Integrals Based on Monotone Measure: Optimization Tools and Special Functionals

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Abstract. Integrals on finite spaces (e.g., sets of criteria in multicriteria decision support) based on capacities are discussed, axiomatized and examplified. We introduce first the universal integrals, covering the Choquet, Shilkret and Sugeno integrals. Based on optimization approach, we discuss decomposition and superdecomposition integrals. We introduce integrals which are universal and decomposition (superdecomposition) ones and integrals constructed by means of copulas. Several distinguished integrals are represented as particular functionals. Finally, we recall also OWA operators and some their generalizations.

Keywords: Capacity; Choquet integral; Copula; Copula–based integral; Decomposition integral; Sugeno integral; Universal integral

1 Introduction

Though integrals are, in general, introduced on general measurable spaces, in many computer–supported applications the finite spaces are considered, such as the sets of criteria, sets of rules, sets of players, etc. Therefore, in this paper we will consider universes $X_n = \{1,...,n\}$, $n \in N$, and σ–algebras 2^{X_n}. More, we restrict our considerations into functions $f : X_n \to [0,1]$, i.e., we will consider n–dimensional vectors $\mathbf{x} = (x_1,...,x_n) = (f(1),...,f(n)) \in [0,1]^n$ as our integrands. Note that these integrands can be seen as membership functions of fuzzy subsets of X_n, too [24]. The information about the weights of subsets of X_n (e.g., groups of players, groups of criteria, etc.) is condensed into a capacity (fuzzy measure) $m : 2^{X_n} \to [0,1]$, which is supported to be monotone and $m(\emptyset) = 0$, $m(X_n) = 1$. Our overview of integrals on X_n, i.e., of discrete integrals is organized as follows. In Section 2, we bring the concept of universal integrals originally introduced in [8]. Section 3 is devoted to the decomposition integrals of Even and Lehrer [3] and the superdecomposition integrals of Mesiar et al [13]. Here we describe also all integrals which are simultaneously universal and decomposition (superdecomposition). In Section 4, we introduce several copula–based universal integrals. Section 5 brings the characterization of some distinguished integrals as particular functionals. Finally, some concluding remarks are added, recalling OWA operators and some of their generalizations.

2 Universal integrals

Though universal integrals were introduced in [8] for any measurable space $(X,\mathcal{A})$, any monotone measure m and all $\mathcal{A}$-measurable functions $f\colon X\to[0,\infty]$, for our purposes we constrain them to act on finite spaces, and we will consider capacities and membership functions of fuzzy sets. For $n\in\mathbb{N}$, let $\mathcal{M}_n$ denote the set of all capacities on X_n.

Definition 1. *A mapping* $I\colon \bigcup_{n=1}^{\infty}\mathcal{M}_n\times[0,1]^n\to[0,1]$ *is called a (discrete) universal integral whenever*

(i) *there is a semicopula* $\otimes\colon[0,1]^2\to[0,1]$ *(i.e., an increasing binary function on* $[0,1]$ *with neutral element* $e=1$*) such that for any* $n\in\mathbb{N}$*,* $m\in\mathcal{M}_n$ *and* $\mathbf{x}=c\cdot 1_A\in[0,1]^n$*,* $c\in[0,1]$*, it holds that*
$$I(m,\mathbf{x})=c\otimes m(A),$$
(ii) *for any* $(m_i,\mathbf{x}_i)\in\mathcal{M}_{n_i}\times[0,1]^{n_i}$*,* $i=1,2$*, such that for any* $t\in[0,1]$*,* $m_1\left(\{i\in X_{n_1}\mid x_i^{(1)}\geq t\}\right)\leq m_2\left(\{j\in X_{n_2}\mid x_j^{(2)}\geq t\}\right)$ *it holds that*
$$I(m_1,\mathbf{x}_1)\leq I(m_2,\mathbf{x}_2).$$

