

# Integrals Based on Monotone Measure: Optimization Tools and Special Functionals

Radko Mesiar

► **To cite this version:**

Radko Mesiar. Integrals Based on Monotone Measure: Optimization Tools and Special Functionals. Khalid Saeed; Wladyslaw Homenda. 14th Computer Information Systems and Industrial Management (CISIM), Sep 2015, Warsaw, Poland. Springer, Lecture Notes in Computer Science, LNCS-9339, pp.48-57, 2015, Computer Information Systems and Industrial Management. <10.1007/978-3-319-24369-6\_4>. <hal-01444500>

**HAL Id: hal-01444500**

**<https://hal.inria.fr/hal-01444500>**

Submitted on 24 Jan 2017

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



# Integrals based on monotone measure: optimization tools and special functionals

Radko Mesiar

Slovak University of Technology, Bratislava, Slovakia  
radko.mesiar@stuba.sk

**Abstract.** Integrals on finite spaces (e.g., sets of criteria in multicriteria decision support) based on capacities are discussed, axiomatized and exemplified. We introduce first the universal integrals, covering the Choquet, Shilkret and Sugeno integrals. Based on optimization approach, we discuss decomposition and superdecomposition integrals. We introduce integrals which are universal and decomposition (superdecomposition) ones and integrals constructed by means of copulas. Several distinguished integrals are represented as particular functionals. Finally, we recall also OWA operators and some their generalizations.

**Keywords:** Capacity; Choquet integral; Copula; Copula-based integral; Decomposition integral; Sugeno integral; Universal integral

## 1 Introduction

Though integrals are, in general, introduced on general measurable spaces, in many computer-supported applications the finite spaces are considered, such as the sets of criteria, sets of rules, sets of players, etc. Therefore, in this paper we will consider universes  $X_n = \{1, \dots, n\}$ ,  $n \in \mathbb{N}$ , and  $\sigma$ -algebras  $2^{X_n}$ . More, we restrict our considerations into functions  $f : X_n \rightarrow [0, 1]$ , i.e., we will consider  $n$ -dimensional vectors  $\mathbf{x} = (x_1, \dots, x_n) = (f(1), \dots, f(n)) \in [0, 1]^n$  as our integrands. Note that these integrands can be seen as membership functions of fuzzy subsets of  $X_n$ , too [24]. The information about the weights of subsets of  $X_n$  (e.g., groups of players, groups of criteria, etc.) is condensed into a capacity (fuzzy measure)  $m : 2^{X_n} \rightarrow [0, 1]$ , which is supposed to be monotone and  $m(\emptyset) = 0$ ,  $m(X_n) = 1$ . Our overview of integrals on  $X_n$ , i.e., of discrete integrals is organized as follows. In Section 2, we bring the concept of universal integrals originally introduced in [8]. Section 3 is devoted to the decomposition integrals of Even and Lehrer [3] and the superdecomposition integrals of Mesiar et al [13]. Here we describe also all integrals which are simultaneously universal and decomposition (superdecomposition). In Section 4, we introduce several copula-based universal integrals. Section 5 brings the characterization of some distinguished integrals as particular functionals. Finally, some concluding remarks are added, recalling OWA operators and some of their generalizations.

## 2 Universal integrals

Though universal integrals were introduced in [8] for any measurable space  $(X, \mathcal{A})$ , any monotone measure  $m$  and all  $\mathcal{A}$ -measurable functions  $f: X \rightarrow [0, \infty]$ , for our purposes we constrain them to act on finite spaces, and we will consider capacities and membership functions of fuzzy sets. For  $n \in \mathbb{N}$ , let  $\mathcal{M}_n$  denote the set of all capacities on  $X_n$ .

