Some Properties of Continuous Yao Graph

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Abstract. Given a set S of points in the plane and an angle $0 < \theta \leq 2\pi$, the *continuous Yao graph* $cY(\theta)$ with vertex set S and angle θ defined as follows. For each $p, q \in S$, we add an edge from p to q in $cY(\theta)$ if there exists a cone with apex p and angular diameter θ such that q is the closest point to p inside this cone.
In this paper, we prove that for $0 < \theta < \pi/3$ and $t \geq \frac{1}{1-2\sin(\theta/2)}$, the continuous Yao graph $cY(\theta)$ is a $\mathcal{C}$-fault-tolerant geometric t-spanner where $\mathcal{C}$ is the family of convex regions in the plane. Moreover, we show that for every $\theta \leq \pi$ and every half-plane h, $cY(\theta) \ominus h$ is connected, where $cY(\theta) \ominus h$ is the graph after removing all edges and points inside h from the graph $cY(\theta)$. Also, we show that there is a set of n points in the plane and a convex region C such that for every $\theta \geq \frac{\pi}{3}$, $cY(\theta) \ominus C$ is not connected.
Given a geometric network G and two vertices x and y of G, we call a path P from x to y a self-approaching path, if for any point q on P, when a point p moves continuously along the path from x to q, it always get closer to q. A geometric graph G is self-approaching, if for every pair of vertices x and y there exists a self-approaching path in G from x to y. In this paper, we show that there is a set P of n points in the plane such that for some angles θ, Yao graph on P with parameter θ is not a self-approaching graph. Instead, the corresponding continuous Yao graph on P is a self-approaching graph. Furthermore, in general, we show that for every $\theta > 0$, $cY(\theta)$ is not necessarily a self-approaching graph.

Keywords: t-spanner, Region-fault tolerant spanner, Continuous Yao graph, Self-approaching graph

1 Introduction

Let S be a set of n points in $\mathbb{R}^d$ and let $t \geq 1$ be a real number. A geometric graph is an edge-weighted graph on $S \subseteq \mathbb{R}^d$ such that the weight of each edge is the Euclidean distance between its endpoints. A geometric graph G with vertex set S is called a t-spanner for S, if for each two points p and q in S, there exists a path Q in G between p and q whose length is at most t times $|pq|$, the Euclidean distance between p and q. The length of a path is defined to be the sum of the lengths, or weight, of all edges on the path. The path Q is called a *t-spanner path* (or *t-path*) between p and q. We denote the length of path Q by $|Q|$. The *stretch factor* (or *dilation*) of G is the smallest value of t for which G is a t-spanner. The t-spanners were introduced by Peleg and Schäffer [14] in the scope of distributed computing and, then, by Chew [6] in the scope of computational geometry. The t-spanners are applicable in many scopes such as graph theory, network topology design, distributed systems, robotics. We refer the reader to [5,8,12,16] for reading about the t-spanners and their applications.

The problem of efficient construction of a t-spanner for a given point set and a constant $t > 1$ has been studied extensively. One can see the major algorithms for building spanners in the book by Narasimhan and Smid [13].

The Yao graph used by Andrew Yao to construct Euclidean minimum spanning tree on high-dimensional Euclidean space [17]. For a set S of points in the plane, the Yao graph Y_k, for $k \geq 2$, is defined as follows. At each point $u \in S$, we draw k cones with apex at u and angle $\frac{2\pi}{k}$. For each point $u \in S$ and cone C, we add an edge between u and the closest point to u in C. If one chooses k sufficiently large, the Yao graph becomes a t-spanner.

In 2014, Barba et al. [3] introduced the *continuous Yao graph* as a variation of Yao graph. The continuous Yao graph $cY(\theta)$ with vertex set S and angle θ defined as follows: For each $p, q \in S$, we add an edge from p to q in $cY(\theta)$ if there exists a *θ-cone*, a cone with aperture θ, with apex at p such that q is the closest point to p inside this θ-cone. In continuous Yao graph, for each $u \in S$, we rotate a θ-cone with apex u around u continuously, and connect u to the closest point inside the θ-cone during this rotation. They showed that $cY(\theta)$ has stretch factor at most $1/(1-2sin(\theta/4))$ for $0 < \theta < 2\pi/3$. Unlike Yao graphs that always have a linear number of edges for any constant k, in the worst case, continuous Yao graphs may have a quadratic number of edges. Since $cY(\theta) \subseteq cY(\gamma)$ for any $\theta \geq \gamma$, the continuous Yao graphs are useful in potential applications that require scalability. Moreover, when some rotations apply on the input point set the continuous Yao graphs are invariant [3].