Note that I is then increasing in both coordinates and $I(m_1,\mathbf{x}_1)=I(m_2,\mathbf{x}_2)$ whenever $m_1\left(\{i\in X_{n_1}\mid x_i^{(1)}\geq t\}\right)=m_2\left(\{j\in X_{n_2}\mid x_j^{(2)}\geq t\}\right)$ for each $t\in[0,1]$ (compare the equality of the expected values of random variables with the same distribution function). Moreover, $I(m,1_A)=m(A)$ and $I(m,c\cdot 1_{X_n})=c$ for all $m\in\mathcal{M}_n$, $A\subseteq X_n$ and $c\in[0,1]$. We recall several distinguished examples:

- For an arbitrary semicopula $\otimes\colon[0,1]^2\to[0,1]$, the smallest universal integral that is linked to $\otimes$, is given, for any $(m,\mathbf{x})\in\mathcal{M}_n\times[0,1]^n$, by
$$\begin{aligned}I_{\otimes}(m,\mathbf{x})&=\sup\left\{x_i\otimes m\left(\left\{j\in X_n\mid x_j\geq x_i\right\}\right)\mid i\in X_n\right\}\\&=\sup\left\{x_{\sigma(i)}\otimes m\left(\{\sigma(i),\ldots,\sigma(n)\}\right)\right\},\end{aligned}\tag{1}$$
where $\sigma\colon X_n\to X_n$ is an arbitrary permutation such that $x_{\sigma(1)}\leq\ldots\leq x_{\sigma(n)}$. In particular, if $\otimes=\wedge$ (minimum) then $I_{\wedge}=Su$ is the Sugeno integral [19], if $\otimes=\cdot$ (product) then $I_{\cdot}=Sh$ is the Shilkret integral [18], and if $\otimes=T$ is a strict norm, then I_T is the Weber integral [20], compare also N-integral of Zhao [23].
- The Choquet integral [1] that is given by
$$Ch(m,\mathbf{x})=\sum_{i=1}^{n}\left(x_{\sigma(i)}-x_{\sigma(i-1)}\right)\cdot m\left(\{\sigma(i),\ldots,\sigma(n)\}\right)\tag{2}$$
($x_{\sigma(0)}=0$ by convention), is a universal integral linked to the product.
- The arithmetic mean of integrals Ch and Su, $I=\frac{1}{2}(Ch+Su)$, is also a universal integral, and it is linked to the semicopula $\otimes\colon[0,1]^2\to[0,1]$, $a\otimes b=\frac{1}{2}(ab+\min\{a,b\})$.

Note that the class of all discrete universal integrals is a convex bounded partially ordered set. Its bottom I_{T_D} is linked to the drastic product t-norm T_D (the smallest semicopula), and, for all $(m,\mathbf{x}) \in \mathcal{M}_n \times [0,1]^n$, is given by

$$I_{T_D}(m,\mathbf{x}) = \max\left\{\max\left\{x_i \mid m\left(\left\{j \in X_n \mid x_j \geq x_i\right\}\right) = 1\right\}, m(\{i \in X_n \mid x_i = 1\})\right\}.$$

Note that if $m(A) < 1$ whenever $A \neq X_n$, then

$$I_{T_D}(m,\mathbf{x}) = \max\left\{\min\left\{x_i \mid i \in X_n\right\}, m(\{i \in X_n \mid x_i = 1\})\right\}.$$

On the other side, the top universal integral $I^{\wedge}$ is linked to the minimum $\wedge$ (which is the greatest semicopula) and given by

$$\begin{aligned} I^{\wedge}(m,\mathbf{x}) &= \mathrm{essup}_m(\mathbf{x}) \wedge m(\mathrm{Supp}(\mathbf{x})) \\ &= \min\left\{\max\left\{x_i \mid m\left(\left\{j \in X_n \mid x_j \geq x_i\right\}\right) > 0\right\}, m(\{i \in X_n \mid x_i > 0\})\right\}. \end{aligned}$$

Several other kinds of universal integrals will be discussed in Sections 3 and 4.