**Definition 1.** A mapping  $I: \bigcup_{n=1}^{\infty} \mathcal{M}_n \times [0, 1]^n \rightarrow [0, 1]$  is called a (discrete) universal integral whenever

- (i) there is a semicopula  $\otimes: [0, 1]^2 \rightarrow [0, 1]$  (i.e., an increasing binary function on  $[0, 1]$  with neutral element  $e = 1$ ) such that for any  $n \in \mathbb{N}$ ,  $m \in \mathcal{M}_n$  and  $\mathbf{x} = c \cdot 1_A \in [0, 1]^n$ ,  $c \in [0, 1]$ , it holds that

$$I(m, \mathbf{x}) = c \otimes m(A),$$

- (ii) for any  $(m_i, \mathbf{x}_i) \in \mathcal{M}_{n_i} \times [0, 1]^{n_i}$ ,  $i = 1, 2$ , such that for any  $t \in [0, 1]$ ,
- $$m_1(\{i \in X_{n_1} \mid x_i^{(1)} \geq t\}) \leq m_2(\{j \in X_{n_2} \mid x_j^{(2)} \geq t\})$$
- it holds that

$$I(m_1, \mathbf{x}_1) \leq I(m_2, \mathbf{x}_2).$$

Note that  $I$  is then increasing in both coordinates and  $I(m_1, \mathbf{x}_1) = I(m_2, \mathbf{x}_2)$  whenever  $m_1(\{i \in X_{n_1} \mid x_i^{(1)} \geq t\}) = m_2(\{j \in X_{n_2} \mid x_j^{(2)} \geq t\})$  for each  $t \in [0, 1]$  (compare the equality of the expected values of random variables with the same distribution function). Moreover,  $I(m, 1_A) = m(A)$  and  $I(m, c \cdot 1_{X_n}) = c$  for all  $m \in \mathcal{M}_n$ ,  $A \subseteq X_n$  and  $c \in [0, 1]$ .

We recall several distinguished examples:

- For an arbitrary semicopula  $\otimes: [0, 1]^2 \rightarrow [0, 1]$ , the smallest universal integral that is linked to  $\otimes$ , is given, for any  $(m, \mathbf{x}) \in \mathcal{M}_n \times [0, 1]^n$ , by

$$\begin{aligned} I_{\otimes}(m, \mathbf{x}) &= \sup \{x_i \otimes m(\{j \in X_n \mid x_j \geq x_i\}) \mid i \in X_n\} \\ &= \sup \{x_{\sigma(i)} \otimes m(\{\sigma(i), \dots, \sigma(n)\})\}, \end{aligned} \quad (1)$$

where  $\sigma: X_n \rightarrow X_n$  is an arbitrary permutation such that  $x_{\sigma(1)} \leq \dots \leq x_{\sigma(n)}$ . In particular, if  $\otimes = \wedge$  (minimum) then  $I_{\wedge} = Su$  is the Sugeno integral [19], if  $\otimes = \cdot$  (product) then  $I_{\cdot} = Sh$  is the Shilkret integral [18], and if  $\otimes = T$  is a strict norm, then  $I_T$  is the Weber integral [20], compare also  $N$ -integral of Zhao [23].

- The Choquet integral [1] that is given by

$$Ch(m, \mathbf{x}) = \sum_{i=1}^n (x_{\sigma(i)} - x_{\sigma(i-1)}) \cdot m(\{\sigma(i), \dots, \sigma(n)\}) \quad (2)$$

( $x_{\sigma(0)} = 0$  by convention), is a universal integral linked to the product.

- The arithmetic mean of integrals  $Ch$  and  $Su$ ,  $I = \frac{1}{2}(Ch + Su)$ , is also a universal integral, and it is linked to the semicopula  $\otimes: [0, 1]^2 \rightarrow [0, 1]$ ,  $a \otimes b = \frac{1}{2}(ab + \min\{a, b\})$ .

Note that the class of all discrete universal integrals is a convex bounded partially ordered set. Its bottom  $I_{T_D}$  is linked to the drastic product t-norm  $T_D$  (the smallest semicopula), and, for all  $(m, \mathbf{x}) \in \mathcal{M}_n \times [0, 1]^n$ , is given by

$$I_{T_D}(m, \mathbf{x}) = \max \left\{ \max \{x_i \mid m(\{j \in X_n \mid x_j \geq x_i\}) = 1\}, m(\{i \in X_n \mid x_i = 1\}) \right\}.$$

