1)}{\arg\min} \|\boldsymbol{v}\|_{K_1}. \tag{6.6}$$

(ii) For all $1 < i \le |\mathcal{T}_{\boldsymbol{a}}|$, we define the spaces

$$\boldsymbol{V}_p(K_i) := \{\boldsymbol{v}_p \in \boldsymbol{RTN}_p(K_i);\ \boldsymbol{v}_p \cdot \boldsymbol{n}_{F_i^{\mathrm{b}}} = r_{F_i^{\mathrm{b}}},\ \nabla \cdot \boldsymbol{v}_p = r_{K_i},\ \boldsymbol{v}_p \cdot \boldsymbol{n}_F = -r_F + \boldsymbol{\zeta}_p^j \cdot \boldsymbol{n}_F\ \forall F \in \mathcal{F}_i^\sharp\}, \tag{6.7a}$$
$$\boldsymbol{V}(K_i) := \{\boldsymbol{v} \in \boldsymbol{H}(\mathrm{div}, K_i);\ \boldsymbol{v} \cdot \boldsymbol{n}_{F_i^{\mathrm{b}}} = r_{F_i^{\mathrm{b}}},\ \nabla \cdot \boldsymbol{v} = r_{K_i},\ \boldsymbol{v} \cdot \boldsymbol{n}_F = -r_F + \boldsymbol{\zeta}_p^j \cdot \boldsymbol{n}_F\ \forall F \in \mathcal{F}_i^\sharp\}, \tag{6.7b}$$

where $j = j(i, F)$ is the index of the cell sharing F with K_i, i.e., $F = \partial K_i \cap \partial K_j$. In (6.7b), the normal trace is prescribed according to (2.5). Consider the following unique minimizers:

$$\boldsymbol{\zeta}_p^i := \underset{\boldsymbol{v}_p \in \boldsymbol{V}_p(K_i)}{\arg\min} \|\boldsymbol{v}_p\|_{K_i}, \qquad \boldsymbol{\zeta}^i := \underset{\boldsymbol{v} \in \boldsymbol{V}(K_i)}{\arg\min} \|\boldsymbol{v}\|_{K_i}. \tag{6.8}$$

The same reasoning as in Sections 6.1 and 6.2 shows that these minimization problems are well-posed. We can also observe that the continuous minimizer $\boldsymbol{\zeta}^i$ of (6.8) is given by $-\nabla \zeta^i$, where $\zeta^i \in H^1(K_i)$ is the weak solution of the problem

$$\begin{aligned} -\Delta \zeta^i &= r_{K_i} && \text{in } K_i,\\ -\nabla \zeta^i \cdot \boldsymbol{n}_F &= -r_F + \boldsymbol{\zeta}_p^j \cdot \boldsymbol{n}_F && \text{on all } F \in \mathcal{F}_i^\sharp,\\ -\nabla \zeta^i \cdot \boldsymbol{n}_{F_i^{\mathrm{b}}} &= r_{F_i^{\mathrm{b}}} && \text{on } F_i^{\mathrm{b}},\\ \zeta^i|_F &= 0 && \text{on all } F \in \mathcal{F}_i^\flat, \end{aligned}$$

whereas $\boldsymbol{\zeta}_p^i$ is its (spectral) mixed finite element approximation.

We will show the following two statements for all $1 \le i \le |\mathcal{T}_{\boldsymbol{a}}|$:

$$\|\boldsymbol{\zeta}_p^i\|_{K_i} \lesssim \|\boldsymbol{\zeta}^i\|_{K_i}, \tag{6.9a}$$
$$\|\boldsymbol{\zeta}^i\|_{K_i} \lesssim \|\boldsymbol{v}^*\|_{\omega_{\boldsymbol{a}}} + \sum_{j<i} \|\boldsymbol{\zeta}_p^j\|_{K_j}, \tag{6.9b}$$

where the sum in (6.9b) is void for $i = 1$ and where $\boldsymbol{v}^* \in \boldsymbol{V}(\mathcal{T}_{\boldsymbol{a}})$ is the global minimizer in (2.10). With these two bounds, we can conclude the proof as in Section 5.3.

Proof of (6.9a). We apply Lemma A.3 on $K = K_i$ with $\mathcal{F}_K^{\mathrm{N}} = \{F_i^{\mathrm{b}}\} \cup \mathcal{F}_i^\sharp$, the p-degree polynomial prescribing the divergence being r_{K_i}, and the p-degree polynomials prescribing the normal components being $r_{F_i^{\mathrm{b}}}$ for $F = F_i^{\mathrm{b}}$ and $r_F - \boldsymbol{\zeta}_p^j \cdot \boldsymbol{n}_F$ for all $F \in \mathcal{F}_i^\sharp$ (recall that $\boldsymbol{n}_F$ points inward K_i since $F = \partial K_j \cap \partial K_i$ with $j < i$). The compatibility condition for these polynomials on the last element follows by the same reasoning as in Section 6.1. □

Proof of (6.9b). The principle of the proof is the same as that of (5.10b) in Section 5.3, the only salient difference being that the pullback by the geometric map has to be replaced by the contravariant Piola transformation. Let us exemplify this modification in the case where $1 < i < |\mathcal{T}_{\boldsymbol{a}}|$ and $|\mathcal{F}_i^\sharp| = 1$, say $\mathcal{F}_i^\sharp = \{F\}$. Let $K_j \in \mathcal{T}_{\boldsymbol{a}}$ be the cell such that $F = \partial K_i \cap \partial K_j$; the definition of $\mathcal{F}_i^\sharp$ implies that $j < i$. Let $\boldsymbol{T} : K_j \to K_i$ be the (unique) bijective affine map leaving F pointwise invariant. Let $\boldsymbol{J}$ be the (constant) Jacobian matrix of $\boldsymbol{T}$ and consider the transformation $\boldsymbol{\psi}(\boldsymbol{v}) = \boldsymbol{A}(\boldsymbol{v} \circ \boldsymbol{T})$ with $\boldsymbol{A} = \det(\boldsymbol{J})\boldsymbol{J}^{-1}$. Then $\boldsymbol{\psi}$ is an isomorphism from $\boldsymbol{H}(\mathrm{div}, K_i)$ to $\boldsymbol{H}(\mathrm{div}, K_j)$, and also from $\boldsymbol{RTN}_p(K_i)$ to $\boldsymbol{RTN}_p(K_j)$. Consider in the cell K_i the function

$$\boldsymbol{v} := \boldsymbol{v}^*|_{K_i} - \boldsymbol{\psi}^{-1}(\boldsymbol{v}^*|_{K_j} - \boldsymbol{\zeta}_p^j), \tag{6.10}$$

and let us prove that $\boldsymbol{v} \in \boldsymbol{V}(K_i)$. It is clear that $\boldsymbol{v} \in \boldsymbol{H}(\mathrm{div}, K_i)$. Concerning the divergence, we use the property $\nabla{\cdot}\boldsymbol{\psi}^{-1}(\boldsymbol{v}_j) = \det(\boldsymbol{J})^{-1}(\nabla{\cdot}\boldsymbol{v}_j)\circ\boldsymbol{T}^{-1}$ in K_i for any function $\boldsymbol{v}_j$ defined in K_j. Applying this identity to the function $\boldsymbol{v}_j = \boldsymbol{v}^*|_{K_j} - \boldsymbol{\zeta}_p^j$ which is divergence-free (since $j < i$), we infer that $\nabla{\cdot}\boldsymbol{v} = \nabla{\cdot}\boldsymbol{v}^*|_{K_i} = r_{K_i}$. Let us now consider the normal component of $\boldsymbol{v}$. Recalling (2.5) and that $\boldsymbol{n}_F$ points inward K_i, we need to prove that

$$(\boldsymbol{v}, \nabla\phi)_{K_i} = -(r_{K_i}, \phi)_{K_i} + (r_{F_i^{\mathrm{b}}}, \phi)_{F_i^{\mathrm{b}}} + (r_F - \boldsymbol{\zeta}_p^j{\cdot}\boldsymbol{n}_F, \phi)_F, \tag{6.11}$$

for all $\phi \in H^1(K_i)$ such that $\phi|_F = 0$ for all $F \in \mathcal{F}_i^\flat$. Let $\widetilde{\phi} : \omega_{\boldsymbol{a}} \to \mathbb{R}$ be such that $\widetilde{\phi}|_{K_i} = \phi$, $\widetilde{\phi}|_{K_j} = \phi \circ \boldsymbol{T}$, and $\widetilde{\phi} = 0$ otherwise, and observe that $\widetilde{\phi} \in H^1(\omega_{\boldsymbol{a}})$. Using $\widetilde{\phi}$ as the test function in (2.4c) for the global minimizer $\boldsymbol{v}^*$, we infer that

$$(\boldsymbol{v}^*, \nabla\phi)_{K_i} + (\boldsymbol{v}^*, \nabla(\phi\circ\boldsymbol{T}))_{K_j} = -(r_{K_i}, \phi)_{K_i} - (r_{K_j}, \phi\circ\boldsymbol{T})_{K_j} + (r_{F_i^{\mathrm{b}}}, \phi)_{F_i^{\mathrm{b}}} + (r_{F_j^{\mathrm{b}}}, \phi\circ\boldsymbol{T})_{F_j^{\mathrm{b}}} + (r_F, \phi)_F.$$

Considering the term $(\boldsymbol{v}, \nabla\phi)_{K_i}$, the definition (6.10), changing variables in the last term in the right-hand side, and employing $\epsilon_{\boldsymbol{J}} = \frac{\det(\boldsymbol{J})}{|\det(\boldsymbol{J})|} = -1$, we obtain

$$\begin{aligned}
(\boldsymbol{v}, \nabla\phi)_{K_i} &= (\boldsymbol{v}^*, \nabla\phi)_{K_i} - (\boldsymbol{\psi}^{-1}(\boldsymbol{v}^*|_{K_j} - \boldsymbol{\zeta}_p^j), \nabla\phi)_{K_i} \\
&= (\boldsymbol{v}^*, \nabla\phi)_{K_i} + (\boldsymbol{v}^*, \nabla(\phi\circ\boldsymbol{T}))_{K_j} - (\boldsymbol{\zeta}_p^j, \nabla(\phi\circ\boldsymbol{T}))_{K_j} \\
&= (\boldsymbol{v}^*, \nabla\phi)_{K_i} + (\boldsymbol{v}^*, \nabla(\phi\circ\boldsymbol{T}))_{K_j} + (r_{K_j}, \phi\circ\boldsymbol{T})_{K_j} - (r_{F_j^{\mathrm{b}}}, \phi\circ\boldsymbol{T})_{F_j^{\mathrm{b}}} - (\boldsymbol{\zeta}_p^j{\cdot}\boldsymbol{n}_F, \phi)_F \\
&= -(r_{K_i}, \phi)_{K_i} + (r_{F_i^{\mathrm{b}}}, \phi)_{F_i^{\mathrm{b}}} + (r_F - \boldsymbol{\zeta}_p^j{\cdot}\boldsymbol{n}_F, \phi)_F,
\end{aligned}$$

where we have used the Green theorem for the term $(\boldsymbol{\zeta}_p^j, \nabla(\phi\circ\boldsymbol{T}))_{K_j}$ and used the prescribed properties of $\boldsymbol{\zeta}_p^j$ together with the above relation satisfied by $\boldsymbol{v}^*$. Hence, (6.11) holds, and $\boldsymbol{v} \in \boldsymbol{V}(K_i)$ as announced.