3 Decomposition and superdecomposition integrals

For $X_n = \{1,\ldots,n\}$, any non-empty subset $B \subseteq 2^{X_n}$ is called a collection, and any non-empty set $\mathcal{H} \subseteq 2^{2^{X_n}\setminus\{\emptyset\}}$ of collections is called a decomposition system. Denote by $\mathbb{X}_n$ the set of all decomposition systems.

We recall several examples of decomposition systems:

- $\mathcal{H}_i = \{(A_1,\ldots,A_i) \mid (A_1,\ldots,A_i) \textit{ is a chain}\}$, $i = 1,\ldots,n$;
- $\mathcal{H}^i = \{(A_1,\ldots,A_i) \mid (A_1,\ldots,A_i) \textit{ is a disjoint system of subsets}\}$, $i = 1,\ldots,n$;
- $\mathcal{H}^* = 2^{X_n} \setminus \{\emptyset\}$;
- $\mathcal{H}_A = \{B \subseteq X_n \mid A \subseteq B\}$.

Recently, Even and Lehrer [3] introduced decomposition integrals based on the idea of decomposition systems. These integrals can be viewed as a modification of the idea of the lower integral (inner measure). Similarly, the upper integral (outer measure) inspired the concept of superdecomposition integrals [13].

Definition 2. *For a given capacity $m\colon 2^{X_n} \to [0,1]$ and a fixed decomposition system $\mathcal{H} \in \mathbb{X}_n$, the corresponding decomposition integral $I_{\mathcal{H},m}\colon [0,1]^n \to [0,\infty[$ is given by*

$$I_{\mathcal{H},m}(\mathbf{x}) = \sup\left\{\sum_{j\in J} a_j m(A_j) \mid (A_j)_{j\in J} \in \mathcal{H},\, a_j \geq 0,\, j \in J,\, \sum_{j\in J} a_j 1_{A_j} \leq \mathbf{x}\right\}, \quad (3)$$

and the corresponding superdecomposition integral $I^{\mathcal{H},m}\colon [0,1]^n \to [0,\infty[$ is given by

$$I^{\mathcal{H},m}(\mathbf{x}) = \inf\left\{\sum_{j\in J} a_j m(A_j) \mid (A_j)_{j\in J} \in \mathcal{H},\, a_j \geq 0,\, j \in J,\, \sum_{j\in J} a_j 1_{A_j} \geq \mathbf{x}\right\}. \quad (4)$$

Note that:

- $I_{\mathcal{H}_1} = I_{\mathcal{H}^1} = Sh$ is the Shilkret integral [18];
- $I_{\mathcal{H}_n} = I^{\mathcal{H}_n} = Ch$ is the Choquet integral [1];
- $I_{\mathcal{H}^n} = PAN$ is the PAN integral of Yang [22];
- $I_{\mathcal{H}^*} = CVE$ is the concave integral of Lehrer [11];
- $I^{\mathcal{H}^*} = CEX$ is the convex integral introduced in [13];
- $I_{\mathcal{H}_{X_n}} = Min$;
- $I^{\mathcal{H}_{X_n}} = Max$.

The relationships between these integrals are visualized in Figure 1.

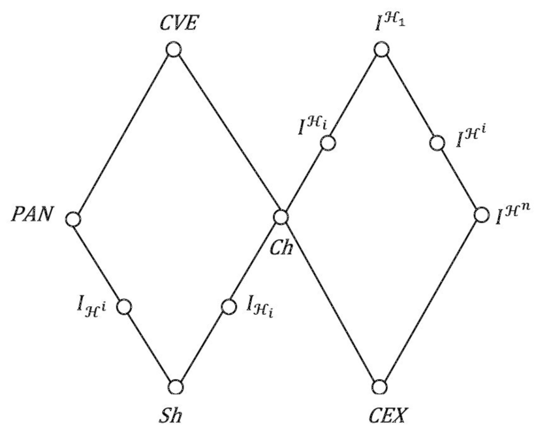

Fig. 1. Hasse diagram of some decomposition and superdecomposition integrals

Example 1.