Note that if  $m(A) < 1$  whenever  $A \neq X_n$ , then

$$I_{T_D}(m, \mathbf{x}) = \max \left\{ \min \{x_i \mid i \in X_n\}, m(\{i \in X_n \mid x_i = 1\}) \right\}.$$

On the other side, the top universal integral  $I^\wedge$  is linked to the minimum  $\wedge$  (which is the greatest semicopula) and given by

$$\begin{aligned} I^\wedge(m, \mathbf{x}) &= \text{essup}_m(\mathbf{x}) \wedge m(\text{Supp}(\mathbf{x})) \\ &= \min \left\{ \max \{x_i \mid m(\{j \in X_n \mid x_j \geq x_i\}) > 0\}, m(\{i \in X_n \mid x_i > 0\}) \right\}. \end{aligned}$$

Several other kinds of universal integrals will be discussed in Sections 3 and 4.

### 3 Decomposition and superdecomposition integrals

For  $X_n = \{1, \dots, n\}$ , any non-empty subset  $B \subseteq 2^{X_n}$  is called a collection, and any non-empty set  $\mathcal{H} \subseteq 2^{2^{X_n} \setminus \{\emptyset\}}$  of collections is called a decomposition system. Denote by  $\mathbb{X}_n$  the set of all decomposition systems.

We recall several examples of decomposition systems:

- $\mathcal{H}_i = \{(A_1, \dots, A_i) \mid (A_1, \dots, A_i) \text{ is a chain}\}, i = 1, \dots, n;$
- $\mathcal{H}^i = \{(A_1, \dots, A_i) \mid (A_1, \dots, A_i) \text{ is a disjoint system of subsets}\}, i = 1, \dots, n;$
- $\mathcal{H}^* = 2^{X_n} \setminus \{\emptyset\};$
- $\mathcal{H}_A = \{B \subseteq X_n \mid A \subseteq B\}.$

Recently, Even and Lehrer [3] introduced decomposition integrals based on the idea of decomposition systems. These integrals can be viewed as a modification of the idea of the lower integral (inner measure). Similarly, the upper integral (outer measure) inspired the concept of superdecomposition integrals [13].

**Definition 2.** For a given capacity  $m: 2^{X_n} \rightarrow [0, 1]$  and a fixed decomposition system  $\mathcal{H} \in \mathbb{X}_n$ , the corresponding decomposition integral  $I_{\mathcal{H}, m}: [0, 1]^n \rightarrow [0, \infty[$  is given by

$$I_{\mathcal{H}, m}(\mathbf{x}) = \sup \left\{ \sum_{j \in J} a_j m(A_j) \mid (A_j)_{j \in J} \in \mathcal{H}, a_j \geq 0, j \in J, \sum_{j \in J} a_j 1_{A_j} \leq \mathbf{x} \right\}, \quad (3)$$

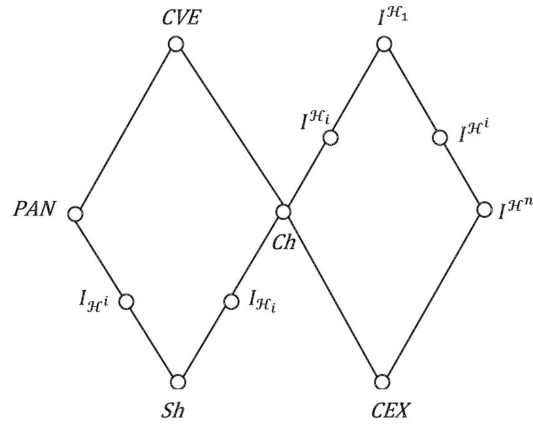
and the corresponding superdecomposition integral  $I^{\mathcal{H}, m}: [0, 1]^n \rightarrow [0, \infty[$  is given by

$$I^{\mathcal{H}, m}(\mathbf{x}) = \inf \left\{ \sum_{j \in J} a_j m(A_j) \mid (A_j)_{j \in J} \in \mathcal{H}, a_j \geq 0, j \in J, \sum_{j \in J} a_j 1_{A_j} \geq \mathbf{x} \right\}. \quad (4)$$

Note that:

- $I_{\mathcal{H}_1} = I_{\mathcal{H}^1} = Sh$  is the Shilkret integral [18];
- $I_{\mathcal{H}_n} = I^{\mathcal{H}_n} = Ch$  is the Choquet integral [1];
- $I_{\mathcal{H}^n} = PAN$  is the PAN integral of Yang [22];
- $I_{\mathcal{H}^*} = CVE$  is the concave integral of Lehrer [11];
- $I^{\mathcal{H}^*} = CEX$  is the convex integral introduced in [13];
- $I_{\mathcal{H}_{X_n}} = Min$ ;
- $I^{\mathcal{H}_{X_n}} = Max$ .