The reasoning is similar when $\mathcal{F}_i^\sharp$ contains two or three interior faces of K_i. Let us still briefly discuss the case where $1 < i < |\mathcal{T}_{\boldsymbol{a}}|$ and $|\mathcal{F}_i^\sharp| = 2$, say $\mathcal{F}_i^\sharp = \{F^1, F^2\}$ with $e = F^1 \cap F^2$. We consider the two-color conforming refinement $\mathcal{T}_e'$ of the rotational path $\mathcal{T}_e$ around e of Lemma B.2 below. Define in the cell K_i the function

$$\boldsymbol{v} := \boldsymbol{v}^*|_{K_i} + \sum_{\kappa\in\mathcal{T}_e'\setminus\{K_i\}} \frac{\epsilon_\kappa}{\epsilon_{K_i}} \boldsymbol{\psi}_\kappa^{-1}((\boldsymbol{v}^* - \boldsymbol{\zeta}_p)|_\kappa), \tag{6.12}$$

where we use the same notation as in Section 5.3, together with the contravariant Piola transformation $\boldsymbol{\psi}_\kappa$ built using the geometric map $\boldsymbol{T}_\kappa$ with Jacobian matrix $\boldsymbol{J}_\kappa$. We need to show that $\boldsymbol{v} \in \boldsymbol{V}(K_i)$. It is again clear that $\boldsymbol{v} \in \boldsymbol{H}(\mathrm{div}, K_i)$ and similarly, remarking that $(\boldsymbol{v}^* - \boldsymbol{\zeta}_p)|_\kappa$ for all $\kappa \in \mathcal{T}_e' \setminus \{K_i\}$ are divergence-free, we infer that $\nabla{\cdot}\boldsymbol{v} = \nabla{\cdot}\boldsymbol{v}^*|_{K_i} = r_{K_i}$. We are thus left to prove that the normal components conditions in (6.7b) are satisfied. Let $F^1 = \partial K_i \cap \partial K_{j_1}$ and $F^2 = \partial K_i \cap \partial K_{j_2}$, where we remark that $j_1, j_2 < i$. Using (2.5), we need to show that

$$(\boldsymbol{v}, \nabla\phi)_{K_i} = -(r_{K_i}, \phi)_{K_i} + (r_{F_i^{\mathrm{b}}}, \phi)_{F_i^{\mathrm{b}}} + (r_{F^1} - \boldsymbol{\zeta}_p^{j_1}{\cdot}\boldsymbol{n}_{F^1}, \phi)_{F^1} + (r_{F^2} - \boldsymbol{\zeta}_p^{j_2}{\cdot}\boldsymbol{n}_{F^2}, \phi)_{F^2} \tag{6.13}$$

for all $\phi \in H^1(K_i)$ such that $\phi|_F = 0$ for all $F \in \mathcal{F}_i^\flat$. Let now $\widetilde{\phi} : \omega_{\boldsymbol{a}} \to \mathbb{R}$ be such that $\widetilde{\phi}|_{K_i} = \phi$, $\widetilde{\phi}|_\kappa = \phi\circ\boldsymbol{T}_\kappa$, $\kappa \in \mathcal{T}_e' \setminus \{K_i\}$, and $\widetilde{\phi} = 0$ otherwise. Observe that $\widetilde{\phi}$ takes value zero on the faces lying on the boundary of the submesh $\mathcal{T}_e'$ and sharing the vertex $\boldsymbol{a}$ and is continuous over the faces in $\mathcal{T}_e'$ sharing the edge e. Consequently, $\widetilde{\phi} \in H^1(\omega_{\boldsymbol{a}})$. Using as above $\widetilde{\phi}$ as the test function in (2.4c), we see that

$$\begin{aligned}
(\boldsymbol{v}^*, \nabla\phi)_{K_i} + \sum_{\kappa\in\mathcal{T}_e'\setminus\{K_i\}} (\boldsymbol{v}^*, \nabla(\phi\circ\boldsymbol{T}_\kappa))_\kappa = &-(r_{K_i}, \phi)_{K_i} - \sum_{\kappa\in\mathcal{T}_e'\setminus\{K_i\}} (r_\kappa, \phi\circ\boldsymbol{T}_\kappa)_\kappa + (r_{F_i^{\mathrm{b}}}, \phi)_{F_i^{\mathrm{b}}} \\
&+ \sum_{\kappa\in\mathcal{T}_e'\setminus\{K_i\}} (r_{F_\kappa^{\mathrm{b}}}, \phi\circ\boldsymbol{T}_\kappa)_{F_\kappa^{\mathrm{b}}} + \sum_{F\in\mathcal{F}_e} (r_F, \phi\circ\boldsymbol{T}_{\kappa_F})_F,
\end{aligned} \tag{6.14}$$

with the obvious association of F_κ^{b} with F_j^{b} and where $\kappa_F \in \mathcal{T}_e'$ is the element sharing F having the lowest enumeration index. Employing the definition (6.12), changing variables, noting that $\epsilon_\kappa\epsilon_{\boldsymbol{J}_\kappa}$ is independent of the two cells sharing $F \in \mathcal{F}_e$ while $\boldsymbol{n}_\kappa$ changes orientation, and using that the jumps of $\boldsymbol{\zeta}_p$ on the faces from $\mathcal{F}_e$ other than F^1 and F^2 are given by r_F whereas $\boldsymbol{\zeta}_p^j$ have no normal component jumps inside of the original simplices K_j from $\mathcal{T}_e$, we deduce

$$
\begin{aligned}
(\boldsymbol{v}, \nabla\phi)_{K_i} &= (\boldsymbol{v}^*, \nabla\phi)_{K_i} - \sum_{\kappa\in\mathcal{T}_e'\setminus\{K_i\}} \Big(\frac{\epsilon_\kappa}{\epsilon_{K_i}} \boldsymbol{\psi}_\kappa^{-1}((\boldsymbol{v}^* - \boldsymbol{\zeta}_p)|_\kappa), \nabla\phi\Big)_{K_i} \\
&= (\boldsymbol{v}^*, \nabla\phi)_{K_i} + \sum_{\kappa\in\mathcal{T}_e'\setminus\{K_i\}} (\boldsymbol{v}^*, \nabla(\phi\circ\boldsymbol{T}_\kappa))_\kappa - \sum_{\kappa\in\mathcal{T}_e'\setminus\{K_i\}} (\boldsymbol{\zeta}_p, \nabla(\phi\circ\boldsymbol{T}_\kappa))_\kappa \\
&= (\boldsymbol{v}^*, \nabla\phi)_{K_i} + \sum_{\kappa\in\mathcal{T}_e'\setminus\{K_i\}} (\boldsymbol{v}^*, \nabla(\phi\circ\boldsymbol{T}_\kappa))_\kappa + \sum_{\kappa\in\mathcal{T}_e'\setminus\{K_i\}} (r_\kappa, \phi\circ\boldsymbol{T}_\kappa)_\kappa \\
&\quad - \sum_{\kappa\in\mathcal{T}_e'\setminus\{K_i\}} (r_{F_\kappa^{\mathrm{b}}}, \phi\circ\boldsymbol{T}_\kappa)_{F_\kappa^{\mathrm{b}}} - \sum_{F\in\mathcal{F}_e\setminus\{F^1,F^2\}} ([\![\boldsymbol{\zeta}_p]\!]{\cdot}\boldsymbol{n}_F, \phi\circ\boldsymbol{T}_{\kappa_F})_F \\
&\quad - (\boldsymbol{\zeta}_p^{j_1}{\cdot}\boldsymbol{n}_{F^1}, \phi\circ\boldsymbol{T}_{\kappa_{F^1}})_{F^1} - (\boldsymbol{\zeta}_p^{j_2}{\cdot}\boldsymbol{n}_{F^2}, \phi\circ\boldsymbol{T}_{\kappa_{F^2}})_{F^2} \\
&= -(r_{K_i}, \phi)_{K_i} + (r_{F_i^{\mathrm{b}}}, \phi)_{F_i^{\mathrm{b}}} + (r_{F^1} - \boldsymbol{\zeta}_p^{j_1}{\cdot}\boldsymbol{n}_{F^1}, \phi)_{F^1} + (r_{F^2} - \boldsymbol{\zeta}_p^{j_2}{\cdot}\boldsymbol{n}_{F^2}, \phi)_{F^2},
\end{aligned}
$$

where we have also employed that both $\boldsymbol{n}_{F^1}$ and $\boldsymbol{n}_{F^2}$ point inwards K_i, and, in the last step, (6.14). Thus (6.13) holds. □

7 Proofs for boundary vertices

In this section we prove Theorems 2.4 and 2.5. We only give details in the H^1 setting; the $\boldsymbol{H}(\mathrm{div})$ setting can be treated similarly. Let $\boldsymbol{a}$ be a boundary vertex as specified in Section 2.4.

7.1 Existence of the minimizers

Let $r \in \mathbb{P}_p(\mathcal{F}_{\boldsymbol{a}}^{\mathrm{i}} \cup \mathcal{F}_{\boldsymbol{a}}^{\mathrm{D}})$ satisfy the compatibility conditions (2.12). We denote respectively by $V_p(\mathcal{T}_{\boldsymbol{a}})$ and $V(\mathcal{T}_{\boldsymbol{a}})$ the corresponding minimization sets in (2.13) and proceed as in Section 5.1. Let $\mathcal{F}_i^{\mathrm{D}} := \mathcal{F}_{K_i} \cap \mathcal{F}_{\boldsymbol{a}}^{\mathrm{D}}$, $1 \le i \le |\mathcal{T}_{\boldsymbol{a}}|$, i.e., $\mathcal{F}_i^{\mathrm{D}}$ is the subset of faces of K_i lying in the (Dirichlet boundary) set $\mathcal{F}_{\boldsymbol{a}}^{\mathrm{D}}$. Consider a function $w_p \in \mathbb{P}_p(\mathcal{T}_{\boldsymbol{a}})$ such that its restrictions $w_p^i = w_p|_{K_i}$, are defined by induction as follows (we again orient all the interior faces $F = \partial K_i \cap \partial K_j \in \mathcal{F}_{\boldsymbol{a}}^{\mathrm{i}}$ so that $\boldsymbol{n}_F$ points from K_j to K_i with $j < i$):

(i) For $i = 1$, w_p^1 is any function in

$$V_p(K_1) := \{v_p \in \mathbb{P}_p(K_1);\ v_p|_{F_1^{\mathrm{b}}} = 0,\ v_p|_F = r_F\ \forall F \in \mathcal{F}_1^{\mathrm{D}}\}.$$

(ii) For all $1 < i \le |\mathcal{T}_{\boldsymbol{a}}|$, w_p^i is any function in

$$V_p(K_i) := \{v_p \in \mathbb{P}_p(K_i);\ v_p|_{F_i^{\mathrm{b}}} = 0,\ v_p|_F = r_F\ \forall F \in \mathcal{F}_i^{\mathrm{D}},\ v_p|_F = -r_F + w_p^j|_F\ \forall F \in \mathcal{F}_i^\sharp\}.$$

We now present the version of Lemma 5.1 for a boundary vertex $\boldsymbol{a}$, ensuring existence of such piecewise polynomial w_p. Here we crucially rely on the enumeration of boundary patches of Assumption C.1:

Lemma 7.1 (Non-emptiness)**.** *For all $1 \le i \le |\mathcal{T}_{\boldsymbol{a}}|$, $V_p(K_i)$ is non-empty.*

Proof. (1) First element, $i = 1$. The condition (2.12a) requests any possibly prescribed Dirichlet datum r_F on $F \in \mathcal{F}_1^{\mathrm{D}}$ to vanish on the edge(s) shared with the face F_1^{b}. Thus, if $\mathcal{F}_1^{\mathrm{D}} \neq \emptyset$, the boundary conditions prescribed in the space $V_p(K_1)$ are compatible (continuous).
(2) For $1 < i \le |\mathcal{T}_{\boldsymbol{a}}|$, we only need to verify that the values prescribed in the definition of the spaces $V_p(K_i)$ on the faces of the element K_i are continuous across any edge e they share. Note that there is always one face F_i^{b}, that $\mathcal{F}_i^{\mathrm{D}}$ can be empty, and that Assumption C.1(iii) gives that $\mathcal{F}_i^\sharp$ contains one or two faces when $1 < i < |\mathcal{T}_{\boldsymbol{a}}|$ and possibly three faces when $i = |\mathcal{T}_{\boldsymbol{a}}|$.
(2.a) "Boundary–interior" or "boundary–Dirichlet edges" e, i.e., edges between the face $F_i^{\mathrm{b}} \subset \partial\omega_{\boldsymbol{a}}^{\mathrm{b}}$ and a face from $\mathcal{F}_i^{\mathrm{D}}$ or $\mathcal{F}_i^\sharp$. The continuity is always ensured here, as all the conditions take the value 0 on $\partial\omega_{\boldsymbol{a}}^{\mathrm{b}}$. Indeed, by (2.12a), all the data r_F vanish on $\partial\omega_{\boldsymbol{a}}^{\mathrm{b}}$, and by induction, w_p^j prescribed in the previously enumerated elements also vanishes on $\partial\omega_{\boldsymbol{a}}^{\mathrm{b}}$.