One of the useful properties of a network is *fault tolerance* that is after one or more vertices or edges fail, the remaining graph is still a good network of alive vertices. In particular, a graph $G = (S, E)$ is called *k-vertex fault-tolerant t-spanner* [10] for S, denoted by (k, t)-VFTS for a given real number $t \geq 1$ and non-negative positive integer k, if for each set $S' \subseteq S$ with cardinality of at most k, the graph $G \backslash S'$ is a t-spanner for $S \backslash S'$. Also, G is called a *k-edge fault-tolerant t-spanner* [10] for S, denoted by (k, t)-EFTS, if for each set $E' \subseteq E$ with cardinality at most k and for each pair of points p and q in S, the graph $G \backslash E'$ contains a path P between p and q with $|P| \leq t|P_S|$ where P_S is the shortest path between p and q in the graph $K_S \backslash E'$ in which that K_S is Euclidean complete graph on S. Levcopoulos et al. [10] for the first time considered the problem of constructing fault-tolerant spanners in Euclidean spaces efficiently. They proposed three algorithms that construct k-vertex fault-tolerant spanners. Some other works on the fault tolerant spanners have been done [7,11].

In 2009, Abam et al. [1] introduced the concept of *region-fault tolerant spanner* for planar point sets. For a fault region F and a geometric graph G on a point set S, assume $G \ominus F$ is the remaining graph after removing the vertices of G that lie inside F and all edges that intersect F. For a set $\mathcal{F}$ of regions in the plane, an $\mathcal{F}$-fault tolerant t-spanner is a geometric graph G on S such that for any region $F \in \mathcal{F}$, the graph $G \ominus F$ is a t-spanner for $\mathcal{G}_c(S) \ominus F$, where $\mathcal{G}_c(S)$ is the complete geometric graph on S. They showed for any set of n points in the plane and any family $\mathcal{C}$ of convex regions, one can construct a $\mathcal{C}$-fault tolerant spanner of size $O(n \log n)$ in $O(n \log^2 n)$ time.

In 2013, Alamdari et al. [2] introduced the concept of *self-approaching* and *increasing-chord* graph drawings. The problem is that, we are given a graph and we need to check if the graph has an self-approaching or increasing-chord embedding in the Euclidean space.

A geometric graph is self-approaching, if for every pair s and t of vertices of the graph, there exists a self-approaching path from s to t, denoted by *st-path.* A path from s to t is a self-approaching path if for each q on the path, not only the vertices, but any place of the path, if a point p starts at s and moves toward q, it always get closer to q in its movement. Also, a graph G is called increasing-chord if, for each pair u and v of its vertices, there exists a path between u and v such that the path is self-approaching both from u to v and from v to u. Obviously, an increasing-chord graph is a self-approaching graph.

In the geometric context, the position of the vertices of the graph is fixed, so we just want to know whether a given geometric graph is self-approaching or increasing-chord. One of the interesting properties of these graph is the following. It is known that the stretch factor (or dilation) of any self-approaching graph is at most 5.3332 [9] and the stretch factor of any increasing-chord graph is at most 2.094 [15].

Our results. In this paper, we prove the following results. For the rest of the paper S denotes a set of n points in the plane.

1. The continuous Yao graph $cY(\theta)$ on S is a $\mathcal{C}$-fault-tolerant geometric t-spanner of S, where $0 < \theta < \pi/3$ and $t \geq \frac{1}{1-2\sin(\theta/2)}$.
2. For any $\theta \leq \pi$ the graph $cY(\theta) \ominus h$ on S is connected, where h is an arbitrary half-plane in the plane.
3. There is a set of n points in the plane and a convex region C such that for every $\theta \geq \frac{\pi}{3}$, $cY(\theta) \ominus C$ is not connected.
4. There is a set P of n points in the plane such that for some angles θ, Yao graph on P with parameter θ is not a self-approaching graph. Instead, the corresponding continuous Yao graph on P is a self-approaching graph.
5. For every $\theta > 0$, $cY(\theta)$ is not necessarily a self-approaching graph.