The relationships between these integrals are visualized in Figure 1.



**Fig. 1.** Hasse diagram of some decomposition and superdecomposition integrals

*Example 1.*

Let  $n = 3$  and let  $m$  be a capacity on  $X_3 = \{1, 2, 3\}$  given by  $m(\{1\}) = 0.3$ ,  $m(\{2\}) = 0.4$ ,  $m(\{3\}) = 0.6$ ,  $m(\{1, 2\}) = m(\{2, 3\}) = 0.7$ ,  $m(\{1, 3\}) = 0.6$  (and obviously,  $m(X) = 1$ ). Consider the score vector  $\mathbf{x} = (0.7, 0.5, 0.4)$ . Then:

- $I_{\mathcal{H}_1, m}(\mathbf{x}) = I_{\mathcal{H}^1, m}(\mathbf{x}) = 0.4$  (Shilkret integral);
- $I_{\mathcal{H}_2, m}(\mathbf{x}) = 0.49$ ;
- $I_{\mathcal{H}_3, m}(\mathbf{x}) = I^{\mathcal{H}_3, m}(\mathbf{x}) = 0.53$  (Choquet integral);

- $I_{\mathcal{H}^2, m}(\mathbf{x}) = 0.59$ ;
- $I_{\mathcal{H}^3, m}(\mathbf{x}) = 0.65$  (PAN integral);
- $I_{\mathcal{H}^*, m}(\mathbf{x}) = 0.65$  (concave integral);
- $I^{\mathcal{H}^*}, m(\mathbf{x}) = 0.49$  (convex integral);
- $I_{\mathcal{H}_3^*}, m(\mathbf{x}) = 0.4$ ;
- $I^{\mathcal{H}_3^*}, m(\mathbf{x}) = 0.7$ .

Decomposition and superdecomposition integrals are positively homogeneous. Hence, if a universal integral  $I$  is also a decomposition (superdecomposition) integral on each  $X_n$ ,  $n \in \mathbb{N}$ , then it is necessarily linked to the product  $\cdot$ ,  $I(m, c \cdot 1_A) = c \cdot m(A)$ . The integrals, which are both universal and decomposition integrals or universal and superdecomposition integrals, were characterized by Mesiar and Stupňanová in [14]. Note that we will use the same notation  $\mathcal{H}_i$ ,  $i \in \mathbb{N}$ , for decomposition systems related to the chains of length at most  $i$ , independently of the underlying space  $X_n$ . Obviously, for a fixed  $n \in \mathbb{N}$ , then  $\mathcal{H}_n = \mathcal{H}_{n+k}$  for each  $k \in \mathbb{N}$ .

**Theorem 1.** *Let  $I$  be a universal integral,  $I \neq Ch$ , which is also a decomposition (superdecomposition) integral on each  $X_n$ ,  $n \in \mathbb{N}$ . Then  $I = I_{\mathcal{H}_k}$  for some  $k \in \mathbb{N}$  ( $I = I^{\mathcal{H}_k}$  for some  $k \in \mathbb{N}$ ).*

Recall that  $I_{\mathcal{H}_1} = Sh$  is the Shilkret integral. Moreover, for a fixed  $n \in \mathbb{N}$ ,  $I_{\mathcal{H}_n} = Ch$  is the Choquet integral on  $X_n$ , and then

$$I_{\mathcal{H}_n} = I_{\mathcal{H}_{n+1}} = I_{\mathcal{H}_k} \text{ for each } k \geq n.$$