(2.b) "Interior edges" e, i.e., edges between two faces in the set $\mathcal{F}_i^\sharp$; note that Assumption C.1(i) gives in this case $\mathcal{F}_e \cap \mathcal{F}_{\boldsymbol{a}}^{\mathrm{N}} \cap \mathcal{F}_{\boldsymbol{a}}^{\mathrm{D}} = \emptyset$. This setting is exactly the same as in Lemma 5.1, case (2). The compatibility condition (2.12b) is crucial.
(2.c) "Dirichlet edges" e, i.e., edges between two faces F^1 and F^2 in the set $\mathcal{F}_i^{\mathrm{D}}$. The compatibility condition (2.12b), noting that $\iota_{F^1,e} + \iota_{F^2,e} = 0$ as $\boldsymbol{n}_{F^1}$ and $\boldsymbol{n}_{F^2}$ are unit exterior normals of the same simplex K_i, immediately gives here $r_{F^1}|_e = r_{F^2}|_e$.
(2.d) "Interior–Dirichlet edges" e, i.e., $F^1 \in \mathcal{F}_i^\sharp$ and $F^2 \in \mathcal{F}_i^{\mathrm{D}}$, covering the remaining possibilities. Here we need to ensure

$$(-r_{F^1} + w_p^j|_{F^1})|_e = r_{F^2}|_e \qquad \forall F^1 \in \mathcal{F}_i^\sharp,\, \forall F^2 \in \mathcal{F}_i^{\mathrm{D}},\, e = F^1 \cap F^2.$$

Now we employ property (vi) of Assumption C.1, stating that in the previously enumerated elements, none has a Neumann face sharing the edge e. This yields that $\mathcal{F}_e \cap \mathcal{F}_{\boldsymbol{a}}^{\mathrm{N}} = \emptyset$ (note that $\mathcal{F}_e \cap \mathcal{F}_{\boldsymbol{a}}^{\mathrm{D}} \neq \emptyset$). Consequently, we can employ condition (2.12b) and prove the desired continuity as in Lemma 5.1, case (2). Properties (ii) and (vi) of Assumption C.1 are central here, showing that when taking the elements $K_{j'}$ from $\mathcal{T}_e$ sharing the edge e in decreasing order according to their number, 1) all the polynomials $w_p^{j'}$ are already defined; 2) on the last last element $K_{j''}$, we come to a Dirichlet face F, where the value of $w_p^{j''}$ has been fixed by the Dirichlet value r_F. $\square$

7.2 Uniqueness of the minimizers

The uniqueness of both minimizers in (2.13) follows as in Section 5.2.

7.3 Proof of the stability bound (2.13)

The proof of the stability bound (2.13) proceeds as in Section 5.3, with the definition of the spaces $V_p(K_i)$ from Section 7.1 and similarly for $V(K_i)$. In particular, in place of problem (5.9), $\zeta^i \in H^1(K_i)$ is the weak solution of the problem

$$\begin{aligned}
-\Delta\zeta^i &= 0 && \text{in } K_i,\\
\zeta^i|_F &= -r_F + \zeta_p^j|_F && \text{on all } F \in \mathcal{F}_i^\sharp,\\
\zeta^i|_F &= 0 && \text{on } F_i^{\mathrm{b}},\\
\zeta^i|_F &= r_F && \text{on all } F \in \mathcal{F}_i^{\mathrm{D}},\\
-\nabla\zeta^i{\cdot}\boldsymbol{n}_{K_i}|_F &= 0 && \text{on all } F \in \mathcal{F}_i^\flat,\\
-\nabla\zeta^i{\cdot}\boldsymbol{n}_{K_i}|_F &= 0 && \text{on all } F \in \mathcal{F}_{\boldsymbol{a}}^{\mathrm{N}} \cap \mathcal{F}_{K_i},
\end{aligned}$$

ζ_p^i is its (spectral) finite element approximation, and we need to show (5.10). Here (5.10a) again immediately follows from Lemma A.1, so that (5.10b) is crucial. Let $K_i \in \mathcal{T}_{\boldsymbol{a}}$, $1 < i \le |\mathcal{T}_{\boldsymbol{a}}|$, be a given simplex, let (5.10) hold on all $K_j \in \mathcal{T}_{\boldsymbol{a}}$, $j < i$, and consider the different cases that can appear taking into account that $\mathcal{T}_{\boldsymbol{a}}$ is a boundary patch and that there are possibly additional Dirichlet conditions on the faces in the set $\mathcal{F}_i^{\mathrm{D}}$:
(1) $|\mathcal{F}_i^\sharp| = 1$ and $\mathcal{F}_i^{\mathrm{D}} = \emptyset$. Here case (1) from Section 5.3 can be used.
(2) $|\mathcal{F}_i^\sharp| = 2$ and $\mathcal{F}_i^{\mathrm{D}} = \emptyset$. Immediately, the last face of the element K_i different from F_i^{b} is a Neumann face from $\mathcal{F}_i^\flat$ or from $\mathcal{F}_{\boldsymbol{a}}^{\mathrm{N}} \cap \mathcal{F}_{K_i}$, so there is no Dirichlet boundary condition to be ensured on this face in the space $V(K_i)$. Noting that Assumption C.1(i) implies that the edge e common to the two faces in $\mathcal{F}_i^\sharp$ lies in the interior of $\mathcal{T}_{\boldsymbol{a}}$, we can apply case (2) from Section 5.3.
(3) $|\mathcal{F}_i^\sharp| = 1$ and $|\mathcal{F}_i^{\mathrm{D}}| = 1$. Similarly to case (2) in Section 5.3, we need to employ a procedure involving all elements $K \in \mathcal{T}_e$ sharing the edge e common to the faces $F^1 \in \mathcal{F}_i^\sharp$ and $F^2 \in \mathcal{F}_i^{\mathrm{D}}$. Properties (ii) and (vi) of Assumption C.1 are central here; the latter namely states that there is no Neumann face from $\mathcal{F}_{\boldsymbol{a}}^{\mathrm{N}}$ in $\mathcal{T}_e$. Here, as $|\mathcal{F}_i^{\mathrm{D}}| = 1$, one can always color the vertices of elements in $\mathcal{T}_e$ not lying on e by two colors; thus no refinement $\mathcal{T}_e'$ of $\mathcal{T}_e$ is necessary. We now use (5.11) with $\mathcal{T}_e' = \mathcal{T}_e$ and notice that 1) $(v^*|_{K_i})|_{F^2} = r_{F^2}$ as $v^* \in V(\mathcal{T}_{\boldsymbol{a}})$; 2) the mapping from the neighbor (through the face F^1) K_j of K_i makes appear the requested boundary condition $-r_{F^1} + \zeta_p^j|_{F^1}$ on the face F^1; 3) concerning the other mappings on the faces F^1 and F^2, one maps a) either a difference of jumps, which is zero as in case (2) in Section 5.3; b) or the polynomial $(v^*|_{K_j} - \zeta_p^j)|_F$ for $j < i$ and $F \in \mathcal{F}_j^{\mathrm{D}}$, which is also zero as the Dirichlet boundary conditions on the face F in the spaces $V(\mathcal{T}_{\boldsymbol{a}})|_{K_j}$ and $V_p(K_j)$ are the same. Thus the function v defined in (5.11) belongs to the

space $V(K_i)$ and one shows (5.10) as in Section 5.3, case (2).
(4) $|\mathcal{F}_i^\sharp| = 1$ and $|\mathcal{F}_i^{\rm D}| = 2$. Note that $\mathcal{F}_i^\flat = \emptyset$, whence K_i is the last element in the patch $\mathcal{T}_{\boldsymbol{a}}$ by the property (iii) of Assumption C.1. Moreover, property (vi) allows to conclude that there is no Neumann boundary condition imposed in the entire patch $\mathcal{T}_{\boldsymbol{a}}$, i.e., $\mathcal{F}_{\boldsymbol{a}}^{\rm N} = \emptyset$. We define a function v by (5.13) as in case (3) in Section 5.3; three coloring is typically easier here and no submesh is necessary. As above in step (3), we see that 1) v^* takes the correct value on the two Dirichlet faces in $\mathcal{F}_i^{\rm D}$; 2) the neighbor through $F^1 \in \mathcal{F}_i^\sharp$ maps the correct contribution on the face F^1; 3) a) any interior face $F \in \mathcal{F}_{\boldsymbol{a}}^{\rm i}$, being shared by two elements, generates a difference of jumps which is zero; b) any Dirichlet boundary face F on an element K_j generates the zero contribution $(v^*|_{K_j} - \zeta_p^j)|_F$.
(5) $|\mathcal{F}_i^\sharp| = 2$ and $|\mathcal{F}_i^{\rm D}| = 1$. Note that as above, $i = |\mathcal{T}_{\boldsymbol{a}}|$ and $\mathcal{F}_{\boldsymbol{a}}^{\rm N} = \emptyset$. We proceed as in step (4), using (5.13) and three coloring, possibly on a submesh $\mathcal{T}_{\boldsymbol{a}}'$ of $\mathcal{T}_{\boldsymbol{a}}$. Note that the given boundary patch $\mathcal{T}_{\boldsymbol{a}}$ can be always completed to a patch $\mathcal{T}_{\boldsymbol{a}}^*$ containing an open ball around the vertex $\boldsymbol{a}$. Then Lemma B.3 with $K_* = K_i$ can be applied to $\mathcal{T}_{\boldsymbol{a}}^*$, yielding a three-color refinement of $\mathcal{T}_{\boldsymbol{a}}^*$. Now removing the added elements outside of $\omega_{\boldsymbol{a}}$, we obtain a three-color refinement $\mathcal{T}_{\boldsymbol{a}}'$ of $\mathcal{T}_{\boldsymbol{a}}$, with a shape regularity parameter $\gamma_{\mathcal{T}_{\boldsymbol{a}}'}$ equivalent, up to a bounded factor, to $\gamma_{\mathcal{T}_{\boldsymbol{a}}}$.
(6) $|\mathcal{F}_i^\sharp| = 3$, so that $\mathcal{F}_i^{\rm D} = \emptyset$. Here again $i = |\mathcal{T}_{\boldsymbol{a}}|$ and $\mathcal{F}_{\boldsymbol{a}}^{\rm N} = \emptyset$. Thus setting (5.13) and proceeding as above is possible.

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A Stable polynomial extensions on a tetrahedron

In this section, we reformulate and extend some recent results by Demkowicz, Gopolakrishnan, and Schöberl [10, 12] and by Costabel and McIntosh [9] on stable polynomial extensions on a single tetrahedron. We first consider H^1-stable extensions in the polynomial space $\mathbb{P}_p(K)$.

Lemma A.1 (H^1-stable polynomial extension on a tetrahedron)**.** *Let K be a tetrahedron with $\mathcal{F}_K^{\rm D} \subset \mathcal{F}_K$ a possibly empty subset of its faces. Let r be a p-degree piecewise polynomial on $\mathcal{F}_K^{\rm D}$, $p \geq 1$, with restriction to each $F \in \mathcal{F}_K^{\rm D}$ denoted by r_F. Assume that r is globally continuous over the Dirichlet boundary given by $\mathcal{F}_K^{\rm D}$. Then there exists a constant $C > 0$ only depending on the shape-regularity parameter γ_K such that*

$$\min_{\substack{v_p \in \mathbb{P}_p(K) \\ v_p|_F = r_F \ \forall F \in \mathcal{F}_K^{\rm D}}} \|\nabla v_p\|_K \leq C \min_{\substack{v \in H^1(K) \\ v|_F = r_F \ \forall F \in \mathcal{F}_K^{\rm D}}} \|\nabla v\|_K. \tag{A.1}$$

Remark A.2 (Equivalent form)**.** *Consider the following problem: find ζ_K such that*

$$\begin{aligned} -\Delta \zeta_K &= 0 && \text{in } K, \\ \zeta_K|_F &= r_F && \text{on all } F \in \mathcal{F}_K^{\rm D}, \\ -\nabla \zeta_K \cdot \boldsymbol{n}_K|_F &= 0 && \text{on all } F \in \mathcal{F}_K \setminus \mathcal{F}_K^{\rm D}, \end{aligned}$$

i.e., in weak form, $\zeta_K \in H^1(K)$ is such that $\zeta_K|_F = r_F$ for all $F \in \mathcal{F}_K^{\rm D}$ and

$$(\nabla \zeta_K, \nabla v)_K = 0 \qquad \forall v \in H^1(K),\ v|_F = 0\ \forall F \in \mathcal{F}_K^{\rm D}.$$

Similarly, the (spectral) finite element method of order p finds $\zeta_{p,K} \in \mathbb{P}_p(K)$ with $\zeta_{p,K}|_F = r_F$ for all $F \in \mathcal{F}_K^{\rm D}$ such that

$$(\nabla \zeta_{p,K}, \nabla v_p)_K = 0 \qquad \forall v_p \in \mathbb{P}_p(K),\ v_p|_F = 0 \text{ for all } F \in \mathcal{F}_K^{\rm D}.$$

As $\zeta_{p,K}$ and ζ_K are, respectively, the unique minimizers from (A.1), Lemma A.1 can be rephrased as a stability result for (spectral) finite elements on a single tetrahedron, i.e., $\|\nabla \zeta_K\|_K \leq \|\nabla \zeta_{p,K}\|_K \leq C \|\nabla \zeta_K\|_K$.