2 $cY(\theta)$ is fault-tolerant

In this section, we show that the continuous Yao graph $cY(\theta)$ for $0 < \theta < \pi/3$ and $t \geq \frac{1}{1-2\sin(\theta/2)}$ is a $\mathcal{C}$-fault-tolerant geometric t-spanner where $\mathcal{C}$ is the family of convex regions in the plane. Furthermore, we show that for every $\theta \leq \pi$ and every half-plane h, $cY(\theta) \ominus h$ is connected. Moreover, we show that there is a set of n points such that for a some convex region C, $cY(\theta) \ominus C$ is not connected for every $\theta \geq \frac{\pi}{3}$. We need the following lemmas.

Lemma 1 ([4]). *Let a, b and c be three points such that $|ac| \leq |ab|$ and $\angle bac \leq \alpha < \pi$. Then*

$$|bc| \leq |ab| - (1 - 2sin(\alpha/2))|ac|.$$

Lemma 2 ([1]). *A geometric graph G on S is a $\mathcal{C}$-fault tolerant t-spanner if and only if it is an $\mathcal{H}$-fault-tolerant t-spanner, where $\mathcal{H}$ is the family of all half-planes.*

Now, we prove the following theorem:

Theorem 1. *Let θ be a real number with $0 < \theta < \pi/3$ and let t be a real number with $t \geq \frac{1}{1-2\sin(\theta/2)}$. For any point set S, the continuous Yao graph $cY(\theta)$ is a $\mathcal{C}$-fault-tolerant geometric t-spanner.*

Proof. By Lemma 2, it is sufficient to prove that $cY(\theta)$ is an $\mathcal{H}$-fault-tolerant geometric t-spanner.

Let h be an arbitrary half-plane in $\mathcal{H}$. We must show that for each pair of points $p, q \in S$ outside h, there is a t-path between p and q in $cY(\theta) \ominus h$. The proof is by induction on the rank of distance $|pq|$.

For the base step, suppose that the pair p and q is the closest pair in $cY(\theta) \ominus h$. Without loss of generality, assume the distance between p and h is less than of equal to the distance between q and h. Since p and q are outside h and $\theta < \pi/2$, there is a θ-cone C_p with apex at p completely outside h such that $q \in C_p$. Since pair p and q is the closest pair, by the construction of $cY(\theta)$, the edge (p, q) must be in $cY(\theta) \ominus h$. Note that if both p and q have same distance to h, then we can choose both p and q as apex, but we have to choose the point such that the corresponding θ-cone does not intersect h.

For the induction hypothesis step, suppose for each pair $u, v \in S$ outside h with $|uv| < |pq|$ there is a t-path between u and v connecting them in $cY(\theta) \ominus h$.

Now we consider the induction step. Since p and q are outside h, there is a θ-cone C_p with apex at p such that $q \in C_p$ (here we assumed that p is closer than q to h).

Suppose that r is the closest point to p inside the cone C_p (see Fig. 1). Since $\theta < \pi/3$, $1 - 2\sin(\theta/2) > 0$, and also since $|pr| \leq |pq|$, by Lemma 1 we have $|rq| < |pq|$. Therefore, by the induction hypothesis, there is a t-path Q between r and q in $cY(\theta) \ominus h$. Now consider the path $P := \{(p, r)\} \cup Q$. Clearly the path P connects p and q, and P is in $cY(\theta) \ominus h$. Now by Lemma 1, we have

$$\begin{aligned}
|P| &= |pr| + |Q| \\
&\leq |pr| + t|rq| \\
&\leq |pr| + t\left(|pq| - (1 - 2\sin(\theta/2))|pr|\right) \\
&= t|pq| + (1 - t(1 - 2\sin(\theta/2)))\,|pr| \\
&\leq t|pq| \qquad \left(\text{since } t \geq \frac{1}{1 - 2\sin(\theta/2)}\right).
\end{aligned}$$

Thus P is a t-path in $cY(\theta) \ominus h$ between p and q. This completes the proof. □