Similarly,

$$I^{\mathcal{H}_n} = Ch = I^{\mathcal{H}_k} \text{ for each } k \geq n.$$

The family  $(I_{\mathcal{H}_n})_{n \in \mathbb{N}} \cup (I^{\mathcal{H}_n})_{n \in \mathbb{N}}$  forms a chain of universal integrals

$$I_{\mathcal{H}_1} \leq I_{\mathcal{H}_2} \leq \dots \leq I_{\mathcal{H}_k} \leq \dots \leq Ch \leq \dots \leq I^{\mathcal{H}_k} \leq \dots \leq I^{\mathcal{H}_2} \leq I^{\mathcal{H}_1},$$

$$Ch = \sup_{k \in \mathbb{N}} I_{\mathcal{H}_k} = \inf_{k \in \mathbb{N}} I^{\mathcal{H}_k}.$$

Note also that the only integral on  $X_n$ , which is both decomposition and superdecomposition integral, is the Choquet integral.

## 4 Copula-based universal integrals

Copulas (of dimension 2) are in fact joint distribution functions (restricted to  $[0, 1]^2$ ) of random vectors  $(X, Y)$  such that both  $X$  and  $Y$  are uniformly distributed over  $[0, 1]$ . Two extremal copulas are the minimum  $M$ ,  $M(x, y) = \min\{x, y\}$  and the Fréchet-Hoeffding lower bound  $W$  given by  $W(x, y) = \max\{0, x + y - 1\}$ . The third distinguished copula is the independence copula  $\Pi$ ,  $\Pi(x, y) = x \cdot y$ . For more details see [16]. Note that any copula  $C: [0, 1]^2 \rightarrow [0, 1]$  is in a one-to-one correspondence with a probability measure  $P_C: \mathcal{B}([0, 1]^2) \rightarrow [0, 1]$  with uniformly distributed margins. In particular, for a Borel subset  $E \subseteq [0, 1]^2$  we have:

- $P_M(E) = \mu_1(\{x \in [0, 1] \mid (x, x) \in E\})$ ,
- $P_W(E) = \mu_1(\{x \in [0, 1] \mid (x, 1-x) \in E\})$ ,
- $P_\Pi(E) = \mu_2(E)$ ,

where  $\mu_i$  is the standard Lebesgue measure on Borel subsets of  $\mathbb{R}^i$ ,  $i = 1, 2$ . Inspired by Imaoka [7], Klement et al. [6] introduced copula-based integrals  $I_C: \bigcup_{n \in \mathbb{N}} \mathcal{M}_n \times [0, 1]^n \rightarrow [0, 1]$  given by

$$\begin{aligned} I_C(m, \mathbf{x}) &= P_C(\{(u, v) \in [0, 1]^2 \mid v \leq m(\{i \in X_n \mid x_i \geq u\})\}) \\ &= \sum_{i=1}^n (C(x_{\sigma(i)}, m(\{\sigma(i), \dots, \sigma(n)\})) - C(x_{\sigma(i-1)}, m(\{\sigma(i), \dots, \sigma(n)\}))), \end{aligned} \tag{5}$$

$\sigma$  being a permutation ensuring  $x_{\sigma(1)} \leq \dots \leq x_{\sigma(n)}$ .

Note that  $I_\Pi = Ch$  is the Choquet integral and  $I_M = Su$  is the Sugeno integral, and that  $I_C$  is a universal integral linked to  $\otimes = C$ . Recently, we introduced [9] hierarchical families of copula-based universal integrals, which extend the results from Section 4.

**Theorem 2.** *Let  $C: [0, 1]^2 \rightarrow [0, 1]$  be a copula and fix  $k \in \mathbb{N}$ . Then the mappings  $I_{C,k}$  and  $I^{C,k}: \bigcup_{n \in \mathbb{N}} \mathcal{M}_n \times [0, 1]^n \rightarrow [0, 1]$ , given by*

$$I_{C,k}(m, \mathbf{x}) = \sup \left\{ \sum_{j=1}^k (C(x_{\sigma(i_j)}, m(A_{i_j})) - C(x_{\sigma(i_{j-1})}, m(A_{i_j}))) \right\},$$