Proof. (1) Let us start by noting that in the case $\mathcal{F}_K^{\rm D} = \emptyset$, both $\zeta_{p,K}$ and ζ_K are zero. Let us henceforth suppose that $\mathcal{F}_K^{\rm D} \neq \emptyset$.

(2) We first establish (A.1) on the unit tetrahedron, say $\widehat{K}$; to this purpose, we proceed in three substeps.
(2.a) Case $K = \widehat{K}$ and $\mathcal{F}_K^{\mathrm{D}} = \mathcal{F}_K$, i.e., the Dirichlet condition is prescribed on the whole boundary ∂K. Then [10, Theorem 6.1] shows that there exists a polynomial $\zeta_p(r) \in \mathbb{P}_p(K)$ such that $\zeta_p(r)|_{\partial K} = r$ and $\|\zeta_p(r)\|_{H^1(K)} \leq C_{\mathrm{DGS}} \|r\|_{H^{1/2}(\partial K)}$, where $\|r\|_{H^{1/2}(\partial K)}$ is defined using the Sobolev–Slobodeckij seminorm as follows:

$$\|r\|^2_{H^{1/2}(\partial K)} = \|r\|^2_{L^2(\partial K)} + \int_{\partial K}\int_{\partial K} \frac{|r(\boldsymbol{x}) - r(\boldsymbol{y})|^2}{\|\boldsymbol{x}-\boldsymbol{y}\|^d_{\ell^2}}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{y},$$

recalling that $d = 3$. Since there exist constants $\underline{C}_{H^{1/2}}$ and $C_{H^{1/2}}$ of order unity so that

$$\underline{C}_{H^{1/2}} \min_{\substack{v \in H^1(K)\\ v|_{\partial K} = r}} \|v\|_{H^1(K)} \leq \|r\|_{H^{1/2}(\partial K)} \leq C_{H^{1/2}} \min_{\substack{v \in H^1(K)\\ v|_{\partial K} = r}} \|v\|_{H^1(K)}, \tag{A.2}$$

and since $\|\zeta_p(r)\|_{H^1(K)} \geq \min \|v_p\|_{H^1(K)}$ over all $v_p \in \mathbb{P}_p(K)$ such that $v_p|_{\partial K} = r$, we infer that

$$\min_{\substack{v_p \in \mathbb{P}_p(K)\\ v_p|_F = r_F\ \forall F \in \mathcal{F}_K^{\mathrm{D}}}} \|v_p\|_{H^1(K)} \leq \widehat{C} \min_{\substack{v \in H^1(K)\\ v|_F = r_F\ \forall F \in \mathcal{F}_K^{\mathrm{D}}}} \|v\|_{H^1(K)}, \tag{A.3}$$

with with $\widehat{C} = C_{\mathrm{DGS}} C_{H^{1/2}}$ and $\|v\|^2_{H^1(K)} = \|v\|^2_K + \|\nabla v\|^2_K$. Note that we are using the H^1-norm in (A.3), and not the H^1-seminorm as in (A.1).
(2.b) Case $K = \widehat{K}$ and $\mathcal{F}_K^{\mathrm{D}} \neq \mathcal{F}_K$. Let us set

$$\tilde{\zeta}_K := \mathop{\arg\min}_{\substack{v \in H^1(K)\\ v|_F = r_F\ F \in \mathcal{F}_K^{\mathrm{D}}}} \|v\|_{H^1(K)}. \tag{A.4}$$

Note that $\tilde{\zeta}_K$ solves in strong form $-\Delta\tilde{\zeta}_K + \tilde{\zeta}_K = 0$ in K, $\tilde{\zeta}_K|_F = r|_F$ for all $F \in \mathcal{F}_K^{\mathrm{D}}$, and $-\nabla\tilde{\zeta}_K{\cdot}\boldsymbol{n}_K|_F = 0$ for all $F \in \mathcal{F}_K \setminus \mathcal{F}_K^{\mathrm{D}}$; note also that $\tilde{\zeta}_K$ is well defined since $\mathcal{F}_K^{\mathrm{D}}$ is assumed to be non-empty. Define a new function $\tilde{r} \in H^{1/2}(\partial K)$ as the trace of $\tilde{\zeta}_K$ on ∂K; this extends the boundary data r originally defined only on the faces in $\mathcal{F}_K^{\mathrm{D}}$ to the whole ∂K, not necessarily by polynomials on the faces in $\mathcal{F}_K \setminus \mathcal{F}_K^{\mathrm{D}}$. The definition (A.4) of $\tilde{\zeta}_K$ combined with (A.2) yields $\|\tilde{r}\|_{H^{1/2}(\partial K)} \leq C_{H^{1/2}} \|\tilde{\zeta}_K\|_{H^1(K)}$. Let us now order the faces of K with the faces in $\mathcal{F}_K^{\mathrm{D}}$ first and consider only the summands corresponding to the faces in $\mathcal{F}_K^{\mathrm{D}}$ instead of the full extension operator of [10, equation (6.1)], applied to the function $\tilde{r}$. Following [10, Theorem 6.1], we obtain a polynomial $\zeta_p(\tilde{r}) \in \mathbb{P}_p(K)$ such that $\zeta_p(\tilde{r})|_F = r_F$ for all $F \in \mathcal{F}_K^{\mathrm{D}}$ and $\|\zeta_p(\tilde{r})\|_{H^1(K)} \leq C_{\mathrm{DGS}} \|\tilde{r}\|_{H^{1/2}(\partial K)}$. Combining with the above bound on $\|\tilde{r}\|_{H^{1/2}(\partial K)}$ and since $\|\zeta_p(\tilde{r})\|_{H^1(K)} \geq \min \|v_p\|_{H^1(K)}$ over all $v_p \in \mathbb{P}_p(K)$ such that $v_p|_F = r_F$ on all $F \in \mathcal{F}_K^{\mathrm{D}}$, we infer that (A.3) also holds if $\mathcal{F}_K^{\mathrm{D}} \neq \mathcal{F}_K$ with $\widehat{C} = C_{\mathrm{DGS}} C_{H^{1/2}}$. This completes the proof of (A.3).
(2.c) Let us prove that (A.1) holds when $K = \widehat{K}$. Let $c \in \mathbb{R}$ be arbitrary and let us set $r' := r + c$; note that r' is also a p-degree piecewise polynomial on $\mathcal{F}_K^{\mathrm{D}}$ that is globally continuous over $\mathcal{F}_K^{\mathrm{D}}$. Still assuming that K is the unit tetrahedron and applying (A.3) with the datum r', we infer that

$$\begin{aligned}
\min_{\substack{v_p \in \mathbb{P}_p(K)\\ v_p|_F = r_F\ \forall F \in \mathcal{F}_K^{\mathrm{D}}}} \|\nabla v_p\|_K &\leq \min_{\substack{v_p \in \mathbb{P}_p(K)\\ v_p|_F = r_F\ \forall F \in \mathcal{F}_K^{\mathrm{D}}}} \|v_p + c\|_{H^1(K)} = \min_{\substack{v_p \in \mathbb{P}_p(K)\\ v_p|_F = r'_F\ \forall F \in \mathcal{F}_K^{\mathrm{D}}}} \|v_p\|_{H^1(K)} \\
&\leq C \min_{\substack{v \in H^1(K)\\ v|_F = r'_F\ \forall F \in \mathcal{F}_K^{\mathrm{D}}}} \|v\|_{H^1(K)} = C \min_{\substack{v \in H^1(K)\\ v|_F = r_F\ \forall F \in \mathcal{F}_K^{\mathrm{D}}}} \|v + c\|_{H^1(K)},
\end{aligned}$$

where the first bounds follows by dropping the L^2-norm of $v_p + c$. Taking the infimum over $c \in \mathbb{R}$ on the right-hand side and using the Poincaré inequality $\inf_{c \in \mathbb{R}} \|v + c\|_K \leq \frac{1}{\pi} h_K \|\nabla v\|_K \leq c \|\nabla v\|_K$ with a constant c of order unity, we infer that (A.1) holds when $K = \widehat{K}$.
(3) Finally, we use a scaling argument to prove (A.1) in any tetrahedron K. Let $\zeta_{p,K}$ and ζ_K be the two minimizers in (A.1) (see Remark A.2). Let $\boldsymbol{T}$ be the geometric map from the unit tetrahedron $\widehat{K}$ to K. Then the pullback by $\boldsymbol{T}$ defined as $\psi(v) = v \circ \boldsymbol{T}$ is an isomorphism from $H^1(K)$ to $H^1(\widehat{K})$ and also from $\mathbb{P}_p(K)$ to $\mathbb{P}_p(\widehat{K})$. Moreover, we have $\|\nabla(\psi(v))\|_{\widehat{K}} \leq C_\psi \|\nabla v\|_K$ and $\|\nabla(\psi^{-1}(\widehat{v}))\|_K \leq C_{\psi^{-1}} \|\nabla \widehat{v}\|_{\widehat{K}}$ with constants such that $C_\psi C_{\psi^{-1}}$ is uniformly bounded by the shape-regularity parameter of K. Let now

$\mathcal{F}_{\widehat{K}}^{\mathrm{D}} := \{\widehat{F} \in \mathcal{F}_{\widehat{K}};\ \boldsymbol{T}(\widehat{F}) \in \mathcal{F}_K^{\mathrm{D}}\}$ and let us introduce the piecewise polynomial $\widehat{r}$ such that $\widehat{r}_{\widehat{F}} = r \circ (\boldsymbol{T}|_{\widehat{F}})$ for all $\widehat{F} \in \mathcal{F}_{\widehat{K}}^{\mathrm{D}}$. Applying the result of Step (2c) to $\widehat{K}$ with polynomial data $\widehat{r}$ and subset $\mathcal{F}_{\widehat{K}}^{\mathrm{D}}$, and introducing the two corresponding minimizers, $\widehat{\zeta}_{p,\widehat{K}}$ and $\widehat{\zeta}_{\widehat{K}}$, we infer that $\|\nabla\widehat{\zeta}_{p,\widehat{K}}\|_{\widehat{K}} \le \widehat{C}\|\nabla\widehat{\zeta}_{\widehat{K}}\|_{\widehat{K}}$ with $\widehat{C}$ of order unity. Finally, we have

$$\begin{aligned}\|\nabla\zeta_{p,K}\|_K &\le \|\nabla(\psi^{-1}(\widehat{\zeta}_{p,\widehat{K}}))\|_K \le C_{\psi^{-1}}\|\nabla\widehat{\zeta}_{p,\widehat{K}}\|_{\widehat{K}} \le C_{\psi^{-1}}\widehat{C}\|\nabla\widehat{\zeta}_{\widehat{K}}\|_{\widehat{K}}\\ &\le C_{\psi^{-1}}\widehat{C}\|\nabla\psi(\zeta_K)\|_{\widehat{K}} \le C_\psi C_{\psi^{-1}}\widehat{C}\|\nabla\zeta_K\|_K,\end{aligned}$$

since $\psi^{-1}(\widehat{\zeta}_{p,\widehat{K}})$ is in the minimization set defining $\zeta_{p,K}$ and $\psi(\zeta_K)$ in that defining $\widehat{\zeta}_{\widehat{K}}$. □

Let us now consider $\boldsymbol{H}(\mathrm{div})$-stable extensions in the polynomial space $\boldsymbol{RTN}_p(K)$. The following lemma rephrases the first two steps of the proof of [1, Theorem 7], while merging them together and extending them to three space dimensions. Recall that the normal trace of a field in $\boldsymbol{H}(\mathrm{div},K)$ is prescribed according to (2.5).