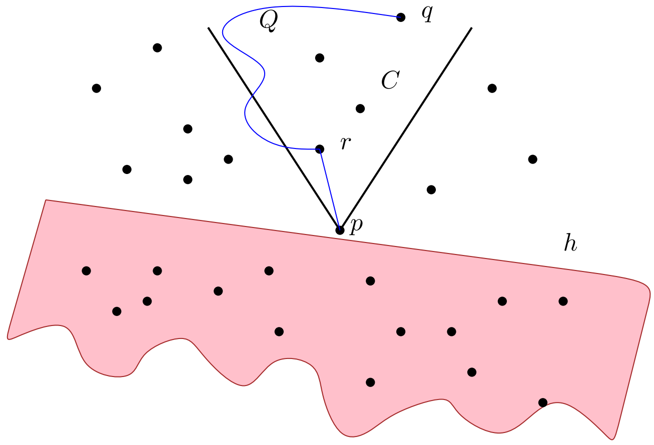

Fig. 1. Illustrating of the proof of Theorem 1.

Note that even though $cY(\theta)$ is a $\mathcal{C}$-fault tolerant spanner, it is not necessarily connected. It is because in fault-tolerant spanners, we compare the graph with the complete graph. So if the complete graph after removing points in some regions becomes disconnected, then some of fault-tolerant spanners of the points set might be disconnected. In the following, we show that for every half-plane h in the plane, the graph $cY(\theta) \ominus h$ is a connected graph for every $\theta \leq \pi$.

Lemma 3. *The continuous Yao graph $cY(\pi)$ on each set S contains $CH(S)$, the convex hull of S.*

Proof. Proof is straightforward. □

Theorem 2. *For any $\theta \leq \pi$ the graph $cY(\theta) \ominus h$ on S is connected, where h is a half-plane.*

Proof. It is easy to see that for every $\alpha, \beta > 0$, if $\alpha \geq \beta$ then $cY(\alpha) \subseteq cY(\beta)$. So, to prove the theorem, it is sufficient to show that $cY(\pi) \ominus h$ is connected for every half-plane h.

Let h be an arbitrary half-plane in the plane. Suppose that $cY(\pi) \ominus h$ is not connected.

Since $cY(\pi) \ominus h$ is disconnected, it has more than one connected component. At least one of the connected components of $cY(\pi) \ominus h$ contains a point from $CH(S)$. Suppose C and C' are two (distinct) connected components of $cY(\pi) \ominus h$ such that at least one of them, say C, contains a point from $CH(S)$. If both of C and C' contains a point from $CH(S)$ then by Lemma 3, part of $CH(S)$ which lies outside h is connected which is a contradiction because we assumed that C and C' are distinct connected components of $cY(\pi) \ominus h$.

Now, assume C' contains no vertices on $CH(S)$. Let x be the intersection of boundary of h and $CH(S)$ and let ℓ be a line through x and tangent to C' (see Fig. 2). We have two cases.

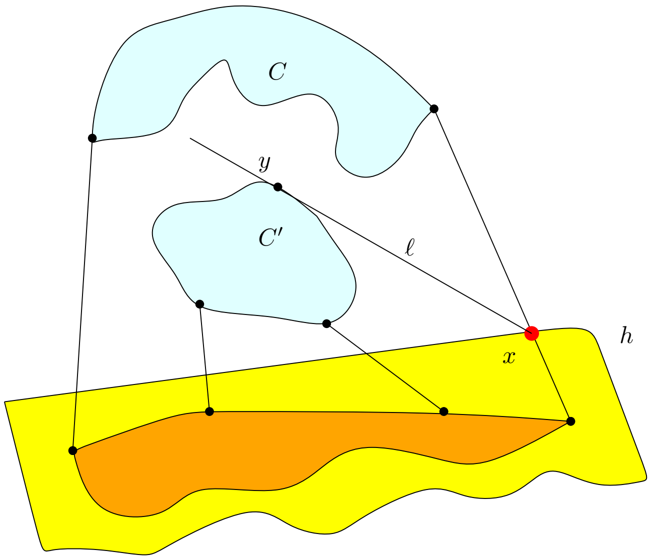

Fig. 2. Illustrating of the proof of Theorem 2: case 1.

Case 1: $\ell \cap C'$ contains exactly one point denoted by y.