Let $n = 3$ and let m be a capacity on $X_3 = \{1,2,3\}$ given by $m(\{1\}) = 0.3$, $m(\{2\}) = 0.4$, $m(\{3\}) = 0.6$, $m(\{1,2\}) = m(\{2,3\}) = 0.7$, $m(\{1,3\}) = 0.6$ (and obviously, $m(X) = 1$). Consider the score vector $\mathbf{x} = (0.7, 0.5, 0.4)$. Then:

- $I_{\mathcal{H}_1,m}(\mathbf{x}) = I_{\mathcal{H}^1,m}(\mathbf{x}) = 0.4$ (Shilkret integral);
- $I_{\mathcal{H}_2,m}(\mathbf{x}) = 0.49$;
- $I_{\mathcal{H}_3,m}(\mathbf{x}) = I^{\mathcal{H}_3,m}(\mathbf{x}) = 0.53$ (Choquet integral);

- $I_{\mathcal{H}^2,m}(\mathbf{x}) = 0.59$;
- $I_{\mathcal{H}^3,m}(\mathbf{x}) = 0.65$ (PAN integral);
- $I_{\mathcal{H}^*,m}(\mathbf{x}) = 0.65$ (concave integral);
- $I^{\mathcal{H}^*,m}(\mathbf{x}) = 0.49$ (convex integral);
- $I_{\mathcal{H}_{X_3},m}(\mathbf{x}) = 0.4$;
- $I^{\mathcal{H}_{X_3},m}(\mathbf{x}) = 0.7$.

Decomposition and superdecomposition integrals are positively homogeneous. Hence, if a universal integral I is also a decomposition (superdecomposition) integral on each X_n, $n \in \mathbb{N}$, then it is necessarily linked to the product $\cdot$, $I(m, c \cdot 1_A) = c \cdot m(A)$. The integrals, which are both universal and decomposition integrals or universal and superdecomposition integrals, were characterized by Mesiar and Stupňanová in [14]. Note that we will use the same notation $\mathcal{H}_i$, $i \in \mathbb{N}$, for decomposition systems related to the chains of length at most i, independently of the underlying space X_n. Obviously, for a fixed $n \in \mathbb{N}$, then $\mathcal{H}_n = \mathcal{H}_{n+k}$ for each $k \in \mathbb{N}$.

Theorem 1. *Let I be a universal integral, $I \neq Ch$, which is also a decomposition (superdecomposition) integral on each X_n, $n \in \mathbb{N}$. Then $I = I_{\mathcal{H}_k}$ for some $k \in \mathbb{N}$ ($I = I^{\mathcal{H}_k}$ for some $k \in \mathbb{N}$).*

Recall that $I_{\mathcal{H}_1} = Sh$ is the Shilkret integral. Moreover, for a fixed $n \in \mathbb{N}$, $I_{\mathcal{H}_n} = Ch$ is the Choquet integral on X_n, and then

$$I_{\mathcal{H}_n} = I_{\mathcal{H}_{n+1}} = I_{\mathcal{H}_k} \text{ for each } k \geq n.$$

Similarly,

$$I^{\mathcal{H}_n} = Ch = I^{\mathcal{H}_k} \text{ for each } k \geq n.$$

The family $\left(I_{\mathcal{H}_n}\right)_{n\in\mathbb{N}} \bigcup \left(I^{\mathcal{H}_n}\right)_{n\in\mathbb{N}}$ forms a chain of universal integrals

$$I_{\mathcal{H}_1} \leq I_{\mathcal{H}_2} \leq \ldots \leq I_{\mathcal{H}_k} \leq \ldots \leq Ch \leq \ldots \leq I^{\mathcal{H}_k} \leq \ldots \leq I^{\mathcal{H}_2} \leq I^{\mathcal{H}_1},$$

$$Ch = \sup_{k\in\mathbb{N}} I_{\mathcal{H}_k} = \inf_{k\in\mathbb{N}} I^{\mathcal{H}_k}.$$

Note also that the only integral on X_n, which is both decomposition and superdecomposition integral, is the Choquet integral.