Lemma A.3 ($\boldsymbol{H}(\mathrm{div})$-stable polynomial extension on a tetrahedron)**.** *Let K be a tetrahedron with unit outward normal $\boldsymbol{n}_K$. Let $\mathcal{F}_K^{\mathrm{N}} \subset \mathcal{F}_K$ be a possibly empty subset of its faces. Let r be a p-degree piecewise polynomial on $\mathcal{F}_K^{\mathrm{N}}$, $p \ge 0$, with restriction to each $F \in \mathcal{F}_K^{\mathrm{N}}$ denoted by r_F. Let r_K be a p-degree polynomial in K. If $\mathcal{F}_K^{\mathrm{N}} = \mathcal{F}_K$, assume that $\sum_{F\in\mathcal{F}_K}(r_F,1)_F = (r_K,1)_K$. Then there exists a constant $C > 0$ only depending on the shape-regularity parameter γ_K such that*

$$\min_{\substack{\boldsymbol{v}_p\in\boldsymbol{RTN}_p(K)\\ \boldsymbol{v}_p\cdot\boldsymbol{n}_K|_F=r_F\ \forall F\in\mathcal{F}_K^{\mathrm{N}}\\ \nabla\cdot\boldsymbol{v}_p=r_K}} \|\boldsymbol{v}_p\|_K \le C \min_{\substack{\boldsymbol{v}\in\boldsymbol{H}(\mathrm{div},K)\\ \boldsymbol{v}\cdot\boldsymbol{n}_K|_F=r_F\ \forall F\in\mathcal{F}_K^{\mathrm{N}}\\ \nabla\cdot\boldsymbol{v}=r_K}} \|\boldsymbol{v}\|_K. \tag{A.5}$$

Remark A.4 (Equivalent form)**.** *Consider the following problem: find ζ_K such that*

$$\begin{aligned} -\Delta\zeta_K &= r_K && \text{in } K, && \text{(A.6a)}\\ \zeta_K|_F &= 0 && \text{on all } F \in \mathcal{F}_K \setminus \mathcal{F}_K^{\mathrm{N}}, && \text{(A.6b)}\\ -\nabla\zeta_K\cdot\boldsymbol{n}_K|_F &= r_F && \text{on all } F \in \mathcal{F}_K^{\mathrm{N}}, && \text{(A.6c)}\end{aligned}$$

i.e., in weak form, $\zeta_K \in H^1(K)$ is such that $\zeta_K|_F = 0$ on all $F \in \mathcal{F}_K \setminus \mathcal{F}_K^{\mathrm{N}}$, $(\zeta_K,1)_K = 0$ if $\mathcal{F}_K^{\mathrm{N}} = \mathcal{F}_K$, and

$$(\nabla\zeta_K,\nabla\phi)_K = (r_K,\phi)_K - \sum_{F\in\mathcal{F}_K^{\mathrm{N}}}(r_F,\phi)_F \qquad \forall\phi\in H^1(K) \text{ with } \phi|_F = 0\ \forall F\in\mathcal{F}_K\setminus\mathcal{F}_K^{\mathrm{N}}. \tag{A.7}$$

The dual weak formulation looks for $\boldsymbol{\xi}_K \in \boldsymbol{H}(\mathrm{div},K)$ with $\nabla\cdot\boldsymbol{\xi}_K = r_K$ and $\boldsymbol{\xi}_K\cdot\boldsymbol{n}_K|_F = r_F$ on all $F \in \mathcal{F}_K^{\mathrm{N}}$ such that

$$(\boldsymbol{\xi}_K,\boldsymbol{v})_K = 0 \qquad \forall\boldsymbol{v}\in\boldsymbol{H}(\mathrm{div},K) \text{ with } \nabla\cdot\boldsymbol{v} = 0 \text{ and } \boldsymbol{v}\cdot\boldsymbol{n}_K|_F = 0,\ \forall F\in\mathcal{F}_K^{\mathrm{N}},$$

and it is well-known that

$$\boldsymbol{\xi}_K = \mathop{\arg\min}_{\substack{\boldsymbol{v}\in\boldsymbol{H}(\mathrm{div},K)\\ \boldsymbol{v}\cdot\boldsymbol{n}_K|_F=r_F\ \forall F\in\mathcal{F}_K^{\mathrm{N}}\\ \nabla\cdot\boldsymbol{v}=r_K}} \|\boldsymbol{v}\|_K = -\nabla\zeta_K. \tag{A.8}$$

Similarly, the dual (or, equivalently, mixed) finite element method (here, a dual spectral method) finds $\boldsymbol{\xi}_{p,K} \in \boldsymbol{RTN}_p(K)$ with $\nabla\cdot\boldsymbol{\xi}_{p,K} = r_K$ and $\boldsymbol{\xi}_{p,K}\cdot\boldsymbol{n}_K|_F = r_F$ on all $F \in \mathcal{F}_K^{\mathrm{N}}$ such that

$$(\boldsymbol{\xi}_{p,K},\boldsymbol{v}_p)_K = 0 \qquad \forall\boldsymbol{v}_p\in\boldsymbol{RTN}_p(K) \text{ with } \nabla\cdot\boldsymbol{v}_p = 0 \text{ and } \boldsymbol{v}_p\cdot\boldsymbol{n}_K|_F = 0,\ \forall F\in\mathcal{F}_K^{\mathrm{N}}.$$

As $\boldsymbol{\xi}_{p,K}$ and $\boldsymbol{\xi}_K$ are, respectively, the unique minimizers from (A.5), Lemma A.3 can be rephrased as a stability result for mixed finite elements on a single tetrahedron, i.e., $\|\boldsymbol{\xi}_K\|_K \le \|\boldsymbol{\xi}_{p,K}\|_K \le C\|\boldsymbol{\xi}_K\|_K$.

Proof. We first establish (A.5) on the unit tetrahedron, say $\widehat{K}$; to this purpose, we proceed in three steps. Then, we establish (A.5) on any tetrahedron by using the contravariant Piola transformation.
(1) Case $K = \widehat{K}$ and $\mathcal{F}_K^{\mathrm{N}} = \emptyset$. We infer from [9, Corollary 3.4] that there is $\boldsymbol{\xi}_p(r_K) \in \boldsymbol{RTN}_p(K)$ such that $\nabla\cdot\boldsymbol{\xi}_p(r_K) = r_K$ and $\|\boldsymbol{\xi}_p(r_K)\|_K \le C_{\mathrm{CM}}\|r_K\|_{H^{-1}(K)}$ where $\|r_K\|_{H^{-1}(K)} := \max(r_K,\phi)_K$ over all $\phi \in H_0^1(K)$

such that $\|\nabla\phi\|_K = 1$. Furthermore, since a Dirichlet boundary condition is prescribed over the whole boundary of K in (A.6) when $\mathcal{F}_K^{\mathrm{N}} = \emptyset$, we infer that $\zeta_K \in H_0^1(K)$ is such that $(\nabla\zeta_K, \nabla\phi)_K = (r_K, \phi)_K$ for all $\phi \in H_0^1(K)$. Then, we have by (A.8),

$$\min_{\substack{\boldsymbol{v}\in\boldsymbol{H}(\mathrm{div},K)\\ \nabla\cdot\boldsymbol{v}=r_K}} \|\boldsymbol{v}\|_K = \|\boldsymbol{\xi}_K\|_K = \|\nabla\zeta_K\|_K = \max_{\substack{\phi\in H_0^1(K)\\ \|\nabla\phi\|_K=1}} (r_K,\phi)_K = \|r_K\|_{H^{-1}(K)}.$$

Altogether,

$$\min_{\substack{\boldsymbol{v}_p\in\boldsymbol{RTN}_p(K)\\ \nabla\cdot\boldsymbol{v}_p=r_K}} \|\boldsymbol{v}_p\|_K \le \|\boldsymbol{\xi}_p(r_K)\|_K \le C_{\mathrm{CM}}\|r_K\|_{H^{-1}(K)} \le C_{\mathrm{CM}} \min_{\substack{\boldsymbol{v}\in\boldsymbol{H}(\mathrm{div},K)\\ \nabla\cdot\boldsymbol{v}=r_K}} \|\boldsymbol{v}\|_K.$$

(2) Case $K = \widehat{K}$, $\mathcal{F}_K^{\mathrm{N}} \neq \emptyset$, and $r_K = 0$. We further distinguish two cases.
(2.a) Assume first that $\mathcal{F}_K^{\mathrm{N}} = \mathcal{F}_K$. Since a Neumann boundary condition is prescribed over the whole boundary of K in (A.6), $\zeta_K \in H^1(K)$ is such that $(\zeta_K, 1)_K = 0$ and $(\nabla\zeta_K, \nabla\phi)_K = (r, \phi)_{\partial K}$ for all $\phi \in H^1(K)$ such that $(\phi, 1)_K = 0$. Since $(r, 1)_{\partial K} = 0$ by assumption, [12, Theorem 7.1] shows that there is $\boldsymbol{\xi}_p(r) \in \boldsymbol{RTN}_p(K)$ (actually, in $[\mathbb{P}_p(K)]^3$) such that $\nabla\cdot\boldsymbol{\xi}_p(r) = 0$, $\boldsymbol{\xi}_p(r)\cdot\boldsymbol{n}_K|_F = r_F$ for all $F \in \mathcal{F}_K$, and $\|\boldsymbol{\xi}_p(r)\|_{\boldsymbol{H}(\mathrm{div},K)} = \|\boldsymbol{\xi}_p(r)\|_K \le C_{\mathrm{DGS}}\|r\|_{H^{-1/2}(\partial K)}$. Here, $H^{-1/2}(\partial K)$ is the dual space of $H^{1/2}(\partial K)$ equipped with the norm $\|r\|_{H^{-1/2}(\partial K)} := \max (r, \phi)_{\partial K}$ over all $\phi \in H^{1/2}(\partial K)$ such that $\|\phi\|_{H^{1/2}(\partial K)} = 1$. On the unit tetrahedron, we can the lower bound in (A.2) and the Poincaré inequality in the space $\{\phi \in H^1(K);\, (\phi,1)_K = 0\}$ to infer that there is a constant $C_{H^{-1/2}}$ of order unity so that

$$\|r\|_{H^{-1/2}(\partial K)} \le C_{H^{-1/2}} \sup_{\substack{\phi\in H^1(K)\\ (\phi,1)_K=0}} \frac{(r,\phi)_{\partial K}}{\|\nabla\phi\|_K}.$$

Owing to (A.8), we infer that

$$\|r\|_{H^{-1/2}(\partial K)} \le C_{H^{-1/2}}\|\nabla\zeta_K\|_K = C_{H^{-1/2}} \min_{\substack{\boldsymbol{v}\in\boldsymbol{H}(\mathrm{div},K)\\ \boldsymbol{v}\cdot\boldsymbol{n}_K|_F=r_F\ \forall F\in\mathcal{F}_K\\ \nabla\cdot\boldsymbol{v}=0}} \|\boldsymbol{v}\|_K.$$

Altogether,

$$\min_{\substack{\boldsymbol{v}_p\in\boldsymbol{RTN}_p(K)\\ \boldsymbol{v}_p\cdot\boldsymbol{n}_K|_F=r_F\ \forall F\in\mathcal{F}_K\\ \nabla\cdot\boldsymbol{v}_p=0}} \|\boldsymbol{v}_p\|_K \le \|\boldsymbol{\xi}_p(r)\|_K \le C_{\mathrm{DGS}}\|r\|_{H^{-1/2}(\partial K)} \le C_{\mathrm{DGS}}C_{H^{-1/2}} \min_{\substack{\boldsymbol{v}\in\boldsymbol{H}(\mathrm{div},K)\\ \boldsymbol{v}\cdot\boldsymbol{n}_K|_F=r_F\ \forall F\in\mathcal{F}_K\\ \nabla\cdot\boldsymbol{v}=0}} \|\boldsymbol{v}\|_K.$$