If $x \notin S$ then clearly by the construction of graph $cY(\pi)$, the vertex y should be connected to a vertex of C that is contradiction, since we assumed that C and C' are distinct connected components of $cY(\pi) \ominus h$. Now, suppose that $x \in S$. Since $\ell \cap C'$ contains exactly one point, we can with continuously rotating the line ℓ around the point y (in Fig. 2, we rotate ℓ counter-clockwise), find a line ℓ' such that $\ell' \cap C'$ contains exactly one point that is y, and also the point in the intersection of ℓ' and boundary h does not belong to S. Hence, by the construction of graph $cY(\pi)$, point y should be connected to a vertex of C. That is a contradiction with that C and C' are distinct connected components of $cY(\pi) \ominus h$.

Case 2: ℓ is tangent to C' in at least two points.

We claim that there is a line ℓ' that is tangent to exactly one point of S on the convex hull of C' (denoted by s). Suppose that s be the farthest point of S on ℓ with respect to x, and suppose that r be the next point after s on the convex hull of C' in counterclockwise order (see Fig. 3). Let ℓ'' be a line that through of s and r, and let z be the intersection of ℓ'' and boundary of h. Now suppose that w be a point on the segment xz ($w \neq x, z$). Let ℓ' be a line that through of w and s. Clearly ℓ' is tangent to exactly one point of S that is s on the convex hull of C'. Now by the construction of $cY(\pi)$, vertex s should be connected to a vertex of connected component of C. This contradicts with the assumption that C and C' are distinct connected components of $cY(\pi) \ominus h$.

According to the above mentioned cases, $cY(\pi) \ominus h$ is connected. This completes the proof. □

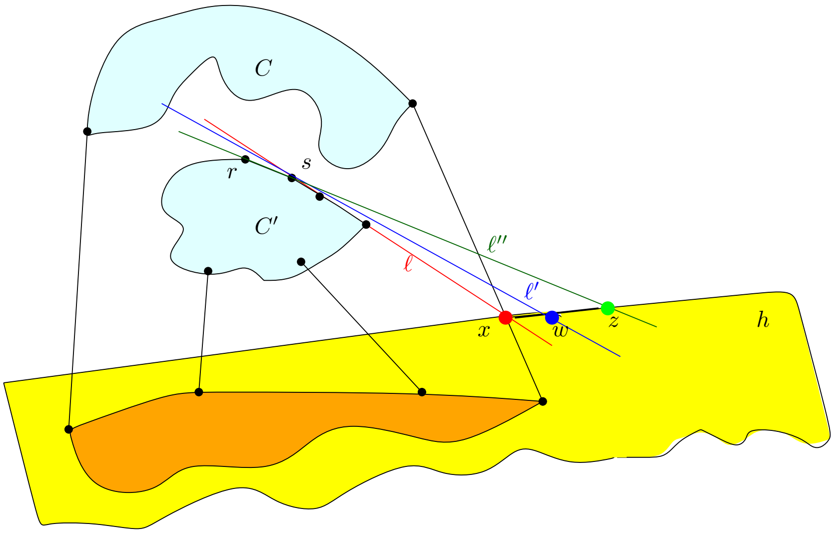

Fig. 3. Illustrating of the proof of Theorem 2: case 2.

In the following, we give an example such that for every $\theta \geq \frac{\pi}{3}$, $cY(\theta) \ominus C$ for some convex region C is not necessarily connected.

Assume $P_0 = \{(0,0),(1,\sqrt{3}),(2,0)\}$ which is the vertices of a equilateral triangle. Let P_i be the translation of P_0 by value $c \times i$ horizontally where c is a sufficiently large constant positive integer. Here, we consider $c = 10$ (see Fig. 4). We choose the set $P := \bigcup_{i=0}^{k-1} P_i$.

Let θ be an angle with $\theta \geq \frac{\pi}{3}$. Consider $cY(\theta)$ on P. Note that for $\theta \geq \frac{\pi}{3}$, $cY(\theta)$ on the vertices of an equilateral triangle is exactly the complete graph of their vertices. Now, let C be a convex region such that C only contains the points $(1,\sqrt{3})$, $(2,0)$, $(10,0)$ and $(11,\sqrt{3})$ from P (see Fig. 4). Since in the $cY(\theta)$ the vertex $(0,0)$ is only connected to the vertices $(1,\sqrt{3})$ and $(2,0)$, clearly in $cY(\theta) \ominus C$ there is no path between $(0,0)$ and $(12,0)$. Hence $cY(\theta) \ominus C$ is not connected.