4 Copula-based universal integrals

Copulas (of dimension 2) are in fact joint distribution functions (restricted to $[0,1]^2$) of random vectors (X,Y) such that both X and Y are uniformly distributed over $[0,1]$. Two extremal copulas are the minimum M, $M(x,y) = \min\{x,y\}$ and the Fréchet-Hoeffding lower bound W given by $W(x,y) = \max\{0, x+y-1\}$. The third distinguished copula is the independence copula Π, $\Pi(x,y) = x \cdot y$. For more details see [16]. Note that any copula $C\colon [0,1]^2 \to [0,1]$ is in a one-to-one correspondence with a probability measure $P_C\colon \mathcal{B}([0,1]^2) \to [0,1]$ with uniformly distributed margins. In particular, for a Borel subset $E \subseteq [0,1]^2$ we have:

- $P_M(E) = \mu_1(\{x \in [0,1] \mid (x,x) \in E\})$,
- $P_W(E) = \mu_1(\{x \in [0,1] \mid (x,1-x) \in E\})$,
- $P_\Pi(E) = \mu_2(E)$,

where μ_i is the standard Lebesgue measure on Borel subsets of $\mathbb{R}^i$, $i = 1,2$. Inspired by Imaoka [7], Klement et al. [6] introduced copula-based integrals $I_C \colon \bigcup_{n\in\mathbb{N}} \mathcal{M}_n \times [0,1]^n \to [0,1]$ given by

$$\begin{aligned} I_C(m,\mathbf{x}) &= P_C\left(\left\{(u,v) \in [0,1]^2 \mid v \le m(\{i \in X_n \mid x_i \ge u\})\right\}\right) \\ &= \sum_{i=1}^{n} \left(C\left(x_{\sigma(i)}, m(\{\sigma(i),\ldots,\sigma(n)\})\right) - C\left(x_{\sigma(i-1)}, m(\{\sigma(i),\ldots,\sigma(n)\})\right)\right), \end{aligned} \tag{5}$$

σ being a permutation ensuring $x_{\sigma(1)} \le \ldots \le x_{\sigma(n)}$.

Note that $I_\Pi = Ch$ is the Choquet integral and $I_M = Su$ is the Sugeno integral, and that I_C is a universal integral linked to $\otimes = C$. Recently, we introduced [9] hierarchical families of copula-based universal integrals, which extend the results from Section 4.

Theorem 2. *Let $C\colon [0,1]^2 \to [0,1]$ be a copula and fix $k \in \mathbb{N}$. Then the mappings $I_{C,k}$ and $I^{C,k}\colon \bigcup_{n\in\mathbb{N}} \mathcal{M}_n \times [0,1]^n \to [0,1]$, given by*

$$I_{C,k}(m,\mathbf{x}) = \sup\left\{\sum_{j=1}^{k}\left(C\left(x_{\sigma(i_j)}, m\left(A_{i_j}\right)\right) - C\left(x_{\sigma(i_{j-1})}, m\left(A_{i_j}\right)\right)\right)\right\},$$

where $A_{i_j} = \left\{\sigma(i_j),\ldots,\sigma(n)\right\}$ and $0 = i_0 < i_1 < \ldots < i_k \le n$,

and

$$I^{C,k}(m,\mathbf{x}) = \inf\left\{\sum_{j=1}^{k}\left(C\left(x_{\sigma(i_j)}, m\left(\tilde{A}_{i_{j-1}}\right)\right) - C\left(x_{\sigma(i_{j-1})}, m\left(\tilde{A}_{i_{j-1}}\right)\right)\right)\right\},$$