(2.b) Assume now that $\emptyset \neq \mathcal{F}_K^{\mathrm{N}} \subsetneq \mathcal{F}_K$. Let us set

$$\tilde{\zeta}_K := \operatorname*{arg\,min}_{\substack{v\in H^1(K)\\ -\nabla v\cdot\boldsymbol{n}_K|_F=r_F\ \forall F\in\mathcal{F}_K^{\mathrm{N}}}} \|\nabla v\|_K,$$

i.e., in weak form $(\nabla\tilde{\zeta}_K, \nabla\phi)_K = \sum_{F\in\mathcal{F}_K^{\mathrm{N}}}(r_F,\phi)_F$ for all $\phi \in H^1(K)$ such that $\phi|_F = 0$ for all $F \in \mathcal{F}_K \setminus \mathcal{F}_K^{\mathrm{N}}$. Since K is a convex polyhedron, elliptic regularity implies that $\tilde{\zeta}_K \in H^2(K)$, so that the normal derivative $\nabla\tilde{\zeta}_K\cdot\boldsymbol{n}_K$ can be given a pointwise meaning on ∂K. Let us call $\tilde{r}$ this normal derivative. We infer that $\|\tilde{r}\|_{H^{-1/2}(\partial K)} \le C\|\nabla\tilde{\zeta}_K\|_K$. Let us now order the faces in $\mathcal{F}_K^{\mathrm{N}}$ first and consider only the summands corresponding to the faces from $\mathcal{F}_K^{\mathrm{N}}$ instead of the full extension operator of [12, equation (7.1)], applied to the function $\tilde{r}$. Following [12, Theorem 7.1], we obtain a polynomial $\boldsymbol{\zeta}_p(\tilde{r}) \in [\mathbb{P}_p(K)]^3$ such that $\boldsymbol{\zeta}_p(\tilde{r})\cdot\boldsymbol{n}_K|_F = r_F$ for all $F \in \mathcal{F}_K^{\mathrm{N}}$ and $\|\boldsymbol{\zeta}_p(\tilde{r})\|_K \le C_{\mathrm{DGS}}\|\tilde{r}\|_{H^{-1/2}(\partial K)}$. Combining the above bounds and reasoning as above, we infer that (A.5) holds.
(3) Proof of (A.5) when $K = \widehat{K}$. Since K is the unit tetrahedron, we can use the bounds established in Steps (1) and (2). Let $\boldsymbol{\xi}'_{p,K} \in \boldsymbol{RTN}_p(K)$ be the discrete arg min with only divergence prescribed by r_K but no boundary flux prescribed. Using the result of Step (1), we infer that

$$\|\boldsymbol{\xi}'_{p,K}\|_K = \min_{\substack{\boldsymbol{v}_p\in\boldsymbol{RTN}_p(K)\\ \nabla\cdot\boldsymbol{v}_p=r_K}} \|\boldsymbol{v}_p\|_K \le C \min_{\substack{\boldsymbol{v}\in\boldsymbol{H}(\mathrm{div},K)\\ \nabla\cdot\boldsymbol{v}=r_K}} \|\boldsymbol{v}\|_K \le C \min_{\substack{\boldsymbol{v}\in\boldsymbol{H}(\mathrm{div},K)\\ \boldsymbol{v}\cdot\boldsymbol{n}_K|_F=r_F\ \forall F\in\mathcal{F}_K^{\mathrm{N}}\\ \nabla\cdot\boldsymbol{v}=r_K}} \|\boldsymbol{v}\|_K = C\|\boldsymbol{\xi}_K\|_K,$$

where the last inequality follows by restricting the minimization set and introducing the unique minimizer $\boldsymbol{\xi}_K$ defined in Remark A.4. Let now $\boldsymbol{\xi}''_{p,K} \in \boldsymbol{RTN}_p(K)$ be the discrete arg min with divergence prescribed to zero and boundary flux prescribed to $r_F - \boldsymbol{\xi}'_{p,K}{\cdot}\boldsymbol{n}_K|_F$ for all $F \in \mathcal{F}_K^{\mathrm{N}}$. In the case where $\mathcal{F}_K^{\mathrm{N}} = \mathcal{F}_K$, the compatibility condition on the prescribed fluxes holds since

$$\sum_{F\in\mathcal{F}_K^{\mathrm{N}}} (r_F - \boldsymbol{\xi}'_{p,K}{\cdot}\boldsymbol{n}_K|_F, 1)_F = \sum_{F\in\mathcal{F}_K^{\mathrm{N}}} (r_F, 1)_F - (\nabla{\cdot}\boldsymbol{\xi}'_{p,K}, 1)_K = \sum_{F\in\mathcal{F}_K^{\mathrm{N}}} (r_F, 1)_F - (r_K, 1)_K = 0,$$

by assumption. Step (2) implies that

$$\|\boldsymbol{\xi}''_{p,K}\|_K = \min_{\substack{\boldsymbol{v}_p\in\boldsymbol{RTN}_p(K)\\ \boldsymbol{v}_p{\cdot}\boldsymbol{n}_K|_F = r_F - \boldsymbol{\xi}'_{p,K}{\cdot}\boldsymbol{n}_K|_F\ \forall F\in\mathcal{F}_K^{\mathrm{N}}\\ \nabla{\cdot}\boldsymbol{v}_p=0}} \|\boldsymbol{v}_p\|_K \le C \min_{\substack{\boldsymbol{v}\in\boldsymbol{H}(\mathrm{div},K)\\ \boldsymbol{v}{\cdot}\boldsymbol{n}_K|_F = r_F - \boldsymbol{\xi}'_{p,K}{\cdot}\boldsymbol{n}_K|_F\ \forall F\in\mathcal{F}_K^{\mathrm{N}}\\ \nabla{\cdot}\boldsymbol{v}=0}} \|\boldsymbol{v}\|_K.$$

Furthermore, a shift by $\boldsymbol{\xi}'_{p,K}$ allows to rewrite equivalently, and then bound by the triangle inequality, the last minimum above as follows:

$$\min_{\substack{\boldsymbol{v}\in\boldsymbol{H}(\mathrm{div},K)\\ \boldsymbol{v}{\cdot}\boldsymbol{n}_K|_F = r_F\ \forall F\in\mathcal{F}_K^{\mathrm{N}}\\ \nabla{\cdot}\boldsymbol{v}=r_K}} \|\boldsymbol{v} - \boldsymbol{\xi}'_{p,K}\|_K \le \min_{\substack{\boldsymbol{v}\in\boldsymbol{H}(\mathrm{div},K)\\ \boldsymbol{v}{\cdot}\boldsymbol{n}_K|_F = r_F\ \forall F\in\mathcal{F}_K^{\mathrm{N}}\\ \nabla{\cdot}\boldsymbol{v}=r_K}} \|\boldsymbol{v}\|_K + \|\boldsymbol{\xi}'_{p,K}\|_K = \|\boldsymbol{\xi}_K\|_K + \|\boldsymbol{\xi}'_{p,K}\|_K,$$

so that $\|\boldsymbol{\xi}''_{p,K}\|_K \le C(\|\boldsymbol{\xi}_K\|_K + \|\boldsymbol{\xi}'_{p,K}\|_K)$. Now $\boldsymbol{\xi}'_{p,K} + \boldsymbol{\xi}''_{p,K}$ belongs to the discrete minimization set in (A.5) and $\|\boldsymbol{\xi}'_{p,K} + \boldsymbol{\xi}''_{p,K}\|_K$ is bounded by $\|\boldsymbol{\xi}_K\|_K$. This proves (A.5) on the unit tetrahedron.
(4) Proof of (A.5) on a general tetrahedron K. We are given a subset $\mathcal{F}_K^{\mathrm{N}}$ of the faces of K, a p-degree piecewise polynomial r on $\mathcal{F}_K^{\mathrm{N}}$, and a p-degree polynomial r_K in K such that, if $\mathcal{F}_K^{\mathrm{N}} = \mathcal{F}_K$, $\sum_{F\in\mathcal{F}_K}(r_F, 1)_F = (r_K, 1)_K$. We are going to prove (A.5) on K by mapping the minimization problems to the unit tetrahedron $\widehat{K}$. Consider an affine bijective map $\boldsymbol{T} : \widehat{K} \to K$ with Jacobian matrix $\boldsymbol{J}$ (note that $\boldsymbol{J}$ is a constant (and invertible) matrix in $\widehat{K}$ since $\boldsymbol{T}$ is an affine (bijective) map). Let $\mathcal{F}_{\widehat{K}}^{\mathrm{N}}$ collect the faces of $\widehat{K}$ that are images by $\boldsymbol{T}^{-1}$ of the faces of K in $\mathcal{F}_K^{\mathrm{N}}$. Let us set

$$\widehat{r}_{\widehat{F}} := \det(\boldsymbol{J})\|\boldsymbol{J}^{-\mathrm{T}}\boldsymbol{n}_{\widehat{F}}\|_{\ell^2}(r_F \circ \boldsymbol{T}|_{\widehat{F}}),\ \forall \widehat{F} \in \mathcal{F}_{\widehat{K}}^{\mathrm{N}}, \qquad \widehat{r}_{\widehat{K}} := \det(\boldsymbol{J})(r_K \circ \boldsymbol{T}),$$

where $\boldsymbol{n}_{\widehat{F}}$ is the unit outward normal to $\widehat{K}$ on the face $\widehat{F}$. Then, $\widehat{r}$ defined by its restrictions to the faces $\widehat{F} \in \mathcal{F}_{\widehat{K}}^{\mathrm{N}}$ is a p-degree piecewise polynomial on $\mathcal{F}_{\widehat{K}}^{\mathrm{N}}$, and $\widehat{r}_{\widehat{K}}$ is a p-degree polynomial on $\widehat{K}$, and in the case where $\mathcal{F}_{\widehat{K}}^{\mathrm{N}} = \mathcal{F}_{\widehat{K}}$, we additionally have

$$\begin{aligned}\sum_{\widehat{F}\in\mathcal{F}_{\widehat{K}}} (\widehat{r}_{\widehat{F}}, 1)_{\widehat{F}} &= \sum_{\widehat{F}\in\mathcal{F}_{\widehat{K}}} (r_F \circ \boldsymbol{T}|_{\widehat{F}}, \det(\boldsymbol{J})\|\boldsymbol{J}^{-\mathrm{T}}\boldsymbol{n}_{\widehat{F}}\|_{\ell^2})_{\widehat{F}}\\ &= \sum_{F\in\mathcal{F}_K} \epsilon_{\boldsymbol{J}}(r_F, 1)_F = \epsilon_{\boldsymbol{J}}(r_K, 1)_K = (r_K \circ \boldsymbol{T}, \det(\boldsymbol{J}))_{\widehat{K}} = (\widehat{r}_{\widehat{K}}, 1)_{\widehat{K}},\end{aligned}$$

with $\epsilon_{\boldsymbol{J}} = \frac{\det(\boldsymbol{J})}{|\det(\boldsymbol{J})|} = \pm 1$. Here, we have used the following classical formulas to change the surface measure and the volume measure: $\mathrm{d}s = |\det(\boldsymbol{J})|\|\boldsymbol{J}^{-\mathrm{T}}\boldsymbol{n}_{\widehat{F}}\|_{\ell^2}\mathrm{d}\widehat{s}$ and $\mathrm{d}\boldsymbol{x} = |\det(\boldsymbol{J})|\mathrm{d}\widehat{\boldsymbol{x}}$. Let us now consider the contravariant Piola transformation such that $\boldsymbol{\psi}(\boldsymbol{v}) = \boldsymbol{A}(\boldsymbol{v} \circ \boldsymbol{T})$ with $\boldsymbol{A} = \det(\boldsymbol{J})\boldsymbol{J}^{-1}$. Then $\boldsymbol{\psi}$ is an isomorphism from $\boldsymbol{H}(\mathrm{div}, K)$ to $\boldsymbol{H}(\mathrm{div}, \widehat{K})$, and also from $\boldsymbol{RTN}_p(K)$ to $\boldsymbol{RTN}_p(\widehat{K})$. Moreover, we have the following key properties:

$$\nabla{\cdot}\boldsymbol{v} = r_K \text{ in } K \iff \nabla{\cdot}(\boldsymbol{\psi}(\boldsymbol{v})) = \widehat{r}_{\widehat{K}} \text{ in } \widehat{K}, \tag{A.9a}$$

$$\boldsymbol{v}{\cdot}\boldsymbol{n}_K|_F = r_F \text{ on all } F \in \mathcal{F}_K^{\mathrm{N}} \iff \boldsymbol{\psi}(\boldsymbol{v}){\cdot}\boldsymbol{n}_{\widehat{F}} = \widehat{r}_{\widehat{F}} \text{ on all } \widehat{F} \in \mathcal{F}_{\widehat{K}}^{\mathrm{N}}, \tag{A.9b}$$

with $K = \boldsymbol{T}(\widehat{K})$ and $F = \boldsymbol{T}(\widehat{F})$. The first equivalence results from $\nabla{\cdot}(\boldsymbol{\psi}(\boldsymbol{v})) = \det(\boldsymbol{J})(\nabla{\cdot}\boldsymbol{v}) \circ \boldsymbol{T}$ and the definition of $\widehat{r}_{\widehat{K}}$. To prove the second equivalence, recalling (2.5), the left-hand side means that

$$(\boldsymbol{v}, \nabla\phi)_K + (\nabla{\cdot}\boldsymbol{v}, \phi)_K = \sum_{F\in\mathcal{F}_K^{\mathrm{N}}} (r_F, \phi)_F,$$

for all $\phi \in H^1(K)$ such that $\phi|_F = 0$ for all $F \in \mathcal{F}_K \setminus \mathcal{F}_K^{\mathrm{N}}$. Changing variables in the volume and surface integrals, the above identity amounts to

$$(\boldsymbol{\psi}(\boldsymbol{v}), \nabla\widehat{\phi})_{\widehat{K}} + (\nabla\cdot(\boldsymbol{\psi}(\boldsymbol{v})), \widehat{\phi})_{\widehat{K}} = \sum_{\widehat{F}\in\mathcal{F}_{\widehat{K}}^{\mathrm{N}}} (\widehat{r}_{\widehat{F}}, \widehat{\phi})_{\widehat{F}},$$

where $\widehat{\phi} = \phi \circ \boldsymbol{T}$. Since the pullback by the geometric map $\boldsymbol{T}$ is an isomorphism from $\{\phi \in H^1(K);\, \phi|_F = 0\,\forall F \in \mathcal{F}_K \setminus \mathcal{F}_K^{\mathrm{N}}\}$ to $\{\widehat{\phi} \in H^1(\widehat{K});\, \widehat{\phi}|_{\widehat{F}} = 0\,\forall \widehat{F} \in \mathcal{F}_{\widehat{K}} \setminus \mathcal{F}_{\widehat{K}}^{\mathrm{N}}\}$, the above identity means that $\boldsymbol{\psi}(\boldsymbol{v})\cdot\boldsymbol{n}_{\widehat{F}} = \widehat{r}_{\widehat{F}}$ on all $\widehat{F} \in \mathcal{F}_{\widehat{K}}^{\mathrm{N}}$.