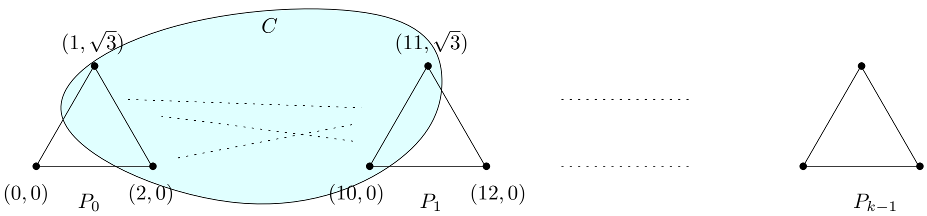

Fig. 4. An example that $cY(\theta)$ is disconnected after convex region fault.

3 $cY(\theta)$ is not self-approaching

In this section, we show that there is a set P of n points in the plane such that for some angles θ, Yao graph on P with parameter θ is not a self-approaching graph. Instead, the corresponding continuous Yao graph on P is a self-approaching graph. Finally, we show that there is a set of n points in the plane such that for every $\theta > 0$, continuous Yao graph $cY(\theta)$ is not a self-approaching graph.

In the following, we assume that the number of cones is even. With making some modifications, similar results holds for odd number of cones.

Now, let $p = (0,0), q = (1,0), r = (r_1, r_2)$ and $s = (s_1, s_2)$ be four points in the plane. For three points x, y and z, let $\angle yxz$ be the angle between the segment xy and xz. Now, assume that $\angle qpr = \alpha_1$, $\angle pqr = \alpha_2$, $\angle prq = \alpha_3$, $\angle qps = \beta_1$, $\angle pqs = \beta_2$, $\angle psq = \beta_3$, for some angles α_i and β_i with $0 < \alpha_i < \pi/2$ and $0 < \beta_i < \pi/2$ for $i = 1, 2, 3$ (see Fig. 5).

We call the quadrilateral on the four points p, q, r and s a *bad quadrilateral* if we have the following properties:

1. $|pq| \cos \alpha_1 < |pr| < |pq|$
2. $|pq| \cos \alpha_2 < |qr| < |pq|$
3. $|pq| \cos \beta_1 < |ps| < |pq|$
4. $|pq| \cos \beta_2 < |qs| < |pq|$

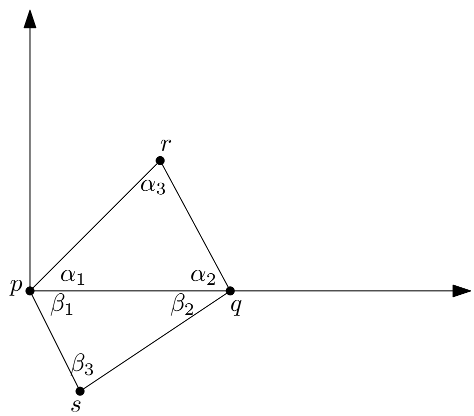

Fig. 5. A bad quadrilateral on the points p, q, r and s.

Now, we consider k cones, generated by the k-rays through origin, where the ith ray has angle $(i-1)\theta$ with the positive x-axis. The ith cones contains the points p such that the angle of line op with positive x-axis is bigger than or equal $(i-1)\theta$ and less than $i\theta$.

Let V be a set of four points $p = (0,0)$, $q = (1,0)$, $r = (r_1, r_2)$ and $s = (s_1, s_2)$ such that the quadrilateral on the vertices p, q, r and s is a bad quadrilateral. It is easy to see that, for every θ with $\alpha_1 < \theta \leq \frac{\pi}{2}$ such that $\frac{2\pi}{\theta}$ is even number and $\beta_2 < \theta$, the Yao graph Y_k on V where $k = \frac{2\pi}{\theta}$ does not contain the edge $\{p, q\}$ but it contains the edges $\{p, r\}, \{p, s\}, \{q, r\}$ and $\{q, s\}$. So the Yao graph Y_k on the point set V is not a self-approaching graph, since every path from p to q is not a pq-path. Indeed, the following lemma shows that $cY(\theta)$ on V is a self-approaching graph.