where $\tilde{A}_{i_{j-1}} = \left\{\sigma(i_{j-1})+1,\ldots,\sigma(n)\right\}$ and $0 = i_0 < i_1 < \ldots < i_k \le n$, with convention $\sigma(i_0) = 0$, are universal integrals linked to $\otimes = C$. Moreover,

$$I_{C,1} \le I_{C,2} \le \ldots \le I_C \le \ldots \le I^{C,2} \le I^{C,1},$$

and, for a fixed $n \in \mathbb{N}$,

$$I_{C,n} = I_C = I^{C,n}.$$

Obviously, if $C = \Pi$, then $I_{\Pi,k} = I_{\mathcal{H}_k}$ and $I^{\Pi,k} = I^{\mathcal{H}_k}$, $k \in \mathbb{N}$.

5 Integrals as special functionals

Recall that the discrete Lebesgue integral in our framework is just the weighted arithmetic mean, $L(\mathbf{x}) = \sum_{i=1}^{n} w_i x_i$, and it is related to the probability measure P on X_n, where $P(\{i\}) = w_i$. L can be seen as an additive aggregation function. Note that $A : [0,1]^n \to [0,1]$ is an aggregation function [5] if and only if A is nondecreasing in each coordinate and $A(0,...,0) = 0$, $A(1,...,1) = 1$. Hence an aggregation function is the Lebesgue integral once $A(\mathbf{x}+\mathbf{y}) = A(\mathbf{x}) + A(\mathbf{y})$ for all $\mathbf{x},\mathbf{y},\mathbf{x}+\mathbf{y} \in [0,1]^n$, and the corresponding capacity (probability) P is given by $P(E) = A(1_E)$. For several other distinguished integrals we have their characterization as special functionals:

- Choquet integral is a comonotone additive aggregation function [17];
- Sugeno integral is a comonotone maxitive and min–homogeneous aggregation function [12];
- Shilkret integral is a comonotone maxitive and positively homogeneous aggregation function [5];
- concave integral is the smallest concave functional on $[0,1]^n$ satisfying $I(1_E) \geq m(E)$ [3];
- convex integral is the greatest convex functional on $[0,1]^n$ such that $I(1_E) \leq m(E)$ [13].

Recall that two vectors $\mathbf{x},\mathbf{y} \in [0,1]^n$ are comonotone if $x_i > x_j$ excludes $y_i < y_j$, i.e., $(x_i - x_j)\,(y_i - y_j) \geq 0$ for each $i, j \in X_n$.

6 Concluding remarks

We have introduced and discussed several kinds of discrete integrals with respect to capacities. As a particular example, often exploited in numerous applications, we recall OWA operators [21] as the mappings $OWA_{\mathbf{w}} \colon [0,1]^n \to [0,1]$, which are defined by

$$OWA_{\mathbf{w}}(\mathbf{x}) = \sum_{i=1}^{n} w_i x_{\sigma(i)},$$

where $\mathbf{w} = (w_1, \ldots, w_n) \in [0,1]^n$ is a normed weighting vector, i.e., $\sum\limits_{i=1}^{n} w_i = 1$.

Considering a capacity $m \colon 2^{X_n} \to [0,1]$ given by $m(A) = \sum\limits_{i=1}^{\operatorname{card}(A)} w_{n-i+1}$, Grabisch [4] showed that $OWA_{\mathbf{w}} = Ch_m$ is the Choquet integral with respect to the capacity m. A capacity, which only depends on the cardinality of measured sets, is called a symmetric capacity. It can be viewed as a basis for generalizations of OWA operators. The first generalization yields OMA (Ordered Modular Average) operators, which were introduced by Mesiar and Zemánková in [15]:

Definition 3. *Let $m \colon 2^{X_n} \to [0,1]$ be a symmetric capacity generated by a normed weighting vector $\mathbf{w} = (w_1, \ldots, w_n)$, and let $C \colon [0,1]^2 \to [0,1]$ be a fixed copula. Then the integral $I_C(m,\cdot) \colon [0,1]^n \to [0,1]$ is called an OMA operator.*

Note that OMA operators were characterized axiomatically as comonotone modular symmetric idempotent aggregation functions, i.e., an $OMA\colon [0,1]^n \to [0,1]$ satisfies the following properties:

(i) $OMA(\mathbf{x}\vee\mathbf{y})+OMA(\mathbf{x}\wedge\mathbf{y}) = OMA(\mathbf{x})+OMA(\mathbf{y})$ for any comonotone couple $\mathbf{x},\mathbf{y} \in [0,1]^n$ (i.e., for $\mathbf{x}$ and $\mathbf{y}$ with the property $(x_i - x_j)(y_i - y_j) \geq 0$ for any $i, j \in X_n$);
(ii) $OMA(\mathbf{x}) = OMA(x_{\sigma(1)},\ldots,x_{\sigma(n)})$ for any permutation $\sigma\colon X_n \to X_n$;
(iii) $OMA(c,\ldots,c) = c$ for any $c \in [0,1]$.

By (5), $OMA_{C,m}$ is given by

$$OMA_{C,m}(\mathbf{x}) = \sum_{i=1}^{n} f_i(x_{\sigma(i)}),$$

where $f_i\colon [0,1] \to [0,1]$ is given by $f_i(t) = C(t, w_i + \ldots + w_n) - C(t, w_{i+1} + \ldots + w_n))$. Thus, f_i is an increasing 1-Lipschitz function that satisfies $f_i(0) = 0$, $f_i(1) = w_i$, and $\sum\limits_{i=1}^{n} f_i(t) = t$ for any $t \in [0,1]$.

If $C = \Pi$ then $f_i(t) = w_i t$, and in that case OMA operators are simply OWA operators (and they are characterized by comonotone additivity, which is a genuine property of the Choquet integrals [17]).

For $C = M$, $f_i(t) = \max\{0, \min\{t - (w_{i+1} + \ldots + w_n), w_i\}\}$, and hence

$$OMA_{M,m}(\mathbf{x}) = \bigvee_{i=1}^{n} x_{\sigma(i)} \wedge (w_i + \ldots + w_n),$$

i.e., $OMA_{M,m}$ is the ordered weighted maximum [2].

Other kinds of OWA generalizations are based on Theorem 2, and again, they can be introduced for an arbitrary copula C. We restrict our considerations to the independence copula Π only. Then, for a fixed $n \in \mathbb{N}$ and a symmetric capacity m on X_n generated by a normed weighting vector $\mathbf{w} = (w_1,\ldots,w_n)$, all integrals $I_{\Pi,k}(m,\cdot)$ and $I^{\Pi,k}(m,\cdot)$, $k \in \{1,\ldots,n\}$, can be viewed as generalizations of OWA operators and

$$I_{\Pi,n}(m,\cdot) = I^{\Pi,n}(m,\cdot) = OWA_{\mathbf{w}}.$$

In general, for a fixed $k \in \{1,\ldots,n\}$, it holds

$$I_{\Pi,k}(m,\mathbf{x}) = \max\left\{\sum_{j=1}^{k}\left(x_{\sigma(i_j)} - x_{\sigma(i_{j-1})}\right)\cdot(w_{i_j} + \ldots + w_n)\right\},$$

where $0 = i_0 < i_1 < \ldots < i_k \leq n$,
and

$$I^{\Pi,k}(m,\mathbf{x}) = \min\left\{\sum_{j=1}^{k}\left(x_{\sigma(i_j)} - x_{\sigma(i_{j-1})}\right)\cdot(w_{i_{j-1}+1} + \ldots + w_n)\right\},$$

with $0 = i_0 < i_1 < \ldots < i_k \leq n$.

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