where  $A_{i_j} = \{\sigma(i_j), \dots, \sigma(n)\}$  and  $0 = i_0 < i_1 < \dots < i_k \leq n$ ,

and

$$I^{C,k}(m, \mathbf{x}) = \inf \left\{ \sum_{j=1}^k (C(x_{\sigma(i_j)}, m(\tilde{A}_{i_{j-1}})) - C(x_{\sigma(i_{j-1})}, m(\tilde{A}_{i_{j-1}}))) \right\},$$

where  $\tilde{A}_{i_{j-1}} = \{\sigma(i_{j-1}) + 1, \dots, \sigma(n)\}$  and  $0 = i_0 < i_1 < \dots < i_k \leq n$ , with convention  $\sigma(i_0) = 0$ , are universal integrals linked to  $\otimes = C$ . Moreover,

$$I_{C,1} \leq I_{C,2} \leq \dots \leq I_C \leq \dots \leq I^{C,2} \leq I^{C,1},$$

and, for a fixed  $n \in \mathbb{N}$ ,

$$I_{C,n} = I_C = I^{C,n}.$$

Obviously, if  $C = \Pi$ , then  $I_{\Pi,k} = I_{\mathcal{H}_k}$  and  $I^{\Pi,k} = I^{\mathcal{H}_k}$ ,  $k \in \mathbb{N}$ .

## 5 Integrals as special functionals

Recall that the discrete Lebesgue integral in our framework is just the weighted arithmetic mean,  $L(\mathbf{x}) = \sum_{i=1}^n w_i x_i$ , and it is related to the probability measure  $P$  on  $X_n$ , where  $P(\{i\}) = w_i$ .  $L$  can be seen as an additive aggregation function. Note that  $A : [0, 1]^n \rightarrow [0, 1]$  is an aggregation function [5] if and only if  $A$  is nondecreasing in each coordinate and  $A(0, \dots, 0) = 0$ ,  $A(1, \dots, 1) = 1$ . Hence an aggregation function is the Lebesgue integral once  $A(\mathbf{x} + \mathbf{y}) = A(\mathbf{x}) + A(\mathbf{y})$  for all  $\mathbf{x}, \mathbf{y}, \mathbf{x} + \mathbf{y} \in [0, 1]^n$ , and the corresponding capacity (probability)  $P$  is given by  $P(E) = A(1_E)$ . For several other distinguished integrals we have their characterization as special functionals:

- Choquet integral is a comonotone additive aggregation function [17];
- Sugeno integral is a comonotone maxitive and min-homogeneous aggregation function [12];
- Shilkret integral is a comonotone maxitive and positively homogeneous aggregation function [5];
- concave integral is the smallest concave functional on  $[0, 1]^n$  satisfying  $I(1_E) \geq m(E)$  [3];
- convex integral is the greatest convex functional on  $[0, 1]^n$  such that  $I(1_E) \leq m(E)$  [13].

Recall that two vectors  $\mathbf{x}, \mathbf{y} \in [0, 1]^n$  are comonotone if  $x_i > x_j$  excludes  $y_i < y_j$ , i.e.,  $(x_i - x_j)(y_i - y_j) \geq 0$  for each  $i, j \in X_n$ .

## 6 Concluding remarks

We have introduced and discussed several kinds of discrete integrals with respect to capacities. As a particular example, often exploited in numerous applications, we recall OWA operators [21] as the mappings  $OWA_{\mathbf{w}} : [0, 1]^n \rightarrow [0, 1]$ , which are defined by

$$OWA_{\mathbf{w}}(\mathbf{x}) = \sum_{i=1}^n w_i x_{\sigma(i)},$$

where  $\mathbf{w} = (w_1, \dots, w_n) \in [0, 1]^n$  is a normed weighting vector, i.e.,  $\sum_{i=1}^n w_i = 1$ .