Lemma 4. *Let $\alpha_1 + \beta_1 \geq \frac{\pi}{2}$ or $\alpha_2 + \beta_2 \geq \frac{\pi}{2}$. For every θ such that*

a) $\alpha_1 < \theta \leq \frac{\pi}{2}$, and
b) $\theta < \alpha_1 + \beta_1$ or $\theta < \alpha_2 + \beta_2$, and
c) $\frac{2\pi}{\theta}$ is even number, and
d) $\beta_2 < \theta$,

continuous Yao graph $cY(\theta)$ on V contains the edge $\{p, q\}$. Furthermore, $cY(\theta)$ on V is a self-approaching graph.

Proof. We prove the theorem for the case that $\theta < \alpha_1 + \beta_1$. Similar argument works for the case $\theta < \alpha_2 + \beta_2$.

Since $\theta < \alpha_1 + \beta_1$, by construction of $cY(\theta)$, clearly there is a θ-cone C with apex p that only contains the point q. Hence the edge $\{p, q\}$ is in $cY(\theta)$.

On the other hand, the Yao graph on V with angle parameter θ is a subgraph of $cY(\theta)$, so the edges $\{p, r\}, \{p, s\}, \{q, r\}$ and $\{q, s\}$ are in $cY(\theta)$. So, we only need to show that there is a self-approaching path from r to s and a self-approaching path from s to r.

Since $\alpha_1 + \beta_1 \geq \frac{\pi}{2}$ or $\alpha_2 + \beta_2 \geq \frac{\pi}{2}$, the path $r \to q \to s$ or $r \to p \to s$ is an rs-path and in the reverse direction is an sr-path. So there is an xy-path between all ordered pairs $\{x, y\}$ for $x, y \in V$. Hence, $cY(\theta)$ is a self-approaching graph on V. This completes the proof. □

Next, we will give another point set such that the Yao graph on the point set is not self-approaching graph, but the continuous Yao graph is a self-approaching graph.

Let P be a set of n points as follows:

$$P = V \cup \{x_1, x_2, \ldots, x_{n-4}\}, \quad (1)$$

with $x_j = (c + j, 0)$ for $1 \leq j \leq n - 4$ where c is a sufficiently large constant positive integer. Here, we consider $c = 19$ (see Fig. 6).

Now, using Lemma 4 we can easily find the following result.

Lemma 5. *Let $\alpha_1 + \beta_1 \geq \frac{\pi}{2}$ or $\alpha_2 + \beta_2 \geq \frac{\pi}{2}$. For every θ such that*

a) $\alpha_1 < \theta \leq \frac{\pi}{2}$, and
b) $\theta < \alpha_1 + \beta_1$ or $\theta < \alpha_2 + \beta_2$, and
c) $\frac{2\pi}{\theta}$ is even number, and
d) $\beta_2 < \theta$,

continuous Yao graph $cY(\theta)$ on P is self-approaching and Y_k is not a self-approaching for $k = \frac{2\pi}{\theta}$.

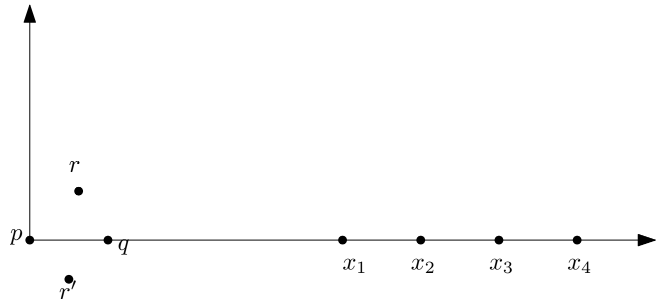

Fig. 6. The set P with $n = 8$.

Note that there are some point set that satisfy the conditions mentioned in above lemmas. For example, let $V = \{p, q, r, s\}$ with $p = (0, 0), q = (1, 0), r = (0.7839, 0.4422), s = (0.2161, -0.4422)$, and suppose that $k = 4$ and $\theta = \frac{\pi}{2}$. We can easily verify that the quadrilateral on the points of V is a bad quadrilateral. Moreover we have

$$\alpha_1 = 29.4271°, \alpha_2 = 63.9535°, \alpha_3 = 86.6198°,$$

$$\beta_1 = 29.4271°, \beta_2 = 63.9535°, \beta_3 = 86.6198°.$$

Now, suppose that $P = V \cup \{x_1, x_2, \ldots, x_{n-4}\}$ where $x_i = (19 + i, 0)$. This point set satisfies Lemma 5.