Let now $\boldsymbol{\xi}_{p,K}$ and $\boldsymbol{\xi}_K$ be the unique minimizers in (A.5) using the polynomial data r and r_K; similarly, let $\widehat{\boldsymbol{\xi}}_{p,\widehat{K}}$ and $\widehat{\boldsymbol{\xi}}_{\widehat{K}}$ be the unique minimizers for the minimization problems posed on $\widehat{K}$ using the polynomial data $\widehat{r}$ and $\widehat{r}_{\widehat{K}}$. We infer from Step (3) that $\|\widehat{\boldsymbol{\xi}}_{p,\widehat{K}}\|_{\widehat{K}} \le \widehat{C}\|\widehat{\boldsymbol{\xi}}_{\widehat{K}}\|_{\widehat{K}}$ with constant C of order unity. Since $\boldsymbol{\psi}^{-1}(\widehat{\boldsymbol{\xi}}_{p,\widehat{K}})$ is in the minimization set defining $\boldsymbol{\xi}_{p,K}$ and since $\boldsymbol{\psi}(\boldsymbol{\xi}_K)$ is in that of $\widehat{\boldsymbol{\xi}}_{\widehat{K}}$, we have

$$\begin{aligned}\|\boldsymbol{\xi}_{p,K}\|_K &\le \|\boldsymbol{\psi}^{-1}(\widehat{\boldsymbol{\xi}}_{p,\widehat{K}})\|_K \le \|\boldsymbol{\psi}^{-1}\|_{\mathcal{L}(L^2,L^2)}\|\widehat{\boldsymbol{\xi}}_{p,\widehat{K}}\|_{\widehat{K}} \le \|\boldsymbol{\psi}^{-1}\|_{\mathcal{L}(L^2,L^2)}\widehat{C}\|\widehat{\boldsymbol{\xi}}_{\widehat{K}}\|_{\widehat{K}}\\ &\le \|\boldsymbol{\psi}^{-1}\|_{\mathcal{L}(L^2,L^2)}\widehat{C}\|\boldsymbol{\psi}(\boldsymbol{\xi}_K)\|_{\widehat{K}} \le \|\boldsymbol{\psi}\|_{\mathcal{L}(L^2,L^2)}\|\boldsymbol{\psi}^{-1}\|_{\mathcal{L}(L^2,L^2)}\widehat{C}\|\boldsymbol{\xi}_K\|_K,\end{aligned}$$

where $\|\boldsymbol{\psi}\|_{\mathcal{L}(L^2,L^2)}$ and $\|\boldsymbol{\psi}^{-1}\|_{\mathcal{L}(L^2,L^2)}$ are the operator norms of $\boldsymbol{\psi}$ and $\boldsymbol{\psi}^{-1}$ as linear maps between $\boldsymbol{L}^2(K)$ and $\boldsymbol{L}^2(\widehat{K})$. This completes the proof since the factor $\|\boldsymbol{\psi}\|_{\mathcal{L}(L^2,L^2)}\|\boldsymbol{\psi}^{-1}\|_{\mathcal{L}(L^2,L^2)}$ is bounded by a constant only depending on the shape-regularity parameter γ_K. □

B On cell enumeration and vertex coloring in patches

We collect in this section some auxiliary results on cell enumeration and vertex coloring in simplicial patches, corresponding to the setting described in Section 2.1. For any cell $K \in \mathcal{T}_{\boldsymbol{a}}$, its interior faces are collected in the set $\mathcal{F}_K^{\mathrm{i}} := \mathcal{F}_K \cap \mathcal{F}_{\boldsymbol{a}}^{\mathrm{i}}$. Let us first observe that any enumeration of the elements in the patch $\mathcal{T}_{\boldsymbol{a}}$ in the form $K_1, \ldots, K_{|\mathcal{T}_{\boldsymbol{a}}|}$ induces a partition of each of the sets $\mathcal{F}_{K_i}^{\mathrm{i}}$, $1 \le i \le |\mathcal{T}_{\boldsymbol{a}}|$, into two disjoint subsets, $\mathcal{F}_{K_i}^{\mathrm{i}} = \mathcal{F}_i^{\flat} \cup \mathcal{F}_i^{\sharp}$, where $\mathcal{F}_i^{\sharp}$ collects all the interior faces of K_i shared by an already enumerated cell K_j with $j < i$, i.e., $\mathcal{F}_i^{\sharp} := \{F \in \mathcal{F}_{\boldsymbol{a}}^{\mathrm{i}},\, F = \partial K_i \cap \partial K_j,\, j < i\}$, and $\mathcal{F}_i^{\flat}$ collects all the other interior faces of K_i, i.e., $\mathcal{F}_i^{\flat} := \{F \in \mathcal{F}_{\boldsymbol{a}}^{\mathrm{i}},\, F = \partial K_i \cap \partial K_j,\, j > i\}$. Note that $|\mathcal{F}_i^{\flat}| + |\mathcal{F}_i^{\sharp}| = 3$. An immediate consequence of this definition is that $\mathcal{F}_1^{\flat} = \mathcal{F}_{K_1}^{\mathrm{i}}$ and $\mathcal{F}_1^{\sharp} = \emptyset$ on the first element, whereas $\mathcal{F}_{|\mathcal{T}_{\boldsymbol{a}}|}^{\flat} = \emptyset$ and $\mathcal{F}_{|\mathcal{T}_{\boldsymbol{a}}|}^{\sharp} = \mathcal{F}_{K_{|\mathcal{T}_{\boldsymbol{a}}|}}^{\mathrm{i}}$ on the last element.

Lemma B.1 (Patch enumeration)**.** *There exists an enumeration of the elements in the patch $\mathcal{T}_{\boldsymbol{a}}$ so that*

(i) For all $1 < i \le |\mathcal{T}_{\boldsymbol{a}}|$, if there are at least two faces in $\mathcal{F}_i^{\sharp}$, intersecting in an edge, then all the elements sharing this edge come sooner in the enumeration, i.e., if $|\mathcal{F}_i^{\sharp}| \ge 2$ with $\{F_i^1, F_i^2\} \subset \mathcal{F}_i^{\sharp}$, then letting $e := F_i^1 \cap F_i^2$, $K_j \in \mathcal{T}_e \setminus \{K_i\}$ implies that $j < i$.

(ii) For all $1 < i < |\mathcal{T}_{\boldsymbol{a}}|$, there are one or two neighbors of K_i which have been already enumerated and correspondingly two or one neighbors of K_i which have not been enumerated yet, i.e., $|\mathcal{F}_i^{\sharp}| \in \{1,2\}$ simultaneously with $|\mathcal{F}_i^{\flat}| \in \{2,1\}$, for all but the first and the last element. In particular, $\mathcal{F}_i^{\sharp}$ contains all the interior faces of K_i if and only if $i = |\mathcal{T}_{\boldsymbol{a}}|$.

Proof. One can see that the requested properties on the enumeration are satisfied if for all $1 \le i < |\mathcal{T}_{\boldsymbol{a}}|$, the boundary of the set $(\cup_{j\le i}K_j)\cap\partial\omega_{\boldsymbol{a}}$ is connected and contains only vertices of degree two (i.e., each vertex is connected by an edge to exactly two other vertices also belonging to this boundary). In turn, this property is satisfied if one considers a shelling of $\mathcal{T}_{\boldsymbol{a}}$ seen as a polytopal complex, see [21, Definition 8.1]. □

Lemma B.2 (Two-color refinement around edges)**.** *Fix a cell $K_* \in \mathcal{T}_{\boldsymbol{a}}$ and an edge e of K_* having $\boldsymbol{a}$ as one endpoint. Recall that $\mathcal{T}_e$ collects all the cells in $\mathcal{T}_{\boldsymbol{a}}$ having e as edge. Then there exists a conforming refinement $\mathcal{T}_e'$ of $\mathcal{T}_e$ composed of tetrahedra such that*

(i) $\mathcal{T}_e'$ contains K_*;

(ii) All the tetrahedra in $\mathcal{T}'_e$ have e as edge, and their two other vertices lie on $\partial\omega_{\boldsymbol{a}}$;

(iii) Collecting all the vertices of $\mathcal{T}'_e$ that are not endpoints of e in the set $\mathcal{V}'_e$, there is a two-color map $\texttt{col} : \mathcal{V}'_e \to \{1,2\}$ so that for all $\kappa \in \mathcal{T}'_e$, the two vertices of κ that are not endpoints of e, say $\{\boldsymbol{a}^n_\kappa\}_{1\le n\le 2}$, satisfy $\texttt{col}(\boldsymbol{a}^n_\kappa) = n$.

Proof. If $|\mathcal{T}_e|$ is even, we can just take $\mathcal{T}'_e = \mathcal{T}_e$ since the vertices of $\mathcal{T}_e$ that are not endpoints of e then form a cycle with an even number of vertices that can be colored using alternating colors. If $|\mathcal{T}_e|$ is odd, we pick one tetrahedron in $\mathcal{T}_e \setminus \{K_*\}$ and subdivide it into two sub-tetrahedra by cutting it along the median plane containing e. By doing so, we obtain a conforming simplicial refinement $\mathcal{T}'_e$ of $\mathcal{T}_e$ that has all the desired properties. □

Lemma B.3 (Three-color patch refinement)**.** *Fix a cell $K_* \in \mathcal{T}_{\boldsymbol{a}}$. There exists a conforming refinement $\mathcal{T}'_{\boldsymbol{a}}$ of $\mathcal{T}_{\boldsymbol{a}}$ composed of tetrahedra such that*

(i) $\mathcal{T}'_{\boldsymbol{a}}$ contains K_*;

(ii) All the tetrahedra in $\mathcal{T}'_{\boldsymbol{a}}$ have $\boldsymbol{a}$ as vertex, and their three other vertices lie on $\partial\omega_{\boldsymbol{a}}$;

(iii) Collecting all the vertices of $\mathcal{T}'_{\boldsymbol{a}}$ distinct from $\boldsymbol{a}$ in the set $\mathcal{V}'_{\boldsymbol{a}}$, there is a three-color map $\texttt{col} : \mathcal{V}'_{\boldsymbol{a}} \to \{1,2,3\}$ so that for all $\kappa \in \mathcal{T}'_{\boldsymbol{a}}$, the three vertices of κ distinct from $\boldsymbol{a}$, say $\{\boldsymbol{a}^n_\kappa\}_{1\le n\le 3}$, satisfy $\texttt{col}(\boldsymbol{a}^n_\kappa) = n$.