Considering a capacity  $m : 2^{X_n} \rightarrow [0, 1]$  given by  $m(A) = \sum_{i=1}^{\text{card}(A)} w_{n-i+1}$ , Grabisch [4] showed that  $OWA_{\mathbf{w}} = Ch_m$  is the Choquet integral with respect to the capacity  $m$ . A capacity, which only depends on the cardinality of measured sets, is called a symmetric capacity. It can be viewed as a basis for generalizations of OWA operators. The first generalization yields OMA (Ordered Modular Average) operators, which were introduced by Mesiar and Zemánková in [15]:

**Definition 3.** Let  $m : 2^{X_n} \rightarrow [0, 1]$  be a symmetric capacity generated by a normed weighting vector  $\mathbf{w} = (w_1, \dots, w_n)$ , and let  $C : [0, 1]^2 \rightarrow [0, 1]$  be a fixed copula. Then the integral  $I_C(m, \cdot) : [0, 1]^n \rightarrow [0, 1]$  is called an OMA operator.



Note that OMA operators were characterized axiomatically as comonotone modular symmetric idempotent aggregation functions, i.e., an  $OMA: [0, 1]^n \rightarrow [0, 1]$  satisfies the following properties:

- (i)  $OMA(\mathbf{x} \vee \mathbf{y}) + OMA(\mathbf{x} \wedge \mathbf{y}) = OMA(\mathbf{x}) + OMA(\mathbf{y})$  for any comonotone couple  $\mathbf{x}, \mathbf{y} \in [0, 1]^n$  (i.e., for  $\mathbf{x}$  and  $\mathbf{y}$  with the property  $(x_i - x_j)(y_i - y_j) \geq 0$  for any  $i, j \in X_n$ );
- (ii)  $OMA(\mathbf{x}) = OMA(x_{\sigma(1)}, \dots, x_{\sigma(n)})$  for any permutation  $\sigma: X_n \rightarrow X_n$ ;
- (iii)  $OMA(c, \dots, c) = c$  for any  $c \in [0, 1]$ .

By (5),  $OMA_{C,m}$  is given by

$$OMA_{C,m}(\mathbf{x}) = \sum_{i=1}^n f_i(x_{\sigma(i)}),$$

where  $f_i: [0, 1] \rightarrow [0, 1]$  is given by  $f_i(t) = C(t, w_i + \dots + w_n) - C(t, w_{i+1} + \dots + w_n)$ . Thus,  $f_i$  is an increasing 1-Lipschitz function that satisfies  $f_i(0) = 0$ ,  $f_i(1) = w_i$ , and  $\sum_{i=1}^n f_i(t) = t$  for any  $t \in [0, 1]$ .

If  $C = \Pi$  then  $f_i(t) = w_i t$ , and in that case OMA operators are simply OWA operators (and they are characterized by comonotone additivity, which is a genuine property of the Choquet integrals [17]).

For  $C = M$ ,  $f_i(t) = \max\{0, \min\{t - (w_{i+1} + \dots + w_n), w_i\}\}$ , and hence

$$OMA_{M,m}(\mathbf{x}) = \bigvee_{i=1}^n x_{\sigma(i)} \wedge (w_i + \dots + w_n),$$

i.e.,  $OMA_{M,m}$  is the ordered weighted maximum [2].

Other kinds of OWA generalizations are based on Theorem 2, and again, they can be introduced for an arbitrary copula  $C$ . We restrict our considerations to the independence copula  $\Pi$  only. Then, for a fixed  $n \in \mathbb{N}$  and a symmetric capacity  $m$  on  $X_n$  generated by a normed weighting vector  $\mathbf{w} = (w_1, \dots, w_n)$ , all integrals  $I_{\Pi,k}(m, \cdot)$  and  $I^{\Pi,k}(m, \cdot)$ ,  $k \in \{1, \dots, n\}$ , can be viewed as generalizations of OWA operators and

$$I_{\Pi,n}(m, \cdot) = I^{\Pi,n}(m, \cdot) = OWA_{\mathbf{w}}.$$

In general, for a fixed  $k \in \{1, \dots, n\}$ , it holds

$$I_{\Pi,k}(m, \mathbf{x}) = \max \left\{ \sum_{j=1}^k (x_{\sigma(i_j)} - x_{\sigma(i_{j-1})}) \cdot (w_{i_j} + \dots + w_n) \right\},$$

where  $0 = i_0 < i_1 < \dots < i_k \leq n$ ,

and

$$I^{\Pi,k}(m, \mathbf{x}) = \min \left\{ \sum_{j=1}^k (x_{\sigma(i_j)} - x_{\sigma(i_{j-1})}) \cdot (w_{i_{j-1}+1} + \dots + w_n) \right\},$$

with  $0 = i_0 < i_1 < \dots < i_k \leq n$ .