At the first view, it seems that the continuous Yao graph is a self-approaching graph, but this is not true in general. Next, we give a point set such that for each θ, the continuous Yao graph $cY(\theta)$ on the point set is not a self-approaching graph.

Theorem 3. *There is a set P of n points such that for every $\theta > 0$, $cY(\theta)$ on P is not a self-approaching graph.*

Proof. We prove the theorem for $0 < \theta \leq \frac{2\pi}{3}$. Since for every α_1 and α_2 with $\alpha_1 \leq \alpha_2$ we have $cY(\alpha_2) \subseteq cY(\alpha_1)$, the theorem holds for every $\theta > 0$.

Consider two points $p = (0, 0)$ and $q = (1, 0)$. Let C be a circle centered at the midpoint of the segment pq, with radius $\frac{1}{2}$. Let D_p and D_q be circles respectively centered at p and q with radius one. Let ℓ be the perpendicular bisector of segment pq (see Fig. 7).

Consider x and y as points outside C and inside $D_p \cap D_q$ such that $\angle xpq < \frac{\theta}{2}$ and $\angle ypq < \frac{\theta}{2}$. Let x' and y' be the symmetries of x and y with respect to line ℓ, respectively.

Now let $V = \{p, q, x, y, x', y'\}$, and consider $cY(\theta)$ on V. Since $\angle xpy = \angle x'qy' < \theta$, $cY(\theta)$ does not contain the edge $\{p, q\}$. So, by our selection of points x, y, x' and y', there is no self-approach path between p and q. Hence, $cY(\theta)$ on V is not self-approaching.

One can extend this point set to a point set with arbitrary number of points such that $cY(\theta)$ is not a self-approaching graph of the point set. To this end, one

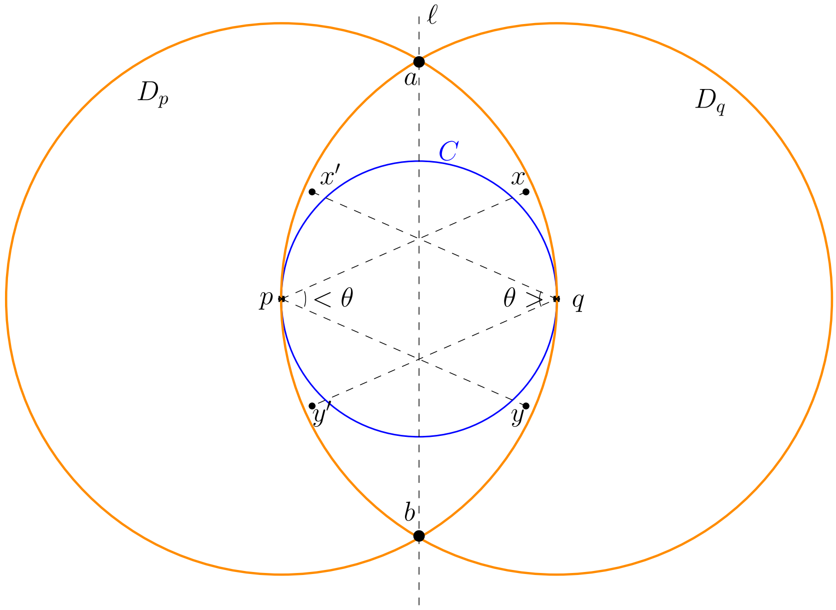

Fig. 7. Illustrating of the proof of Theorem 3.

can add any point to V which is sufficiently far from the points in the original set V, because adding points which are far can not help in making a self-approaching path from p to q. □

4 Concluding remarks

We proved that for $0 < \theta < \pi/3$ and $t \geq \frac{1}{1-2\sin(\theta/2)}$, the continuous Yao graph $cY(\theta)$ is a $\mathcal{C}$-fault-tolerant geometric t-spanner. Furthermore, we showed that for every $\theta \leq \pi$ and every half-plane h, $cY(\theta) \ominus h$ is connected. Finally, we showed that for every $\theta > 0$, $cY(\theta)$ is not necessarily a self-approaching graph. The question whether $cY(\theta)$ for $\frac{\pi}{3} \leq \theta \leq \pi$ is a $\mathcal{C}$-fault-tolerant geometric spanner with constant stretch factor remains open.

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