Proof. Since all the cells in $\mathcal{T}_{\boldsymbol{a}}$ and in $\mathcal{T}'_{\boldsymbol{a}}$ have $\boldsymbol{a}$ as vertex and their three other vertices lie on $\partial\omega_{\boldsymbol{a}}$, we will reason on the trace of $\mathcal{T}_{\boldsymbol{a}}$ on $\partial\omega_{\boldsymbol{a}}$. Using a homeomorphism, we can map $\cup_{K'\in\mathcal{T}_{\boldsymbol{a}}\setminus\{K_*\}}\{K' \cap \partial\omega_{\boldsymbol{a}}\}$ to an interior triangulation, say $\mathfrak{T}$, of the unit triangle T in $\mathbb{R}^2$ with the particularity that the three sides of T are edges of cells in $\mathfrak{T}$ (these three triangular cells are the images by the above homeomorphism of the trace on $\partial\omega_{\boldsymbol{a}}$ of the three tetrahedra sharing a face with K_*); see Figure 1. We now devise a conforming triangular

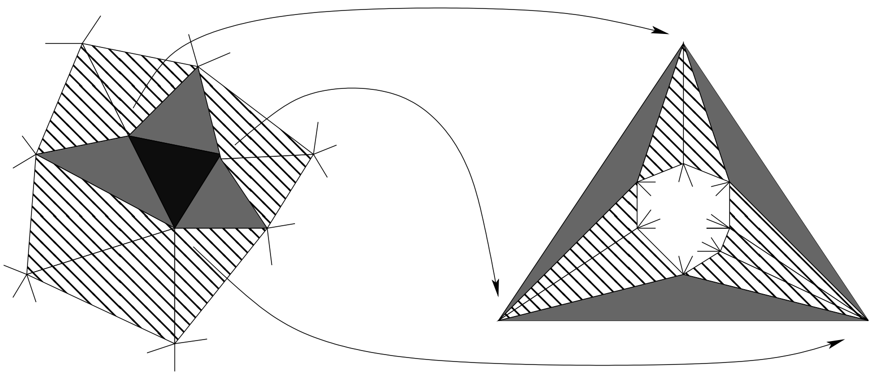

Figure 1: Left: original patch $\mathcal{T}_{\boldsymbol{a}} \cap \partial\omega_{\boldsymbol{a}}$ locally around $K_* \cap \partial\omega_{\boldsymbol{a}}$ (highlighted in dark grey), the three triangles $K' \cap \partial\omega_{\boldsymbol{a}}$ for which the tetrahedron K' shares an interior face with K are highlighted in light grey, and the triangles $K' \cap \partial\omega_{\boldsymbol{a}}$ for which the tetrahedron K' shares only an interior edge with K are dashed; Right: mapping by the homeomorphism to a triangulation of the unit triangle T in $\mathbb{R}^2$; the polygon at the heart of T is the image by the homeomorphism of all the triangles $K' \cap \partial\omega_{\boldsymbol{a}}$ where K' only shares the vertex $\boldsymbol{a}$ with K_*.

refinement of $\mathfrak{T}$ that does not refine the three sides of T and such that all the vertices in this refinement are connected to an even number of other vertices (the number of connections is called the degree of the vertex). The existence of a three-coloring map on the vertices of this refinement will then follow from [20]. To this purpose, we proceed in several steps. Let us call $\{\boldsymbol{z}_1, \boldsymbol{z}_2, \boldsymbol{z}_3\}$ the three vertices of T. The three triangles in $\mathfrak{T}$ supported on the three edges of T are denoted by $\{\tau_1, \tau_2, \tau_3\}$ in such a way that τ_n does not touch the vertex $\boldsymbol{z}_n$, for all $n = 1, 2, 3$. Let $\boldsymbol{z}'_n$ denote the barycenter of τ_n.
(1) We subdivide all the triangles in $\mathfrak{T}$ by barycentric subdivision into six sub-triangles. By doing so, we create new vertices, namely the barycenter of each triangle in $\mathfrak{T}$ (with degree 6), and the midpoint of each edge in $\mathfrak{T}$ (with degree 4). Moreover, all the original vertices of $\mathfrak{T}$ have now even degree, except for

$\{z_1, z_2, z_3\}$ which have odd degree. To avoid refining the three edges of T, we remove for all $n \in \{1, 2, 3\}$ the connection between the barycenter z'_n and the midpoint of the edge of T supporting the triangle τ_n. By doing so, the degree of the three barycenters $\{z'_1, z'_2, z'_3\}$ changes from six to five. At this stage, we have a conforming, triangular refinement preserving the three sides of T, but which contains six vertices of odd degree, namely $\{z_1, z_2, z_3\}$ and $\{z'_1, z'_2, z'_3\}$.
(2) We subdivide the triangle with vertices $\{z_1, z'_2, z_3\}$ into three triangles by joining its barycenter, say z''_2, to the three vertices. The degree of z_1, z'_2, and z_3 is now even as desired, but we have created the new vertex z''_2 with degree three. This new triangulation is illustrated in the central panel of Figure 2 in a slightly simplified setting with respect to Figure 1 since we have reduced to one the number of original dashed triangles at each of the three vertices of T (see the left panel of Figure 2 for the original triangulation).
(3) We now subdivide all the triangles having z_1 as vertex, except the newly created one, into two subtriangles as depicted in the right panel of Figure 2. The vertices z''_2 and z_2 now have even degree, and we have created additional vertices at some edge midpoints that all have degree four while we have also increased by two the degree of some vertices.
(4) Finally, we use a similar process, as depicted also in the right panel of Figure 2, so that the vertices z'_1 and z'_3 now have even degree, while we create additional vertices at some edge midpoints that all have degree four. We now have a triangulation where all the vertices have even degree. $\square$

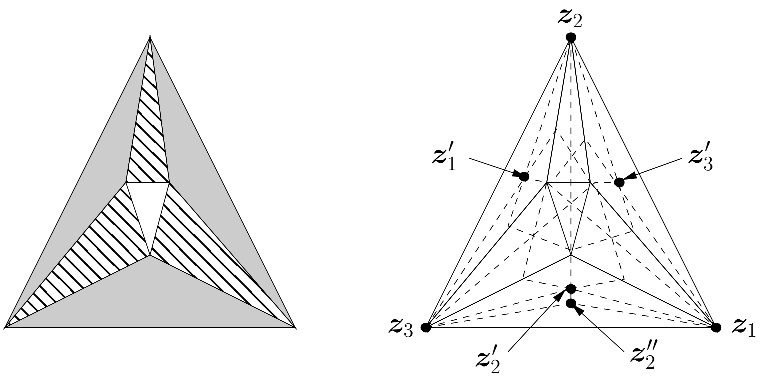

Figure 2: Left: original triangulation in the simple case where there is only one dashed triangle at each of the three vertices $\{z_1, z_2, z_3\}$; Center: refined triangulation at the end of Step (2) showing the newly created barycenters $\{z'_1, z'_2, z'_3\}$ and z''_2; Right: final refined triangulation where now all the vertices have even degree.

C On boundary patches enumeration

Let $\boldsymbol{a}$ be a boundary vertex as specified in Section 2.4. Decompose $\mathcal{F}^{\mathrm{i}}_{K_i} = \mathcal{F}^{\flat}_i \cup \mathcal{F}^{\sharp}_i$ as in Appendix B; note, however, that $|\mathcal{F}^{\flat}_i| + |\mathcal{F}^{\sharp}_i| \neq 3$ here in general; there rather holds $|\mathcal{F}^{\flat}_i| + |\mathcal{F}^{\sharp}_i| + |\mathcal{F}_{K_i} \cap \mathcal{F}^{\mathrm{D}}_{\boldsymbol{a}}| + |\mathcal{F}_{K_i} \cap \mathcal{F}^{\mathrm{N}}_{\boldsymbol{a}}| = 3$, where all the sets are disjoint. We now give a variant of Lemma B.1 on enumeration of the cells in the patch $\mathcal{T}_{\boldsymbol{a}}$. We state it as assumption to cover as general as possible geometrical situations but note that it can be immediately verified in most cases of practical interest. We also note that two of the three properties below are only requested in the H^1 setting; in particular property (vi) seems to request that the interior of the Dirichlet set $\mathcal{F}^{\mathrm{D}}_{\boldsymbol{a}}$ is connected and that, when both $\mathcal{F}^{\mathrm{D}}_{\boldsymbol{a}}$ and $\mathcal{F}^{\mathrm{N}}_{\boldsymbol{a}}$ are non-empty, the enumeration is started in a simplex $K \in \mathcal{T}_{\boldsymbol{a}}$ containing a Dirichlet face from $\mathcal{F}^{\mathrm{D}}_{\boldsymbol{a}}$, with a neighbor containing a Neumann face from $\mathcal{F}^{\mathrm{N}}_{\boldsymbol{a}}$.

Assumption C.1 (Boundary patch enumeration)**.** *There exists an enumeration of the patch $\mathcal{T}_{\boldsymbol{a}}$ so that*

(i) For all $1 < i \leq |\mathcal{T}_{\boldsymbol{a}}|$, if there are at least two faces in $\mathcal{F}^{\sharp}_i$, intersecting in an edge e, then this edge lies in the interior of the patch $\omega_{\boldsymbol{a}}$, i.e., if $|\mathcal{F}^{\sharp}_i| \geq 2$, then $\mathcal{F}_e \cap \mathcal{F}^{\mathrm{N}}_{\boldsymbol{a}} \cap \mathcal{F}^{\mathrm{D}}_{\boldsymbol{a}} = \emptyset$.

(ii) (H^1 setting only) For all $1 < i \leq |\mathcal{T}_{\boldsymbol{a}}|$, if there are at least two faces in $\mathcal{F}_i^\sharp$ or if there is at least one face in $\mathcal{F}_i^\sharp$ and at least one face in $\mathcal{F}_{K_i} \cap \mathcal{F}_{\boldsymbol{a}}^{\mathrm{D}}$, intersecting in an edge, then all the elements sharing this edge come sooner in the enumeration, i.e., if $|\mathcal{F}_i^\sharp| + |\mathcal{F}_{K_i} \cap \mathcal{F}_{\boldsymbol{a}}^{\mathrm{D}}| \geq 2$ with $\{F_i^1, F_i^2\} \subset \mathcal{F}_i^\sharp \cup (\mathcal{F}_{K_i} \cap \mathcal{F}_{\boldsymbol{a}}^{\mathrm{D}})$, then for all $e := F_i^1 \cap F_i^2$, $K_j \in \mathcal{T}_e \setminus \{K_i\}$ implies that $j < i$.

(iii) For all $1 < i < |\mathcal{T}_{\boldsymbol{a}}|$, there are one or two neighbors of K_i which have been already enumerated and one or two neighbors of K_i which have not been enumerated yet, i.e., $|\mathcal{F}_i^\sharp| \in \{1,2\}$ and $|\mathcal{F}_i^\flat| \in \{1,2\}$, for all but the first and the last element. In particular, $\mathcal{F}_i^\flat$ is empty if and only if $i = |\mathcal{T}_{\boldsymbol{a}}|$.

(iv) (H^1 setting only) The last element contains at least one (Neumann) face from $\mathcal{F}_{\boldsymbol{a}}^{\mathrm{N}}$ if $\mathcal{F}_{\boldsymbol{a}}^{\mathrm{N}}$ is not empty, i.e., $\mathcal{F}_{\boldsymbol{a}}^{\mathrm{N}} \neq \emptyset$ implies $\mathcal{F}_{K_{|\mathcal{T}_{\boldsymbol{a}}|}} \cap \mathcal{F}_{\boldsymbol{a}}^{\mathrm{N}} \neq \emptyset$.

(v) ($\boldsymbol{H}(\mathrm{div})$ setting only) The last element contains at least one (Dirichlet) face from $\mathcal{F}_{\boldsymbol{a}}^{\mathrm{D}}$ if $\mathcal{F}_{\boldsymbol{a}}^{\mathrm{D}}$ is not empty, i.e., $\mathcal{F}_{\boldsymbol{a}}^{\mathrm{D}} \neq \emptyset$ implies $\mathcal{F}_{K_{|\mathcal{T}_{\boldsymbol{a}}|}} \cap \mathcal{F}_{\boldsymbol{a}}^{\mathrm{D}} \neq \emptyset$.

(vi) (H^1 setting only) For all edges e having $\boldsymbol{a}$ as vertex, all the elements sharing this edge and containing a (Dirichlet) face from the set $\mathcal{F}_{\boldsymbol{a}}^{\mathrm{D}}$ come sooner in the enumeration than elements sharing e and containing a (Neumann) face from the set $\mathcal{F}_{\boldsymbol{a}}^{\mathrm{N}}$, i.e., for all $e \in \mathcal{E}_{\boldsymbol{a}}$, if $K_j \in \mathcal{T}_e$ is such that $\mathcal{F}_{K_j} \cap \mathcal{F}_{\boldsymbol{a}}^{\mathrm{D}} \neq \emptyset$ and $K_i \in \mathcal{T}_e$, $K_i \neq K_j$, is such that $\mathcal{F}_{K_i} \cap \mathcal{F}_{\boldsymbol{a}}^{\mathrm{N}} \neq \emptyset$, than $j < i$.

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