**Acknowledgment.** The support of grant APVV-0013-14 is kindly acknowledged.

## References

1. G. Choquet, Theory of capacities. *Ann. Inst. Fourier* **5** (1953-54) 131–292.
2. D. Dubois, H. Prade, A review of fuzzy aggregation connectives. *Inform. Sci.* **36** (1985) 85–121.
3. Y. Even, E. Lehrer, Decomposition-integral: unifying Choquet and concave integrals. *Econ. Theory* **56**(1) (2014) 33–58.
4. M. Grabisch, Fuzzy integral in multicriteria decision making. *Fuzzy Sets and Systems* **69** (1995) 279–298.
5. M. Grabisch, J.-L. Marichal, R. Mesiar, E. Pap, *Aggregation Functions*. Cambridge University Press, Cambridge, 2009.
6. E.P. Klement, R. Mesiar, E. Pap, Measure-based aggregation operators. *Fuzzy Sets and Systems* **142** (2004) 3–14.
7. H. Imaoka, On a subjective evaluation model by a generalized fuzzy integral. *Int. J. Uncertain. Fuzziness Knowledge-Based Systems* **5** (1997) 517–529.
8. E.P. Klement, R. Mesiar, E. Pap, A universal integral as common frame for Choquet and Sugeno integral. *IEEE Trans. Fuzzy Systems* **18** (2010) 178–187.
9. E.P. Klement, R. Mesiar, F. Spizzichino, A. Stupňanová, Universal integrals based on copulas. *Fuzzy Optimization and Decision Making* **13** (2014) 273–286.
10. A. Kolesárová, A. Stupňanová, J. Beganová, Aggregation-based extensions of fuzzy measures. *Fuzzy Sets and Systems* **194** (2012) 1–14.
11. E. Lehrer, A new integral for capacities. *Econ. Theory* **39** (2009) 157–176.
12. J.-L. Marichal, An axiomatic approach of the discrete Sugeno integral as a tool to aggregate interacting criteria in a qualitative framework. *IEEE Transactions on Fuzzy Systems* **9**(1) (2001) 164–172.
13. R. Mesiar, J. Li, E. Pap, Superdecomposition integrals. *Fuzzy Sets and Systems* **259** (2015) 3–11.
14. R. Mesiar, A. Stupňanová, Decomposition integrals. *Int. J. Approximate Reasoning* **54**(8) (2013) 1252–1259.
15. R. Mesiar, A. Mesiarová-Zemánková, The Ordered Modular Averages. *IEEE Transactions on Fuzzy Systems* **19**(1) (2011) 42–50.
16. R.B. Nelsen, *An Introduction to Copulas*. Second Edition. Springer Verlag, New York, 2006.
17. D. Schmeidler, Integral representation without additivity. *Proc. Amer. Math. Soc.* **97** (1986) 255–261.
18. N. Shilkret, Maxitive measure and integration. *Indag. Math.* **33** (1971) 109–116.
19. M. Sugeno, *Theory of Fuzzy Integrals and its Applications*. PhD thesis, Tokyo Institute of Technology, 1974.
20. S. Weber, Two integrals and some modified versions - critical remarks. *Fuzzy Sets and Systems* **20** (1986) 97–105.
21. R.R. Yager, On ordered weighted averaging aggregation operators in multi-criteria decision making. *IEEE Transactions on Systems, Man and Cybernetics* **18** (1988) 183–190.
22. Q. Yang, The pan-integral on fuzzy measure space (in Chinese). *Fuzzy Mathematics* **3** (1985) 107–114.
23. R. Zhao, (N) fuzzy integral. *J. Mathematical Research and Exposition* **2** (1981) 55–72.
24. L.A. Zadeh, Fuzzy Sets. *Information Control* **8** (1965) 338–353.